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► **To cite this version:**

Elena Di Bernardino, Didier Rullière. A note on upper-patched generators for Archimedean copulas. 2016. hal-01347869v1

**HAL Id: hal-01347869**

**<https://hal.science/hal-01347869v1>**

Preprint submitted on 21 Jul 2016 (v1), last revised 1 Mar 2017 (v2)

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# A NOTE ON UPPER-PATCHED GENERATORS FOR ARCHIMEDEAN COPULAS

ELENA DI BERNARDINO AND DIDIER RULLIÈRE

ABSTRACT. The class of multivariate Archimedean copulas is defined by using a real-valued function called the generator of the copula. This generator satisfies some properties, including  $d$ -monotony. We propose here a new basic transformation of this generator, preserving these properties, thus ensuring the validity of the transformed generator and inducing a proper valid copula. This transformation acts only on a specific portion of the generator, it allows both the non-reduction of the likelihood on a given dataset, and the choice of the upper tail dependence coefficient of the transformed copula. Numerical illustrations show the utility of this construction, which can improve the fit of a given copula both on its central part and its tail.

Keywords: *Archimedean copulas; transformations; distortions; tail dependence coefficients; likelihood.*

## 1. INTRODUCTION

The class of Archimedean copulas is a well-known class of copulas, indexed by a function  $\phi : \mathbb{R}^+ \rightarrow [0, 1]$  called the generator of the copula (see McNeil and Nešlehová (2009)). In practice, depending on the problem at hand and the data, many procedures are available to choose a suitable generator, either by selecting a parametric one, or by trying non-parametric estimation of the latter (see, among many references, Genest and Rivest (1993); Genest et al. (1995); Lambert (2007); Kim et al. (2007); Genest et al. (2011); Di Bernardino and Rullièrre (2013b); Dimitrova et al. (2008)). A problem is that it can be difficult to fit both the global shape of the copula and its tail dependence.

In this article we aim at proposing a basic transformation of a given initial generator such that the transformed generator is guaranteed to be a valid Archimedean generator, such that the upper tail dependence can be chosen, and which ensures that the likelihood of a dataset is not reduced by the transformation. In other words, we are looking for valid distortions that are changing mainly the upper tail dependence behaviour of a given initial copula, i.e. that would be able to exhibit any chosen tail dependence coefficient, and that would also preserve the shape of a copula on its central part.

We focus on Archimedean copulas. We consider an initial  $d$ -dimensional Archimedean copula with a generator  $\phi$ . This generator is a real function,  $\phi : [0, \infty) \rightarrow [0, 1]$ , non-increasing and continuous, such that  $\phi(0) = 1$  and  $\lim_{x \rightarrow +\infty} \phi(x) = 0$ . From Theorem 2.2 in McNeil and Nešlehová (2009),

$$C(u_1, \dots, u_d) := \phi(\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d))$$

is a  $d$ -dimensional copula if and only if its generator  $\phi$  is  $d$ -monotone on  $[0, \infty)$ , where the  $d$ -monotony definition is recalled hereafter. In the following, a  $d$ -monotone generator, will be called *valid generator* in the dimension  $d$ . We write  $\psi = \phi^{-1}$  the inverse function

of a given generator  $\phi$ , with by convention  $\psi(0) = \inf \{x \in \mathbb{R}^+ : \phi(x) = 0\}$ . This function  $\psi : [0, 1] \rightarrow \mathbb{R}^+$  will be called an *inverse generator*.

Furthermore, we always write  $f^{(k)}(t) = \frac{d^k}{dt^k} f(t)$  the derivative of order  $k$  of a  $k$ -differentiable function  $f$ .

Let us recall the definition of the  $d$ -monotony, as given in McNeil and Nešlehová (2009).

**Definition 1.1** (*d-monotone function*). *A real function  $f$  is called  $d$ -monotone in  $(a, b)$ , where  $a, b \in \bar{\mathbb{R}}$  and  $d \geq 2$ , if it is differentiable there up to the order  $d-2$  and the derivatives satisfy*

$$(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, 1, \dots, d-2,$$

for any  $x \in (a, b)$  and further if  $(-1)^{d-2} f^{(d-2)}$  is non-increasing and convex in  $(a, b)$ . For  $d = 1$ ,  $f$  is called *1-monotone* in  $(a, b)$  if it is nonnegative and non-increasing there.

