



**HAL**  
open science

# Moving bottlenecks for the Aw-Rascle-Zhang traffic flow model

Stefano Villa, Paola Goatin, Christophe Chalons

► **To cite this version:**

Stefano Villa, Paola Goatin, Christophe Chalons. Moving bottlenecks for the Aw-Rascle-Zhang traffic flow model. *Discrete and Continuous Dynamical Systems - Series B*, 2017, 22 (10), pp.3921-3952. hal-01347925

**HAL Id: hal-01347925**

**<https://hal.science/hal-01347925v1>**

Submitted on 21 Jul 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Moving bottlenecks for the Aw-Rascle-Zhang traffic flow model

STEFANO VILLA\*      PAOLA GOATIN†      CHRISTOPHE CHALONS‡

July 21, 2016

## Abstract

We introduce a second order model for traffic flow with moving bottlenecks. The model consists of the  $2 \times 2$  Aw-Rascle-Zhang system with a point-wise flow constraint whose trajectory is governed by an ordinary differential equation. We define two Riemann solvers, characterize the corresponding invariant domains and propose numerical strategies, which are effective in capturing the non-classical shocks due to the constraint activation.

**Key words:** Traffic flow models; Moving bottlenecks; Unilateral flux constraints; Riemann Solvers; Finite Volume schemes.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>The ARZ model with moving constraints</b>	<b>3</b>
<b>3</b>	<b>The Riemann problem</b>	<b>5</b>
3.1	The first Riemann solver $\mathcal{RS}_1^\alpha$ . . . . .	7
3.2	The second Riemann solver $\mathcal{RS}_2^\alpha$ . . . . .	8
<b>4</b>	<b>Invariant domains</b>	<b>9</b>
4.1	$\mathcal{D}_{v_1, v_2, w_1, w_2} \subseteq \mathcal{U}$ . . . . .	10
4.2	$\mathcal{D}_{v_1, v_2, w_1, w_2} \subseteq \mathcal{V}$ . . . . .	11
4.2.1	Proof of Theorem 1 (ii) . . . . .	11
4.2.2	Proof of Theorem 2 (ii) . . . . .	12
4.3	$\mathcal{D}_{v_1, v_2, w_1, w_2}$ intersects both $\mathcal{U}$ and $\mathcal{V}$ . . . . .	14
<b>5</b>	<b>Numerical schemes</b>	<b>15</b>
5.1	The constrained Godunov scheme for $\mathcal{RS}_1^\alpha$ . . . . .	16
5.2	The constrained Godunov schemes for $\mathcal{RS}_2^\alpha$ . . . . .	20
5.3	Treatment of the contact discontinuities . . . . .	24
	<b>Appendices</b>	<b>26</b>
<b>A</b>	<b>A Front-Tracking technique to compute the bus trajectory</b>	<b>26</b>
A.0.1	Case (i) . . . . .	27
A.0.2	Case (ii) . . . . .	29

---

\*University of Milano-Bicocca, Italy, & Inria Sophia Antipolis - Méditerranée, France, E-mail: s.villa33@campus.unimib.it

†Inria Sophia Antipolis - Méditerranée, France, E-mail: paola.goatin@inria.fr

‡Université de Versailles St. Quentin, France, E-mail: christophe.chalons@uvsq.fr

# 1 Introduction

Conservation laws with pointwise unilateral constraints on the flux have been studied widely in the last decade. Their peculiarity and main challenge from the analytical and numerical points of view is the possible presence of a non-classical shock (i.e. violating the classical Kruřkov [18] or Lax [19] entropy admissibility conditions) at the constraint location.

Scalar conservation laws with *fixed* flux constraints have been introduced in [12], where an existence result is established for data with bounded total variation (*BV*). Further results have been obtained by [4, 11, 13]. In particular, [4] provides the first description of the numerical treatment of the flux constraint in a finite volume setting. In the articles above, the problem formulation is intended to provide a general mathematical framework to model local constraints in traffic flow, such as toll gates, construction works and traffic lights. The approach has been extended to systems in the case of the Aw-Rascle-Zhang (ARZ) second order model [6, 23]. [16] provides the definition and analysis of two Riemann solvers, as well as the corresponding numerical strategies to compute approximate solutions by a modified Godunov scheme. Existence results for the ARZ model with fixed constraints are provided in [3, 5, 17].

More recently, the approach has been extended to *moving* constraints that, in the framework of traffic flow models, are meant to describe the dynamics of moving bottlenecks caused by large slow moving vehicles like trucks or buses. The resulting model consists of a strongly coupled PDE-ODE system, where the PDE is a constrained scalar conservation law describing the global traffic evolution and the ODE gives the trajectory of the slow vehicle, which in turn is affected by downstream traffic conditions through the speed law. The existence of solutions with *BV* initial data has been proved in [15], while two numerical strategies have been proposed in [9, 14] to compute approximate solutions, which correctly capture the non-classical shock arising when the constraint is enforced.

In this article, we consider the ARZ traffic flow model subject to moving constraints. We define two constrained Riemann solvers, characterize the corresponding invariant domains and propose numerical strategies, which are effective in capturing the non-classical shocks due to the constraint activation. The present contribution is a significant step forward compared to the previous contributions since we propose here a both theoretical and numerical comprehensive treatment of a *system* of conservation laws with a *moving* constraint. Note also that one of the two constrained Riemann solvers is non conservative.

The numerical approximation of non-classical shocks is well-known to be very challenging since these discontinuities are very sensitive to the underlying numerical diffusion of the scheme, see for instance [20] for more details. In [7], the authors define a conservative scheme which is based on in-cell discontinuous reconstructions of the non-classical shocks for approximating the solutions of a scalar conservation law with a non convex flux. The striking feature of the strategy is to allow for a perfect control of the numerical diffusion associated with the non-classical discontinuities. More precisely, it allows for the *exact* computation of such isolated simple waves. In [9], the authors succeeded in extending this approach based on in-cell reconstructions to a constrained but still *scalar* conservation law. Here, we extend it to the *system* case of the ARZ constrained traffic flow model, which is much more difficult to deal with since in-cell reconstructions have to be proposed for the two components of the unknowns vector. Note in addition that a *non-conservative* Riemann solver associated with the ARZ model will be studied in details. At last, let us mention that the extension of the in-cell discontinuous reconstruction strategy to the system case has already been considered for *non-constrained* problems in the recent paper [2] for the non diffusive computation of classical solutions of the barotropic gas dynamics equations in Lagrangian coordinates, and in [1] for the computation of non-classical solutions associated with a model of elastodynamics.

## 2 The ARZ model with moving constraints

Let  $V$  and  $R$  be respectively the maximal speed and the maximal density of the vehicles on a stretch of road and let  $\alpha \in (0, 1)$  be the coefficient expressing the reduction of the road capacity at the bus position, so that  $\alpha R$  is the maximal density at  $x = y(t)$  for which the velocity of the vehicles is  $v = 0$ . We denote by  $F_\alpha$  the upper bound of the flux at the bus position and we consider the following PDE-ODE system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho v w) = 0, \end{cases} \quad x \in \mathbb{R}, t > 0, \quad (2.1a)$$

$$\rho(t, y(t))(v(t, y(t)) - \dot{y}(t)) \leq F_\alpha, \quad (2.1b)$$

$$\dot{y}(t) = \omega(v(t, y(t)+)), \quad (2.1c)$$

with initial conditions

$$\rho(0, x) = \rho_0(x), \quad (2.2a)$$

$$v(0, x) = v_0(x), \quad (2.2b)$$

$$y(0) = y_0. \quad (2.2c)$$

The phase space is defined by the domain

$$\mathcal{D} = \{(\rho, v) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 \leq v \leq V, 0 \leq v + p(\rho) \leq p(R)\}$$

away from the bus position, and

$$\mathcal{D}_\alpha = \{(\rho, v) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 \leq v \leq V, 0 \leq v + p(\rho) \leq p(\alpha R)\}$$

at the bus position. Above,  $\rho = \rho(t, x)$  and  $v = v(t, x)$  denote respectively the density and the mean velocity of traffic. Note that the quantity  $w = v + p(\rho)$ , usually referred to as *Lagrangian marker*, is transported at velocity  $v$  and depends on the density through a pressure law  $p \in \mathcal{C}^2([0, +\infty[; [0, +\infty[)$  satisfying the following hypotheses:

$$p(0) = 0, \quad (2.3a)$$

$$p'(\rho) > 0 \quad \text{for every } \rho > 0, \quad (2.3b)$$

$$\rho \mapsto \rho p(\rho) \quad \text{is strictly convex.} \quad (2.3c)$$

**Basics.** We recall in this paragraph the basic properties of system (2.1a). If we denote the flux function by

$$f(\rho, v) = \begin{pmatrix} f_1(\rho, v) \\ f_2(\rho, v) \end{pmatrix} = \begin{pmatrix} \rho v \\ \rho v(v + p(\rho)) \end{pmatrix},$$

the eigenvalues of the Jacobian matrix  $Df$  in the  $(\rho, v)$  coordinates are given by

$$\lambda_1(\rho, v) = v - \rho p'(\rho), \quad \lambda_2(\rho, v) = v,$$

the eigenvectors are

$$r_1(\rho, v) = \begin{pmatrix} -1 \\ p'(\rho) \end{pmatrix}, \quad r_2(\rho, v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and we observe that

$$\nabla \lambda_1 \cdot r_1 = 2p'(\rho) + \rho p''(\rho) > 0, \quad \nabla \lambda_2 \cdot r_2 = 0,$$

which means that the first characteristic field is genuinely non-linear and the second is linearly degenerate. We also recall that system (2.1a) belongs to the Temple class [21], and its Riemann invariants are given by

$$W_1 = v + p(\rho), \quad W_2 = v.$$

The Lax curves through a point  $(\bar{\rho}, \bar{v})$  in the  $(\rho, v)$ -plane are therefore given by

$$v = L_1(\rho; \bar{\rho}, \bar{v}) := \bar{v} + p(\bar{\rho}) - p(\rho), \quad v = L_2(\rho; \bar{\rho}, \bar{v}) := \bar{v}.$$

At last, let us briefly recall that the solution to the Riemann problem associated with the usual system (2.1a) (with no constraint) is given by two simple waves separated by a single intermediate state. The first simple wave is a shock wave or a rarefaction wave associated with the first characteristic field and such that the intermediate state belongs to the first Lax curve  $L_1$  passing through the left initial state, while the second one is a contact discontinuity associated with the second characteristic field and such that the intermediate state belongs to the second Lax curve  $L_2$  passing through the right initial state.

**Bus trajectory and flux constraint.** In (2.1b), (2.1c),  $y = y(t)$  denotes the bus trajectory, which moves at speed  $\omega : \mathbb{R}^+ \rightarrow [0, V_b]$  given by

$$\omega(v) := \begin{cases} V_b & \text{if } V_b < v, \\ v & \text{if } V_b \geq v, \end{cases}$$

with  $0 < V_b < V$ . In case of no flux restriction due to the presence of the bus, namely when  $v \leq V_b$ , the solution to the Riemann problem associated with (2.1) is expected to be the same as the one associated with (2.1a) (with no constraint) and briefly described above. On the contrary, when the bus acts on the flow, namely when  $v > V_b$  around the bus position, the general idea to construct the solution to the Riemann problem is similar in the sense that the left and right initial states will be joined by simple waves associated with the two characteristic fields, but one has to check carefully that the phase space  $\mathcal{D}_\alpha$  is preserved, namely that  $0 \leq v \leq V$  and  $v + p(\rho) \leq p(\alpha R)$  at the bus position. The first constraint is already satisfied when the bus is not present, so that one has to take care of the second inequality which is also equivalent to

$$\rho(v - \dot{y}) \leq \rho(p(\alpha R) - p(\rho)) - \rho \dot{y}.$$

Under this form, it is clear that the constraint reads as a constraint on the relative flux at the bus position. In order to go further in that direction, let us determine under which condition the quantity  $v + p(\rho)$  equals a constant  $K$  such that  $K \leq p(\alpha R)$ , or equivalently

$$\rho(v - \dot{y}) = \rho(K - p(\rho)) - \rho \dot{y},$$

with  $K$  such that  $K \leq p(\alpha R)$ . The function  $\rho \rightarrow \phi(\rho; K) = \rho(K - p(\rho)) - \rho \dot{y}$  is strictly concave in  $\rho$  by assumption (2.3c) and non-decreasing in  $K$  since  $\rho \geq 0$ . Moreover, we have  $\phi(0; K) = 0$  for all  $K$ . Therefore, the maximal possible value  $F_\alpha$  of the relative flux corresponds to  $K = p(\alpha R)$  and

$$F_\alpha = \phi(\rho_\alpha; p(\alpha R)) = \rho_\alpha(p(\alpha R) - p(\rho_\alpha)) - \rho_\alpha \dot{y}$$

with  $\rho_\alpha$  such that  $\phi'(\rho_\alpha; p(\alpha R)) = 0$ , that is to say

$$p(\alpha R) - \rho_\alpha p'(\rho_\alpha) - p(\rho_\alpha) - \dot{y} = 0.$$

Easy calculations allow to find that the largest admissible flux is given by  $F_\alpha = \rho_\alpha^2 p'(\rho_\alpha)$ . Therefore, the classical solution will remain admissible provided that the relative flux does not exceed the upper bound  $F_\alpha$ . This criterion will be the key ingredient to determine the two possible Riemann solutions devised in the next section. As we will see, these solutions necessarily develop non-classical shocks, in addition to the usual simple waves associated with the first and second characteristic fields. To conclude this section, Figure 1 below illustrates the above considerations with the notations  $w_\alpha := p(\alpha R)$  and  $w_{max} := p(R)$ .

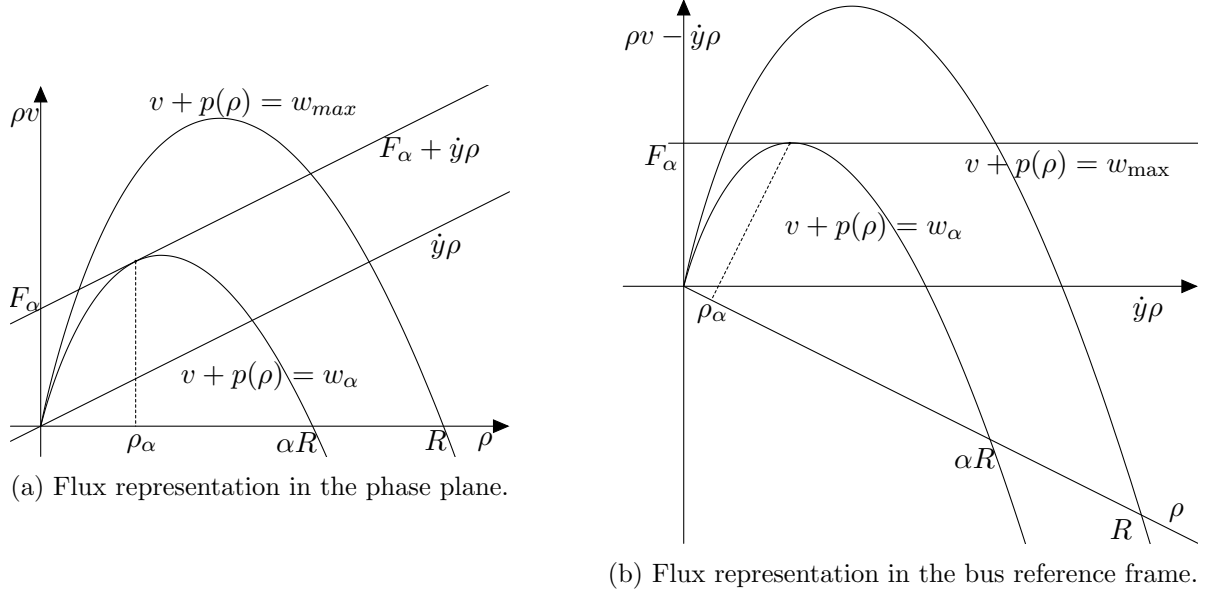


Figure 1: Representation of the phase plane in the fixed and in the bus reference frame.

