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An Energy Preserving Monolithic Eulerian Fluid-Structure Numerical Scheme *

Olivier Pironneau

Sorbonne Universités, UPMC (Paris VI)

Laboratoire Jacques-Louis Lions

Place Jussieu, Boite 187, Paris 75252, France

E-mail: Olivier.Pironneau@upmc.fr

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Abstract

The conservation laws of continuum mechanic written in an Eulerian frame make no difference between fluids and solids except in the expression of the stress tensors, usually with Newton’s hypothesis for the fluids and Helmholtz potentials of energy for hyperelastic solids. By taking the velocities as unknown, monolithic methods for fluid structure interactions (FSI) are built. In this article such a formulation is analyzed when the fluid is compressible and the fluid is incompressible. The idea is not new but the progress of mesh generators and numerical schemes like the Characteristics-Galerkin method render this approach feasible and reasonably robust. In this article the method and its discretization are presented, stability is discussed by through an energy estimate. A numerical section discusses implementation issues and presents a few simple tests.

AMS classification 65M60 (74F10 74S30 76D05 76M25).

Introduction

Currently two methods dominate FSI science: Arbitrary Lagrangian Eulerian methods especially for thin structures [28][32] and immersed boundary methods (IBM)[29][10], for which the mathematical analysis is more advanced[5] but the numerical implementations lack behind. ALE for large displacements have meshing difficulties [25] and to a lesser extent with the matching conditions at the fluid-solid interface[23]. Furthermore, iterative solvers for ALE-based FSI methods which rely on alternative solutions of the fluid and the structure parts are subject to the added mass effect and require special solvers[15][7].

Alternatives to ALE and IBM are few. One old method [2][3] has resurfaced recently, the so-called *actualized Lagrangian methods* for computing structures [22] [26] (see also [9] although different from the present study because it deals mostly with membranes).

*Written in honor of Philippe Ciarlet for his 80th birthday.

Continuum mechanics doesn't distinguish between solids and fluids till it comes to the constitutive equations. This has been exploited numerically in several studies but most often in the context of ALE[24][20][34].

In the present study, which is a follow up of [31] and [18], we investigate what Stephan Turek[20] Heil[19] and Wang[36] call a monolithic formulation but here in an Eulerian framework, as in [12][13][33][14], following the displaced geometry of the fluid and the solid. In [12] the authors obtained excellent results with the fully Eulerian formulation adopted here but at the cost of meshing difficulties to handled the Lagrangian derivatives. Here we advocated the Characteristic-Galerkin method and obtain an energy estimate which is not a proof of stability but a prerequisite for it.

1 Conservation Laws

Let the time dependent computational domain Ω_t be made of a fluid region Ω_t^f and a solid region Ω_t^s with no overlap: $\bar{\Omega}_t = \bar{\Omega}_t^f \cup \bar{\Omega}_t^s$, $\Omega_t^f \cap \Omega_t^s = \emptyset$ at any times $t \in (0, T)$. At initial time Ω_0^f and Ω_0^s are prescribed.

Let the fluid-structure interface be $\Sigma_t = \bar{\Omega}_t^f \cap \bar{\Omega}_t^s$ and the boundary of Ω_t be $\partial\Omega_t$. The part of $\partial\Omega_t$ on which either the structure is clamped or there is a no slip condition on the fluid, that part is denoted by Γ and assumed to be independent of time.

The following standard notations are used (see [8],[27],[2],[20],[24],[1]):

- $\mathbf{X} : \Omega_0 \times (0, T) \mapsto \Omega_t$: $\mathbf{X}(x^0, t)$, the Lagrangian position at t of x^0 .
- $\mathbf{u} = \partial_t \mathbf{X}$, the velocity of the deformation,
- $\mathbf{F} = \nabla^T \mathbf{X} = ((\partial_{x_i^0} \mathbf{X}_j))$, the Jacobian of the deformation,
- $J = \det_{\mathbf{F}}$.

We denote by tr_A and \det_A the trace and determinant of A . To describe the fluid structure system we need the following:

- $\rho = \mathbf{1}_{\Omega_t^f} \rho^f + \mathbf{1}_{\Omega_t^s} \rho^s$, the density,
- $\sigma = \mathbf{1}_{\Omega_t^f} \sigma^f + \mathbf{1}_{\Omega_t^s} \sigma^s$, the stress tensor,
- $\mathbf{f}(x, t)$ the density of volumic forces at x, t .
- $\mathbf{d} = \mathbf{X}(x^0, t) - x^0$, the displacement.

Finally and unless specified all spatial derivatives are with respect to $x \in \Omega_t$ and not with respect to $x^0 \in \Omega_0$. If ϕ is a function of $x = \mathbf{X}(x^0, t)$, $x^0 \in \Omega_0$,

$$\nabla_{x^0} \phi = [\partial_{x_i^0} \phi] = [\partial_{x_i^0} \mathbf{X}_j \partial_{x_j} \phi] = \mathbf{F}^T \nabla \phi.$$

When \mathbf{X} is one-to-one and invertible, \mathbf{d} and \mathbf{F} can be seen as functions of (x, t) instead of (x^0, t) . They are related by

$$\mathbf{F}^T = \nabla_{x^0} \mathbf{X} = \nabla_{x^0} (\mathbf{d} + x^0) = \nabla_{x^0} \mathbf{d} + \mathbf{I} = \mathbf{F}^T \nabla \mathbf{d} + \mathbf{I}, \quad \Rightarrow \quad \mathbf{F} = (\mathbf{I} - \nabla \mathbf{d})^{-T}$$

Time derivatives are related by

$$\mathbb{D}_t \phi := \frac{d}{dt} \phi(\mathbf{X}(x_0, t), t)|_{x=\mathbf{X}(x_0, t)} = \partial_t \phi(x, t) + \mathbf{u} \cdot \nabla \phi(x, t).$$

