Generic Properties of Dynamical Systems

Christian Bonatti

Generic Properties of Dynamical Systems

C Bonatti, Université de Bourgogne, Dijon, France

Introduction

The state of a concrete system (from physics, chemistry, ecology, or other sciences) is described using (finitely many, say n) observable quantities (e.g., positions and velocities for mechanical systems, population densities for echological systems, etc.). Hence, the state of a system may be represented as a point x in a geometrical space $\mathbb{R}^n$. In many cases, the quantities describing the state are related, so that the phase space (space of all possible states) is a submanifold $M \subset \mathbb{R}^n$. The time evolution of the system is represented by a curve $x_t, t \in \mathbb{R}$ drawn on the phase space M, or by a sequence $x_n \in M$, $n \in \mathbb{Z}$, if we consider discrete time (i.e., every day at the same time, or every January 1st).

Believing in determinism, and if the system is isolated from external influences, the state x_0 of the system at the present time determines its evolution. For continuous-time systems, the infinitesimal evolution is given by a differential equation or vector field $dx/dt = X(x)$; the vector $X(x)$ represents velocity and direction of the evolution. For a discrete-time system, the evolution rule is a function $F : M \to M$; if x is the state at time t, then $F(x)$ is the state at the time $t + 1$. The evolution of the system, starting at the initial data x_0, is described by the orbit of x_0, that is, the sequence $\{(x_n)_{n\in\mathbb{Z}} \mid x_{n+1} = F(x_n)\}$ (discrete time) or the maximal solution x_t of the differential equation $ax/dt = X(x)$ (continuous time).

General problem *Knowing the initial data and the infinitesimal evolution rule, what can we tell about the long-time evolution of the system?*

The dynamics of a dynamical system (differential equation or function) is the behavior of the orbits, when the time tends to infinity. The aim of "dynamical systems" is to produce a general procedure for describing the dynamics of any system. For example, Conley's theory presented in the next section organizes the global dymamics of a general system using regions concentrating the orbit accumulation and recurrence and splits these regions in elementary pieces: the chain recurrence classes.

We focus our study on C^r-diffeomorphisms F (i.e., F and F^{-1} are r times continuously derivable) on a compact smooth manifold M (most of the notions and results presented here also hold for vector fields). Even for very regular systems (F algebraic) of a low-dimensional space ($\dim(M) = 2$), the dynamics may be chaotic and very unstable: one cannot hope for a precise description of all systems. Furthermore, neither the initial data of a concrete system nor the infinitesimal-evolution rule are known exactly: fragile properties describe the evolution of the theoretical model, and not of the real system. For these reasons, we are mostly interested in properties that are persistent, in some sense, by small perturbations of the dynamical system. The notion of small perturbations of the system requires a topology on the space $\mathrm{Diff}^r(M)$ of C^r-diffeomorphisms: two diffeomorphisms are close for the C^r-topology if all their partial derivatives of order $\leq r$ are close at each point of M. Endowed with this topology, $\mathrm{Diff}^r(M)$ is a complete metric space.

The open and dense subsets of $\mathrm{Diff}^r(M)$ provide the natural topological notion of "almost all" F. Genericity is a weaker notion: by Baire's theorem, if $\mathcal{O}_i, i \in \mathbb{N}$, are dense and open subsets, the intersection $\bigcap_{i\in\mathbb{N}} \mathcal{O}_i$ is a dense subset. A subset is called residual if it contains such a countable intersection of dense open subsets. A property $\mathcal{P}$ is generic if it is verified on a residual subset. By a practical abuse of language, one says:

"C^r-generic diffeomorphisms verify $\mathcal{P}$"

A countable intersection of residual sets is a residual set. Hence, if $\{\mathcal{P}_i\}, i \in \mathbb{N}$, is a countable family of generic properties, generic diffeomorphisms verify simultanuously all the properties $\mathcal{P}_i$.

A property $\mathcal{P}$ is C^r-robust if the set of diffeomorphisms verifying $\mathcal{P}$ is open in $\mathrm{Diff}^r(M)$. A property $\mathcal{P}$ is locally generic if there is an (nonempty) open set $\mathcal{O}$ on which it is generic, that is, there is residual set $\mathcal{R}$ such that $\mathcal{P}$ is verified on $\mathcal{R} \cap \mathcal{O}$.

The properties of generic dynamical systems depend mostly on the dimension of the manifold M and of the C^r-topology considered, $r \in \mathbb{N} \cup \{+\infty\}$ (an important problem is that C^r-generic diffeomorphisms are not C^{r+1}):

- On very low dimensional spaces (diffeomorphisms of the circle and vector fields on compact surfaces) the dynamics of generic systems (indeed in a open and dense subset of systems) is very simple (called Morse–Smale) and well understood; see the subsection "Generic properties of the low-dimensional systems."
- In higher dimensions, for C^r-topology, $r > 1$, one has generic and locally generic properties related to the periodic orbits, like the Kupka–Smale property (see the subsection "Kupka–Smale theorem") and the Newhouse phenomenon (see the subsection "Local C^2-genericity of wild behavior for surface diffeomorphisms"). However, we still do not know if the dynamics of C^r-generic diffeomorphisms is well approached by their periodic orbits, so that one is still far from a global understanding of C^r-generic dynamics.
- For the C^1-topology, perturbation lemmas show that the global dynamics is very well approximated by periodic orbits (see the section "C^1-generic systems: global dynamics and periodic orbits"). One then divides generic systems in "tame" systems, with a global dynamics analoguous to hyperbolic dynamics, and "wild" systems, which present infinitely many dynamically independent regions. The notion of dominated splitting (see the section "Hyperbolic properties of C^1-generic diffeormorphisms") seems to play an important role in this division.