### 3 The Riemann problem

Let  $(\rho^l, v^l)$  and  $(\rho^r, v^r)$  be two points in the domain  $\mathcal{D}$ . We consider the Riemann problem for (2.1) corresponding to the initial data

$$\rho(0, x) = \begin{cases} \rho^l & \text{if } x < 0, \\ \rho^r & \text{if } x > 0, \end{cases} \quad (3.1a)$$

$$v(0, x) = \begin{cases} v^l & \text{if } x < 0, \\ v^r & \text{if } x > 0, \end{cases} \quad (3.1b)$$

$$y(0) = 0. \quad (3.1c)$$

We look for self-similar solutions of (2.1), (3.1). Therefore, we assume that the bus speed will be constant:  $\dot{y}(t) = V_b$  for all  $t > 0$ .

Let  $I$  be the set

$$\begin{aligned} I &= \{\rho \in [0, R] : \rho L_1(\rho, \rho^l, v^l) = \rho(v^l + p(\rho^l) - p(\rho)) = F_\alpha + \rho V_b\} = \\ &= \{\rho \in [0, R] : \rho(L_1(\rho, \rho^l, v^l) - V_b) = F_\alpha\}. \end{aligned}$$

If  $I \neq \emptyset$ , let  $(\hat{\rho}, \hat{v})$  and  $(\check{\rho}_1, \check{v}_1)$  be the points defined by

$$\hat{\rho} = \max I, \quad \hat{v} = \frac{F_\alpha}{\hat{\rho}} + V_b, \quad \check{\rho}_1 = \min I \quad \text{and} \quad \check{v}_1 = \frac{F_\alpha}{\check{\rho}_1} + V_b. \quad (3.2)$$

These are respectively the points with maximal and minimal density of the Lax curve of the first family passing through  $(\rho^l, v^l)$  which satisfy the condition (2.1b) on the flux. Moreover, we define the point  $(\check{\rho}_2, \check{v}_2)$  as

$$\check{\rho}_2 = \frac{F_\alpha}{v^r - V_b} \quad \text{and} \quad \check{v}_2 = v^r. \quad (3.3)$$

This is the point of maximal density of the Lax curve of the second family passing through  $(\rho^r, v^r)$  for which (2.1b) is satisfied. Finally we denote by  $(\rho^m, v^m)$  the middle state of the classical Riemann solver for (2.1a), (3.1a), (3.1b):

$$v^m = v^r, \quad \rho^m = \max \left\{ p^{-1} \left( v^l + p(\rho^l) - v^r \right), 0 \right\}.$$

All these points are depicted in Figure 2.

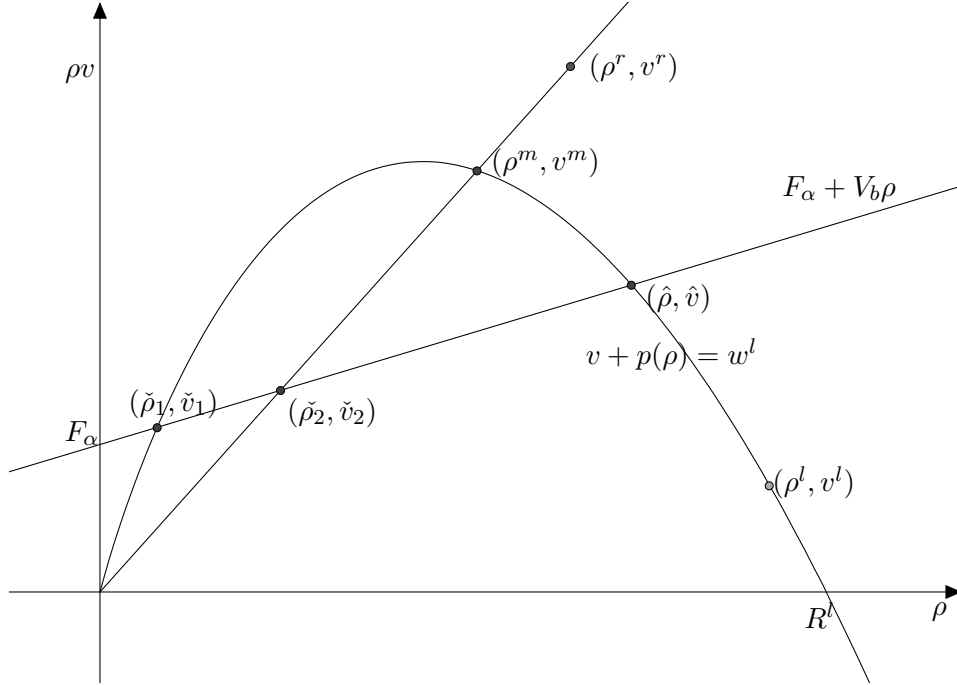


Figure 2: Notations used in the definition of the Riemann solvers.

Let  $\mathcal{RS}$  be the standard Riemann solver for (2.1a), see [6], and let

$$\bar{\rho}((\rho^l, v^l), (\rho^r, v^r))(\cdot) \quad \text{and} \quad \bar{v}((\rho^l, v^l), (\rho^r, v^r))(\cdot)$$

be respectively the  $\rho$  and  $v$  components of the classical solution  $\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(\cdot)$ .

**Lemma 1** *Let conditions (2.3) hold. If  $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(V_b)) > F_\alpha + V_b \bar{\rho}((\rho^l, v^l), (\rho^r, v^r))(V_b)$ , then  $I = \{\check{\rho}_1, \hat{\rho}\}$ .*

*Proof.* Let us consider the plane  $(\rho, \rho v)$  and let  $w^l = v^l + p(\rho^l)$  be the value of the Riemann invariant  $w$  at  $(\rho^l, v^l)$ . In these coordinates the set of points  $(\rho, v)$  such that  $v + p(\rho) = w^l$  is the graph of the function

$$\psi(\rho) := \rho w^l - \rho p(\rho).$$

This function is strictly concave, by (2.3c). Then the cardinality of the set  $I$  is at most 2. Since  $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(V_b)) > F_\alpha + V_b \bar{\rho}((\rho^l, v^l), (\rho^r, v^r))(V_b)$ , there will be exactly two points  $(\hat{\rho}, \hat{v})$  and  $(\check{\rho}_1, \check{v}_1)$  belonging to the curve  $\psi$  and such that

$$\check{\rho}_1 < \bar{\rho}((\rho^l, v^l), (\rho^r, v^r))(V_b), \quad \hat{\rho} > \bar{\rho}((\rho^l, v^l), (\rho^r, v^r))(V_b), \quad \psi(\check{\rho}_1) = F_\alpha + V_b \check{\rho}_1 \quad \text{and} \quad \psi(\hat{\rho}) = F_\alpha + V_b \hat{\rho}.$$

□

**Lemma 2** *Let  $w^l \in [0, p(R)]$  be fixed and let  $\rho v = F_\alpha + V_b \rho$  be the constraint. Let us consider the function  $\rho \rightarrow \psi(\rho) = \rho w^l - \rho p(\rho)$ . Let  $(\rho^\sigma, v^\sigma)$  be a point such that  $\rho^\sigma v^\sigma = \psi(\rho^\sigma)$ . Under the hypotheses of Lemma 1, we have*

$$\rho^\sigma > \hat{\rho} \quad \text{or} \quad \rho^\sigma < \check{\rho}_1 \quad \text{if and only if} \quad \rho^\sigma v^\sigma < F_\alpha + V_b \rho^\sigma.$$

*Proof.* By the hypotheses of Lemma 1, there exists a point  $(\rho^m, v^m)$  such that  $v^m = L_1(\rho^m; \rho^l, v^l)$  and  $\rho^m v^m > F_\alpha + V_b \rho^m$ . Moreover we have  $I = \{\check{\rho}_1, \hat{\rho}\}$ .

If  $\rho^\sigma > \hat{\rho}$ , there exists  $\gamma \in (0, 1)$  such that  $\hat{\rho} = \gamma \rho^\sigma + (1 - \gamma) \check{\rho}_1$ . Assume by contradiction that  $\psi(\rho^\sigma) \geq F_\alpha + V_b \rho^\sigma$ , then by the strict concavity of  $\psi$ , we would have

$$\begin{aligned} \psi(\hat{\rho}) &= \psi(\gamma \rho^\sigma + (1 - \gamma) \check{\rho}_1) > \gamma \psi(\rho^\sigma) + (1 - \gamma) \psi(\check{\rho}_1) \geq \\ &\geq \gamma (F_\alpha + V_b \rho^\sigma) + (1 - \gamma) (F_\alpha + V_b \check{\rho}_1) = \\ &= \gamma F_\alpha + \gamma V_b \rho^\sigma + F_\alpha + V_b \check{\rho}_1 - \gamma F_\alpha - \gamma V_b \check{\rho}_1 = \\ &= F_\alpha + V_b (\gamma \rho^\sigma + (1 - \gamma) \check{\rho}_1) = \\ &= F_\alpha + V_b \hat{\rho}, \end{aligned}$$

which contradicts the definition of  $(\hat{\rho}, \hat{v})$ . Similarly for  $\rho^\sigma < \check{\rho}_1$ .

Conversely, if  $\rho^\sigma \in [\check{\rho}_1, \hat{\rho}]$ , there exists  $\gamma \in [0, 1]$  such that  $\rho^\sigma = \gamma \check{\rho}_1 + (1 - \gamma) \hat{\rho}$ . Hence

$$\begin{aligned} \psi(\rho^\sigma) &\geq \gamma \psi(\check{\rho}_1) + (1 - \gamma) \psi(\hat{\rho}) = \gamma (F_\alpha + \check{\rho}_1 V_b) + (1 - \gamma) (F_\alpha + \hat{\rho} V_b) = \\ &= F_\alpha + V_b (\gamma \check{\rho}_1 + (1 - \gamma) \hat{\rho}) = F_\alpha + V_b \rho^\sigma. \end{aligned}$$

□

### 3.1 The first Riemann solver $\mathcal{RS}_1^\alpha$

Let us introduce the first Riemann solver for the constrained problem (2.1).

**Definition 1** *The Riemann solver*

$$\mathcal{RS}_1^\alpha : \mathcal{D} \times \mathcal{D} \rightarrow L^1(\mathbb{R}, \mathbb{R}^+ \times \mathbb{R}^+)$$

*is defined as follows.*



1. If  $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(V_b)) > F_\alpha + V_b \bar{\rho}((\rho^l, v^l), (\rho^r, v^r))(V_b)$ , then

$$\mathcal{RS}_1^\alpha((\rho^l, v^l), (\rho^r, v^r))(x/t) = \begin{cases} \mathcal{RS}((\rho^l, v^l), (\hat{\rho}, \hat{v}))(x/t) & \text{if } x < y(t), \\ \mathcal{RS}((\check{\rho}_1, \check{v}_1), (\rho^r, v^r))(x/t) & \text{if } x > y(t), \end{cases}$$

and  $y(t) = V_b t$ .

2. If  $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(V_b)) \leq F_\alpha + V_b \bar{\rho}((\rho^l, v^l), (\rho^r, v^r))(V_b)$  and  $V_b < \bar{v}((\rho^l, v^l), (\rho^r, v^r))(V_b)$ , then

$$\mathcal{RS}_1^\alpha((\rho^l, v^l), (\rho^r, v^r))(x/t) = \mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(x/t) \quad \text{and } y(t) = V_b t.$$

3. If  $\bar{v}((\rho^l, v^l), (\rho^r, v^r))(V_b) \leq V_b$ , then

$$\mathcal{RS}_1^\alpha((\rho^l, v^l), (\rho^r, v^r))(x/t) = \mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(x/t) \quad \text{and } y(t) = v(t, y(t)+) t.$$

Note that, due to the Rankine Hugoniot relations, the inequalities above are well defined also along jump discontinuities, being equivalent along the right and left traces of the solution.

The first case refers to a situation in which the traffic is influenced by the bus and the bus travels with its own velocity; in the second case the bus and the traffic do not influence each other; the third case represents a road where the traffic is so slow that the bus travels with the speed of the downstream traffic.

Remark that the solution  $\mathcal{RS}_1^\alpha$  is conservative for both density and momentum of the vehicles. Moreover, in case 1., the solution given by  $\mathcal{RS}_1^\alpha$  does not satisfy the Lax entropy condition between the states  $(\hat{\rho}, \hat{v})$  and  $(\check{\rho}_1, \check{v}_1)$ , because we have  $\check{\rho}_1 < \hat{\rho}$ . Therefore, we say that  $(\hat{\rho}, \hat{v})$  and  $(\check{\rho}_1, \check{v}_1)$  are connected by a *non-classical shock*.

### 3.2 The second Riemann solver $\mathcal{RS}_2^\alpha$

Let us introduce the second Riemann solver for the constrained problem (2.1).

**Definition 2** *The Riemann solver*

$$\mathcal{RS}_2^\alpha : \mathcal{D} \times \mathcal{D} \rightarrow L^1(\mathbb{R}, \mathbb{R}^+ \times \mathbb{R}^+)$$

is defined as follows.

1. If  $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(V_b)) > F_\alpha + V_b \bar{\rho}((\rho^l, v^l), (\rho^r, v^r))(V_b)$ , then

$$\mathcal{RS}_2^\alpha((\rho^l, v^l), (\rho^r, v^r))(x/t) = \begin{cases} \mathcal{RS}((\rho^l, v^l), (\hat{\rho}, \hat{v}))(x/t) & \text{if } x < y(t), \\ \mathcal{RS}((\check{\rho}_2, \check{v}_2), (\rho^r, v^r))(x/t) & \text{if } x > y(t), \end{cases}$$

and  $y(t) = V_b t$ .

2. If  $f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(V_b)) \leq F_\alpha + V_b \bar{\rho}((\rho^l, v^l), (\rho^r, v^r))(V_b)$  and  $V_b < \bar{v}((\rho^l, v^l), (\rho^r, v^r))(V_b)$ , then

$$\mathcal{RS}_2^\alpha((\rho^l, v^l), (\rho^r, v^r))(x/t) = \mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(x/t) \quad \text{and } y(t) = V_b t.$$

3. If  $\bar{v}((\rho^l, v^l), (\rho^r, v^r))(V_b) \leq V_b$ , then

$$\mathcal{RS}_2^\alpha((\rho^l, v^l), (\rho^r, v^r))(x/t) = \mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(x/t) \quad \text{and } y(t) = v(t, y(t)+) t.$$

The Riemann solver  $\mathcal{RS}_2^\alpha$  conserves only the number of the vehicles. Indeed, along the line  $x = y(t)$  the Rankine-Hugoniot condition holds for the first flux component, because both  $(\hat{\rho}, \hat{v})$  and  $(\check{\rho}_2, \check{v}_2)$  belong to the line  $\rho v = F_\alpha + \bar{V} \rho$ , but not for the second component.

## 4 Invariant domains

The study of invariant domains for  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  follows closely [16, Section 3]. First of all, let us recall that the sets

$$\mathcal{D}_{v_1, v_2, w_1, w_2} = \{(\rho, v) \in \mathbb{R}^+ \times \mathbb{R}^+ : v_1 \leq v \leq v_2, w_1 \leq v + p(\rho) \leq w_2\}$$

are invariant for (2.1a) for any  $0 < v_1 < v_2$  and  $0 < w_1 < w_2$  with  $v_2 < w_2$ . Moreover, we define the function

$$\begin{aligned} h_\alpha : ]V_b, +\infty[ &\longrightarrow \mathbb{R}^+ \\ h(v) &= v + p\left(\frac{F_\alpha}{v - V_b}\right), \end{aligned} \quad (4.1)$$

which gives the value of the invariant  $w$  at the point  $(\rho, v) \in ]0, +\infty[ \times ]V_b, +\infty[$  such that  $\rho(v - V_b) = F_\alpha$ .

The following Lemma is a direct consequence of the strict convexity of the function  $\rho \rightarrow \rho p(\rho)$ , see (2.3c).