If  $f$  has derivatives of all orders in  $(a, b)$  and if  $(-1)^k f^{(k)}(x) \geq 0$ , for any  $x \in (a, b)$ , then  $f$  is called *completely monotone*.

As announced, the aim of this work is to change a part of a given generator in order to change the tail dependence behaviour of the resulting copula. We thus recall hereafter some classical indicators of the tail dependence of a copula. Among them one can propose, for assessing the tail behaviour of a copula, the so-called tail dependence coefficients (TDC). In the general multivariate case, they can be expressed as follows (as adapted from definitions in De Luca and Rieviccio (2012), Li (2009)).

**Definition 1.2** (Multivariate tail dependence coefficients). *Assume that the considered copula  $C$  is the distribution of some random vector  $\mathbf{U} := (U_1, \dots, U_d)$ . Denote  $I = \{1, \dots, d\}$  and consider two non-empty subsets  $I_h \subset I$  and  $\bar{I}_h = I \setminus I_h$  of respective cardinal  $h \geq 1$  and  $d-h \geq 1$ . Let us define, for  $u \in (0, 1)$ ,*

$$\begin{aligned} \lambda_L^{I_h, \bar{I}_h}(u) &= \mathbb{P}[U_i \leq u, i \in I_h \mid U_i \leq u, i \in \bar{I}_h] \\ \lambda_U^{I_h, \bar{I}_h}(u) &= \mathbb{P}[U_i \geq u, i \in I_h \mid U_i \geq u, i \in \bar{I}_h]. \end{aligned}$$

A multivariate version of classical bivariate tail dependence coefficients, when the limits exist, is given by

$$\lambda_L^{I_h, \bar{I}_h} = \lim_{u \rightarrow 0^+} \lambda_L^{I_h, \bar{I}_h}(u), \quad \lambda_U^{I_h, \bar{I}_h} = \lim_{u \rightarrow 1^-} \lambda_U^{I_h, \bar{I}_h}(u).$$

If for all  $I_h \subset I$ ,  $\lambda_L^{I_h, \bar{I}_h} = 0$ , (resp.  $\lambda_U^{I_h, \bar{I}_h} = 0$ ) then we say  $\mathbf{U}$  is *lower tail independent* (resp. *upper tail independent*).

**Definition 1.3** (Multivariate tail dependence coefficients for Archimedean copulas). *For Archimedean copulas the multivariate lower and upper tail dependence coefficients, when the limit exist, are respectively:*

$$\lambda_L^{I_h, \bar{I}_h} = \lambda_L^{(h, d-h)} = \lim_{u \rightarrow 0^+} \lambda_L^{(h, d-h)}(u), \quad \lambda_U^{I_h, \bar{I}_h} = \lambda_U^{(h, d-h)} = \lim_{u \rightarrow 1^-} \lambda_U^{(h, d-h)}(u),$$

where  $\lambda_L^{(h, d-h)}(u) = \frac{\psi^{-1}(d\psi(u))}{\psi^{-1}((d-h)\psi(u))}$  and  $\lambda_U^{(h, d-h)}(u) = \frac{\sum_{i=0}^d (-1)^i C_d^i \psi^{-1}(i\psi(u))}{\sum_{i=0}^{d-h} (-1)^i C_{d-h}^i \psi^{-1}(i\psi(u))}$ , for  $u \in (0, 1)$ , and where  $C_d^i$  and  $C_{d-h}^i$  are binomial coefficients.

One can show that, when it exists, the lower tail dependence coefficient  $\lambda_L^{I_h, \bar{I}_h}$  is linked with the asymptotic behaviour of  $\phi(x)$  when  $x \rightarrow +\infty$ , and that  $\lambda_U^{I_h, \bar{I}_h}$  is linked with the behaviour of  $\phi(x)$  when  $x \rightarrow 0$  (see e.g., Charpentier and Segers (2009), Di Bernardino and

Rullière (2016)). Thus modifying a generator  $\phi(x)$  for small values of  $x$  allows to change the upper tail dependence coefficient of the copula.