It is convenient to introduce the notation

$$\mathbf{D}\mathbf{u} = \nabla \mathbf{u} + \nabla^T \mathbf{u}.$$

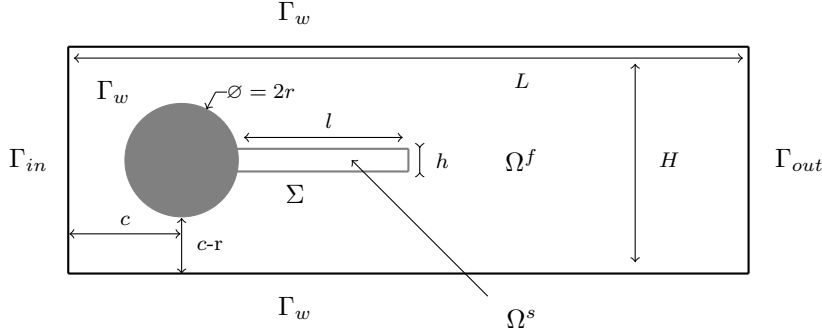


Figure 1.1: *The geometry of the FLUSTRUK test[14]. The cylinder (in black) is fixed but the flagella is a compressible Mooney-Rivlin material clamped to the cylinder by its left boundary; the outer rectangle is filled with a fluid which enters from the left Γ_{in} and leaves on the right Γ_{out} ; the horizontal boundaries of the outer rectangle are walls, so they form together with the cylinder the boundary Γ_w . The flagella is at time zero a rectangle of size $l \times h$. The outer rectangle has $L \times H$ for dimensions. The center of the circle representing the cylinder is at (c, c) when the lower left corner of the outer boundary is $(0, 0)$; the cylinder radius is r .*

Conservation of momentum and conservation of mass take the same form for the fluid and the solid:

$$\rho \mathbb{D}_t \mathbf{u} = \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}, \quad \frac{d}{dt}(J\rho) = 0,$$

So $J\rho = \rho_0$ at all times and

$$J^{-1} \rho_0 \mathbb{D}_t \mathbf{u} = \mathbf{f} + \nabla \cdot \boldsymbol{\sigma} \text{ in } \Omega_t, \quad \forall t \in (0, T), \quad (1.1)$$

with continuity of \mathbf{u} and of $\boldsymbol{\sigma} \cdot \mathbf{n}$ at the fluid-structure interface Σ in absence of interface constraints. There are also unwritten constraints pertaining to the realizability of the map \mathbf{X} (see [8],[27]). Note that incompressibility in the fluid implies $J|_{\Omega_t^f} = 1$ and so $\rho^f = \rho_0^f$ constant in Ω_t^f .

1.1 Constitutive Equations

- For a Newtonian incompressible fluid : $\boldsymbol{\sigma}^f = -p^f \mathbf{I} + \mu^f \mathbf{D}\mathbf{u}$
- For an hyperelastic material : $\boldsymbol{\sigma}^s = \rho^s \partial_{\mathbf{F}} \Psi \mathbf{F}^T$

where Ψ is the Helmholtz potential which, in the case of a S^t-Venant-Kirchhoff material, is [8]

$$\Psi(\mathbf{F}) = \frac{\lambda^s}{2} \text{tr}_{\mathbf{E}}^2 + \mu^s \text{tr}_{\mathbf{E}^2}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}) \quad (1.2)$$

It is easy to see that $\text{tr}_{\mathbf{E}} = \frac{1}{2} \text{tr}_{\mathbf{F}^T \mathbf{F}} - 1$ and

$$\partial_{\mathbf{F}} \text{tr}_{\mathbf{F}^T \mathbf{F}} = ((\partial_{\mathbf{F}_{ij}} \sum_{m,n} F_{m,n}^2)) = 2\mathbf{F} \Rightarrow \partial_{\mathbf{F}} \text{tr}_{\mathbf{E}} = \mathbf{F}$$

$$\partial_{\mathbf{F}} \text{tr}(\mathbf{F}^T \mathbf{F})^2 = ((\partial_{\mathbf{F}})_{ij} \sum_{n,m,p,k} F_{n,k} F_{n,m} F_{p,m} F_{p,k}) = 4\mathbf{F}\mathbf{F}^T \mathbf{F} \quad (1.3)$$

which implies that $\partial_{\mathbf{F}} \text{tr}_{\mathbf{E}^2} = 2\mathbf{F}\mathbf{E}$. Therefore

$$\partial_{\mathbf{F}} \Psi(\mathbf{F}) \mathbf{F}^T = (\lambda^s \text{tr}_{\mathbf{E}} \mathbf{F} + 2\mu^s \mathbf{F}\mathbf{E}) \mathbf{F}^T$$

which in turn implies that

$$\sigma^s = \rho^s \mathbf{F} (\lambda^s \text{tr}_{\mathbf{E}} + 2\mu^s \mathbf{E}) \mathbf{F}^T = J^{-1} \rho_0^s \mathbf{F} (\lambda^s \text{tr}_{\mathbf{E}} + 2\mu^s \mathbf{E}) \mathbf{F}^T$$

Proposition 1. *Let*

$$\gamma = \text{tr}_{\mathbf{F}\mathbf{F}^T} = (2 - 2\nabla \cdot \mathbf{d} + |\nabla \mathbf{d}|^2) J^2, \quad \tilde{\gamma} = \gamma J^{-2}$$

where $|\mathbf{A}| = \sum_{ij} A_{ij}^2$. The following holds

$$\begin{aligned} \sigma^s &= \rho^s \left(a\mathbf{I} + 2b(\mathbf{D}\mathbf{d} - \nabla \mathbf{d} \nabla^T \mathbf{d}) \right), \quad \text{with} \\ a &= \lambda^s \left(\frac{1}{2} \gamma - 1 \right) (\tilde{\gamma} - 1) + \mu^s (\gamma - J^2 - 1) \tilde{\gamma}, \\ b &= \frac{1}{2} \left(\frac{\lambda^s}{2} + \mu^s \right) (\gamma - 1) - \frac{\lambda^s}{4} \end{aligned} \quad (1.4)$$

Proof

First note that if $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ then

$$\sigma^s = \rho^s \left[\left[\lambda^s \left(\frac{1}{2} \gamma - 1 \right) - \mu^s \right] \mathbf{B} + \mu^s \mathbf{B}^2 \right] \quad (1.5)$$

Now by the Cayley- Hamilton theorem in 2 dimensions, $\mathbf{B}^2 - \gamma \mathbf{B} + J^2 \mathbf{I} = 0$. As $\mathbf{B}^{-1} = \mathbf{I} - \mathbf{D}\mathbf{d} + \nabla \mathbf{d} \nabla^T \mathbf{d}$ let $\mathbf{C} = \mathbf{I} - \mathbf{B}^{-1} = \mathbf{D}\mathbf{d} - \nabla \mathbf{d} \nabla^T \mathbf{d}$. Then

$$\mathbf{B} = \gamma \mathbf{I} - J^2 \mathbf{B}^{-1} = (\gamma - J^2) \mathbf{I} + J^2 \mathbf{C}, \quad \mathbf{B}^2 = (\gamma^2 - (1 + \gamma) J^2) \mathbf{I} + \gamma J^2 \mathbf{C}. \quad (1.6)$$