**Lemma 3** *Let us suppose that the hypotheses (2.3) hold. For any  $\alpha \in ]0, 1[$ , there exists  $\bar{v} = \bar{v}(\alpha) \in ]V_b, +\infty[$  such that  $h_\alpha(v)$  is strictly decreasing in  $]V_b, \bar{v}[$  and strictly increasing in  $]\bar{v}, +\infty[$ .*

**Proposition 1** *Let  $V_b \leq v_1 < v_2$ ,  $0 < w_1 < w_2$ ,  $v_2 < w_2$  and  $\alpha \in ]0, 1[$  be fixed. If  $h_\alpha(v) \geq w_2$  for every  $v$  in  $[v_1, v_2]$ , then the Riemann solvers  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  coincide with the standard Riemann solver  $\mathcal{RS}$ . Therefore, the domain  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for both  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$ .*

*Proof.* Let  $(\rho, v) \in \mathcal{D}_{v_1, v_2, w_1, w_2}$ . By the hypotheses we have

$$v + p(\rho) \leq w_2 \leq h_\alpha(v) = p\left(\frac{F_\alpha}{v - V_b}\right) + v \quad \implies \quad p(\rho) \leq p\left(\frac{F_\alpha}{v - V_b}\right),$$

which gives  $\rho(v - V_b) \leq F_\alpha$  by monotonicity of  $p$ . Therefore,

$$\sup\{\rho(v - V_b) : (\rho, v) \in \mathcal{D}_{v_1, v_2, w_1, w_2}\} \leq F_\alpha,$$

and for any  $(\rho^l, v^l)$  and  $(\rho^r, v^r)$  in  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ , the classical solution  $\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))$  satisfies the constraint for  $x = V_b t$ . Hence the solutions given by the two Riemann solvers  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  coincide with the classical solution.  $\square$

The next Theorems characterize the invariant domains of the Riemann solvers  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  when they are different from the standard Riemann solver. Figure 3 gives two examples of these domains.

**Theorem 1** *Let  $0 < v_1 < v_2$ ,  $0 < w_1 < w_2$ ,  $v_2 < w_2$  and  $\alpha \in ]0, 1[$  be fixed.*

(i) *If  $v_2 \leq V_b$ , the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_1^\alpha$ .*

(ii) *If  $v_1 \geq V_b$ , let us assume that there exists  $\bar{v} \in [v_1, v_2]$  such that  $h_\alpha(\bar{v}) < w_2$ . Then the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_1^\alpha$  if and only if*

$$h_\alpha(v_1) \geq w_2 \quad \text{and} \quad h_\alpha(v_2) \geq w_2. \quad (4.2)$$

(iii) If  $v_1 < V_b < v_2$ , let us assume that there exists  $\bar{v} \in [V_b, v_2]$  such that  $h_\alpha(\bar{v}) < w_2$ . Then the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_1^\alpha$  if and only if

$$h_\alpha(v_2) \geq w_2. \quad (4.3)$$

**Theorem 2** Let  $0 < v_1 < v_2$ ,  $0 < w_1 < w_2$ ,  $v_2 < w_2$  and  $\alpha \in ]0, 1[$  be fixed.

(i) If  $v_2 \leq V_b$ , the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_2^\alpha$ .

(ii) If  $v_1 \geq V_b$ , let us suppose that there exists  $\bar{v} \in [v_1, v_2]$ , such that  $h_\alpha(\bar{v}) < w_2$ . Then the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_2^\alpha$  if and only if

$$h_\alpha(v_1) \geq w_2, \quad h_\alpha(v_2) \leq w_2 \quad \text{and} \quad h_\alpha(v) \geq w_1 \quad (4.4)$$

for every  $v \in [v_1, v_2]$ .

(iii) If  $v_1 < V_b < v_2$ , let us suppose that there exists  $\bar{v} \in [V_b, v_2]$ , such that  $h_\alpha(\bar{v}) < w_2$ . Then the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_2^\alpha$  if and only if

$$h_\alpha(v_2) \leq w_2 \quad \text{and} \quad h_\alpha(v) \geq w_1 \quad (4.5)$$

for every  $v \in [V_b, v_2]$ .

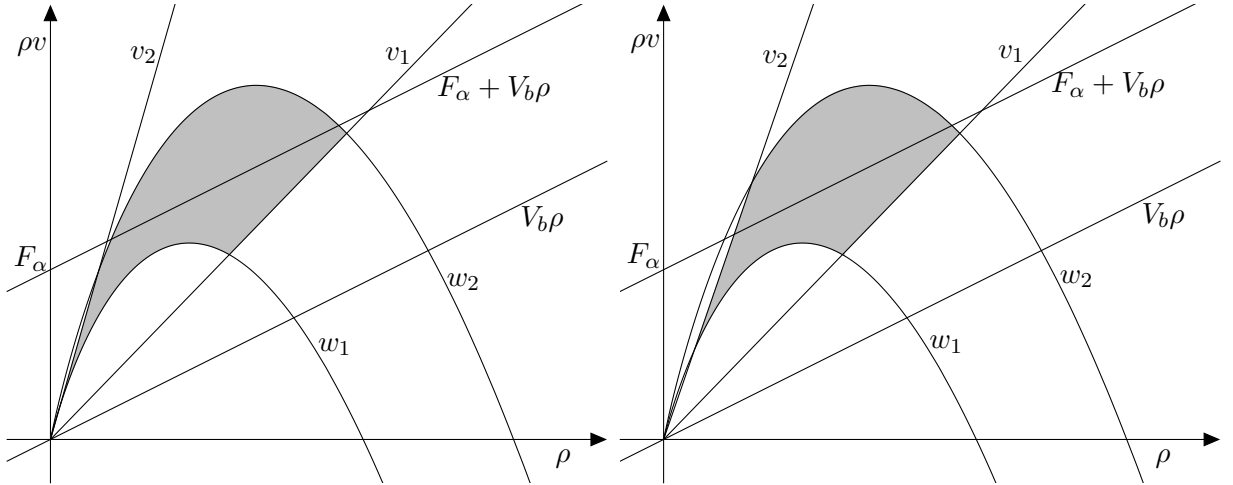


Figure 3: Example of an invariant domain (the shaded area) for  $\mathcal{RS}_1^\alpha$  (left) and  $\mathcal{RS}_2^\alpha$  (right), for  $v_1 > V_b$ .

Let us define the sets

$$\begin{aligned} \mathcal{U} &:= \{(\rho, v) \in (0, +\infty) \times (0, +\infty) : v \leq V_b\}, \\ \mathcal{V} &:= \{(\rho, v) \in (0, +\infty) \times (0, +\infty) : v \geq V_b\}. \end{aligned}$$

The proofs of Theorems 1 and 2 will be developed in the next subsections.

#### 4.1 $\mathcal{D}_{v_1, v_2, w_1, w_2} \subseteq \mathcal{U}$

If  $v_2 \leq V_b$ , then  $\mathcal{D}_{v_1, v_2, w_1, w_2} \subseteq \mathcal{U}$ . Let  $(\rho^l, v^l)$  and  $(\rho^r, v^r)$  be two points in  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ . Since  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for the standard Riemann solver  $\mathcal{RS}$ , we have

$$f_1(\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(V_b)) \leq V_b \bar{\rho}((\rho^l, v^l), (\rho^r, v^r))(V_b).$$

Therefore  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  coincide with  $\mathcal{RS}$  and  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for both, proving points (i) of Theorems 1 and 2.

## 4.2 $\mathcal{D}_{v_1, v_2, w_1, w_2} \subseteq \mathcal{V}$

We remark that  $\mathcal{D}_{v_1, v_2, w_1, w_2} \subseteq \mathcal{V}$  if and only if  $v_1 \geq \bar{V}$ .

### 4.2.1 Proof of Theorem 1 (ii)

1. Let us suppose that  $h_\alpha(v_1) \geq w_2$  and  $h_\alpha(v_2) \geq w_2$ .

Since  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}$ , it is sufficient to prove that, for every Riemann data  $(\rho^l, v^l)$  and  $(\rho^r, v^r)$  in  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ , even  $(\hat{\rho}, \hat{v})$  and  $(\check{\rho}_1, \check{v}_1)$  are in  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ .

Since  $\hat{v} + p(\hat{\rho}) = v^l + p(\rho^l)$ , we find

$$w_1 \leq \hat{v} + p(\hat{\rho}) \leq w_2. \quad (4.6)$$

Moreover, from  $\hat{\rho}\hat{v} = F_\alpha + V_b\hat{\rho} > V_b\hat{\rho}$ , we get  $\hat{v} > V_b$ , hence the function  $h_\alpha$  is well defined in  $\hat{v}$  and its value is

$$h_\alpha(\hat{v}) = \hat{v} + p\left(\frac{F_\alpha}{\hat{v} - V_b}\right) = \hat{v} + p(\hat{\rho}).$$

Let  $\tilde{v} \in ]V_b, +\infty[$  be the minimum of the function  $h_\alpha$  which exists by Lemma 3. If  $v_1 < v_2 \leq \tilde{v}$ , then  $h_\alpha(v_1) > h_\alpha(v_2) \geq h_\alpha(\tilde{v})$ . By hypothesis, there exists  $\bar{v} \in [v_1, v_2]$  such that  $h_\alpha(\bar{v}) < w_2$ , but since  $\bar{v} \leq v_2$ , we find  $w_2 > h_\alpha(\bar{v}) \geq h_\alpha(v_2)$ , in contradiction with (4.2).

Similarly, if  $\tilde{v} < v_1 < v_2$ ,  $h_\alpha$  is increasing in  $[v_1, v_2]$  and hence since  $\bar{v} \geq v_1$ , we find  $w_2 > h_\alpha(\bar{v}) \geq h_\alpha(v_1)$ , again in contradiction with (4.2).

Therefore it must hold  $v_1 \leq \tilde{v} \leq v_2$ . If  $\hat{v} < v_1 \leq \tilde{v}$ , then we would have  $h_\alpha(\hat{v}) > h_\alpha(v_1) \geq w_2$ , which is a contradiction with (4.6). Similarly, if  $\hat{v} > v_2 \geq \tilde{v}$ , then  $h_\alpha(\hat{v}) > h_\alpha(v_2) \geq w_2$ , again in contradiction with (4.6).

In conclusion we have  $v_1 \leq \hat{v} \leq v_2$  and  $(\hat{\rho}, \hat{v}) \in \mathcal{D}_{v_1, v_2, w_1, w_2}$ .

Since  $(\check{\rho}_1, \check{v}_1)$  satisfies the same hypotheses of  $(\hat{\rho}, \hat{v})$ , the proof is similar.

2. Let us suppose that  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_1^\alpha$  and, by contradiction, that  $h_\alpha(v_1) < w_2$ . Let  $(\rho^*, v^*) \in \mathcal{D}_{v_1, v_2, w_1, w_2}$  be the solution to the system

$$\begin{cases} v + p(\rho) = w_2, \\ v = v_1. \end{cases}$$

The point  $(\rho^*, v^*)$  satisfies the inequality  $\rho^*v^* > F_\alpha + V_b\rho^*$  (see Figure 4a), indeed by the hypotheses (2.3) we find

$$h_\alpha(v_1) = v_1 + p\left(\frac{F_\alpha}{v_1 - V_b}\right) < w_2 = v^* + p(\rho^*) = v_1 + p(\rho^*).$$

Therefore the left trace of  $\mathcal{RS}_1^\alpha((\rho^*, v^*), (\rho^*, v^*))$  at  $\lambda = V_b$  is  $(\hat{\rho}, \hat{v})$ , which does not belong to  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ . Indeed

$$\hat{v} = v_1 + p(\rho^*) - p(\hat{\rho}) < v_1,$$

therefore the inequality  $h_\alpha(v_1) \geq w_2$  must hold.

3. Let us suppose that  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_1^\alpha$  and, by contradiction, that  $h_\alpha(v_2) < w_2$ . Let  $(\rho^*, v^*) \in \mathcal{D}_{v_1, v_2, w_1, w_2}$  be the solution to the system

$$\begin{cases} v + p(\rho) = w_2, \\ v = v_2. \end{cases}$$

The point  $(\rho^*, v^*)$  is such that  $\rho^* v^* > F_\alpha + V_b \rho^*$  (see Figure 4b), indeed by (2.3) we find

$$h_\alpha(v_2) = v_2 + p\left(\frac{F_\alpha}{v_2 - V_b}\right) < w_2 = v^* + p(\rho^*) = v_2 + p(\rho^*).$$

Hence the right trace of  $\mathcal{RS}_1^\alpha((\rho^*, v^*), (\rho^*, v^*))$  at  $\lambda = V_b$  is  $(\check{\rho}_1, \check{v}_1)$ , which cannot belong to  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ . Indeed,

$$\check{v}_1 = v_2 + p(\rho^*) - p(\check{\rho}_1) > v_2,$$

hence the inequality  $h_\alpha(v_2) \geq w_2$  must hold.

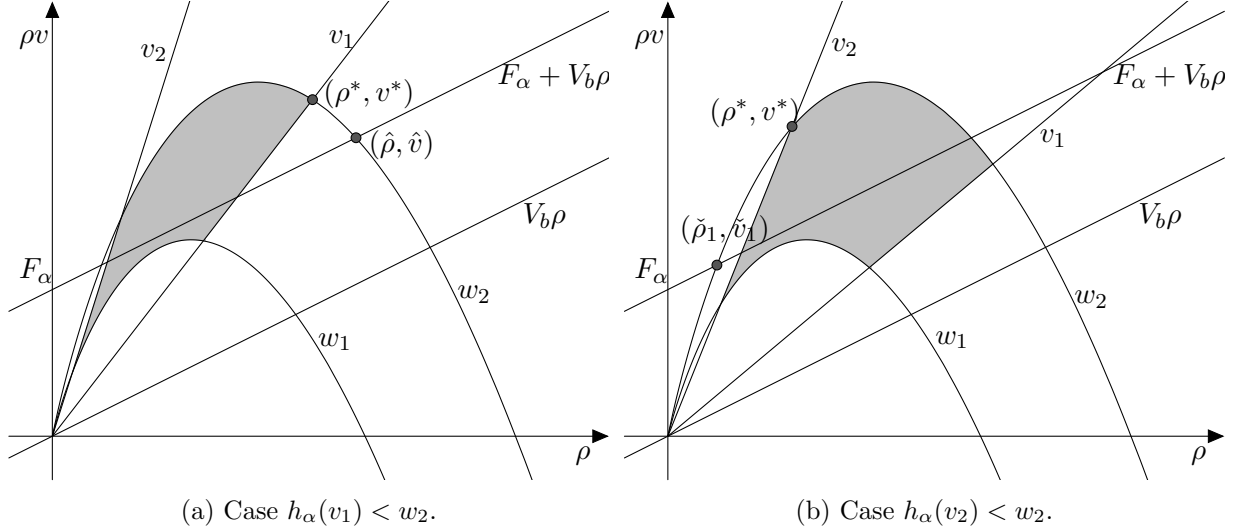


Figure 4: Representation of the points used in Section 4.2.1. The invariant domain is the colored area. If by contradiction  $h_\alpha(v_1) < w_2$  (case (a)), then the two points  $(\rho^*, v^*)$  and  $(\hat{\rho}, \hat{v})$  cannot both belong to  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ . The same is for  $(\rho^*, v^*)$  and  $(\check{\rho}_1, \check{v}_1)$  if  $h_\alpha(v_2) < w_2$  (case (b)).

#### 4.2.2 Proof of Theorem 2 (ii)

**Lemma 4** *Let  $V_b \leq v_1 < v_2$ ,  $0 < w_1 < w_2$ ,  $v_2 < w_2$  and  $\alpha \in ]0, 1[$  be fixed. Let us suppose that there exists  $\bar{v} \in [v_1, v_2]$  such that  $h_\alpha(\bar{v}) < w_2$ . If the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_2^\alpha$ , then  $h_\alpha(v_1) \geq w_2$ .*

The proof is the same of part 2 of Section 4.2.1.