In the literature, some transformations of copulas correspond to the creation of a new Archimedean generator  $T \circ \phi$ , by composition of a distortion function  $T$  with an initial generator  $\phi$ , see for example Durrleman et al. (2000), Valdez and Xiao (2011), Klement et al. (2005), Morillas (2005). In previous works, we investigated the use of hyperbolic distortions of Archimedean generator, for example in a logit scale, see Di Bernardino and Rullière (2013a,b) for the estimation of some transformations and Di Bernardino and Rullière (2015) for illustrations with specific real data in the dimension  $d = 5$ . These constructions have the advantage of being readily invertible, i.e. an explicit parametric expression of both the generator and its inverse function are available. This eases in particular calculations that are done using the transformed copula. However, the validity of the transformed generator requires the calculation of the signs of the  $d$  first derivatives of the transformed generator (in the case it is  $d$  times differentiable), which can be difficult, especially when  $d$  is greater than two, when Faa Di Bruno's formula is involved. This is why we propose here another approach. Notice that the approach here does not aim at leaving the Archimedean class of copulas: the distorted copula will stay within this class.

The structure of the paper is as follows. We give in Section 2 the definition of the proposed upper-patched generator. In particular, in Section 2.1 we show the automatic validity of this generator, which induces a valid distorted copula. In Section 2.2 we investigate the upper tail dependence behaviour of the distorted copula. In Section 2.3 we show that one can choose a upper tail dependence coefficient, without reducing the likelihood of a given dataset. At last, Section 3 gives some illustrations where a copula exhibiting the wrong dependence behaviour is transformed using a upper patched generator.

## 2. UPPER PATCHED GENERATOR

In this section, we introduce a new construction, that we have chosen to call upper-patched generator (the adjective upper refers to the fact that the transformation will modify the upper-tail dependence coefficient of the copula, the word patched is because it makes only a local change on the generator). We show that this construction necessarily yields a valid Archimedean generator in a chosen dimension  $d \geq 2$ . We also show that any upper-tail dependence coefficient can be obtained using this construction, without reducing the likelihood of a given dataset.

We will distort the initial Archimedean copula  $C$  having generator  $\phi$ . To this aim we use another  $d$ -dimensional Archimedean copula having generator  $\phi_D$ . Both generators will be assumed to be valid. We give first some conditions on these generators  $\phi$  and  $\phi_D$ .

**Definition 2.1** (Initial and distortion generators). *Let  $t_0 \in (0, +\infty)$  be a given real value, and  $d \geq 2$  be a given integer dimension. The initial generator  $\phi : \mathbb{R}^+ \rightarrow [0, 1]$  and distortion generator  $\phi_D : \mathbb{R}^+ \rightarrow [0, 1]$  are such that*

- $\phi$  is a valid  $d$ -dimensional Archimedean generator, thus  $d$ -monotone on  $\mathbb{R}^+$ .
- $\phi_D$  is a valid  $d$ -dimensional Archimedean generator, thus  $d$ -monotone on  $\mathbb{R}^+$ , such that  $k$ -th derivatives  $\phi_D^{(k)}(d_0) = 0$ , for all  $k = 0, \dots, d-2$  and for some  $d_0 \in (0, t_0]$ . This implies in particular that  $\phi_D$  is a non-strict generator with end-point  $d_0 = \inf \{t \in \mathbb{R}^+ : \phi_D(t) = 0\} \leq t_0$ .

Some examples of non-strict generators from Table 4.1 in Nelsen (1999) are given in Table 1. Remark that, when a generator  $\phi_D^0$  has an end-point  $d_0$ ,  $\phi_D(t) = \phi_D^0(\frac{d_0}{t_0}t)$  is a valid  $d$ -dimensional generator with end-point  $t_0$ . Remark also that, for instance, Copula 4.1.11 in Table 1 satisfies:  $\phi_D^{(k)}(d_0) = 0$ , for all  $k = 0, \dots, d - 2$  with  $d_0 = \ln(2)$ .