Therefore

$$\begin{aligned} \sigma^s &= \rho^s \left[\left[\lambda^s \left(\frac{1}{2} \gamma - 1 \right) - \mu^s \right] [(\gamma - J^2) \mathbf{I} + J^2 \mathbf{C}] + \mu^s [(\gamma^2 - (1 + \gamma) J^2) \mathbf{I} + \gamma J^2 \mathbf{C}] \right] \\ &= \rho^s \left[\left[\lambda^s \left(\frac{1}{2} \gamma - 1 \right) \right] (\gamma - J^2) \right. \\ &\quad \left. + \mu^s \gamma (\gamma - 1 - J^2) \right] \mathbf{I} + \left[\lambda^s \left(\frac{1}{2} \gamma - 1 \right) + \mu^s (\gamma - 1) \right] J^2 \mathbf{C} \end{aligned} \quad (1.7)$$

◇

1.2 Variational Monolithic Eulerian Formulation

Consequently one must find (\mathbf{u}, p) with $\mathbf{u}|_{\Gamma} = 0$, \mathbf{d} and Ω_t^r , $r = s, f$, solution for all $(\hat{\mathbf{u}}, \hat{p})$ with $\hat{\mathbf{u}}|_{\Gamma} = 0$ of

$$\begin{cases} \int_{\Omega_t^f} \left[\rho^f \mathbb{D}_t \mathbf{u} \cdot \hat{\mathbf{u}} - p \nabla \cdot \hat{\mathbf{u}} - \hat{p} \nabla \cdot \mathbf{u} + \frac{\mu^f}{2} \mathbf{D}\mathbf{u} : \mathbf{D}\hat{\mathbf{u}} \right] \\ + \int_{\Omega_t^s} \rho^s \left[\mathbb{D}_t \mathbf{u} \cdot \hat{\mathbf{u}} + b(\mathbf{D}\mathbf{d} - \nabla \mathbf{d} \nabla^T \mathbf{d}) : \mathbf{D}\hat{\mathbf{u}} + a \nabla \cdot \hat{\mathbf{u}} \right] = \int_{\Omega_t} f \cdot \hat{\mathbf{u}} \quad (1.8) \\ \mathbb{D}_t \mathbf{d} = \mathbf{u}, \text{ and } \{\dot{x}(t) = \mathbf{u}(x(t), t), x(0) = x_0 \in \Omega_0^r \Rightarrow x(t) \in \Omega_t^r\}. \end{cases}$$

Existence of solution up to time T^* is shown in [4] (see also [11][35]) for a non clamped structure provided a regularization term is added to insure that the solution satisfies $\partial_t \mathbf{d} \in \mathbf{H}(\Omega)$; T^* is such that the solid does not touch the boundary and Σ_t does not buckle.

2 Numerical Scheme

For the stability of the numerical scheme one must extract from $a\nabla \cdot \hat{\mathbf{u}}$ in the above variational formulation a term proportional to $(\nabla \cdot \mathbf{d})(\nabla \cdot \hat{\mathbf{u}})$.

Notice that

$$\begin{aligned} J^2 &= 1 + 2\nabla \cdot \mathbf{d} - 2\det\nabla\mathbf{d} + 3(\nabla \cdot \mathbf{d})^2 + o(|\nabla\mathbf{d}|^2) \\ \gamma &= 2(1 + \nabla \cdot \mathbf{d} + (\nabla \cdot \mathbf{d})^2 + \frac{1}{2}|\nabla\mathbf{d}|^2 - 2\det\nabla\mathbf{d}) + o(|\nabla\mathbf{d}|^2) \\ \left(\frac{\gamma}{2} - 1\right)(\tilde{\gamma} - 1) &= \nabla \cdot \mathbf{d} - (\nabla \cdot \mathbf{d})^2 - \frac{1}{2}|\nabla\mathbf{d}|^2 - 2\det\nabla\mathbf{d} + o(|\nabla\mathbf{d}|^2) \end{aligned} \quad (2.1)$$

So it makes sense to define

$$c = a - \lambda^s \nabla \cdot \mathbf{d} \quad (2.2)$$

To prepare the time discretization of (1.8) with a given time step δt , let

$$\tilde{\mathbf{d}} = \mathbf{d} - \delta t \mathbf{u}$$

Then (1.8) becomes

$$\left\{ \begin{aligned} &\int_{\Omega_t^f} \left[\rho^f \mathbb{D}_t \mathbf{u} \cdot \hat{\mathbf{u}} - p \nabla \cdot \hat{\mathbf{u}} - \hat{p} \nabla \cdot \mathbf{u} + \frac{\mu^f}{2} \mathbf{D} \mathbf{u} : \mathbf{D} \hat{\mathbf{u}} \right] \\ &+ \int_{\Omega_t^s} \rho^s \delta t \left[b(\mathbf{D} \mathbf{u} - \nabla \tilde{\mathbf{d}} \nabla^T \mathbf{u} - \nabla \mathbf{u} \nabla^T \tilde{\mathbf{d}} + \delta t \nabla \mathbf{u} \nabla^T \mathbf{u}) : \mathbf{D} \hat{\mathbf{u}} + \lambda^s \nabla \cdot \mathbf{u} \nabla \cdot \hat{\mathbf{u}} \right] \\ &+ \int_{\Omega_t^s} \rho^s \left[\mathbb{D}_t \mathbf{u} \cdot \hat{\mathbf{u}} + b(\mathbf{D} \tilde{\mathbf{d}} - \nabla \tilde{\mathbf{d}} \nabla^T \tilde{\mathbf{d}}) : \mathbf{D} \hat{\mathbf{u}} + (c + \lambda^s \nabla \cdot \tilde{\mathbf{d}}) \nabla \cdot \hat{\mathbf{u}} \right] = \int_{\Omega_t} f \cdot \hat{\mathbf{u}} \\ &\mathbb{D}_t \mathbf{d} = \mathbf{u}, \end{aligned} \right.$$