**Lemma 5** *Let  $V_b \leq v_1 < v_2$ ,  $0 < w_1 < w_2$ ,  $v_2 < w_2$  and  $\alpha \in ]0, 1[$  be fixed. Let us suppose that there exists  $\bar{v} \in [v_1, v_2]$  such that  $h_\alpha(\bar{v}) < w_2$ . If the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_2^\alpha$ , then  $h_\alpha(v) \geq w_1$  for every  $v \in [v_1, v_2]$ .*

*Proof.* Assume by contradiction that there exists  $\tilde{v} \in [v_1, v_2]$  such that  $h_\alpha(\tilde{v}) < w_1$ . Let  $(\rho^*, v^*) \in \mathcal{D}_{v_1, v_2, w_1, w_2}$  be the solution to the system

$$\begin{cases} v + p(\rho) = w_2, \\ v = \tilde{v}. \end{cases}$$

We note that

$$h_\alpha(\tilde{v}) = \tilde{v} + p\left(\frac{F_\alpha}{\tilde{v} - V_b}\right) < w_1 < w_2 = v^* + p(\rho^*),$$

therefore  $\rho^*(v^* - V_b) > F_\alpha$ . Hence the right trace of  $\mathcal{RS}_2^\alpha((\rho^*, v^*), (\rho^*, v^*))$  in  $\lambda = V_b$  is  $(\check{\rho}_2, \check{v}_2)$  which does not belong to  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  (see Figure 5a). Indeed by definition we have

$$\check{v}_2 = v^* = \tilde{v} \implies h_\alpha(\check{v}_2) = h_\alpha(\tilde{v}) < w_1.$$

This is in contradiction with the invariance of the domain  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ .  $\square$

**Lemma 6** *Let  $V_b \leq v_1 < v_2$ ,  $0 < w_1 < w_2$ ,  $v_2 < w_2$  and  $\alpha \in ]0, 1[$  be fixed. Let us suppose that there exists  $\tilde{v} \in [v_1, v_2]$  such that  $h_\alpha(\tilde{v}) < w_2$ . If the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_2^\alpha$ , then  $h_\alpha(v_2) \leq w_2$ .*

*Proof.* Assume by contradiction that  $h_\alpha(v_2) > w_2$ . Let  $(\rho^r, v^r) \in \mathcal{D}_{v_1, v_2, w_1, w_2}$  be the solution to the system (see Figure 5b)

$$\begin{cases} v + p(\rho) = w_2, \\ v = v_2, \end{cases}$$

and let  $(\rho^l, v^l) \in \mathcal{D}_{v_1, v_2, w_1, w_2}$  be the solution to

$$\begin{cases} v + p(\rho) = w_2, \\ v = v_1. \end{cases}$$

The points  $(\rho^l, v^l)$  and  $(\rho^r, v^r)$  are connected by the standard Riemann solver with a rarefaction, because  $v^l = v_1 < v_2 = v^r$ . We claim that  $\rho^l \geq \hat{\rho}$  and  $\rho^r < \check{\rho}_1$ .

Indeed, let  $\tilde{v}$  be the minimum of the function  $h_\alpha$ . We have  $v_1 \leq \tilde{v}$ , otherwise for every  $v \in [v_1, v_2]$  we would have  $h_\alpha(v) \geq h_\alpha(v_1) \geq w_2$  by Lemma 4.

If  $\hat{v} \geq \tilde{v}$ , then we clearly have  $v_1 \leq \hat{v}$ . By the definition of  $(\rho^l, v^l)$ , we have  $w_2 = v^l + p(\rho^l) = \hat{v} + p(\hat{\rho}) = h_\alpha(\hat{v})$ . Hence if  $\hat{v} < \tilde{v}$ , we find

$$h_\alpha(\hat{v}) = w_2 \leq h_\alpha(v_1) \implies \hat{v} \geq v_1,$$

because  $h_\alpha(v_1) \geq w_2$ . In both cases we have  $\hat{v} \geq v^l$  which is equivalent to  $\rho^l \geq \hat{\rho}$ .

Similarly, if  $v_2 \leq \tilde{v}$ , for every  $v \in [v_1, v_2]$  we would have  $h_\alpha(v) \geq h_\alpha(v_2) > w_2$ , which is a contradiction. Therefore  $v_2 > \tilde{v}$ . If  $\check{v}_1 \leq \tilde{v}$ , then clearly  $v_2 > \check{v}_1$ , while if  $\check{v}_1 > \tilde{v}$ , since

$$h_\alpha(\check{v}_1) = \check{v}_1 + p(\check{\rho}_1) = v^l + p(\rho^l) = w_2 < h_\alpha(v_2),$$

we find  $v^r = v_2 > \check{v}_1$  which is equivalent to  $\rho^r < \check{\rho}_1$ , because

$$v^r + p(\rho^r) = w_2 = v^l + p(\rho^l) = \check{v}_1 + p(\check{\rho}_1).$$

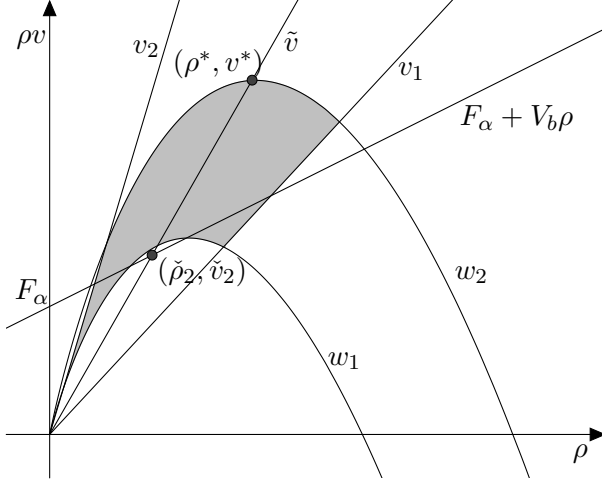
This proves the claim.

Since the classical solution  $\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))$  does not satisfy the constraint for Lemma 2, the right trace of  $\mathcal{RS}_2^\alpha((\rho^l, v^l), (\rho^r, v^r))$  at  $\lambda = V_b$  is  $(\check{\rho}_2, \check{v}_2)$ .

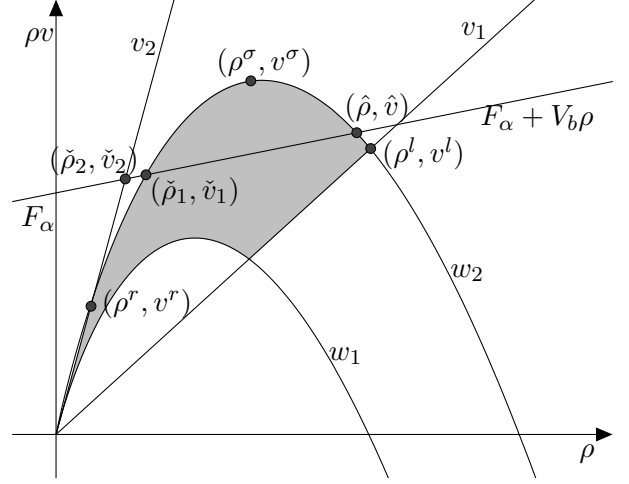
Since  $\check{v}_2 = v^r = v_2$ , we have

$$\check{v}_2 + p(\check{\rho}_2) = h_\alpha(\check{v}_2) = h_\alpha(v_2) > w_2.$$

Therefore  $(\check{\rho}_2, \check{v}_2)$  does not belong to  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ , in contradiction with the hypothesis of invariance of this domain for the Riemann solver  $\mathcal{RS}_2^\alpha$ .  $\square$



(a) Representation of the points used in Lemma 5. The invariant domain is the colored area. If, by contradiction, there was  $\tilde{v} \in [v_1, v_2]$  such that  $h_\alpha(\tilde{v}) < w_1$ , the point  $(\check{\rho}_2, \check{v}_2)$  would not be in the domain  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ .



(b) Representation of the points used in the proof of Lemma 6. The invariant domain is the colored area. If, by contradiction, the inequality  $h_\alpha(v_2) > w_2$  held, the point  $(\check{\rho}_2, \check{v}_2)$  would not be in the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ .

**Lemma 7** Let  $V_b \leq v_1 < v_2$ ,  $0 < w_1 < w_2$ ,  $v_2 < w_2$  and  $\alpha \in ]0, 1[$  be fixed and (4.4) hold. Then the set  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for the Riemann solver  $\mathcal{RS}_2^\alpha$ .

*Proof.* Since  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}$ , we have only to show that the left and right traces of  $\mathcal{RS}_2^\alpha((\rho^l, v^l), (\rho^r, v^r))$  at  $\lambda = V_b$  belong to  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ .

Let  $(\rho^l, v^l)$  and  $(\rho^r, v^r) \in \mathcal{D}_{v_1, v_2, w_1, w_2}$ . If  $\mathcal{RS}_2^\alpha((\rho^l, v^l), (\rho^r, v^r))$  coincides with  $\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))$ , there is nothing to prove. Otherwise, let  $(\check{\rho}_2, \check{v}_2)$  and  $(\hat{\rho}, \hat{v})$  be respectively the right and left traces of the solution at  $V_b$ .

If  $(\hat{\rho}, \hat{v}) \notin \mathcal{D}_{v_1, v_2, w_1, w_2}$ , let  $(\rho^*, v^*) \in \mathcal{D}_{v_1, v_2, w_1, w_2}$  be a point of  $L_1(\rho; \rho^l, v^l)$ . Then by (4.4) we have  $\rho^* v^* \leq F_\alpha + V_b \rho^*$ , and the Riemann solver gives the classical solution.

If  $(\check{\rho}_2, \check{v}_2) \notin \mathcal{D}_{v_1, v_2, w_1, w_2}$ , then each point  $(\rho^*, v^*) \in \mathcal{D}_{v_1, v_2, w_1, w_2}$  belonging to  $L_2(\rho; \rho^r, v^r)$  satisfies  $\rho^* v^* \leq F_\alpha + V_b \rho^*$ . Therefore the Riemann solver gives again the classical solution.  $\square$

### 4.3 $\mathcal{D}_{v_1, v_2, w_1, w_2}$ intersects both $\mathcal{U}$ and $\mathcal{V}$

If  $v_1 < V_b < v_2$ , let us suppose that there exists  $\bar{v} \in [V_b, v_2]$  such that  $h_\alpha(\bar{v}) < w_2$ .

We can restate the results that we have obtained until now as follows.

1. The domain  $\mathcal{D}_{v_1, V_b, w_1, w_2}$  is invariant for  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$ .
2.  $h_\alpha(v_2) \geq w_2$  if and only if  $\mathcal{D}_{V_b, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_1^\alpha$ .
3.  $h_\alpha(v) \geq w_1$  for every  $v \in [V_b, v_2]$  and  $h_\alpha(v_2) \leq w_2$  hold if and only if  $\mathcal{D}_{V_b, v_2, w_1, w_2}$  is invariant for  $\mathcal{RS}_2^\alpha$ .

**Proposition 2** Let  $0 < v_1 < v_2$ ,  $0 < w_1 < w_2$ ,  $v_2 < w_2$  and  $\alpha \in ]0, 1[$  be fixed. If the domains  $\mathcal{D}_{v_1, V_b, w_1, w_2}$  and  $\mathcal{D}_{V_b, v_2, w_1, w_2}$  are invariant for  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$ , then also  $\mathcal{D}_{v_1, v_2, w_1, w_2}$  is invariant for both Riemann solvers.

*Proof.* The domain  $\mathcal{D}_{v_1, V_b, w_1, w_2}$  and  $\mathcal{D}_{V_b, v_2, w_1, w_2}$  coincide respectively with

$$\mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{U}} := \mathcal{D}_{v_1, v_2, w_1, w_2} \cap \mathcal{U} \quad \text{and} \quad \mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{V}} := \mathcal{D}_{v_1, v_2, w_1, w_2} \cap \mathcal{V}.$$

If the solutions given by the two Riemann solvers  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  coincide with the classic one, then the result is trivial.

Moreover we observe that, for every pair of initial data  $((\rho^l, v^l), (\rho^r, v^r)) \in (\mathbb{R}^+ \times \mathbb{R}^+)^2$ , the points  $(\check{\rho}_1, \check{v}_1)$ ,  $(\check{\rho}_2, \check{v}_2)$  and  $(\hat{\rho}, \hat{v})$  belong to  $\mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{V}}$ . Therefore, if  $(\rho^l, v^l), (\rho^r, v^r) \in \mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{V}}$ , the solutions given by the Riemann solvers  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  are contained in  $\mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{V}}$ .

If  $v^r < V_b$ , we have  $\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(V_b) = (\rho^r, v^r) \in \mathcal{U}$ . Therefore, the solutions given by the Riemann solvers  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  coincide with the classical one.

If  $v^r = V_b$ , then the points  $(\rho^l, v^l)$  and  $(\rho^r, v^r)$  are both in  $\mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{U}}$  or both in  $\mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{V}}$ . Hence the solutions  $\mathcal{RS}_1^\alpha((\rho^l, v^l), (\rho^r, v^r))$  and  $\mathcal{RS}_2^\alpha((\rho^l, v^l), (\rho^r, v^r))$  are entirely contained in one of the two sub-domains and therefore also in  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ .

If  $v^r > V_b$ ,  $\mathcal{RS}((\rho^l, v^l), (\rho^r, v^r))(V_b)$  can be equal to  $(\rho^m, v^m)$ ,  $(\rho^l, v^l)$  or, if  $\rho^l \geq \rho^m$ , one of the points of the rarefaction curve which connects  $(\rho^m, v^m)$  to  $(\rho^l, v^l)$ . Both  $(\rho^m, v^m)$  and  $(\rho^r, v^r)$  are in  $\mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{V}}$ . If also  $(\rho^l, v^l)$  belongs to  $\mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{V}}$ , we can conclude by Theorem 1 (ii) and Theorem 2 (ii).

If  $(\rho^l, v^l) \in \mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{U}}$ , we must have  $\rho^l \geq \rho^m$ , because  $v^l \leq V_b \leq v^m$ , and the standard Riemann solver connects these two points with a rarefaction. If  $(\rho^m, v^m)$  satisfies the constraint, then the same holds for each point  $(\rho^\sigma, v^\sigma)$  of the rarefaction, because  $\rho^\sigma \geq \rho^m$  and Lemma 2. Hence  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  coincide with the standard Riemann solver and the solutions are contained in  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ . If  $(\rho^m, v^m)$  does not satisfy the constraint, let  $(\rho^*, v^*)$  be the solution to the system

$$\begin{cases} v + p(\rho) = v^l + p(\rho^l), \\ v = V_b. \end{cases}$$

By consistency, we can decompose the solution to the Riemann problem (2.1), (3.1) given by  $\mathcal{RS}_1^\alpha$ , as

$$\mathcal{RS}_1^\alpha((\rho^l, v^l), (\rho^r, v^r))(x/t) = \begin{cases} \mathcal{RS}_1^\alpha((\rho^l, v^l), (\rho^*, v^*))(x/t) & \text{if } x \leq \lambda_1(\rho^*, v^*)t, \\ \mathcal{RS}_1^\alpha((\rho^*, v^*), (\rho^r, v^r))(x/t) & \text{if } x > \lambda_1(\rho^*, v^*)t, \end{cases}$$

and similarly for the second Riemann solver

$$\mathcal{RS}_2^\alpha((\rho^l, v^l), (\rho^r, v^r))(x/t) = \begin{cases} \mathcal{RS}_2^\alpha((\rho^l, v^l), (\rho^*, v^*))(x/t) & \text{if } x \leq \lambda_1(\rho^*, v^*)t, \\ \mathcal{RS}_2^\alpha((\rho^*, v^*), (\rho^r, v^r))(x/t) & \text{if } x > \lambda_1(\rho^*, v^*)t. \end{cases}$$

If the classical solution does not satisfy the constraint in  $x = V_b t$ , then the left trace of  $\mathcal{RS}_i^\alpha$  for  $i = 1, 2$  at  $\lambda = V_b$  is  $(\hat{\rho}, \hat{v})$ . The solutions given by the two Riemann solvers  $\mathcal{RS}_i^\alpha$  for  $i = 1, 2$  present a rarefaction between  $(\rho^l, v^l)$  and  $(\rho^*, v^*)$  and another rarefaction linking  $(\rho^*, v^*)$  to  $(\hat{\rho}, \hat{v})$ . Moreover, since  $\mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{U}}$  and  $\mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{V}}$  are invariant for both Riemann solvers, the point  $(\rho^*, v^*)$  belongs to the intersection of the two domains and since  $\mathcal{D}_{v_1, v_2, w_1, w_2} = \mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{U}} \cup \mathcal{D}_{v_1, v_2, w_1, w_2}^{\mathcal{V}}$ , the solution is contained in the domain  $\mathcal{D}_{v_1, v_2, w_1, w_2}$ , which is therefore invariant.  $\square$

This concludes the proofs of Theorems 1 and 2.

## 5 Numerical schemes

The aim of this section is to propose a numerical strategy to compute the solutions to the constrained problem (2.1). We will more precisely propose two variants of the strategy which will



correspond to the two Riemann solvers  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$ . Importantly, a particular attention will be given to the numerical treatment of the non-classical shocks arising at the bus position since the numerical approximation of such moving discontinuities is well-known to be challenging. We refer the reader to the introduction of this paper for more details.