Copula	$\phi_D(t)$	$\phi_D^{-1}(t)$	parameter $\theta$	$d_0$	$\lambda_L$	$\lambda_U$
4.1.2	$1 - t^{1/\theta}$	$(1 - t)^\theta$	$\theta \in [1, \infty)$	1	0	$2 - 2^{\frac{1}{\theta}}$
4.1.7	$\frac{1}{\theta}(\theta + \exp(-t) - 1)$	$-\ln(\theta t + (1 - \theta))$	$\theta \in (0, 1]$	$-\ln(-\theta + 1)$	0	0
4.1.8	$\frac{t-1}{-\theta t - t - 1}$	$\frac{1-t}{1+(1+\theta)t}$	$\theta \in [1, \infty)$	1	0	0
4.1.11	$(2 - \exp(t))^{\frac{1}{\theta}}$	$\ln(2 - t^\theta)$	$\theta \in (0, 1/2]$	$\ln(2)$	0	0
4.1.15	$(1 - t^{\frac{1}{\theta}})^\theta$	$(1 - t^{1/\theta})^\theta$	$\theta \in [1, \infty)$	1	0	$2 - 2^{\frac{1}{\theta}}$
4.1.18	$\frac{\ln(t)+\theta}{\ln(t)}$	$\exp(\frac{\theta}{t-1})$	$\theta \in [2, \infty)$	$\exp(-\theta)$	0	1
4.1.21	$1 - [-(t+1)^\theta + 1]^{1/\theta}$	$1 - [1 - (1-t)^\theta]^{1/\theta}$	$\theta \in [1, \infty)$	1	0	$2 - 2^{\frac{1}{\theta}}$
4.1.22	$(-\sin(t) + 1)^{1/\theta}$	$\arcsin(1 - t^\theta)$	$\theta \in (0, 1]$	$\frac{\pi}{2}$	0	0

TABLE 1. Some examples of non-strict Archimedean generators from Table 4.1 in Nelsen (1999).  $\lambda_L$  and  $\lambda_U$  are here the classical bivariate tail dependence coefficients and  $d_0$  the end-point of the generator.

**Definition 2.2** (Patched generator). *Consider an initial generator  $\phi$  and distortion one  $\phi_D$  as in Definition 2.1, where  $d_0$  is the end-point of  $\phi_D$ . Let  $t_0 \geq d_0$ . A patched generator  $\tilde{\phi}$ , built from an initial generator  $\phi$  and using a distortion generator  $\phi_D$ , is given by*

$$(1) \quad \tilde{\phi}(t) = \begin{cases} p_{d-1}(t) + (1 - p_{d-1}(0))\phi_D(t), & \text{if } t < t_0, \\ \phi(t), & \text{if } t \geq t_0; \end{cases}$$

for all  $t \in \mathbb{R}^+$ , where  $p_{d-1}(t) = \sum_{i=0}^{d-1} \frac{\phi_+^{(i)}(t_0)}{i!} (t - t_0)^i$  is the Taylor expansion at order  $d - 1$  (with a right derivative at order  $d - 1$ , since  $\phi$  is  $d - 2$  times differentiable and  $d - 1$  time right differentiable, by convexity of  $\phi^{(d-2)}$ ).

The validity of this patched generator will be shown in Proposition 2.2 below. Its principle is that the initial generator remains unchanged on  $[t_0, +\infty]$ , so that the change acts only on the upper tail dependence behavior the copula. On  $[0, t_0]$ , due to constraints on derivatives at  $t = t_0$  and convexity conditions, the patched generator must be above the Taylor expansion  $p_{d-1}(t)$ , so that we just add a new distortion generator  $\phi_D$ , with suitable normalisation constant  $1 - p_{d-1}(0)$  to ensure that  $\tilde{\phi}(0) = 1$ .

Using a non-strict generator with  $\phi_D(t_0) = 0$  ensures the continuity of  $\tilde{\phi}$ , but the supplementary constraints on derivatives of  $\phi_D$ , in Definition 2.1, are needed in order to ensure the  $d$ -monotony of  $\tilde{\phi}$ , as discussed in Proposition 2.2 below.

We give in Figure 1 an illustration of the patched generator  $\tilde{\phi}$  proposed in Definition 2.2 when  $d = 2$  (left panel) and  $d = 3$  (right panel). As one can see in Figure 1 (left panel), the patched generator in  $d = 2$  is continuous decreasing and convex, so that in particular it can be used as a valid 2-dimensional generator. One can also verify that the proposed