Results on General Systems

Notions of Recurrence

Some regions of M are considered as the heart of the dynamics:

- $\mathrm{Per}(F)$ denotes the set of periodic points $x \in M$ of F, that is, $F^n(x) = x$ for some $n > 0$.
- A point x is recurrent if its orbit comes back arbitrarily close to x, infinitely many times. $\mathrm{Rec}(F)$ denotes the set of recurrent points.
- The limit set $\mathrm{Lim}(F)$ is the union of all the accumulation points of all the orbits of F.

- A point x is "wandering" if it admits a neighborhood $U_x \subset M$ disjoint from all its iterates $F^n(U_x), n > 0$. The nonwandering set $\Omega(F)$ is the set of the nonwandering points.
- $\mathcal{R}(F)$ is the set of chain recurrent points, that is, points $x \in M$ which look like periodic points if we allow small mistakes at each iteration: for any $\varepsilon > 0$, there is a sequence $x = x_0, x_1, \ldots, x_k = x$ where $d(f(x_i), x_{i+1}) < \varepsilon$ (such a sequence is an ε-pseudo-orbit).

A periodic point is recurrent, a recurrent point is a limit point, a limit point is nonwandering, and a nonwandering point is chain recurrent:

$$\mathrm{Per}(F) \subset \mathrm{Rec}(F) \subset \mathrm{Lim}(F) \subset \Omega(F) \subset \mathcal{R}(F)$$

All these sets are invariant under F, and $\Omega(F)$ and $\mathcal{R}(F)$ are compact subsets of M. There are diffeomorphisms F for which the closures of these sets are distinct:

- A rotation $x \mapsto x + \alpha$ with irrational angle $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ has no periodic points but every point is recurrent.
- The map $x \mapsto x + (1/4\pi)(1 + \cos(2\pi x))$ induces on the circle S^1 a diffeomorphism F having a unique fixed point at $x = 1/2$; one verifies that $\Omega(F) = \{1/2\}$ and $\mathcal{R}(F)$ is the whole circle S^1.

An invariant compact set $K \subset M$ is transitive if there is $x \in K$ whose forward orbit is dense in K. Generic points $x \in K$ have their forward and backward orbits dense in K: in this sense, transitive sets are dynamically indecomposable.

Conley's Theory: Pairs Attractor/Repeller and Chain Recurrence Classes

A trapping region $U \subset M$ is a compact set whose image $F(U)$ is contained in the interior of U. By definition, the intersection $A = \bigcap_{n \geq 0} F^n(U)$ is an attractor of F: any orbit in U "goes to A." Denote by V the complement of the interior of U: it is a trapping region for F^{-1} and the intersection $R = \bigcap_{n \geq 0} F^{-n}(V)$ is a repeller. Each orbit either is contained in $A \cup R$, or "goes from the repeller to the attractor." More precisely, there is a smooth function $\psi : M \to [0,1]$ (called Lyapunov function) equal to 1 on R and 0 on A, and strictly decreasing on the other orbits:

$$\psi(F(x)) < \psi(x) \quad \text{for } x \notin A \cup R$$

So, the chain recurrent set is contained in $A \cup R$. Any compact set contained in U and containing the interior of $F(U)$ is a trapping region inducing the same attracter and repeller pair (A, R); hence, the set of attracter/repeller pairs is countable. We denote by $(A_i, R_i, \psi_i), i \in \mathbb{N}$, the family of these pairs endowed with an associated Lyapunov function. Conley (1978) proved that

$$\mathcal{R}(F) = \bigcap_{i \in \mathbb{N}} (A_i \cup R_i)$$

This induces a natural partition of $\mathcal{R}(F)$ in equivalence classes: $x \sim y$ if $x \in A_i \Leftrightarrow y \in A_i$. Conley proved that $x \sim y$ iff, for any $\varepsilon > 0$, there are ε-pseudo orbits from x to y and vice versa. The equivalence classes for $\sim$ are called chain recurrence classes.

Now, considering an average of the Lyapunov functions ψ_i one gets the following result: there is a continuous function $\varphi : M \to \mathbb{R}$ with the following properties:

- $\varphi(F(x)) \leq \varphi(x)$ for every $x \in M$, (i.e., φ is a Lyapunov function);
- $\varphi(F(x)) = \varphi(x) \Leftrightarrow x \in \mathcal{R}(F)$;
- for $x, y \in \mathcal{R}(F)$, $\quad \varphi(x) = \varphi(y) \Leftrightarrow x \sim y$; and
- the image $\varphi(\mathcal{R}(F)$ is a compact subset of $\mathbb{R}$ with empty interior.