Let  $h$  and  $k^n$  be positive real numbers representing the increments for space and time discretizations and let us define the mesh points  $(t^n, x_{j+1/2})$  by

$$x_{j+1/2} = jh \text{ for every } j \in \mathbb{Z} \text{ and } t^{n+1} = t^n + k^n \text{ for every } n \in \mathbb{N}.$$

We divide the  $x$ -axis in a sequence  $\{C_j\}_{j \in \mathbb{Z}}$  of cells such that  $C_j = [x_{j-1/2}, x_{j+1/2})$ . We also define the cell centers as

$$x_j = \left(j - \frac{1}{2}\right)h \in C_j \text{ for every } j \in \mathbb{Z}.$$

For every  $n \in \mathbb{N}$ , we aim at constructing a piecewise constant approximation  $x \mapsto \mathbf{u}(t^n, x)$  of the conserved variables  $\mathbf{u} = (\rho, z) = (\rho, \rho w) = (\rho, \rho(v + p(\rho)))$  given by

$$\mathbf{u}(t^n, x) = \mathbf{u}_j^n \text{ for all } x \in C_j, j \in \mathbb{Z}, n \in \mathbb{N}.$$

For completeness, we recall that the classical Godunov scheme can be written as

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{k^n}{h} \left(F_{j+1/2}^n - F_{j-1/2}^n\right) \quad (5.1)$$

for all  $j \in \mathbb{Z}$ , where  $F_{j+1/2}^n = F(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) = f(\mathcal{RS}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n)(0))$ , under the classical CFL stability condition

$$2k^n \max_{j \in \mathbb{Z}} \{|\lambda_1(\mathbf{u}_j^n)|, |\lambda_2(\mathbf{u}_j^n)|\} \leq h.$$

This classical Godunov scheme will be used except near the non-classical shocks since such discontinuities require specific treatments, which are described in the next two sections for both solvers  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$ . Note that above, and in the following, we denote by  $\mathcal{RS}$ ,  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  the Riemann solvers defined on the conserved variables  $(\rho, z)$ . At last, the proposed numerical approximation of the bus trajectory is described in Appendix A and we emphasize that it is exact when the bus velocity is given by  $V_b$ .

## 5.1 The constrained Godunov scheme for $\mathcal{RS}_1^\alpha$

First of all, we define for every  $j \in \mathbb{Z}$  the approximation  $\mathbf{u}_j^0$  of the initial data  $\mathbf{u}^0$  on the cell  $C_j$  as the mean

$$\mathbf{u}_j^0 := \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}_0(x) dx.$$

In this section, we consider the Riemann solver  $\mathcal{RS}_1^\alpha$  and our aim is to modify the classical Godunov scheme near the non-classical shocks that may develop at the moving constraint location. We follow the approach proposed in [1, 7, 9], where an in-cell discontinuous reconstruction strategy is developed and allows a sharp computation of the non-classical shocks. By sharp, we mean that such isolated discontinuities are exactly computed with only one point of numerical diffusion, the value of which corresponds to the mean value of the exact solution in the corresponding cell. It is clearly the best result achievable by a conservative scheme.

Let us first consider the case in which the bus is not influenced by the preceding vehicles, so that its position at the time  $t^n$  is  $y^n := y(t^n) = y_0 + V_b t^n \in C_m$  for some  $m \in \mathbb{Z}$  and the value of

the approximate solution at  $y^n$  is  $\mathbf{u}_m^n = (\rho_m^n, z_m^n)$ .

Following [9] and referring to the definition of the Riemann solver  $\mathcal{RS}_1^\alpha$ , a first idea to detect whether a non-classical shock is expected to develop around  $y^n$  or not is to check if the inequality

$$f_1(\mathbf{u}_m^n) > F_\alpha + V_b \rho_m^n \quad (5.2)$$

holds. If this inequality actually holds true, one is naturally tempted to consider that a non-classical shock arises as the solution given by  $\mathcal{RS}_1^\alpha$  to the Riemann problem with initial datum

$$\mathbf{u}(0, x) = \begin{cases} \mathbf{u}_{m-1}^n & \text{if } x \leq y(t^n), \\ \mathbf{u}_{m+1}^n & \text{if } x > y(t^n). \end{cases}$$

In this case, it is also natural to check whether also the inequality

$$f_1(\mathcal{RS}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V_b)) > F_\alpha + V_b \bar{\rho}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V_b) \quad (5.3)$$

holds true or not.

Let us assume for a moment that (5.2) and (5.3) are relevant to detect whether a non-classical shock occurs at the bus position or not (notice from now on that it will not be the case). The proposed strategy to properly compute the non-classical discontinuity which is expected to occur at the bus position consists in making a reconstruction of this non-classical discontinuity within the  $m$ -th cell, and to modify the classical Godunov scheme accordingly. More precisely, the proposed reconstruction will lead to a new definition of the left and right numerical fluxes  $F_{m\pm 1/2}^n$ , which we describe below.

We introduce in the  $m$ -th cell a left state  $\mathbf{u}_{m,l}^n = (\rho_{m,l}^n, z_{m,l}^n)$  and a right state  $\mathbf{u}_{m,r}^n = (\rho_{m,r}^n, z_{m,r}^n)$  as

$$\mathbf{u}_{m,l}^n = \hat{\mathbf{u}} \quad \text{and} \quad \mathbf{u}_{m,r}^n = \check{\mathbf{u}}_1,$$

where  $\hat{\mathbf{u}}$  and  $\check{\mathbf{u}}_1$  are the points defined by (3.2), (3.3). Then we replace  $\mathbf{u}_m^n$  with the function  $\mathbf{u}_{\text{rec}}^n = (\rho_{\text{rec}}^n, z_{\text{rec}}^n)$  defined by

$$\begin{aligned} \rho_{\text{rec}}^n &= \rho_{m,l}^n \mathbf{1}_{[x_{m-1/2}, \bar{x}_m^\rho]} + \rho_{m,r}^n \mathbf{1}_{[\bar{x}_m^\rho, x_{m+1/2})}, \\ z_{\text{rec}}^n &= z_{m,l}^n \mathbf{1}_{[x_{m-1/2}, \bar{x}_m^z]} + z_{m,r}^n \mathbf{1}_{[\bar{x}_m^z, x_{m+1/2})}, \end{aligned}$$

where we have used the two points

$$\bar{x}_m^\rho = x_{m-1/2} + h d_m^{\rho,n} \quad \text{and} \quad \bar{x}_m^z = x_{m-1/2} + h d_m^{z,n}$$

defined for two suitable constants  $d_m^{\rho,n}$  and  $d_m^{z,n}$  such that

$$d_m^{\rho,n} \in [0, 1], \quad d_m^{z,n} \in [0, 1]. \quad (5.4)$$

Conditions (5.4) ensure that the reconstructed discontinuity is located in the cell  $C_m$ ; see Figure 6. Since the Riemann solver  $\mathcal{RS}_1^\alpha$  is conservative with respect to  $\mathbf{u} = (\rho, z)$ , we also aim at preserving this property in the process of reconstruction. With this in mind, we require the average value of the reconstructed discontinuity to be equal to  $\mathbf{u}_m^n$ , which writes

$$\begin{aligned} \rho_{m,l}^n d_m^{\rho,n} + \rho_{m,r}^n (1 - d_m^{\rho,n}) &= \rho_m^n, \\ z_{m,l}^n d_m^{z,n} + z_{m,r}^n (1 - d_m^{z,n}) &= z_m^n. \end{aligned}$$

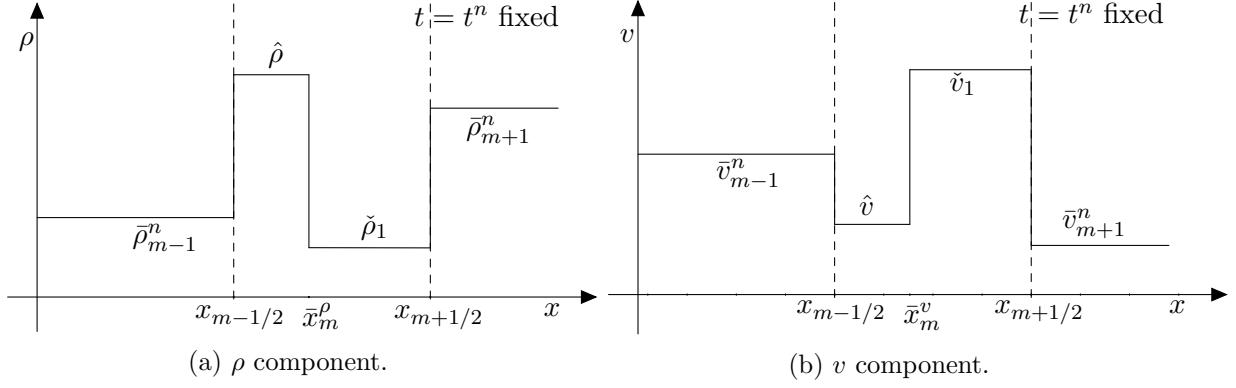


Figure 6: An example of a discontinuity reconstruction for the Riemann solver  $\mathcal{RS}_1^\alpha$ .

Therefore, we get the following definition of the two constants  $d_m^{\rho,n}$  and  $d_m^{z,n}$ , namely

$$d_m^{\rho,n} = \frac{\rho_m^n - \rho_{m,r}^n}{\rho_{m,l}^n - \rho_{m,r}^n} \quad \text{and} \quad d_m^{z,n} = \frac{z_m^n - z_{m,r}^n}{z_{m,l}^n - z_{m,r}^n}. \quad (5.5)$$

Note that these two constants can be different in general, which means that the reconstructed discontinuities of the two components of  $\mathbf{u}$  can be located at different positions.

The non-classical shock travels with the bus speed  $V_b > 0$ . As a consequence, if we denote  $\Delta t_{m+1/2}^\rho$  and  $\Delta t_{m+1/2}^z$  respectively the time needed by the  $\rho$  and the  $z$  component of the discontinuity to reach the interface  $x_{m+1/2}$ , we have

$$\begin{aligned} \Delta t_{m+1/2}^\rho V_b = h(1 - d_m^{\rho,n}) &\iff \Delta t_{m+1/2}^\rho = h \frac{1 - d_m^{\rho,n}}{V_b}, \\ \Delta t_{m+1/2}^z V_b = h(1 - d_m^{z,n}) &\iff \Delta t_{m+1/2}^z = h \frac{1 - d_m^{z,n}}{V_b}. \end{aligned}$$

Therefore, considering the proposed reconstruction and since the waves emerging from the Riemann problem associated with  $\mathbf{u}_{m,r}^n$  and  $\mathbf{u}_{m+1}^n$  at the interface  $x_{m+1/2}$  propagate to the right, the flux at interface  $x_{m+1/2}$  equals  $f(\mathbf{u}_{m,r}^n)$  until  $t^n + \Delta t_{m+1/2}^{\rho,z}$  (depending on the  $\rho$  and  $z$  components) and  $f(\mathbf{u}_{m,l}^n)$  afterwards. One then proposes to replace the classical Godunov flux  $F_{m+1/2}^n$  by a new numerical flux, whose components are denoted by  $F_1(\mathbf{u}_m^n, \mathbf{u}_{m+1}^n)$  and  $F_2(\mathbf{u}_m^n, \mathbf{u}_{m+1}^n)$  and given by

$$\begin{aligned} F_1(\mathbf{u}_m^n, \mathbf{u}_{m+1}^n) &= \frac{1}{k^n} \left[ \min(\Delta t_{m+1/2}^\rho, k^n) f_1(\mathbf{u}_{m,r}^n) + \max(k^n - \Delta t_{m+1/2}^\rho, 0) f_1(\mathbf{u}_{m,l}^n) \right], \\ F_2(\mathbf{u}_m^n, \mathbf{u}_{m+1}^n) &= \frac{1}{k^n} \left[ \min(\Delta t_{m+1/2}^z, k^n) f_2(\mathbf{u}_{m,r}^n) + \max(k^n - \Delta t_{m+1/2}^z, 0) f_2(\mathbf{u}_{m,l}^n) \right]. \end{aligned} \quad (5.6)$$

It is also natural to modify the left flux at interface  $x_{m-1/2}$  according to the left state  $\mathbf{u}_{m-1}^n$  and the new right state  $\mathbf{u}_{m,l}^n = \hat{\mathbf{u}}$ , i.e. we propose to replace  $F_{m-1/2}$  with  $F(\mathbf{u}_{m-1}^n, \mathbf{u}_{m,l}^n)$ , which preserves the consistency of the Godunov method. This concludes the description of the reconstruction method.

Let us now briefly go back to the proposed procedure (5.2), (5.3) to detect whether a non-classical shock appears or not. It turns out that as it stands, it may introduce undesirable oscillations in some cases. The example showed in Figure 7 is one of these situations: at some iterations the exact solution does not satisfy the constraint, while the numerical solution does, leading to an incorrect computation of the non-classical shock and mild oscillations. In view of this example, we

propose to remove the first condition (5.2) and to keep only the inequality (5.3) as necessary to enforce the reconstruction procedure, together with (5.4) of course.

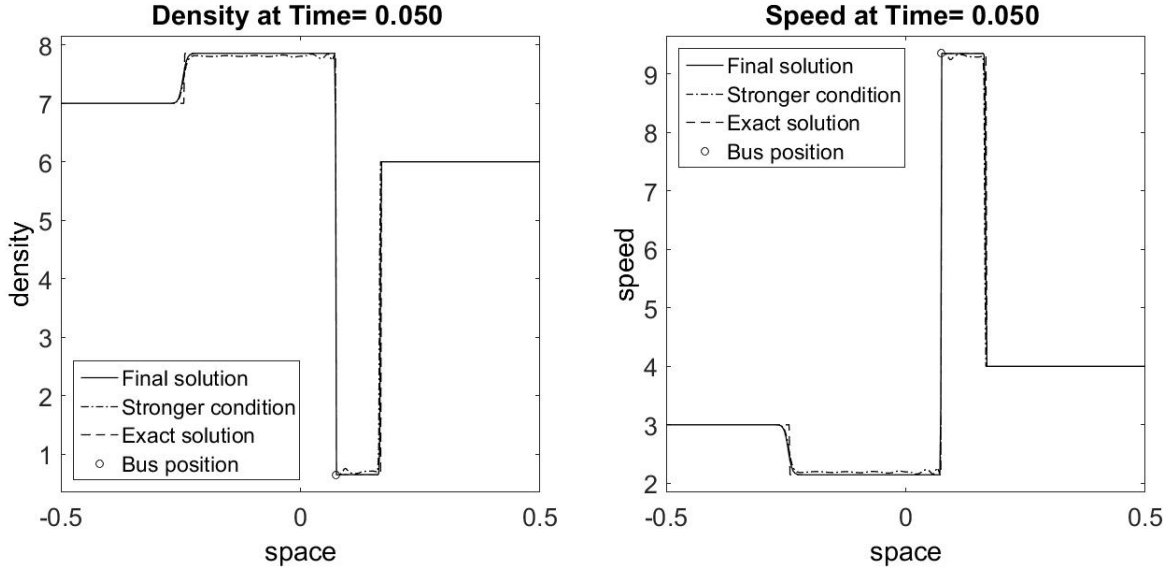


Figure 7: Comparison between the solution obtained with both conditions (5.2) and (5.3) (the dot-dashed line) and the one obtained when only (5.3) is enforced (the continuous line): the first has undesirable oscillation, while the latter is correct. The initial data are:  $(\rho^l, v^l) = (7, 3)$ ,  $(\rho^r, v^r) = (6, 4)$ ,  $\alpha = 0.4$ ,  $V_b = 1.5$ ,  $R = 15$  and  $y_0 = 0$ .

To sum up, our numerical scheme is defined by (5.1) together with

$$F_{j+1/2}^n = \begin{cases} F(\mathbf{u}_{m-1}^n, \mathbf{u}_{m,l}^n) & \text{if } j = m - 1 \text{ and (5.3) - (5.4) hold true,} \\ (5.6) & \text{if } j = m \text{ and (5.3) - (5.4) hold true,} \\ f(\mathcal{RS}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n)(0)) & \text{otherwise.} \end{cases} \quad (5.7)$$

Let us now establish a nice property, which states that the scheme is able to exactly compute isolated non-classical discontinuities and thus justifies the proposed reconstruction procedure. In particular, this property also makes clear why the proposed numerical scheme performs so well in computing the non-classical discontinuities, since such moving shocks are proved to be computed with no extra numerical diffusion by construction.