2.1 Discretization of Total Derivatives

Let $\Omega \subset \mathbb{R}^d$, $\mathbf{u} \in \mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^d$, ($d = 2$ here), $t \in (0, T)$ and $x \in \Omega$. Then let $\chi_{\mathbf{u}, x}^t(\tau)$ be the solution at time τ of

$$\dot{\chi}(\tau) = \mathbf{u}(\chi(\tau), \tau) \text{ with } \chi(t) = x.$$

If \mathbf{u} is Lipschitz in space and continuous in time the solution exists. The Characteristics-Galerkin method relies on the concept of total derivative:

$$\mathbb{D}_t \mathbf{v}(x, t) := \frac{d}{d\tau} \mathbf{v}(\chi(\tau), \tau) |_{\tau=t} = \partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v}.$$

Given a time step δt , let us approximate

$$\chi_{\mathbf{u}^{n+1}, x}^{(n+1)\delta t}(n\delta t) \approx \mathbb{Y}^{n+1}(x) := x - \mathbf{u}^{n+1}(x)\delta t$$

Remark 1. Note that

$$\mathbb{J}_{n+1} := \det_{\nabla \mathbb{Y}^{n+1}} = 1 - \nabla \cdot \mathbf{u}^{n+1} \delta t + \det_{\nabla \mathbf{u}^{n+1}} \delta t^2.$$

Remark 2. Note also that, as ρ^r is convected by \mathbf{u} , $r = f, s$, that is $\rho^r(\chi_{\mathbf{u}, x}^t(\tau), \tau) = \rho^r(x, t)$, so a consistent approximation is

$$\rho_n^r \circ \mathbb{Y}^{n+1}(x) = \rho_{n+1}^r(x), \quad x \in \Omega_{n+1}^r.$$

Thus discretizing the total derivative of \mathbf{u} or $\rho \mathbf{u}$ will give the same scheme.

$$\rho_{n+1} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n \circ \mathbb{Y}^{n+1}}{\delta t} \Big|_x = \frac{\rho_{n+1} \mathbf{u}^{n+1} - (\rho_n \mathbf{u}^n) \circ \mathbb{Y}^{n+1}}{\delta t} \Big|_x$$

2.2 Updating the fluid and solid domain

From the definition of \mathbb{Y} notice that the only way to be consistent is to define Ω_{n+1} with \mathbf{u}^{n+1} , i.e. implicitly since the later is defined also on Ω_{n+1} :

$$\Omega_{n+1} = (\mathbb{Y}^{n+1})^{-1}(\Omega_n) = \{x : \mathbb{Y}^{n+1}(x) := x - \mathbf{u}^{n+1}(x)\delta t \in \Omega_n\}$$

2.3 The Fully Time Discretized Scheme

Let

$$\mathbf{d}^{n+1} = \tilde{\mathbf{d}}^n + \delta t \mathbf{u}^{n+1}, \quad \text{with } \tilde{\mathbf{d}}^n := \mathbf{d}^n \circ \mathbb{Y}^{n+1} \quad (2.3)$$

Let \tilde{b}_n, \tilde{c}_n be given by (1.4,2.2) computed with \tilde{d}^n . The following defines $\mathbf{u}^{n+1}, p^{n+1}$: $\mathbf{u}^{n+1}|_\Gamma = 0$ and $\forall \hat{\mathbf{u}}, \hat{p}$, with $\hat{\mathbf{u}}|_\Gamma = 0$,

$$\left\{ \begin{array}{l} \int_{\Omega_{n+1}} \rho_{n+1} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n \circ \mathbb{Y}^{n+1}}{\delta t} \cdot \hat{\mathbf{u}} \\ + \int_{\Omega_{n+1}^f} \left[-p^{n+1} \nabla \cdot \hat{\mathbf{u}} - \hat{p} \nabla \cdot \mathbf{u}^{n+1} + \frac{\mu^f}{2} \text{Du}^{n+1} : \text{D}\hat{\mathbf{u}} \right] \\ + \int_{\Omega_{n+1}^s} \rho_{n+1}^s \delta t \left[\tilde{b}_n (\text{Du}^{n+1} - \nabla \tilde{\mathbf{d}}^n \nabla^T \mathbf{u}^{n+1} - \nabla \mathbf{u}^{n+1} \nabla^T \tilde{\mathbf{d}}^n) : \text{D}\hat{\mathbf{u}} \right. \\ \quad \left. + \lambda^s \nabla \cdot \mathbf{u}^{n+1} \nabla \cdot \hat{\mathbf{u}} + \delta t \tilde{b}_n \nabla \mathbf{u}^{n+1} \nabla^T \mathbf{u}^{n+1} : \text{D}\hat{\mathbf{u}} \right] \\ + \int_{\Omega_{n+1}^s} \left[\tilde{b}_n (\text{D}\tilde{\mathbf{d}}^n - \nabla \tilde{\mathbf{d}}^n \nabla^T \tilde{\mathbf{d}}^n) : \text{D}\hat{\mathbf{u}} + (\tilde{c}_n + \lambda^s \nabla \cdot \tilde{\mathbf{d}}^n) \nabla \cdot \hat{\mathbf{u}} \right] \\ = \int_{\Omega_{n+1}} f \cdot \hat{\mathbf{u}} \end{array} \right. \quad (2.4)$$

2.4 Iterative Solution by Fixed Point

The most natural method to solve the above is to freeze some coefficients so as to obtain a well posed linear problem and iterate:

1. Set $\rho = \rho_n$, $\Omega = \Omega_n$, $\mathbf{u} = \mathbf{u}^n$, $\mathbb{Y}(x) = x - \mathbf{u}\delta t$.
2. Solve