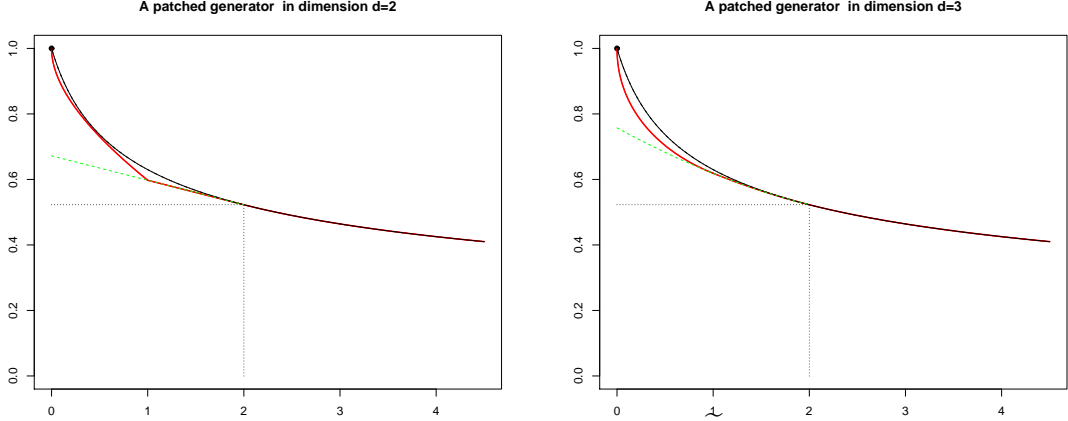


FIGURE 1. **Left panel:** A patched generator  $\tilde{\phi}$  (red line) in the case where  $d = 2$ . We consider here the distortion generator  $\phi_D(t) = 1 - t^{\frac{1}{\theta_D}}$  with  $\theta_D = 2$  and end-point  $d_0 = 1$  (see Copula 4.1.2 in Table 1). **Right panel:** A patched generator  $\tilde{\phi}$  (red line) in the case where  $d = 3$ . Here  $\phi_D(t) = (1 - t^{\frac{1}{\theta_D}})^{\theta_D}$  with  $\theta_D = 2$  and end-point  $d_0 = 1$  (see Copula 4.1.15 in Table 1). In both plots, the function  $p_{d-1}(t)$  for  $t \leq t_0$  is the blue dashed tangent line of  $\phi(t)$  at abscissa  $t_0 = 2$ . The considered initial Clayton generator  $\phi(t) = (1 + \theta t)^{\frac{-1}{\theta}}$  with  $\theta = 3$  is represented by the black curve in both panels.

patched generator  $\tilde{\phi}$  in Figure 1 (right panel) is a valid 3-dimensional generator.

Notice that on the domain  $[0, t_0]$ , as  $t \leq t_0$  and as  $\phi$  is  $d$ -monotone,  $\phi^{(i)}(t)(t - t_0)^i \geq 0$ , so that  $p_{d-1}$  can be written as a sum of positive terms

$$p_{d-1}(t) = \sum_{i=0}^{d-1} \left| \frac{\phi^{(i)}(t_0)}{i!} (t - t_0)^i \right|.$$

As  $\phi$  is  $d$ -monotone, one can also check that

$$p_{d-1}^{(k)}(t) = (-1)^k 1_{\{d-1 \geq k\}} \left| \sum_{j=0}^{d-1-k} \frac{\phi^{(j+k)}(t_0)}{j!} (t - t_0)^j \right|,$$

so that  $p_{d-1}$  is also  $d$ -monotone and even completely monotone (see Definition 1.1).

The patched generator  $\tilde{\phi}$  was constructed in two steps. Firstly, the shape of the initial generator  $\phi$  was preserved on  $[t_0, \infty)$ . Secondly, the generator  $\phi$  was replaced by another function, say  $f$ , on  $[0, t_0]$ . The following result analyses the bounds of a candidate function  $f$ , in order to preserve the  $d$ -monotone shape of the obtained patched generator  $\tilde{\phi}$ . As a consequence of this result, it is natural to build  $\tilde{\phi}$  from  $p_{d-1}(t)$ , as  $p_{d-1}(t)$  can be seen as the smallest  $d$ -monotone candidate for the prolongation of  $\phi$  on  $[0, t_0]$ .