**Proposition 3** *The scheme (5.1)-(5.7) is exact when the Riemann initial datum is made of a single nonclassical shock between the left state  $\hat{\mathbf{u}}$  and the right state  $\check{\mathbf{u}}_1$  and provided that the bus position is computed exactly (which is the case with the front tracking method introduced in Appendix A). In particular, let  $d_m^{\rho,n}$  and  $d_m^{z,n}$  be the two constants defined in (5.5). If  $\mathbf{u}_{m-1}^n = \hat{\mathbf{u}}$ ,  $\mathbf{u}_{m+1}^n = \check{\mathbf{u}}_1$  and there exists  $\gamma \in [0, 1]$  such that*

$$\mathbf{u}_m^n = \gamma \mathbf{u}_{m-1}^n + (1 - \gamma) \mathbf{u}_{m+1}^n, \quad (5.8)$$

then  $d_m^{\rho,n} = d_m^{z,n} = \gamma$ .

*Proof.* Let us start with an initial condition such that  $\mathbf{u}_j^0 = \hat{\mathbf{u}}$  if  $j < m$  and  $\mathbf{u}_j^0 = \check{\mathbf{u}}_1$  if  $j \geq m$ . It is clear on the one hand that (5.8) holds true with  $n = 0$  and  $\gamma = 0$ , and on the other hand that

$$\mathbf{u}_j^n = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}^{exa}(t^n, x) dx \quad \text{for all } x \in \mathbb{Z} \quad (5.9)$$

with  $n = 0$  and  $\mathbf{u}^{exa} = \mathbf{u}_0$ . In order to prove that the scheme is exact for such an initial datum, it is thus sufficient to assume that (5.8) and (5.9) hold true for a given time  $t^n$ , and to show that

$$\mathbf{u}_j^{n+1} = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}^{exa}(t^{n+1}, x) dx \quad \text{for all } x \in \mathbb{Z}. \quad (5.10)$$

Note that since the exact solution is a discontinuity propagating with velocity  $V_b$ , (5.10) implies immediately the validity of a convex combination like (5.8) at time  $t^{n+1}$ .

Let us first notice that  $F_{m-1/2}^n = F(\hat{\mathbf{u}})$ , and that (5.8) holds true at time  $t^{n+1}$  for  $j \neq m, m+1$ , since our scheme is equivalent to the usual Godunov's scheme and we are in the constant regions of the exact solution. Next, since condition (5.8) holds, we have

$$\begin{pmatrix} \rho_m^n \\ z_m^n \end{pmatrix} = \begin{pmatrix} \gamma \hat{\rho} + (1 - \gamma) \check{\rho}_1 \\ \gamma \hat{z} + (1 - \gamma) \check{z}_1 \end{pmatrix}.$$

Hence we find

$$\begin{aligned} d_m^{\rho, n} &= \frac{\rho_m^n - \rho_{m, \rho}^n}{\rho_{m, l}^n - \rho_{m, r}^n} = \frac{\gamma \hat{\rho} + (1 - \gamma) \check{\rho}_1 - \check{\rho}_1}{\hat{\rho} - \check{\rho}_1} = \frac{\gamma(\hat{\rho} - \check{\rho}_1)}{\hat{\rho} - \check{\rho}_1} = \gamma, \\ d_m^{z, n} &= \frac{z_m^n - z_{m, r}^n}{z_{m, l}^n - z_{m, r}^n} = \frac{\gamma \hat{z} + (1 - \gamma) \check{z}_1 - \check{z}_1}{\hat{z} - \check{z}_1} = \frac{\gamma(\hat{z} - \check{z}_1)}{\hat{z} - \check{z}_1} = \gamma. \end{aligned}$$

This means that the reconstructed discontinuity is located at the same position as the discontinuity of the exact solution. As a consequence, the numerical flux  $F_{m+1/2}^n$  defined by (5.7) is also nothing but the exact flux passing through the interface  $x_{m+1/2}$  so that  $\mathbf{u}_m^{n+1}$  and  $\mathbf{u}_{m+1}^{n+1}$  also exactly coincide with the average of the exact solution in the corresponding cell. Which concludes the proof.  $\square$

Finally, note that if

$$V_b \bar{\rho}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V_b) < f_1(\mathcal{RS}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V_b)) \leq F_\alpha + V_b \bar{\rho}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V_b),$$

which corresponds to the second case of Definition 1, the bus and the vehicles do not influence each other. Hence, the bus position at the time  $t^{n+1}$  is  $y^{n+1} = y^n + V_b k^n$  and the solution of the ARZ system can be computed with the standard Godunov's method. In the third case of Definition 1, the solution of the ARZ system can also be computed with the standard Godunov's method but the bus position will be updated taking into account the flow velocity  $v$  using the front tracking technique described in Appendix A.

## 5.2 The constrained Godunov schemes for $\mathcal{RS}_2^\alpha$

Let  $V^n$  be the bus speed at time  $t^n$ . The Riemann solvers  $\mathcal{RS}_1^\alpha$  and  $\mathcal{RS}_2^\alpha$  give the same solution whenever the constraint is satisfied. Hence, if

$$f_1(\mathcal{RS}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V^n)) \leq F_\alpha + V^n \bar{\rho}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V^n),$$

we can apply to  $\mathcal{RS}_2^\alpha$  the techniques described for  $\mathcal{RS}_1^\alpha$ . When the constraint is enforced, something special has to be done to properly compute the non-classical shocks, in the spirit of the method proposed in the previous section for  $\mathcal{RS}_1^\alpha$ . Note that the non-classical shocks now join  $\hat{\mathbf{u}}$  and  $\check{\mathbf{u}}_2$  instead of  $\check{\mathbf{u}}_1$ . Actually, we will follow exactly the same approach based on in-cell discontinuous reconstructions but since the Riemann solver  $\mathcal{RS}_2^\alpha$  is not conservative on the second component of  $\mathbf{u}$  (unlike  $\mathcal{RS}_1^\alpha$ ), the scheme will be based on the evolution of  $\rho$  and  $v$ , instead of  $\rho$  and  $z = \rho w$ . Results not reported here show that considering the evolution of the conservative variable  $\mathbf{u} = (\rho, \rho w)$  does not lead to a good approximation of the non-classical shocks joining  $\hat{\mathbf{u}}$  to  $\check{\mathbf{u}}_2$ .

First of all, we introduce the vector of non conservative variables  $\mathbf{v} = (\rho, v)$  and the natural changes of variables  $\mathbf{v} = \mathbf{v}(\mathbf{u})$  and  $\mathbf{u} = \mathbf{u}(\mathbf{v})$ , with obvious notations. We then define for every  $j \in \mathbb{Z}$  the approximation  $\mathbf{u}_j^0$  of the initial data  $\mathbf{u}^0$  on the cell  $C_j$  as follows,

$$\mathbf{u}_j^0 := \mathbf{u}(\mathbf{v}_j^0) \quad \text{with} \quad \mathbf{v}_j^0 = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}(\mathbf{u}_0)(x) dx.$$

Again, let  $y^n := y(t^n)$  be the bus position at time  $t^n$  and let  $m \in \mathbb{Z}$  be such that  $y^n \in C_m$ . If the Riemann solver  $\mathcal{RS}_2^\alpha$  does not give the classical solution, a non-classical shock is expected to appear at  $x = y(t)$  and arise as the solution given by  $\mathcal{RS}_2^\alpha$  to the Riemann problem with initial datum

$$\mathbf{u}(0, x) = \begin{cases} \mathbf{u}_{m-1}^n & \text{if } x \leq y^n, \\ \mathbf{u}_{m+1}^n & \text{if } x > y^n. \end{cases}$$

As before, we will make a reconstruction of the discontinuity if the inequality

$$f_1(\mathcal{RS}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V^n)) > F_\alpha + V^n \bar{\rho}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V^n) \quad (5.11)$$

holds. In this case we modify the Godunov's scheme as follows.

We introduce in the  $m$ -th cell a left state  $\mathbf{u}_{m,l}^n = (\rho_{m,l}^n, z_{m,l}^n)$  and a right state  $\mathbf{u}_{m,r}^n = (\rho_{m,r}^n, z_{m,r}^n)$  defined by

$$\mathbf{u}_{m,l}^n = \hat{\mathbf{u}} \quad \text{and} \quad \mathbf{u}_{m,r}^n = \check{\mathbf{u}}_2,$$

where  $\hat{\mathbf{u}}$  and  $\check{\mathbf{u}}_2$  are given by (3.2), (3.3). We then replace  $\mathbf{u}_m^n$  by the function  $\mathbf{u}_{\text{rec}}^n = (\rho_{\text{rec}}^n, z_{\text{rec}}^n)$  defined by means of  $\rho_{\text{rec}}^n$  and  $v_{\text{rec}}^n$  (recall that  $z = \rho(v + p(\rho))$ )

$$\rho_{\text{rec}}^n = \rho_{m,l}^n \mathbf{1}_{[x_{m-1/2}, x_m^{\rho,n})} + \rho_{m,r}^n \mathbf{1}_{[x_m^{\rho,n}, x_{m+1/2})}, \quad (5.12a)$$

$$v_{\text{rec}}^n = v_{m,l}^n \mathbf{1}_{[x_{m-1/2}, x_m^{v,n})} + v_{m,r}^n \mathbf{1}_{[x_m^{v,n}, x_{m+1/2})}, \quad (5.12b)$$

where we have used the two points

$$x_m^{\rho,n} = x_{m-1/2} + h d_m^{\rho,n} \quad \text{and} \quad x_m^{v,n} = x_{m-1/2} + h d_m^{v,n}$$

defined for  $d_m^{\rho,n}$  and  $d_m^{v,n}$  in  $[0, 1]$  such that

$$\begin{aligned} \rho_{m,l}^n d_m^{\rho,n} + \rho_{m,r}^n (1 - d_m^{\rho,n}) &= \rho_m^n, \\ v_{m,l}^n d_m^{v,n} + v_{m,r}^n (1 - d_m^{v,n}) &= v_m^n. \end{aligned} \quad (5.13)$$

We emphasize here that our in-cell reconstruction procedure is fully based on the  $\rho$  and  $v$  variables ( $z = \rho w$  just follows from the relation  $z = \rho(v + p(\rho))$ ) and therefore generally does not just consists in a single discontinuity) while the positions of the discontinuities in  $\rho$  and  $v$  are defined in such way

that the average values of the reconstructed discontinuities equal  $\rho_m^n$  and  $v_m^n$  respectively. More precisely, solving (5.13) for  $d_m^{\rho,n}$  and  $d_m^{v,n}$ , we find

$$d_m^{\rho,n} = \frac{\rho_m^n - \rho_{m,r}^n}{\rho_{m,l}^n - \rho_{m,r}^n} \quad \text{and} \quad d_m^{v,n} = \frac{v_m^n - v_{m,r}^n}{v_{m,l}^n - v_{m,r}^n}. \quad (5.14)$$

Clearly, the conditions  $d_m^{\rho,n} \in [0, 1]$  and  $d_m^{v,n} \in [0, 1]$  are necessary to reconstruct the discontinuity in the cell  $C_m$ . Notice again that these two constants are in general different.

As we did in the previous section for the first Riemann solver, we naturally assume that the discontinuities in (5.12) propagate at the same speed of the non-classical shock. Therefore their positions at time  $t^{n+1}$  are

$$x_m^{n+1,\rho} = x_m^{\rho,n} + V^n k^n \quad \text{and} \quad x_m^{n+1,v} = x_m^{v,n} + V^n k^n.$$

Depending on the values of  $V^n$  and  $k^n$ , the  $\rho$  and  $v$  reconstructed discontinuities at time  $t^{n+1}$  may be located in either the  $m$ -th or  $(m+1)$ -th cell, that is to say  $x_m^{n+1,\rho}$  and  $x_m^{n+1,v}$  can be less or larger than  $x_{m+1/2}$ . Since our objective is to define the solution at time  $t^{n+1}$  by averaging  $\rho$  and the non conservative variable  $v$  in the vicinity of the bus position (and to carry on using the conservative variables  $\rho$  and  $z = \rho v$  “away” from the bus position), one is led to distinguish between the following two cases in order to make the  $\rho$  and  $v$  variables evolve (note that  $\rho$  is a conservative variable so that we can use a flux formulation).

**Reconstruction of the  $\rho$ -component.** Since the first variable  $\rho$  is still conserved by  $\mathcal{RS}_2^\alpha$ , we can apply the same strategy used in Section 5.1 for  $\mathcal{RS}_1^\alpha$ . Denoting  $\Delta t_{m+1/2}^\rho = h(1 - d_m^{\rho,n})/V_b$ , we set

$$\begin{aligned} F_1(\mathbf{u}_{m-1}^n, \mathbf{u}_m^n) &= F_1(\mathbf{u}_{m-1}^n, \mathbf{u}_{m,l}^n), \\ F_1(\mathbf{u}_m^n, \mathbf{u}_{m+1}^n) &= \frac{1}{k^n} \left[ \min(\Delta t_{m+1/2}^\rho, k^n) f_1(\mathbf{u}_{m,r}^n) + \max(k^n - \Delta t_{m+1/2}^\rho, 0) f_1(\mathbf{u}_{m,l}^n) \right], \end{aligned}$$

and we apply the usual conservation formula

$$\rho_j^{n+1} = \rho_j^n - \frac{k^n}{h} (F_1(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) - F_1(\mathbf{u}_{j-1}^n, \mathbf{u}_j^n)) \quad \forall j \in \mathbb{Z}.$$

**Reconstruction of the  $v$ -component.** We distinguish two cases:

- (i) If  $x_m^{n+1,v} < x_{m+1/2}$  (see Figure 8a), the average of  $v$  in the  $m$ -th cell at time  $t^{n+1}$  is given by

$$v_m^{n+1} = \frac{1}{h} [(h d_m^{v,n} + V^n k^n) \hat{v} + (h - (h d_m^{v,n} + V^n k^n)) \check{v}_2], \quad (5.15)$$

The corresponding value of  $z$  is

$$z_m^{n+1} = \rho_m^{n+1} (v_m^{n+1} + p(\rho_m^{n+1})), \quad (5.16)$$

and to update the solution in the  $(m+1)$ -th cell, we use the right trace value  $\check{\mathbf{u}}_2$  to compute the numerical flux at the interface  $x_{m+1/2}$ , i.e.

$$\mathbf{u}_{m+1}^{n+1} = \mathbf{u}_{m+1}^n - \frac{k^n}{h} (F(\mathbf{u}_{m+1}^n, \mathbf{u}_{m+2}^n) - F(\check{\mathbf{u}}_2, \mathbf{u}_{m+1}^n)).$$

The other cells are updated in the usual conservative way.

(ii) If  $x_m^{n+1,v} \geq x_{m+1/2}$  (see Figure 8b), we first recall that  $v_{m+1}^n = \check{v}_2$ . Therefore the  $v$  component of the solution at time  $t^{n+1}$  is:

$$\begin{aligned} v_m^{n+1} &= \hat{v}, \\ v_{m+1}^{n+1} &= \frac{1}{h} [(h(2 - d_m^{v,n}) - V^n k^n) \check{v}_2 + (h(d_m^{v,n} - 1) + V^n k^n) \hat{v}] \end{aligned} \quad (5.17)$$

The corresponding values of  $z$  are

$$z_m^{n+1} = \rho_m^{n+1} (v_m^{n+1} + p(\rho_m^{n+1})) \quad \text{and} \quad z_{m+1}^{n+1} = \rho_{m+1}^{n+1} (v_{m+1}^{n+1} + p(\rho_{m+1}^{n+1})). \quad (5.18)$$

The other cells are updated in the usual conservative way.

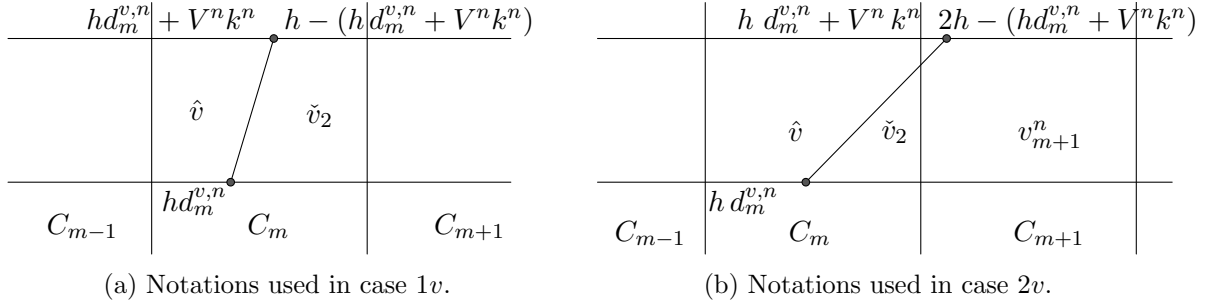


Figure 8: Representation of the reconstruction method ( $v$ -component).