This result is called the "fundamental theorem of dynamical systems" by several authors (see Robinson (1999)).

Any orbit is φ-decreasing from a chain recurrence class to another chain reccurence class (the global dynamics of F looks like the dynamics of the gradient flow of a function ϕ, the chain recurrence classes supplying the singularities of ϕ). However, this description of the dynamics may be very rough: if F preserves the volume, Poincaré's recurrence theorem implies that $\Omega(F) = \mathcal{R}(F) = M$; the whole M is the unique chain recurrence class and the function φ of Conley's theorem is constant.

Conley's theory provides a general procedure for describing the global topological dynamics of a system: one has to characterize the chain recurrence classes, the dynamics in restriction to each class, the stable set of each class (i.e., the set of points whose positive orbits goes to the class), and the relative positions of these stable sets.

Hyperbolicity

Smale's hyperbolic theory is the first attempt to give a global vision of almost all dynamical systems. In this section we give a very quick overview of this theory. For further details, *see* Hyperbolic Dynamical Systems.

Hyperbolic Periodic Orbits

A fixed point x of F is hyperbolic if the derivative $DF(x)$ has no (neither real nor complex) eigenvalue with modulus equal to 1. The tangent space at x

splits as $T_xM = E^s \oplus E^u$, where E^s and E^u are the $DF(x)$-invariant spaces corresponding to the eigenvalues of moduli <1 and >1, respectively. There are C^r-injectively immersed F-invariant submanifolds $W^s(x)$ and $W^u(x)$ tangent at x to E^s and E^u; the stable manifold $W^s(x)$ is the set of points y whose forward orbit goes to x. The implicit-function theorem implies that a hyperbolic fixed point x varies (locally) continuously with F; (compact parts of) the stable and unstable manifolds vary continuously for the C^r-topology when F varies with the C^r-topology.

A periodic point x of period n is hyperbolic if it is a hyperbolic fixed point of F^n and its invariant manifolds are the corresponding invariant manifolds for F^n. The stable and unstable manifold of the orbit of x, $W^s_{\text{orb}}(x)$ and $W^u_{\text{orb}}(x)$, are the unions of the invariant manifolds of the points in the orbit.

Homoclinic Classes

Distinct stable manifolds are always disjoint; however, stable and unstable manifolds may intersect. At the end of the nineteenth century, Poincaré noted that the existence of transverse homoclinic orbits, that is, transverse intersection of $W^s_{\text{orb}}(x)$ with $W^u_{\text{orb}}(x)$ (other than the orbit of x), implies a very rich dynamical behavior: indeed, Birkhoff proved that any transverse homoclinic point is accumulated by a sequence of periodic orbits (see Figure 1). The homoclinic class $H(x)$ of a periodic orbit is the closure of the transverse homoclinic point associated to x:

$$H(p) = \overline{W^s_{\text{orb}}(x) \pitchfork W^u_{\text{orb}}(x)}$$

There is an equivalent definition of the homoclinic class of x: we say that two hyperbolic periodic points x and y are homoclinically related if $W^s_{\text{orb}}(x)$ and $W^u_{\text{orb}}(x)$ intersect transversally $W^u_{\text{orb}}(y)$ and $W^s_{\text{orb}}(y)$, respectively; this defines an equivalence relation in $\text{Per}_{\text{hyp}}(F)$ and the homoclinic classes are the closure of the equivalence classes.

The homoclinic classes are transitive invariant compact sets canonically associated to the periodic

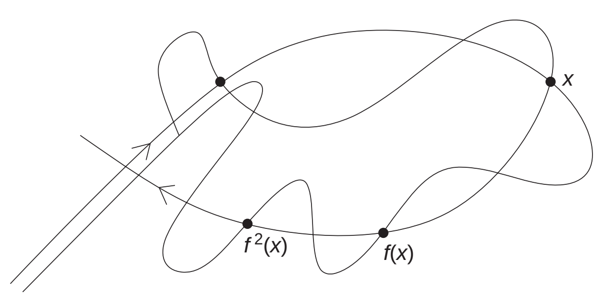

Figure 1 A transverse homoclinic orbit.

orbits. However, for general systems, homoclinic classes are not necessarily disjoint.

For more details, *see* Homoclinic Phenomena.