**Proposition 2.1** (Smallest and largest suitable candidate). *Let  $\phi$  be an initial  $d$ -dimensional generator as in Definition 2.1. Let  $f$  be a valid  $d$ -dimensional generator such that  $f(t) = \phi(t)$  for all  $t \geq t_0$  and such that  $f(0) = 1$ . It holds that*

- i.  $f(t) \geq p_{d-1}(t)$ ,  $t \in [0, t_0]$ .

$$\text{ii. } f(t) \leq p_{d-1}(t) + (1 - p_{d-1}(0)) \cdot \left(\frac{t_0-t}{t_0}\right), \quad t \in [0, t_0].$$

Thus  $f(t) \leq p_{d-1}(t) + (1 - p_{d-1}(0)) \cdot \phi_D^{\max}(t)$ , where  $\phi_D^{\max}(t) = \left(\frac{t_0-t}{t_0}\right)_+$  is the function associated to the lower Fréchet-Hoeffding bound. In the particular case where  $d = 2$ ,  $f$  is necessarily below the line joining the two points  $(0, 1)$  and  $(t_0, \phi(t_0))$ .

*Proof.* We firstly prove the *i.* item. The function  $f$  is a valid  $d$ -dimensional generator, so that  $f$  is continuous, non-increasing and  $d$ -monotone on  $\mathbb{R}^+$ , thus  $(d-2)$  times differentiable, in particular at abscissa  $t_0$ . As a consequence,  $f^{(k)}(t_0) = \phi^{(k)}(t_0)$  for all  $k = 0, \dots, d-2$ .

Let  $I = [t_0, t_0 + \varepsilon]$ . Remark that, since  $\phi^{(d-2)}$  is a convex function, then in particular it is right-differentiable in  $t_0$ . As  $f(t) = \phi(t)$  for any  $t \geq t_0$ , then  $\lim_{a \rightarrow t_0^+, a \in I} f^{(d-1)}(a) = \phi_+^{(d-1)}(t_0)$ .

Let us now consider  $a \in I$ . Then  $f$  can be written by Taylor's theorem with the Lagrange form of the remainder, as

$$f(t) = \sum_{k=0}^{d-2} \frac{(t-a)^k}{k!} f^{(k)}(a) + \frac{(t-a)^{d-1}}{(d-1)!} f^{(d-1)}(\xi_t), \quad \text{for } t \in [0, t_0],$$

where  $\xi_t \in [t, a]$ . As  $f$  is  $d$ -monotone, then  $(-1)^{d-2} f^{(d-2)}$  is non-negative (also non-increasing and convex), which implies  $(-1)^{d-1} f^{(d-1)}$  is non-increasing. Thus

$$f(t) \geq \sum_{k=0}^{d-2} \frac{(t-a)^k}{k!} f^{(k)}(a) + \frac{(t-a)^{d-1}}{(d-1)!} f^{(d-1)}(a).$$

Recall that  $f^{(k)}(t_0) = \phi^{(k)}(t_0)$ ,  $k = 0, \dots, d-2$ . Now, by passing to the limit for  $a \rightarrow t_0^+$  the last inequality and by recalling that  $p_{d-2}(t) = \sum_{k=0}^{d-2} \frac{(t-t_0)^k}{k!} \phi^{(k)}(t_0)$ , one gets,

$$f(t) \geq p_{d-2}(t) + \frac{(t-t_0)^{d-1}}{(d-1)!} \phi_+^{(d-1)}(t_0), \quad \text{for } t \in [0, t_0].$$

Hence the first result. We now prove the second statement of this result (see item *ii.*). Let  $t \in [0, t_0]$ . As  $f$  is  $d$ -monotone and thus convex,  $f(t) \leq \frac{t}{t_0} f(t_0) + \left(1 - \frac{t}{t_0}\right) f(0)$ . As  $f(0) = 1$  and  $p_0(0) = f(t_0)$ , one easily shows that  $f(t) \leq p_0(0) + (1 - p_0(0)) \frac{t_0-t}{t_0}$ . Now consider a given integer  $k \leq d-2$ , so that  $p_{k+1}(t)$  exists, and define the assumption  $H_k$ : for all  $t \in [0, t_0]$ ,  $f(t) \leq p_k(0) + (1 - p_k(0)) \frac{t_0-t}{t_0}$ . We have shown that  $H_0$  holds. Now, define the function  $g_{k+1}(t)$  such that  $f(t) = p_{k+1}(0) + (1 - p_{k+1}(0)) g_{k+1}(t)$ , for all  $t \in [0, t_0]$ . If  $H_k$  holds, we easily get  $(1 - p_{k+1}(0)) g_{k+1}(t) \leq p_k(0) - p_{k+1}(0) + (1 - p_k(0)) \frac{t_0-t}{t_0}$  for all  $t \in [0, t_0]$ . Defining  $\Delta_k(0) = p_{k+1}(0) - p_k(0)$ , we get  $(1 - p_{k+1}(0)) \left(g_{k+1}(t) - \frac{t_0-t}{t_0}\right) \leq -\Delta_k(0) + \frac{t_0-t}{t_0} \Delta_k(0)$ , and finally  $(1 - p_{k+1}(0)) \left(\frac{t_0-t}{t_0} - g_{k+1}(t)\right) \geq \Delta_k(0) \frac{t}{t_0}$ . One can check that  $p_{k+1}(0) \leq 1$  and, by  $d$ -monotonicity of  $\phi$ ,  $\Delta_k(0) \geq 0$ . It follows that  $\frac{t_0-t}{t_0} - g_{k+1}(t) \geq 0$ , so that  $H_{k+1}$  holds. By induction, the result holds.  $\square$