$$\left\{ \begin{array}{l} \int_{\Omega} \rho \frac{\mathbf{u}^{n+1} - \mathbf{u}^n \circ \mathbb{Y}}{\delta t} \cdot \hat{\mathbf{u}} \\ + \int_{\Omega^f} \left[-p^{n+1} \nabla \cdot \hat{\mathbf{u}} - \hat{p} \nabla \cdot \mathbf{u}^{n+1} + \frac{\mu^f}{2} \text{Du}^{n+1} : \text{D}\hat{\mathbf{u}} \right] \\ + \int_{\Omega^s} \rho^s \delta t \left[\tilde{b}_n (\text{Du}^{n+1} - \nabla \tilde{\mathbf{d}}^n \nabla^T \mathbf{u}^{n+1} - \nabla \mathbf{u}^{n+1} \nabla^T \tilde{\mathbf{d}}^n) : \text{D}\hat{\mathbf{u}} \right. \\ \quad \left. + \lambda^s \nabla \cdot \mathbf{u}^{n+1} \nabla \cdot \hat{\mathbf{u}} + \delta t \tilde{b}_n \nabla \mathbf{u}^{n+1} \nabla^T \mathbf{u}^{n+1} : \text{D}\hat{\mathbf{u}} \right] \\ + \int_{\Omega^s} \left[\tilde{b}_n (\text{D}\tilde{\mathbf{d}}^n - \nabla \tilde{\mathbf{d}}^n \nabla^T \tilde{\mathbf{d}}^n) : \text{D}\hat{\mathbf{u}} + (\tilde{c}_n + \lambda^s \nabla \cdot \tilde{\mathbf{d}}^n) \nabla \cdot \hat{\mathbf{u}} \right] \\ = \int_{\Omega} f \cdot \hat{\mathbf{u}} \end{array} \right. \quad (2.5)$$

3. Set $\mathbf{u} = \mathbf{u}^{n+1}$, $\mathbb{Y}(x) = x - \mathbf{u}\delta t$, $\Omega^r = \mathbb{Y}^{-1}(\Omega_n^r)$, $r = s, f$, $\Omega = \Omega^f \cup \Omega^s$, recompute ρ, b, c with \mathbf{u}^{n+1} .
4. If not converged return to Step 2.

Notice that (2.5) is a well posed linear problem whenever

$$A(\mathbf{u}, \hat{\mathbf{u}}) = \int_{\Omega^s} \left[b (\text{Du} - \nabla \tilde{\mathbf{d}}^n \nabla^T \mathbf{u} - \nabla \mathbf{u} \nabla^T \tilde{\mathbf{d}}^n) : \text{D}\hat{\mathbf{u}} + \lambda^s \nabla \cdot \mathbf{u} \nabla \cdot \hat{\mathbf{u}} \right]$$

is coercive. Then (2.5) gives a solution bounded in $\mathbf{H}^1(\Omega)$ and converging subsequences can be extracted from $\rho_{n+1}, \mathbf{u}^{n+1}, \Omega_{n+1}^r$ when $\bar{\Omega} = \overline{\Omega_n^f \cup \Omega_n^s}$ is fixed. Then convergence would occur if we could prove that Ω_{n+1}^r converges to an open set.

2.5 Spatial Discretization with Finite Elements

Let \mathcal{T}_h^0 be a triangulation of the initial domain. Spatial discretization can be done with the most popular finite element for fluids: the Lagrangian triangular elements of degree 2 for the space V_h of velocities and displacements and Lagrangian triangular elements of degree 1 for the pressure space Q_h provided that the pressure be different in the structure and the fluid because the pressure is discontinuous at the interface Σ ; therefore Q_h is the space of piecewise linear functions on the triangulation continuous in Ω_{n+1}^r , $r = s, f$. A small penalization with parameter ϵ must be added to impose uniqueness of the pressure.

This leads us to find $\mathbf{u}_h^{n+1} \in V_{h0\Gamma}$, $p_h^{n+1} \in Q_h$, Ω_{n+1} such that for all $\hat{\mathbf{u}}_h, \hat{p}_h \in V_{h0\Gamma} \times Q_h$ with

$$\tilde{\mathbf{d}}_h^n := \mathbf{d}_h^n \circ \mathbb{Y}^{n+1}, \quad \text{where } \mathbb{Y}^{n+1}(x) = x - \mathbf{u}_h^{n+1}(x)\delta t,$$

the following holds:

$$\left\{ \begin{array}{l} \mathbf{a}(\tilde{b}_n, \tilde{c}_n; \mathbf{u}^{n+1}, \hat{\mathbf{u}}) := \int_{\Omega_{n+1}} \rho_{n+1} \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n \circ \mathbb{Y}^{n+1}}{\delta t} \cdot \hat{\mathbf{u}}_h \\ \quad + \int_{\Omega_{n+1}^f} \left[-p^{n+1} \nabla \cdot \hat{\mathbf{u}}_h - \hat{p} \nabla \cdot \mathbf{u}_h^{n+1} + \frac{\mu^f}{2} \text{D}\mathbf{u}_h^{n+1} : \text{D}\hat{\mathbf{u}}_h \right] \\ \quad + \int_{\Omega_{n+1}^s} \rho_{n+1}^s \delta t \left[\tilde{b}_n (\text{D}\mathbf{u}_h^{n+1} - \nabla \tilde{\mathbf{d}}_h^n \nabla^T \mathbf{u}_h^{n+1} - \nabla \mathbf{u}_h^{n+1} \nabla^T \tilde{\mathbf{d}}_h^n) : \text{D}\hat{\mathbf{u}}_h \right. \\ \quad \quad \left. + \lambda^s \nabla \cdot \mathbf{u}_h^{n+1} \nabla \cdot \hat{\mathbf{u}}_h \right] \\ \quad + \int_{\Omega_{n+1}^s} \left[\tilde{b}_n (\text{D}\tilde{\mathbf{d}}_h^n - \nabla \tilde{\mathbf{d}}_h^n \nabla^T \tilde{\mathbf{d}}_h^n) : \text{D}\hat{\mathbf{u}}_h + (\tilde{c}_n + \lambda^s \nabla \cdot \tilde{\mathbf{d}}_h^n) \nabla \cdot \hat{\mathbf{u}}_h \right] \\ = \int_{\Omega_{n+1}} \mathbf{f} \cdot \hat{\mathbf{u}}_h, \quad \Omega_{n+1} = (\mathbb{Y}^{n+1})^{-1}(\Omega_n) = \{x : \mathbb{Y}^{n+1}(x) \in \Omega_n\}. \end{array} \right. \quad (2.6)$$

Then \mathbf{d} is updated by

$$\mathbf{d}_h^{n+1} = \tilde{\mathbf{d}}_h^n + \delta t \mathbf{u}_h^{n+1},$$

2.6 Implementation

The various tests we made lead us to recommend the following:

- Move the vertices of the mesh in the structure by the velocity:

$$q_i^{n+1} = q_i^n + \mathbf{u}_h^{n+1}(q_i^{n+1})\delta t \quad (2.7)$$

which, as explained above has to be implemented through an iterative process.

- Remesh the fluid part at each iteration with a Delaunay-Voronoi mesh generator from the boundary vertices.