Smale's Hyperbolic Theory

A diffeomorphism F is *Morse–Smale* if $\Omega(F) = \text{Per}(F)$ is finite and hyperbolic, and if $W^s(x)$ is tranverse to $W^u(y)$ for any $x, y \in \text{Per}(F)$. Morse–Smale diffeomorphisms have a very simple dynamics, similar to the one of the gradient flow of a Morse function; apart from periodic points and invariant manifolds of periodic saddles, each orbit goes from a source to a sink (hyperbolic periodic repellers and attractors). Furthermore, Morse–Smale diffeomorphisms are C^1-*structurally stable*, that is, any diffeomorphism C^1-close to F is conjugated to F by a homeomorphism: the topological dynamics of F remains unchanged by small C^1-perturbation. Morse–Smale vector fields were known (Andronov and Pontryagin, 1937) to characterize the structural stability of vector fields on the sphere S^2. However, a diffeomorphism having transverse homoclinic intersections is robustly not Morse–Smale, so that Morse–Smale diffeomorphisms are not C^r-dense, on any compact manifold of dimension ≥ 2. In the early 1960s, Smale generalized the notion of hyperbolicity for nonperiodic sets in order to get a model for homoclinic orbits. The goal of the theory was to cover a whole dense open set of all dynamical systems.

An invariant compact set K is hyperbolic if the tangent space $TM|_K$ of M over K splits as the direct sum $TM_K = E^s \oplus E^u$ of two DF-invariant vector bundles, where the vectors in E^s and E^u are uniformly contracted and expanded, respectively, by F^n, for some $n > 0$. Hyperbolic sets persist under small C^1-perturbations of the dynamics: any diffeomorphism G which is C^1-close enough to F admits a hyperbolic compact set K_G close to K and the restrictions of F and G to K and K_G are conjugated by a homeomorphism close to the identity. Hyperbolic compact sets have well-defined invariant (stable and unstable) manifolds, tangent (at the points of K) to E^s and E^u and the (local) invariant manifolds of K_G vary locally continuously with G.

The existence of hyperbolic sets is very common: if y is a transverse homoclinic point associated to a hyperbolic periodic point x, then there is a transitive hyperbolic set containing x and y.

Diffeomorphisms for which $\mathcal{R}(F)$ is hyperbolic are now well understood: the chain recurrence classes are homoclinic classes, finitely many, and transitive, and admit a combinatorical model (subshift of finite type). Some of them are attractors or repellers, and the basins of the attractors cover a dense open subset of M. If, furthermore, all the stable and unstable manifolds of points in $\mathcal{R}(f)$ are transverse, the diffeomorphism is C^1-structurally stable (Robbin 1971, Robinson 1976); indeed, this condition, called "axiom A + strong transversality," is equivalent to the C^1-structural stability (Mañé 1988).

In 1970, Abraham and Smale built examples of robustly non-axiom A diffeomorphisms, when $\dim M \geq 3$: the dream of a global understanding of dynamical systems was postponed. However, hyperbolicity remains a key tool in the study of dynamical systems, even for nonhyperbolic systems.

C^r-Generic Systems

Kupka–Smale Theorem

Thom's transversality theorem asserts that two submanifolds can always be put in tranverse position by a C^r-small perturbations. Hence, for F in an open and dense subset of $\mathrm{Diff}^r(M), r \geq 1$, the graph of F in $M \times M$ is transverse to the diagonal $\Delta = \{(x,x), x \in M\}$: F has finitely many fixed points x_i, depending locally continuously on F, and 1 is not an eigenvalue of the differential $DF(x_i)$. Small local perturbations in the neighborhood of the x_i avoid eigenvalue of modulus equal to 1: one gets a dense and open subset $\mathcal{O}_1^r$ of $\mathrm{Diff}^r(M)$ such that every fixed point is hyperbolic. This argument, adapted for periodic points, provides a dense and open set $\mathcal{O}_n^r \subset \mathrm{Diff}^r(M)$, such that every periodic point of period n is hyperbolic. Now $\bigcap_{n\in\mathbb{N}} \mathcal{O}_n^r$ is a residual subset of $\mathrm{Diff}^r(M)$, for which every periodic point is hyperbolic.

Similarly, the set of diffeomorphisms $F \in \bigcap_{i=0}^n \mathcal{O}_i^r(M)$ such that all the disks of size n, of invariant manifolds of periodic points of period less that n, are pairwise transverse, is open and dense. One gets the Kupka–Smale theorem (see Palis and de Melo (1982) for a detailed exposition): *for C^r-generic diffeomorphisms $F \in \mathrm{Diff}^r(M)$, every periodic orbit is hyperbolic and $W^s(x)$ is transverse to $W^u(y)$ for $x, y \in Per(F)$.*

Generic Properties of Low-Dimensional Systems

Poincaré–Denjoy theory describes the topological dynamics of all diffeomorphisms of the circle S^1 (*see* Homeomorphisms and Diffeomorphisms of the Circle). Diffeomorphisms in an open and dense subset of $\mathrm{Diff}_+^r(S^1)$ have a nonempty finite set of periodic orbits, all hyperbolic, and alternately attracting (sink) or repelling (source). The orbit of a nonperiodic point comes from a source and goes to a sink. Two C^r-generic diffeomorphisms of S^1 are conjugated iff they have same rotation number and same number of periodic points.

This simple behavior has been generalized in 1962 by Peixoto for vector fields on compact orientable surfaces S. Vector fields X in a C^r-dense and open subset are Morse–Smale, hence structurally stable (see Palis and de Melo (1982) for a detailed proof). Peixoto gives a complete classification of these vector fields, up to topological equivalence.