Like in the previous section, the next proposition states that if the initial datum is a non-classical shock, then the solution given by the in-cell discontinuous reconstruction method on density and velocity is the non-classical shock itself.

**Proposition 4** *The scheme (5.1), (5.11)-(5.18) is exact when the Riemann initial datum is made of a single non-classical shock between the left state  $\hat{\mathbf{u}}$  and the right state  $\check{\mathbf{u}}_2$  and provided that the bus position is computed exactly (which is the case with the front tracking method introduced in Appendix A). In particular, let  $d_m^{\rho,n}$  and  $d_m^{v,n}$  be the two constants defined in (5.14). If  $\mathbf{v}_{m-1}^n = \hat{\mathbf{v}}$ ,  $\mathbf{v}_{m+1}^n = \check{\mathbf{v}}_2$  and there exists  $\gamma \in [0, 1]$  such that*

$$\mathbf{v}_m^n = \gamma \mathbf{v}_{m-1}^n + (1 - \gamma) \mathbf{v}_{m+1}^n, \quad (5.19)$$

then  $d_m^{\rho,n} = d_m^{v,n} = \gamma$ .

*Proof.* The proof follows the same steps as the one associated with  $\mathcal{RS}_1^\alpha$ , but is adapted to the non conservative treatment of the  $v$  variable. Let us consider an initial condition such that  $\mathbf{v}_j^0 = \hat{\mathbf{v}}$  if  $j < m$  and  $\mathbf{v}_j^0 = \check{\mathbf{v}}_2$  if  $j \geq m$ , so that (5.19) holds true with  $n = 0$  and  $\gamma = 0$ , and

$$\mathbf{v}_j^n = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}^{exa}(t^n, x) dx \quad \text{for all } x \in \mathbb{Z} \quad (5.20)$$

for  $n = 0$  and  $\mathbf{v}^{exa} = \mathbf{v}_0$ . In order to prove that the scheme is exact for such an isolated non-classical shock, let us assume that (5.19) and (5.20) hold true for a given time  $t^n$ , and let us show that

$$\mathbf{v}_j^{n+1} = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}^{exa}(t^{n+1}, x) dx \quad \text{for all } x \in \mathbb{Z}. \quad (5.21)$$



This is clearly true away from the non-classical shock, namely for  $j \neq m, m+1$ , since it corresponds to the constant regions of the solution, where the proposed scheme coincides with the usual Godunov's scheme. Moreover, since condition (5.19) holds, we have

$$\begin{pmatrix} \rho_m^n \\ v_m^n \end{pmatrix} = \begin{pmatrix} \gamma \hat{\rho} + (1-\gamma) \check{\rho}_2 \\ \gamma \hat{v} + (1-\gamma) \check{v}_2 \end{pmatrix}.$$

Hence we find  $d_m^{\rho,n} = d_m^{v,n} = \gamma$ , meaning that the reconstructed discontinuity is located at the same position as the discontinuity of the exact solution in the  $\rho$  and  $v$  variables. The proposed update formulas for  $\rho$  and  $v$  then clearly coincide with the averages of the exact solutions in the corresponding cells. Indeed, the averages are made in the non conservative variables  $\rho$  and  $v$  and follow the reconstructed discontinuity. This concludes the proof.  $\square$

### 5.3 Treatment of the contact discontinuities

The Godunov scheme is known to fail in capturing correctly contact discontinuities, see [10]. Therefore, we propose to modify the method as follows.

**Basic notations.** Let us denote

$$\sigma(\mathbf{u}^l, \mathbf{u}^r) = \frac{\rho^r v^r - \rho^l v^l}{\rho^r - \rho^l}$$

the propagation speed of the shock joining two points  $\mathbf{u}^l$  and  $\mathbf{u}^r$ . For all  $j \in \mathbb{Z}$  such that  $v_{j-1}^n \neq v_j^n$ , let  $\mathbf{u}_{j-1/2}^n$  be the intermediate state of the standard solution to the Riemann problem centred in  $x_{j-1/2}$ ,  $(\rho_{j-1/2}^n, v_{j-1/2}^n)$  being the corresponding coordinates in the non conserved variables. Let us define the value

$$\lambda_j^{l,n} = \begin{cases} \lambda_1(\rho_{j-1/2}^n, v_{j-1/2}^n) & \text{if } \rho_{j-1/2}^n \leq \rho_{j-1}^n, \\ \sigma(\mathbf{u}_{j-1}^n, \mathbf{u}_{j-1/2}^n) & \text{if } \rho_{j-1/2}^n > \rho_{j-1}^n, \end{cases}$$

which gives either the right propagation speed of a rarefaction wave or the propagation speed of a shock centred in  $x_{j-1/2}$ .

Similarly let us assume that  $v_j^n \neq v_{j+1}^n$  and consider the Riemann problem centred in  $x_{j+1/2}$ . Let  $\mathbf{u}_{j+1/2}^n$  be the intermediate state of the standard solution and let  $(\rho_{j+1/2}^n, v_{j+1/2}^n)$  be the corresponding point in the non conserved variables. Let us define the speed

$$\lambda_j^{r,n} = \begin{cases} \lambda_1(\rho_j^n, v_j^n) & \text{if } \rho_j^n \geq \rho_{j+1/2}^n, \\ \sigma(\mathbf{u}_j^n, \mathbf{u}_{j+1/2}^n) & \text{if } \rho_{j+1/2}^n > \rho_j^n, \end{cases}$$

which gives either the left propagation speed of a rarefaction wave or the propagation speed of a shock centred in  $x_{j+1/2}$ .

**The proposed modification.** Let us first notice that a contact discontinuity appears in the Riemann problems centred respectively in  $x_{j-1/2}$  or  $x_{j+1/2}$  whenever

$$v_{j-1}^n + p(\rho_{j-1}^n) \neq v_j^n + p(\rho_j^n) \quad \text{or} \quad v_{j+1}^n + p(\rho_{j+1}^n) \neq v_j^n + p(\rho_j^n). \quad (5.22)$$

In order to capture exactly contact discontinuities, we fix the value of the velocity in the cell  $C_j$  between  $t^n$  and  $t^{n+1}$ , i.e. we take

$$v_j^{n+1} = v_j^n$$

if one of the following conditions holds.

- (i) A wave of the first family with negative speed joins  $\mathbf{u}_{j-1}^n$  and  $\mathbf{u}_j^n$  and a contact discontinuity connects  $\mathbf{u}_j^n$  to  $\mathbf{u}_{j+1}^n$ :

$$v_{j-1}^n \neq v_j^n, \lambda_j^{l,n} \leq 0 \text{ and } v_j^n = v_{j+1}^n.$$

- (ii) A contact discontinuity joins  $\mathbf{u}_{j-1}^n$  to  $\mathbf{u}_j^n$  and a wave of the first family with positive speed joins  $\mathbf{u}_j^n$  to  $\mathbf{u}_{j+1}^n$ :

$$v_{j-1}^n = v_j^n, v_j^n \neq v_{j+1}^n \text{ and } \lambda_j^{r,n} \geq 0.$$

- (iii) Two contact discontinuities join respectively  $\mathbf{u}_{j-1}^n$  to  $\mathbf{u}_j^n$  and  $\mathbf{u}_j^n$  to  $\mathbf{u}_{j+1}^n$ :

$$v_{j-1}^n = v_j^n \text{ and } v_j^n = v_{j+1}^n.$$

- (iv) A wave of the first family with negative speed joins  $\mathbf{u}_{j-1}^n$  and  $\mathbf{u}_j^n$  and a wave of the first family with positive speed joins  $\mathbf{u}_j^n$  to  $\mathbf{u}_{j+1}^n$ :

$$v_{j-1}^n \neq v_j^n, \lambda_j^{l,n} \leq 0, v_j^n \neq v_{j+1}^n \text{ and } \lambda_j^{r,n} \geq 0.$$

It is important to notice that the classical Godunov's scheme does not provide us with such update values on  $v$  since it does average  $\rho$  and  $z$ . We also remark that, due to Lax entropy inequalities, the previous conditions can be summarized in the following ones:

$$M := \max\{\sigma(\mathbf{u}_{j-1}^n, \mathbf{u}_{j-1/2}^n), \lambda_1(\mathbf{u}_{j-1/2}^n)\} \leq 0,$$

$$m := \min\{\sigma(\mathbf{u}_j^n, \mathbf{u}_{j+1/2}^n), \lambda_1(\mathbf{u}_j^n)\} \geq 0.$$

Let us impose

$$\sigma(\mathbf{u}_{j-1}^n, \mathbf{u}_{j-1/2}^n) = \lambda_1(\mathbf{u}_{j-1/2}^n) = 0 \text{ and } \sigma(\mathbf{u}_j^n, \mathbf{u}_{j+1/2}^n) = \lambda_1(\mathbf{u}_j^n) = 0$$

respectively if  $v_{j-1}^n = v_j^n$  and  $v_j^n = v_{j+1}^n$ . Applying such a correction to the proposed method for  $\mathcal{RS}_1^\alpha$  does not raise difficulties. To apply the above correction to  $\mathcal{RS}_2^\alpha$ , assume that at time  $t^n$  the constraint is violated, i.e.

$$f_1(\mathcal{RS}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V^n)) > F_\alpha + V^n \bar{\rho}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V^n).$$

Then at time  $t^{n+1}$  a non-classical shock is expected at the bus position in the  $m$ -th or  $(m+1)$ -th cell, where we have a convex combination of  $\hat{\mathbf{u}}$  and  $\check{\mathbf{u}}_2$ , as a result of the discontinuity reconstruction method. Now, we have to distinguish two cases as in the previous section.

**Case 1:** At time  $t^{n+1}$  the discontinuity is in the  $m$ -th cell.

In all cells but the  $(m-1)$ -th, the  $m$ -th and the  $(m+1)$ -th, we check the condition (5.22). After the bus, we know that the solution given by  $\mathcal{RS}_2^\alpha$  contains a contact discontinuity on the right of the non-classical shock. The condition (5.22) is too strong for the  $(m+1)$ -th cell, because the velocity in the  $m$ -th cell is an average of  $\hat{v}$  and  $\check{v}_2$ . Therefore we check the weaker condition

$$v_{m+1}^n = v_{m+2}^n$$

and in this case we fix the value of the velocity in the  $(m+1)$ -th cell:  $v_{m+1}^{n+1} = v_{m+1}^n$ .

The solution in the cells  $C_{m-1}$  and  $C_m$  at time  $t^{n+1}$  is obtained with the discontinuity reconstruction method and is kept unchanged.

**Case 2:** At time  $t^{n+1}$  the discontinuity is in the  $(m+1)$ -th cell.

The argument is similar to the previous case. In order to capture the contact discontinuities, we check the condition (5.22) and correct the Godunov method everywhere, except for the cells  $C_m$ ,  $C_{m+1}$  and  $C_{m+2}$ .

For the cell  $C_{m+2}$  we use the weaker condition

$$v_{m+2}^n = v_{m+3}^n$$

and in this case we fix the value of the velocity in the  $(m+2)$ -th cell:  $v_{m+2}^{n+1} = v_{m+2}^n$ .

The solution in the  $m$ -th and in the  $(m+1)$ -th cell is given by the discontinuity reconstruction method and the velocity are respectively  $\hat{v}$  and a convex combination between  $\hat{v}$  and  $\check{v}_2$ .

Figure 9 illustrates the capability of this strategy in capturing non-classical shocks.

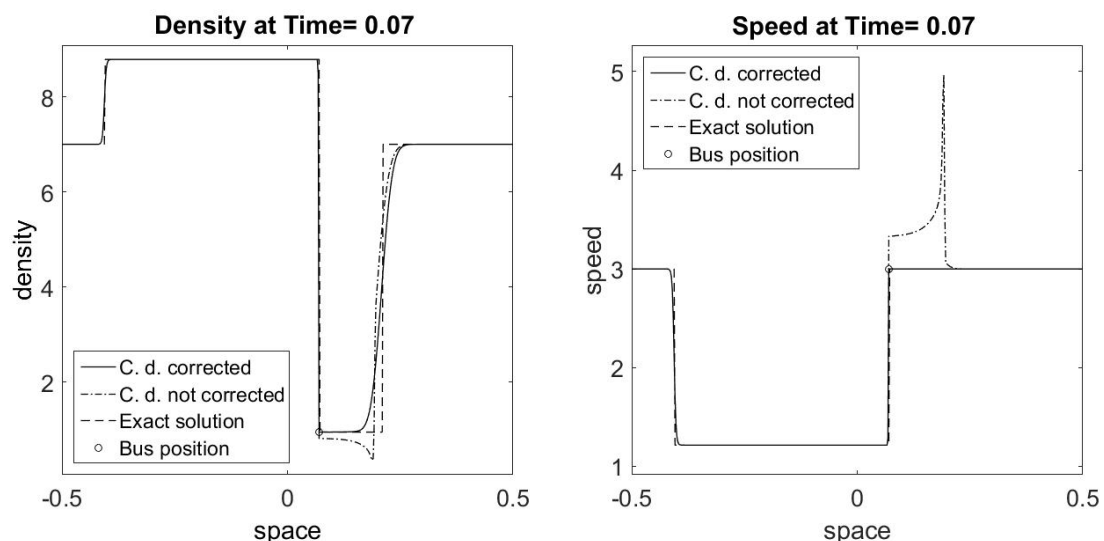


Figure 9: Solutions obtained with the discontinuity reconstruction method for the  $(\rho, v)$  coordinates. The dot-dashed line is obtained without the correction for the contact discontinuity, while the continuous line is obtained with the correction: in the first case, the velocity downstream the shock is overestimated, in the latter it is correct. The initial data are  $(\rho^l, v^l) = (\rho^r, v^r) = (7, 3)$ ,  $V_b = 1$ ,  $\alpha = 0.25$  and  $y_0 = 0$ .

## Appendices

### A A Front-Tracking technique to compute the bus trajectory

The bus trajectory is given by the solution of the following ODE

$$\dot{y}(t) = \begin{cases} V_b & \text{if } V_b \leq v(t, y(t)+), \\ v(t, y(t)+) & \text{otherwise.} \end{cases}$$

Fix  $n \in \mathbb{N}$ . Let  $y^n = y(t^n)$  be the bus position at time  $t^n$  and let us define  $m \in \mathbb{Z}$  such that  $y^n \in C_m = [x_{m-1/2}, x_{m+1/2}]$ . Let us assume that

$$V^n \bar{\rho}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V^n) \geq f_1(\mathcal{RS}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V^n)), \quad (\text{A.1})$$

where

$$V^n = \dot{y}(t^n) = \begin{cases} V_b & \text{if } V_b \leq v_m^n, \\ v_m^n & \text{if } V_b > v_m^n. \end{cases}$$

The inequality (A.1) implies that

$$V^n \geq \bar{v}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(V^n),$$

where we recall that  $\bar{v}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)$  is the  $v$  component of the classical solution  $\mathcal{RS}(\mathbf{u}_{m-1}^n, \mathbf{u}_{m+1}^n)(\bar{V}^n)$ . Since the bus would travel faster than the preceding vehicles, it has to adapt its speed to the traffic and it will keep this speed until the traffic will not change its velocity. This is due to an interaction between the bus trajectory and a wave coming from the local Riemann problems centered at  $x_{m\pm 1/2}$ . We distinguish the two cases:

- (i) the bus trajectory interacts with a wave coming from  $x_{m+1/2}$ ;
- (ii) the bus trajectory interacts with a wave coming from  $x_{m-1/2}$ .

We adapt the algorithms introduced in [8, 9].

### A.0.1 Case (i)

Let us consider the classical Riemann problem for (2.1a) with initial data

$$(\rho, z)(t^n, x) = \begin{cases} \mathbf{u}_m^n & \text{if } x \leq x_{m+1/2}, \\ \mathbf{u}_{m+1}^n & \text{if } x > x_{m+1/2}, \end{cases}$$

and let  $\mathbf{u}_{m+1/2}^{\text{int}} = (\rho_{m+1/2}^{\text{int}}, \rho_{m+1/2}^{\text{int}}(v_{m+1/2}^{\text{int}} + p(\rho_{m+1/2}^{\text{int}})))$  be the intermediate state of the classical solution  $\mathcal{RS}(\mathbf{u}_m^n, \mathbf{u}_{m+1}^n)(\cdot)$ . Since the bus trajectory is only affected by speed changes, it keeps constant speed across contact discontinuities. Therefore, we focus on interactions with waves of the first family.