This required the development of a specific module to identify computationally the vertices of the fluid-structure interface Σ , which are then input to the fluid mesh generator.

- In doing so, the discrete topological properties of the structural part are preserved and we have the important property that the value $\mathbf{d}[i]$ of \mathbf{d} at vertex q_i in the computer implementation of \mathbf{d} by an array of values at the nodes, satisfies

$$\mathbf{d}^{n+1}[i] = \mathbf{d}^n[i] + \delta t \mathbf{u}^{n+1}[i], \quad \forall i.$$

In other words $\mathbf{d}^n \circ \mathbb{Y}^{n+1}$ is $\mathbf{d}^n[i]$ when the vertices are moved by (2.7).

3 Energy Estimate

3.1 Stability of the Scheme Discretized in Time

To conserve energy we need to change the scheme (2.6) slightly

$$\begin{aligned} \mathbf{a}(\tilde{b}_n, \tilde{c}_n; \mathbf{u}^{n+1}, \hat{\mathbf{u}}) &= \int_{\Omega_{n+1}} \mathbf{f} \cdot \hat{\mathbf{u}}_h \rightarrow \\ \mathbf{a}(b_{n+1}, c_{n+1}; \mathbf{u}^{n+1}, \hat{\mathbf{u}}) &+ \delta t^2 \int_{\Omega_{n+1}^s} \rho_{n+1}^s b_{n+1} \nabla \mathbf{u}_h^{n+1} \nabla^T \mathbf{u}_h^{n+1} : \mathbf{D} \hat{\mathbf{u}}_h \\ &= \int_{\Omega_{n+1}} \mathbf{f} \cdot \hat{\mathbf{u}}_h \end{aligned} \quad (3.1)$$

Lemma 3.1. *The mapping $\mathbf{X}^n : \Omega_0 \mapsto \Omega_n$ is given by $\mathbf{X}^{n+1} = (\mathbb{Y}^{n+1})^{-1} \circ \mathbf{X}^n$, $n \geq 1$ and the jacobian of the transformation is $\mathbf{F}^n := \nabla_{x_0}^T \mathbf{X}^n = (\mathbf{I} - \nabla \mathbf{d}^n)^{-T}$.*

Proof

Notice that $\mathbb{Y}^1(\mathbb{Y}^2(\dots \mathbb{Y}^{n-1}(\mathbb{Y}^n(\Omega_n)) \dots)) = \Omega_0$ Hence

$$\mathbf{X}^{n+1} = [\mathbb{Y}^1(\mathbb{Y}^2(\dots \mathbb{Y}^n(\mathbb{Y}^{n+1})))^{-1}]^{-1} = (\mathbb{Y}^{n+1})^{-1} \circ \mathbf{X}^n.$$

By definition of \mathbf{d}^{n+1} in (2.3)

$$\begin{aligned} \mathbf{d}^{n+1}(\mathbf{X}^{n+1}(x_0)) &= \mathbf{d}^n(\mathbb{Y}^{n+1}(\mathbf{X}^{n+1}(x_0))) + \mathbf{u}^{n+1}(\mathbf{X}^{n+1}(x_0)) \delta t \\ &= \mathbf{d}^n(\mathbf{X}^n(x_0)) + \mathbf{u}^{n+1}(\mathbf{X}^{n+1}(x_0)) \delta t, \end{aligned} \quad (3.2)$$

so $\mathbf{X}^{n+1}(x_0) = \mathbf{d}^{n+1}(\mathbf{X}^{n+1}(x_0)) + x_0$ and therefore

$$\begin{aligned} \mathbf{F}^{n+1} &= \nabla_{x_0}^t (\mathbf{d}^{n+1}(\mathbf{X}^{n+1}(x_0)) + x_0), \\ &= \nabla \mathbf{d}^{n+1 T} \mathbf{F}^{n+1} + \mathbf{I} \Rightarrow \mathbf{F}^{n+1} = (\mathbf{I} - \nabla \mathbf{d}^{n+1})^{-T} \end{aligned} \quad (3.3)$$

◇

Note that (3.2) shows also that

$$\mathbf{F}^{n+1} = \mathbf{F}^n + \delta t \nabla_{x_0}^T \mathbf{u}^{n+1} \quad (3.4)$$

Lemma 3.2. *With Ψ defined by (1.2),*

$$\begin{aligned} \int_{\Omega_{n+1}^s} \rho_{n+1}^s \left[b^{n+1} (\mathbf{D} \mathbf{d}^{n+1} - \nabla \mathbf{d}^{n+1} \nabla^T \mathbf{d}^{n+1}) : \mathbf{D} \hat{\mathbf{u}} + a^{n+1} \nabla \cdot \hat{\mathbf{u}} \right] \\ = \int_{\Omega_0^s} \partial_{\mathbf{F}} \Psi^{n+1} : \nabla_{x_0} \hat{\mathbf{u}} \end{aligned} \quad (3.5)$$

Proof By Propositions 3.1 and 1

$$\int_{\Omega_{n+1}^s} \rho_{n+1}^s \left(a^{n+1} \mathbf{I} + 2b^{n+1} (\mathbf{D} \mathbf{d}^{n+1} - \nabla \mathbf{d}^{n+1} \nabla^T \mathbf{d}^{n+1}) \right) : \nabla \hat{\mathbf{u}}$$

$$\begin{aligned}
&= \int_{\Omega_{n+1}^s} \sigma_{n+1}^s : \nabla \hat{\mathbf{u}} = \int_{\Omega_{n+1}^s} \left[\rho_{n+1}^s \partial_{\mathbf{F}} \Psi \mathbf{F}^T \right] |_{n+1} : \nabla \hat{\mathbf{u}} \\
&= \int_{\Omega_{n+1}^s} \left[J_{n+1}^{-1} \rho_0^s \partial_{\mathbf{F}} \Psi \mathbf{F}^T \right] |_{n+1} : \nabla \hat{\mathbf{u}} = \int_{\Omega_0^s} \rho_0^s \partial_{\mathbf{F}} \Psi^{n+1} : \nabla \hat{\mathbf{u}} \quad (3.6)
\end{aligned}$$