Peixoto's argument uses the fact that the return maps of the vector field on transverse sections are increasing functions: this helped control the effect on the dynamics of small "monotonous" perturbations, and allowed him to destroy any nontrivial recurrences. Peixoto's result remains true on nonorientable surfaces for the C^1-topology but remains an open question for $r > 1$: is the set of Morse–Smale vector fields C^2-dense, for S nonorientable closed surface?

Local C^2-Genericity of Wild Behavior for Surface Diffeomorphisms

The generic systems we have seen above have a very simple dynamics, simpler than the general systems. This is not always the case. In the 1970s, Newhouse exhibited a C^2-open set $\mathcal{O} \subset \mathrm{Diff}^2(S^2)$ (where S^2 denotes the two-dimensional sphere), such that C^2-generic diffeomorphisms $F \in \mathcal{O}$ have infinitely many hyperbolic periodic sinks. In fact, C^2-generic diffeomorphisms in $\mathcal{O}$ present many other pathological properties: for instance, it has been recently noted that they have uncountably many chain recurrence classes without periodic orbits. Densely (but not generically) in $\mathcal{O}$, they present many other phenomena, such as strange (Henon-like) attractors (*see* Lyapunov Exponents and Strange Attractors).

This phenomenon appears each time that a diffeomorphism F_0 admits a hyperbolic periodic point x whose invariant manifolds $W^s(x)$ and $W^u(x)$ are tangent at some point $p \in W^s(x) \cap W^u(x)$ (p is a homoclinic tangency associated to x). Homoclinic tangencies appear locally as a codimension-1 submanifold of $\mathrm{Diff}^2(S^2)$; they are such a simple phenomenon that they appear in very natural contexts. When a small perturbation transforms the tangency into tranverse intersections, a new hyperbolic set K with very large fractal dimensions is created. The local stable and unstable manifolds of K, each homeomorphic to the product of a Cantor set by a segment, present tangencies in a C^2-robust way, that is, for F in some C^2-open set $\mathcal{O}$ (see **Figure 2**). As a consequence, for a C^2-dense subset of $\mathcal{O}$, the

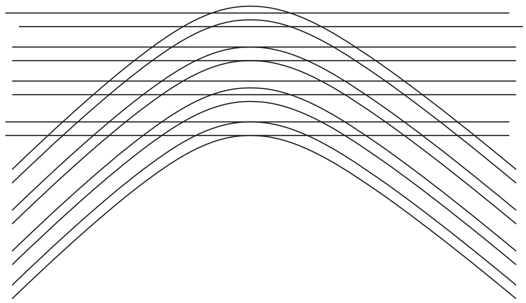

Figure 2 Robust tangencies.

invariant manifolds of the point x present some tangency (this is not generic, by Kupka–Smale theorem). If the Jacobian of F at x is <1, each tangency allows to create one more sink, by an arbitrarily small perturbation. Hence, the sets of diffeomorphisms having more than n hyperbolic sinks are dense open subsets of $\mathcal{O}$, and the intersection of all these dense open subsets is the announced residual set. See Palis and Takens (1993) for details on this deep argument.

C^1-Generic Systems: Global Dynamics and Periodic Orbits

See Bonatti *et al.* (2004), Chapter 10 and Appendix A, for a more detailed exposition and precise references.

Perturbations of Orbits: Closing and Connecting Lemmas

In 1968, Pugh proved the following Lemma.

Closing lemma *If x is a nonwandering point of a diffeomorphism F, then there are diffeomorphisms G arbitrarily C^1-close to F, such that x is periodic for G.*

Consider a segment $x_0, \ldots, x_n = F^n(x_0)$ of orbit such that x_n is very close to $x_0 = x$; one would like to take G close to F such that $G(x_n) = x_0$, and $G(x_i) = F(x_i) = x_{i+1}$ for $i \neq n$. This idea works for the C^0-topology (so that the C^0-closing lemma is easy). However, if one wants G ε-C^1-close to F, one needs that the points $x_i, i \in \{1, \ldots, n-1\}$, remain at distance $d(x_i, x_0)$ greater than $C(d(x_n, x_0)/\varepsilon)$, where C bounds $\|Df\|$ on M. If C/ε is very large, such a segment of orbit does not exist. Pugh solved this difficulty in two steps: the perturbation is first spread along a segment of orbit of x in order to decrease this constant; then a subsegment $y_0, \ldots, y_k$ of $x_0, \ldots, x_n$ is selected, verifying the geometrical condition.

For the C^2 topology, the distances $d(x_i, x_0)$ need to remain greater than $\sqrt{d(x_n, x_0)/\varepsilon} \gg d(x_n, x_0)$. This new difficulty is why the C^2-closing lemma remains an open question.