1. Let us consider a shock centered at  $(t^n, x_{m+1/2})$ .

This case happens whenever  $\rho_m^n < \rho_{m+1/2}^{\text{int}}$  (and  $v_m^n > v_{m+1/2}^{\text{int}}$ , therefore the bus will slow down). The propagation speed  $\sigma_{m+1/2}$  of the shock is given by the Rankine-Hugoniot condition, namely

$$\sigma_{m+1/2} = \frac{\rho_m^n v_m^n - \rho_{m+1/2}^{\text{int}} v_{m+1/2}^{\text{int}}}{\rho_m^n - \rho_{m+1/2}^{\text{int}}}.$$

Since  $V^n > \sigma_{m+1/2}$ , let  $(t^*, x^*)$  be the interaction point between the shock and the bus trajectory. Solving in  $t^*$  the equation

$$y^n + (t^* - t^n)V^n = x_{m+1/2} + \sigma_{m+1/2}(t^* - t^n),$$

we find

$$t^* = \frac{x_{m+1/2} - y^n}{V^n - \sigma_{m+1/2}} + t^n, \quad x^* = y^n + t^*V^n.$$

If  $t^* \geq k^n$ , then no interaction between the bus and the shock wave occurs within the interval  $[t^n, t^{n+1})$ . Otherwise, the new position of the bus at time  $t^{n+1}$  is

$$y^{n+1} = y^n + V^n t^* + \min(v_{m+1}^n, V_b)(k^n - t^*).$$

2. Let us consider the case of a rarefaction wave:  $\rho_m^n \geq \rho_{m+1/2}^{\text{int}}$ . During the motion, the bus takes the speed of the vehicles in front of it until their speed is lower than the maximal velocity of the bus. Therefore, setting

$$\xi(t, x) = \lambda_1(\rho(t, x), v(t, x)) = \frac{x - x_{m+1/2}}{t - t^n},$$

we have

$$\dot{y}(t) = \begin{cases} V^n = \min(v_m^n, V_b) & \text{if } \xi(t, y(t)) \leq \lambda_1(\rho_m^n, v_m^n), \\ \min(v(t, y(t)), V_b) & \text{if } \lambda_1(\rho_m^n, v_m^n) < \xi(t, y(t)) < \lambda_1(\rho_{m+1/2}^{\text{int}}, v_{m+1/2}^{\text{int}}), \\ \min(v_{m+1}^n, V_b) & \text{if } \xi(t, y(t)) \geq \lambda_1(\rho_{m+1/2}^{\text{int}}, v_{m+1/2}^{\text{int}}). \end{cases} \quad (\text{A.2})$$

Let  $(t^*, x^*)$  be the first point of interaction between the bus and the rarefaction. Solving with respect to  $t^*$  the equation

$$y^n + V^n(t^* - t^n) = x_{m+1/2} + \lambda_1(\rho_m^n, v_m^n)(t^* - t^n),$$

we find

$$t^* = \frac{x_{m+1/2} - y^n}{V^n - \lambda_1(\rho_m^n, v_m^n)} + t^n, \quad x^* = y^n + V^n t^*. \quad (\text{A.3})$$

To compute explicitly the bus trajectory along the rarefaction, we focus on a specific pressure function  $p$ :

$$p(\rho) = \rho^\gamma \quad \text{for } \gamma \geq 1,$$

see [6]. Long but tedious computations lead to the following:

**Proposition 5** *If the bus interacts with a rarefaction wave centered in  $(t^n, x_{m+1/2})$ , then its trajectory is*

$$y(t) = x_{m+1/2} + \omega_m^n(t - t^n) + C^*(t - t^n)^{\frac{1}{\gamma+1}} \quad (\text{A.4})$$

for  $t \in [t^*, t^{**}]$ , where  $w_m^n = v_m^n + p(\rho_m^n)$ ,

$$C^* = \frac{x^* - x_{m+1/2} - \omega_m^n(t^* - t^n)}{(t^* - t^n)^{\frac{1}{\gamma+1}}}$$

and  $t^*, x^*$  are defined in (A.3) and

$$t^{**} = \left[ (\gamma + 1) \frac{\min\{V_b, v_{m+1}^n\} - \omega_m^n}{C^*} \right]^{-\frac{\gamma+1}{\gamma}} \quad (\text{A.5})$$

is the last interaction time between the bus and the rarefaction.

*Proof.* Deriving the equation (A.4), we find

$$\dot{y}(t) = \omega_m^n + \frac{C^*}{\gamma + 1} (t - t^n)^{-\frac{\gamma}{\gamma+1}}.$$

Let  $(t^{**}, x^{**})$  be the last point of interaction between the bus trajectory and the rarefaction wave. By (A.2), the bus will take the speed  $v \in [v_m^n, v_{m+1}^n]$  of the vehicles in front of him, until this speed is lower than its maximal speed  $V_b$ . Hence the point  $(t^{**}, x^{**})$  satisfies the equation

$$\dot{y}(t^{**}) = \min\{V_b, v_{m+1}^n\},$$

which gives (A.5). For more details, see [22].  $\square$

If  $t^* \geq k^n$ , then no interaction between the bus and the rarefaction wave occurs in the interval  $[t^n, t^{n+1})$  and the bus position at time  $t^{n+1}$  is

$$y^{n+1} = y^n + V^n k^n.$$

If  $t^* < k^n$ , then we have to consider two cases: whether  $t^{**} \geq k^n$  the bus position at time  $t^{n+1}$  is

$$y^{n+1} = y(t^{n+1}),$$

where  $y(t)$  is the trajectory (A.4); otherwise

$$y^{n+1} = x^{**} + (k^n - t^{**}) \max\{\min(V_b, v) : v \in [v_m^n, v_{m+1}^n]\}.$$

### A.0.2 Case (ii)

Let us consider the classical Riemann problem for (2.1a) with initial data

$$(\rho, z)(t^n, x) = \begin{cases} \mathbf{u}_{m-1}^n & \text{if } x \leq x_{m-1/2}, \\ \mathbf{u}_m^n & \text{if } x > x_{m-1/2}. \end{cases} \quad (\text{A.6})$$

Let  $\mathbf{u}_{m-1/2}^{\text{int}} = (\rho_{m-1/2}^{\text{int}}, \rho_{m-1/2}^{\text{int}}(v_{m-1/2}^{\text{int}} + p(\rho_{m-1/2}^{\text{int}})))$  be the intermediate state of the classical solution  $\mathcal{RS}(\mathbf{u}_{m-1}^n, \mathbf{u}_m^n)(\cdot)$ .

Again, we concentrate on interactions with waves belonging to the first family. Moreover, a wave solution to (2.1a), (A.6), can interact with the trajectory of the bus only if its propagation speed is positive.

1. Let us assume that  $\rho_{m-1}^n < \rho_{m-1/2}^{\text{int}}$ , so that the wave coming from  $x_{m-1/2}$  is a shock with propagation speed  $\sigma_{m-1/2}$  given by

$$\sigma_{m-1/2} = \frac{\rho_{m-1/2}^{\text{int}} v_{m-1/2}^{\text{int}} - \rho_{m-1}^n v_{m-1}^n}{\rho_{m-1/2}^{\text{int}} - \rho_{m-1}^n}. \quad (\text{A.7})$$

Let us suppose that the bus speed  $V^n = \min(V_b, v_m^n)$  at time  $t^n$  is equal to  $v_m^n$ . This means that  $v_m^n \leq V_b$ . Since  $y^n \geq x_{m+1/2}$ , an interaction with the shock can happen if and only if  $v_m^n \leq \sigma_{m-1/2}$ , which is absurd. On the other hand the case  $V^n = V_b$  holds when  $V_b \leq v_m^n$ . Since  $v_m^n < v_{m-1}^n$ , we have

$$V_{\text{new}}^n = \min(V_b, v_{m-1}^n) = V_b.$$

2. Let us suppose that  $\rho_{m-1}^n \geq \rho_{m-1/2}^{\text{int}}$ , so that  $\mathbf{u}_{m-1}^n$  and  $\mathbf{u}_{m-1/2}^{\text{int}}$  are connected by a rarefaction wave centered at  $x_{m-1/2}$ .

Let us recall that the bus speed at the instant  $t^n$  is  $V^n = \min(V_b, v_m^n)$ .

**Proposition 6** *If  $V^n < \lambda_1(\rho_{m-1/2}^{\text{int}}, v_{m-1/2}^{\text{int}})$ , so that an interaction occurs between the bus and the rarefaction wave centered in  $x_{m-1/2}$ , then the bus speed before the interaction is  $V^n = V_b$  and the bus keeps its maximal speed during all the interaction.*

*Proof.* We have  $V^n = v_m^n$  if and only if  $v_m^n \leq V_b$ , which is in contradiction with the assumption  $V^n < \lambda_1(\rho_{m-1/2}^{\text{int}}, v_{m-1/2}^{\text{int}})$ . Therefore we must have  $V^n = V_b > v_m^n$ .

Let  $(\bar{\rho}, \bar{v})$  be the density and speed values at the last interaction point of the bus with the rarefaction: we must have

$$\lambda_1(\bar{\rho}, \bar{v}) = \min\{V_b, \bar{v}\}.$$

Since  $\bar{v} > \lambda_1(\bar{\rho}, \bar{v})$ , when the interaction occurs the bus cannot take the speed of the vehicles but it keeps its maximal speed  $V_b$ . Moreover the bus interacts only with the points  $(\rho, v)$  on the rarefaction wave such that

$$\lambda_1(\bar{\rho}, \bar{v}) \leq \lambda_1(\rho, v) \leq \lambda_1(\rho_{m-1/2}^{\text{int}}, v_{m-1/2}^{\text{int}}).$$

Since for all these points we have  $v > \bar{v} > V_b$ , the bus keeps its maximal speed  $V_b$  during the whole interaction.  $\square$

Therefore in both cases the bus speed remains constant and the bus position at time  $t^{n+1}$  is given by

$$y^{n+1} = y^n + k^n \min(V_b, v_m^n).$$

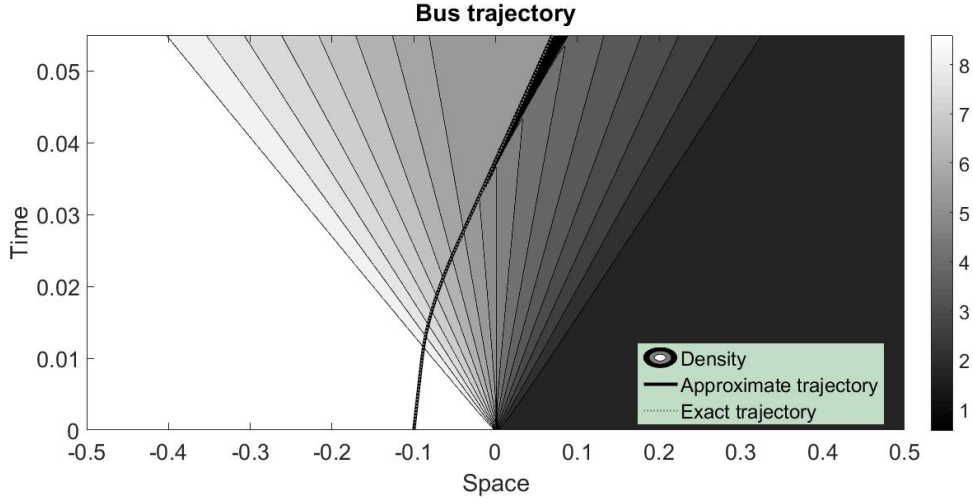


Figure 10: Bus trajectory corresponding to the data  $(\rho^l, v^l) = (9, 1)$ ,  $(\rho^r, v^r) = (2, 8)$ ,  $V_b = 4$ ,  $\alpha = 0.5$ ,  $R = 15$  and  $y_0 = -0.1$ .

## Acknowledgments

This research was partially supported by the European Research Council under the European Union’s Seventh Framework Program (FP/2007-2013) ERC Grant Agreement 257661. S. Villa was also partially supported by the Erasmus+ for Traineeship grant (2014/2015).

## References

- [1] N. Aguillon. Capturing nonclassical shocks in nonlinear elastodynamic with a conservative finite volume scheme. Feb. 2015.
- [2] N. Aguillon and C. Chalons. Nondiffusive conservative schemes based on approximate riemann solvers for lagrangian gas dynamics. *ESAIM: M2AN*, to appear.
- [3] B. Andreianov, C. Donadello, and M. D. Rosini. A second-order model for vehicular traffics with local point constraints on the flow. *Math. Models Methods Appl. Sci.*, 26(4):751–802, 2016.
- [4] B. Andreianov, P. Goatin, and N. Seguin. Finite volume schemes for locally constrained conservation laws. *Numer. Math.*, 115(4):609–645, 2010. With supplementary material available online.
- [5] B. P. Andreianov, C. Donadello, U. Razafison, J. Y. Rolland, and M. D. Rosini. Solutions of the Aw-Rascle-Zhang system with point constraints. *Netw. Heterog. Media*, 11(1):29–47, 2016.
- [6] A. Aw and M. Rascle. Resurrection of “second order” models of traffic flow. *SIAM J. Appl. Math.*, 60(3):916–938 (electronic), 2000.
- [7] B. Boutin, C. Chalons, F. Lagoutire, and P. G. LeFloch. A convergent and conservative scheme for nonclassical solutions based on kinetic relations. I. *Interfaces and Free Boundaries*, 10(3):399–421, 2008.
- [8] G. Bretti and B. Piccoli. A tracking algorithm for car paths on road networks. *SIAM J. Appl. Dyn. Syst.*, 7(2):510–531, 2008.
- [9] C. Chalons, M. L. Delle Monache, and P. Goatin. A conservative scheme for non-classical solutions to a strongly coupled PDE-ODE problem. 2014.
- [10] C. Chalons and P. Goatin. Transport-equilibrium schemes for computing contact discontinuities in traffic flow modeling. *Commun. Math. Sci.*, 5(3):533–551, 2007.
- [11] C. Chalons, P. Goatin, and N. Seguin. General constrained conservation laws. Application to pedestrian flow modeling. *Netw. Heterog. Media*, 8(2):433–463, 2013.
- [12] R. M. Colombo and P. Goatin. A well posed conservation law with a variable unilateral constraint. *J. Differential Equations*, 234(2):654–675, 2007.
- [13] R. M. Colombo, P. Goatin, and M. D. Rosini. On the modelling and management of traffic. *ESAIM Math. Model. Numer. Anal.*, 45(5):853–872, 2011.
- [14] M. L. Delle Monache and P. Goatin. A front tracking method for a strongly coupled PDE-ODE system with moving density constraints in traffic flow. *Discrete Contin. Dyn. Syst. Ser. S*, 7(3):435–447, 2014.



- [15] M. L. Delle Monache and P. Goatin. Scalar conservation laws with moving constraints arising in traffic flow modeling: an existence result. *J. Differential Equations*, 257(11):4015–4029, 2014.
- [16] M. Garavello and P. Goatin. The Aw-Rascle traffic model with locally constrained flow. *J. Math. Anal. Appl.*, 378(2):634–648, 2011.
- [17] M. Garavello and S. Villa. The Cauchy problem for the Aw-Rascle-Zhang traffic model with locally constrained flow. 2016.
- [18] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.
- [19] P. D. Lax. *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*. CBMF-NSF Regional Conf. Series in Appl. Math. SIAM, Philadelphia, PA, 1973.
- [20] P. LeFloch. *Hyperbolic Systems of Conservation Laws. The theory of classical and nonclassical shock waves*. Lectures in Mathematics. ETH Zrich, Birkhuser, 2002.
- [21] B. Temple. Systems of conservation laws with invariant submanifolds. *Trans. Amer. Math. Soc.*, 280(2):781–795, 1983.
- [22] S. Villa. *The Aw-Rascle-Zhang model with constraints*. Master thesis, Università degli Studi di Milano - Bicocca, 2015.
- [23] H. Zhang. A non-equilibrium traffic model devoid of gas-like behavior. *Transportation Research Part B: Methodological*, 36(3):275 – 290, 2002.