◇

Theorem 3.3. *When $f = 0$ and ρ is constant in each domain Ω_n^r , $r = s, f$, the numerical scheme (3.1) has the following property:*

$$\int_{\Omega_n} \frac{\rho_n}{2} |\mathbf{u}^n|^2 + \delta t \sum_{k=1}^n \int_{\Omega_k^f} \frac{\nu}{2} |\mathbf{D}\mathbf{u}^k|^2 + \int_{\Omega_0^s} \Psi^n \leq \int_{\Omega_0} \frac{\rho_0}{2} |\mathbf{u}^0|^2 + \int_{\Omega_0^s} \Psi^0 \quad (3.7)$$

Proof Let $r = s$ or f . Let us choose $\hat{\mathbf{u}} = \mathbf{u}^{n+1}$ in (2.6). By Schwartz inequality

$$\int_{\Omega_{n+1}^r} (\rho_n^r \mathbf{u}^n) \circ \mathbb{Y}^{n+1} \cdot \mathbf{u}^{n+1} \leq \left(\int_{\Omega_{n+1}^r} (\sqrt{\rho_n^r} (\mathbf{u}^n)^2 \circ \mathbb{Y}^{n+1}) \right)^{\frac{1}{2}} \left(\int_{\Omega_{n+1}^r} \rho_{n+1}^r \mathbf{u}^{n+1} \right)^{\frac{1}{2}}$$

because $\rho_n^r \circ \mathbb{Y}^{n+1}(x) = \rho_{n+1}^r(x)$, $x \in \Omega_{n+1}^r$. So by a change of variable

$$\int_{\Omega_{n+1}^r} \rho_{n+1}^r (\mathbf{u}^n \circ \mathbb{Y}^{n+1})^2 = \int_{\Omega_{n+1}^r} (\sqrt{\rho_n^r} \mathbf{u}^n)^2 \circ \mathbb{Y}^{n+1} = \int_{\Omega_n^r} \rho_n^r \mathbf{u}^{n2}$$

Consequently, using $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$,

$$\int_{\Omega_{n+1}^r} \rho_{n+1}^r \mathbf{u}^n \circ \mathbb{Y}^{n+1} \cdot \mathbf{u}^{n+1} \leq \frac{1}{2} \int_{\Omega_n^r} \rho_n^r \mathbf{u}^{n2} + \frac{1}{2} \int_{\Omega_{n+1}^r} \rho_{n+1}^r \mathbf{u}^{n+12}$$

Finally,

$$\begin{aligned}
\int_{\Omega_{n+1}} \frac{\rho_{n+1}}{2} |\mathbf{u}^{n+1}|^2 + \delta t \int_{\Omega_{n+1}^f} \frac{\nu}{2} |\mathbf{D}\mathbf{u}^{n+1}|^2 + \int_{\Omega_0} \Psi^{n+1} \\
\leq \int_{\Omega_n} \frac{\rho_n}{2} |\mathbf{u}^n|^2 + \int_{\Omega_0} \Psi^n \quad (3.8)
\end{aligned}$$

◇

3.2 Conservation of Energy for the Time and Space Discrete Scheme

The proof for the spatially continuous case will work for the discrete case if

$$\mathbf{X}^n = \mathbf{X}^{n+1} \circ \mathbb{Y}^{n+1}. \quad (3.9)$$

As discussed in [18] it may be possible to program an isoparametric $P^2 - P^1$ element for which (3.9) but it is certainly far from easy. On the other hand, consider the $P_3^1 - P^1$ element: the fluid pressure and the solid pressure are continuous and piecewise linear on the triangulation and each "pressure-triangle" is divided into 3 smaller triangles by a fourth vertex anywhere inside the triangle; on this refined triangulation the velocity is continuous piecewise linear.

Then (3.9) holds and the proof of the spatially continuous case can be adapted. The inner vertex used to construct the fluid mesh will be moved by \mathbb{Y}^{n+1} but $\mathbf{X}^{n+1} \circ \mathbb{Y}^{n+1}$ remains linear and for each triangle $T_n^k = \mathbb{Y}^{n+1}(T_{n+1}^k)$.

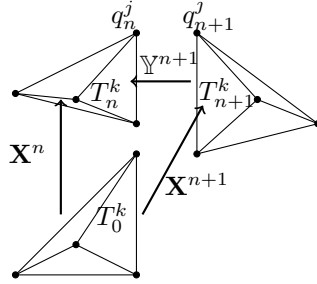


Figure 3.1: Sketch to understand if $\mathbf{X}^n = \mathbb{Y}^{n+1} \circ \mathbf{X}^{n+1}$ holds with the $P_3^1 - P1$ element where each triangle is divided into four subtriangles on which the velocities are P^1 and continuous. A triangle T_0^k in the reference domain (chosen here to be its initial position at time zero) becomes triangles T_n^k and T_{n+1}^k : $T_n^k = \mathbf{X}^n(T_0^k)$ and $T_{n+1}^k = \mathbf{X}^{n+1}(T_0^k)$. Vertices are preserved by these transformations. However the inner vertex does not stay in the center of the pressure triangle, which is alright so long as it stays with the triangle.

Remark 3. Because of energy preservation scheme (3.1), implemented via a fixed point algorithm as in (2.5), generates bounded sequences ρ, \mathbf{u}, q^i ; it seems safe to assess that out of these bounded subsequences will converge to a solution of the problem discretized in space but continuous in time when $\delta t \rightarrow 0$.

4 Numerical Tests

In our tests we have used the $P^2 - P^1$ element, confident that it will behave as well as the $3P^1 - P1$ element as indicated in [18].

4.1 The Cylinder-Flagella Test

A compressible hyperelastic Mooney-Rivlin material, shaped as a rectangular flagella of size $[0, l] \times [0, h]$, is attached behind a cylinder of radius r and beats in tune with the Karman vortices of the wake behind the cylinder; the fluid in the computational rectangular domain $[0, L] \times [0, H]$ enters from the left and is free to leave on the right. The center of the cylinder is at (c, c) (see figure 1.1). In [12] the following numerical values are suggested:

Geometry $l = 0.35, h = 0.02, L = 2.5, H = 0.41, c = 0.2$ which puts the cylinder slightly below the symmetry line.

Fluid density $\rho^f = 10^3 \text{ kg/m}^3$ and a reduced viscosity $\nu^f = \frac{\mu^f}{\rho^f} = 10^{-3} \text{ m}^2/\text{s}$;

inflow horizontal velocity $\mathbf{u}(0, y) = \bar{U} \left(\frac{6}{H^2} y(H - y), 0 \right)^T$ is a parabolic profile with flux $\bar{U}H$. Top and bottom boundaries are walls with no-slip conditions.