Pugh's argument does not suffice to create homoclinic point for a periodic orbit whose unstable manifold accumulates on the stable one. In 1998, Hayashi solved this problem proving the

Connecting lemma (Hayashi 1997) *Let y and z be two points such that the forward orbit of y and the backward orbit of z accumulate on the same nonperiodic point x. Fix some $\varepsilon > 0$. There is $N > 0$ and a ε-C^1-perturbation G of F such that $G^n(y) = z$ for some $n > 0$, and $G \equiv F$ out of an arbitrary small neighborhood of $\{x, F(x), \ldots, F^N(x)\}$.*

Using Hayashi's arguments, we (with Crovisier) proved the following lemma:

Connecting lemma for pseudo-orbits (Bonatti and Crovisier 2004) *Assume that all periodic orbits of F are hyperbolic; consider $x, y \in M$ such that, for any $\varepsilon > 0$, there are ε-pseudo-orbits joining x to y; then there are arbitrarily small C^1-perturbations of F for which the positive orbit of x passes through y.*

Densities of Periodic Orbits

As a consequence of the perturbations lemma above, we (Bonatti and Crovisier 2004) proved that for F C^1-generic,

$$\mathcal{R}(F) = \Omega(F) = \overline{\mathrm{Per}_{\mathrm{hyp}}(F)}$$

where $\overline{\mathrm{Per}_{\mathrm{hyp}}(F)}$ denotes the closure of the set of hyperbolic periodic points.

For this, consider the map $\Psi: F \mapsto \Psi(F) = \overline{\mathrm{Per}_{\mathrm{hyp}}(F)}$ defined on $\mathrm{Diff}^1(M)$ and with value in $\mathcal{K}(M)$, space of all compact subsets of M, endowed with the Hausdorff topology. $\mathrm{Per}_{\mathrm{hyp}}(F)$ may be approximated by a finite set of hyperbolic periodic points, and this set varies continuously with F; so $\mathrm{Per}_{\mathrm{hyp}}(F)$ varies lower-semicontinuously with F: for G very close to F, $\mathrm{Per}_{\mathrm{hyp}}(G)$ cannot be very much smaller than $\mathrm{Per}_{\mathrm{hyp}}(F)$. As a consequence, a result from general topology asserts that, for C^1-generic F, the map Ψ is continuous at F. On the other hand, C^1-generic diffeomorphisms are Kupka–Smale, so that the connecting lemma for pseudo-orbits may apply: if $x \in \mathcal{R}(F), x$ can be turned into a hyperbolic periodic point by a C^1-small perturbation of F. So, if $x \notin \overline{\mathrm{Per}_{\mathrm{hyp}}(F)}, F$ is not a continuity point of Ψ, leading to a contradiction.

Furthermore, Crovisier proved the following result: "for C^1-generic diffeomorphisms, each chain recurrence class is the limit, for the Hausdorff distance, of a sequence of periodic orbits."

This good approximation of the global dynamics by the periodic orbits will now allow us to better understand the chain recurrence classes of C^1-generic diffeomorphisms.

Chain Recurrence Classes/Homoclinic Classes of C^1-Generic Systems

Tranverse intersections of invariant manifolds of hyperbolic orbits are robust and vary locally continuously with the diffeomorphisms F. So, the homoclinic class $H(x)$ of a periodic point x varies lower-semicontinuously with F (on the open set where the continuation of x is defined). As a consequence, for C^r-generic diffeomorphisms ($r \geq 1$), each homoclinic class varies continuously with F. Using the connecting lemma, Arnaud (2001) proved the following result: "for Kupka–Smale diffeomorphisms, if the closures $\overline{W^u_{orb}}(x)$ and $\overline{W^s_{orb}}(x)$ have some intersection point z, then a C^1-pertubation of F creates a tranverse intersection of $\overline{W^u_{orb}}(x)$ and $\overline{W^s_{orb}}(x)$ at z." So, if $z \notin H(x)$, then F is not a continuity point of the function $F \mapsto H(x, F)$. Hence, for C^1-generic diffeomorphisms F and for every periodic point x,

$$H(x) = \overline{W^u_{orb}}(x) \cap \overline{W^s_{orb}}(x)$$

In the same way, $\overline{W^u_{orb}}(x)$ and $\overline{W^s_{orb}}(x)$ vary locally lower-semicontinuously with F so that, for F C^r-generic, the closures of the invariant manifolds of each periodic point vary locally continuously. For Kupka–Smale diffeomorphisms, the connecting lemma for pseudo-orbits implies: "if z is a point in the chain recurrence class of a periodic point x, then a C^1-small perturbation of F puts z on the unstable manifold of x"; so, if $z \notin \overline{W^u_{orb}}(x)$, then F is not a continuity point of the function $F \mapsto \overline{W^u_{orb}}(x, F)$. Hence, for C^1-generic diffeomorphisms F and for every periodic point x, the chain recurrence class of x is contained in $\overline{W^u_{orb}}(x) \cap \overline{W^s_{orb}}(x)$, and, therefore, coincides with the homoclinic class of x. This argument proves:

> For a C^1-generic diffeomorphism F, each homoclinic class $H(x)$ is a chain recurrence class of F (of Conley's theory): a chain recurrence class containing a periodic point x coincides with the homoclinic class $H(x)$. In particular, two homoclinic classes are either disjoint or equal.

Tame and Wild Systems

For generic diffeomorphisms, the number $N(F) \in \mathbb{N} \cup \{\infty\}$ of homoclinic classes varies lower-semicontinuously with F. One deduces that $N(F)$ is locally constant on a residual subset of $\mathrm{Diff}^1(M)$ (Abdenur 2003).

A local version (in the neighborhood of a chain recurrence class) of this argument shows that, for C^1-generic diffeomorphisms, any isolated chain recurrence classe C is robustly isolated: for any diffeomorphism G, C^1-close enough to F, the intersection of $\mathcal{R}(G)$ with a small neighborhood of C is a unique chain recurrence class C_G close to C.

One says that a diffeomorphism is "tame" if each chain recurrence class is robustly isolated. We denote by $\mathcal{T}(M) \subset \mathrm{Diff}^1(M)$ the (C^1-open) set of tame diffeomorphisms and by $\mathcal{W}(M)$ the complement of the closure of $\mathcal{T}(M)$. C^1-generic diffeomorphisms in $\mathcal{W}(M)$ have infinitely many disjoint homoclinic classes, and are called "wild" diffeomorphisms.

Generic tame diffeomorphisms have a global dynamics analogous to hyperbolic systems: the chain recurrence set admits a partition into finitely many homoclinic classes varying continuously with the dynamics. Every point belongs to the stable set of one of these classes. Some of the homoclinic classes are (transitive) topological attractors, and the union of the basins covers a dense open subset of M, and the basins vary continuously with F (Carballo Morales 2003). It remains to get a good description of the dynamics in the homoclinic classes, and particularly in the attractors. As we shall see in the next section, tame behavior requires some kind of weak hyperbolicity. Indeed, in dimension 2, tame diffeomorphisms satisfy axiom A and the noncycle condition.

As of now, very little is known about wild systems. One knows some semilocal mechanisms generating locally C^1-generic wild dynamics, therefore proving their existence on any manifold with dimension $\dim(M) \geq 3$ (the existence of wild diffeomorphisms in dimension 2, for the C^1-topology, remains an open problem). Some of the known examples exhibit a universal dynamics: they admit infinitely many disjoint periodic disks such that, up to renormalization, the return maps on these disks induce a dense subset of diffeomorphisms of the disk. Hence, these locally generic diffeomorphisms present infinitely many times any robust property of diffeomorphisms of the disk.

Ergodic Properties

A point x is well closable if, for any $\varepsilon > 0$ there is G ε-C^1-close to F such that x is periodic for G and $d(F^i(x), G^i(x)) < \varepsilon$ for $i \in \{0, \ldots, p\}$, p being the period of x. As an important refinement of Pugh's closing lemma, Mañé proved the following lemma:

Ergodic closing lemma *For any F-invariant probability, almost every point is well closable.*

As a consequence, "for C^1-generic diffeomophisms, any ergodic measure μ is the weak limit of a sequence of Dirac measures on periodic orbits, which converges also in the Hausdorff distance to the support of μ."

It remains an open problem to know if, for C^1-generic diffeomorphisms, the ergodic measures supported in a homoclinic class are approached by periodic orbits in this homoclinic class.

Conservative Systems

The connecting lemma for pseudo-orbits has been adapted for volume preserving and symplectic diffeomorphisms, replacing the condition on the periodic orbits by another generic condition on the eigenvalues. As a consequence, one gets: "C^1-generic volume-preserving or symplectic diffeomorphisms are transitive, and M is a unique homoclinic class."

Notice that the KAM theory implies that this result is wrong for C^4-generic diffeomorphisms, the persistence of invariant tori allowing to break robustly the transitivity.

The Oxtoby–Ulam (1941) theorem asserts that C^0-generic volume-preserving homeomorphisms are ergodic. The ergodicity of C^1-generic volume-preserving diffeomorphisms remains an open question.

Hyperbolic Properties of C^1-Generic Diffeomorphisms

For a more detailed exposition of hyperbolic properties of C^1-generic diffeomorphisms, the reader is referred to Bonatti *et al.* (2004, chapter 7 and appendix B).

Perturbations of Products of Matrices

The C^1-topology enables us to do small perturbations of the differential DF at a point x without perturbing either $F(x)$ or F out of an arbitrarily small neighborhood of x. Hence, one can perturb the differential of F along a periodic orbit, without changing this periodic orbit (Frank's lemma). When x is a periodic point of period n, the differential of F^n at x is fundamental for knowing the local behavior of the dynamics. This differential is (up to a choice of local coordinates) a product of the matrices $DF(x_i)$, where $x_i = F^i(x)$. So, the control of the dynamical effect of local perturbations along a periodic orbit comes from a problem of linear algebra: "consider a product $\mathcal{A} = A_n \circ A_{n-1} \circ \cdots \circ A_1$ of $n \gg 0$ bounded linear ismorphisms of $\mathbb{R}^d$; how do the eigenvalues and the eigenspaces of $\mathcal{A}$ vary under small perturbations of the A_i?"