Solid $E = 2\mu(1 + \sigma)/\rho^s, \sigma = 0.4, \lambda = \frac{E\sigma}{(1 + \sigma)(1 - 2\sigma)}$.

Initial velocities and displacements are zero. In all cases the same mesh is used initially with 2500 vertices. The time step is 0.005.

4.1.1 Free Fall of the Flagella

When $\bar{U} = 0$, $\mu = 0.13510^6$ and $\rho^s = 20\rho^f$, the flagella falls under its own weight; it comes to touch the lower boundary with zero velocity at time 0.49 and then moves up under its spring effect. This test is named FLUSTRUK-FSI-2* in [12] but we have used a different value for μ because the one reported in [12] seems unlikely.

Figure 4.1 shows a zoom around the flagella at the time when it has stopped to descent and started to move upward. Pressure lines are drawn in the flow region together with the mesh and the velocity vectors in the flagella and drawn at each vertex. Figure 4.2 shows the coordinates of the upper right tip of the flagella versus time.

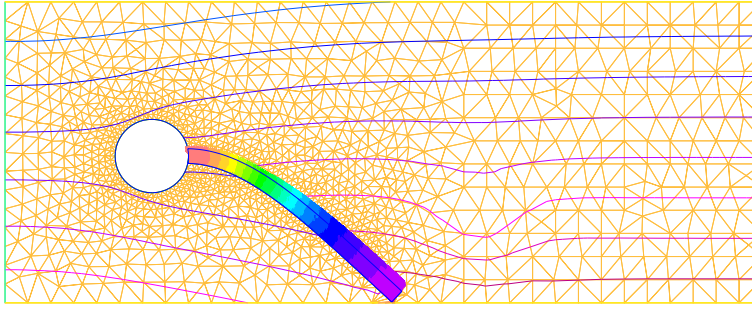


Figure 4.1: *FLUSTRUK-FSI-2** [12]. Zoom near the flagella at $t=0.495$ just as it begins to move up after the fall under its own weight in a flow initially at rest. Mesh and Pressure lines are shown in the fluid and velocity vectors in the solid.

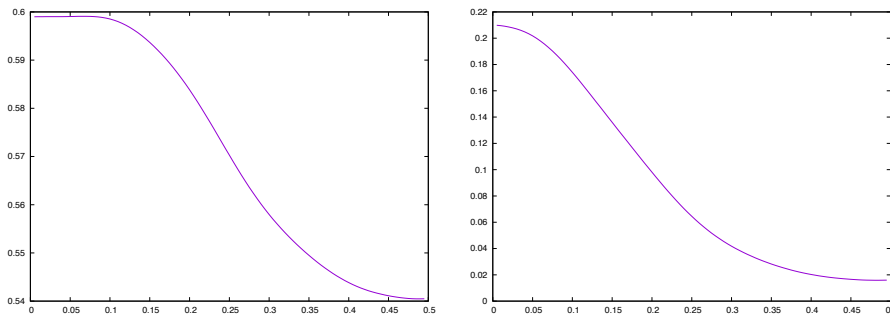


Figure 4.2: *FLUSTRUK-FSI-2** test of [12]. Position of the upper right corner of the flagella versus time: x vs t on the left and y vs t on the right.

4.1.2 Flow past a Cylinder with a Flagella Attached

This test is known as FLUSTRUK-FSI-3 in [12]. The geometry is the same as above but now $\bar{U} = 2$, $\mu = 210^6$ and $\rho^s = \rho^f$. After some time a Karman-Vortex alley develops and the flagella beats accordingly. Results are shown

on figures 4.3 and 4.4; the first one displays a snapshot of the velocity vector norms and the second the y-coordinate versus time of the top right corner of the flagella.

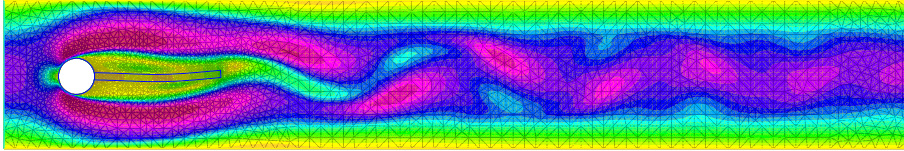


Figure 4.3: *FLUSTRUK-FSI-3 Test. Color map based on the norm of the fluid and solid velocity vectors*

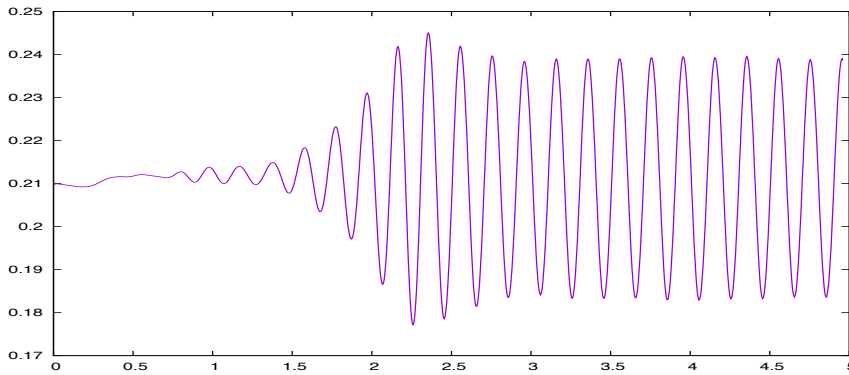


Figure 4.4: *FLUSTRUK-FSI-3 Test. Vertical position of the upper right tip of the flagella versus time shown up to $t=5$*

These numerical results compare reasonably well with those of [12]. The frequency is $5s^{-1}$ compared to 5.04 and the maximum amplitude 0.031 compared with 0.032. However the results are sensitive to the time step.

Conclusion

A fully Eulerian fluid-structure formulation has been presented for compressible materials with large displacements, discretized by an implicit first order Euler Scheme and the $3P^1 - P1$ or $P2 - P1$ elements. An energy estimate has been obtained which guarantees the stability of the scheme so long as the motion of the vertices does not flip-over a triangle. The method has been implemented with `FreeFem++` [17]. It is reasonably robust when the vertices in the structure are moved by their velocities and the fluid is remeshed with an automatic Delaunay mesh generator. The method is first order in time and therefore somewhat too diffusive for delicate tests. It needs to be extended to 3D and to second order in time discretization.

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