A partial answer to this general problem uses the notion of dominated splitting. Let $X \subset M$ be an F-invariant set such that the tangent space of M at the points $x \in X$ admits a DF-invariant splitting $T_x(M) = E_1(x) \oplus \cdots E_k(x)$, the dimensions $\dim(E_i(x))$ being independent of x. This splitting is dominated if the vectors in E_{i+1} are uniformly more expanded than the vectors in E_i: there exists $\ell > 0$ such that, for any $x \in X$, any $i \in \{1, \ldots, k-1\}$ and any unit vectors $u \in E_i(x)$ and $v \in E_{i+1}(x)$, one has

$$\|DF^\ell(u)\| < \tfrac{1}{2}\|DF^\ell(v)\|$$

Dominated splittings are always continuous, extend to the closure of X, and persist and vary continuously under C^1-perturbation of F.

Dominated Splittings versus Wild Behavior

Let $\{\gamma_i\}$ be a set of hyperbolic periodic orbits. On $X = \bigcup \gamma_i$ one considers the natural splitting $TM|_X = E^s \oplus E^u$ induced by the hyperbolicity of the γ_i. Mañé (1982) proved: "if there is a C^1-neighborhood of F on which each γ_i remains hyperbolic, then the splitting $TM|_X = E^s \oplus E^u$ is dominated."

A generalization of Mañé's result shows: "if a homoclinic class $H(x)$ has no dominated splitting, then for any $\varepsilon > 0$ there is a periodic orbit γ in $H(x)$ whose derivative at the period can be turned into an homothety, by an ε-small perturbation of the derivative of F along the points of γ"; in particular, this periodic orbit can be turned into a sink or a source. As a consequence, one gets: "for C^1-generic diffeomorphisms F, any homoclinic class either has a dominated splitting or is contained in the closure of the (infinite) set of sinks and sources."

This argument has been used in two directions:

- Tame systems must satisfy some hyperbolicity. In fact, using the ergodic closing lemma, one proves that the homoclinic classes $H(x)$ of tame diffeomorphisms are volume hyperbolic, that is, there is a dominated splitting $TM = E_1 \oplus \cdots \oplus E_k$ over $H(x)$ such that DF contracts uniformly the volume in E_1 and expands uniformly the volume in E_k.
- If F admits a homoclinic class $H(x)$ which is robustly without dominated splittings, then generic diffeomorphisms in the neighborhood of F are wild: at this time this is the unique known way to get wild systems.

See also: Cellular Automata; Chaos and Attractors; Fractal Dimensions in Dynamics; Homeomorphisms and Diffeomorphisms of the Circle; Homoclinic Phenomena; Hyperbolic Dynamical Systems; Lyapunov Exponents and Strange Attractors; Polygonal Billiards; Singularity and Bifurcation Theory; Synchronization of Chaos.

Further Reading

Abdenur F (2003) Generic robustness of spectral decompositions. *Annales Scientifiques de l'Ecole Normale Superieure IV* 36(2): 213–224.

Andronov A and Pontryagin L (1937) Systèmes grossiers. *Dokl. Akad. Nauk. USSR* 14: 247–251.

Arnaud M-C (2001) Création de connexions en topologie C^1. *Ergodic Theory and Dynamical Systems* 21(2): 339–381.

Bonatti C and Crovisier S (2004) Récurrence et généricité. *Inventiones Mathematical* 158(1): 33–104 (French) (see also the short English version: (2003) Recurrence and genericity. *Comptes Rendus Mathematique de l'Academie des Sciences Paris* 336 (10): 839–844).

Bonatti C, Dìaz LJ, and Viana M (2004) Dynamics Beyond Uniform Hyperbolicity. *Encyclopedia of Mathematical Sciences*, vol. 102. Berlin: Springer.

Carballo CM and Morales CA (2003) Homoclinic classes and finitude of attractors for vector fields on *n*-manifolds. *Bulletins of the London Mathematical Society* 35(1): 85–91.

Hayashi S (1997) Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows. *Annals of Mathematics* 145: 81–137.

Mañé R (1988) A proof of the C^1 stability conjecture. *Inst. Hautes Études Sci. Publ. Math.* 66: 161–210.

Palis J and de Melo W (1982) *Gemetric Theory of Dynamical Systems, An Introduction.* New York–Berlin: Springer.

Palis J and Takens F (1993) *Hyperbolicity and Sensitive-Chaotic Dynamics at Homoclinic Bifurcation.* Cambridge: Cambridge University Press.

Robbin JW (1971) A structural stability theorem. *Annals of Mathematics* 94(2): 447–493.

Robinson C (1974) Structural stability of vector fields. *Annals of Mathematics* 99(2): 154–175 (errata. *Annals of Mathematics* 101(2): 368 (1975)).

Robinson C (1976) Structural stability of C^1 diffeomorphisms. *Journal of Differential Equations* 22(1): 28–73.

Robinson C (1999) Dynamical Systems. *Studies in Advanced Mathematics*. Boca Raton, FL: CRC Press.