Notice that in the particular case where  $d = 2$ ,  $f$  is necessarily below the line joining the two points  $(0, 1)$  and  $(t_0, \phi(t_0))$ .

From Proposition 2.1, it follows that  $p_{d-1}(t)$  is the smallest candidate function for  $\tilde{\phi}(t)$  on  $[0, t_0]$ , and that it is thus natural to replace  $\phi(t)$  on  $[0, t_0]$  by the sum of  $p_{d-1}(t)$  and another positive  $d$ -monotone function (but other constructions could be imagined, for example using a product instead of a sum). Notice that if  $\phi_D$  has an end-point  $d_0$  that tends to

0, the integrated distance  $\int_0^{t_0} |\tilde{\phi}(t) - p_{d-1}(t)| dt$  between  $\tilde{\phi}$  and its lower bound can be as small as desired. As  $\phi$  is also a  $d$ -monotone function, another consequence of previous Proposition 2.1 is that  $p_{d-1}(t) \leq \phi(t)$ , so that in particular when  $t = 0$  we get  $p_{d-1}(0) \leq 1$ . We will see in the further section that, due to Proposition 2.2, the quantity  $p_{d-1}(t) + (1 - p_{d-1}(0)) \cdot \phi_D^{\max}(t)$  is a  $d$ -monotone function (at least in dimension  $d = 2$ ), then the patched generator  $\tilde{\phi}$  can be equal to this upper bound.

**2.1. Validity.** We give here conditions under which the patched generator is a valid Archimedean copula generator.

**Proposition 2.2** (Validity of the patched generator). *Consider two generators  $\phi$  and  $\phi_D$  as in Definition 2.1, and let  $\tilde{\phi}$  be a patched distorted generator as in Definition 2.2. Then  $\tilde{\phi}$  is a valid  $d$ -dimensional Archimedean generator.*

*Proof.* The calculation of the  $k$ -derivatives of the patched generator, for  $k \geq 1$ , gives

$$(2) \quad \tilde{\phi}^{(k)}(t) = 1_{\{t \geq t_0\}} \phi^{(k)}(t) + 1_{\{t < t_0\}} \left( p_{d-1}^{(k)}(t) + (1 - p_{d-1}(0)) \phi_D^{(k)}(t) \right)$$

One first shows that, up to order  $d - 2$ ,  $\tilde{\phi}$  is differentiable on  $(0, +\infty)$ . When  $t > t_0$ , this is obvious as  $\phi$  is  $d$ -monotone. When  $t < t_0$ , this is clear since both  $p_{d-1}$  and  $\phi_D$  are  $d$ -monotone. Now, if  $\phi_D^{(k)}(t_0) = 0$  for all  $k = 0, \dots, d - 2$ , then checking that  $p_{d-1}^{(k)}(t_0) = \phi^{(k)}(t_0)$ , we get  $\tilde{\phi}^{(k)}(t_0) = \phi^{(k)}(t_0)$ . In particular when  $k = 0$ ,  $\tilde{\phi}$  is clearly continuous. We have shown that for all  $t \in (0, +\infty)$ , the  $k$ -th derivative  $\tilde{\phi}^{(k)}(t)$  exists for all  $k = 0, \dots, d - 2$ .

One then show that  $(-1)^k \tilde{\phi}^{(k)} \geq 0$ , for all  $k = 0, \dots, d - 2$ . Again, this is clear for  $t < t_0$  and for  $t \geq t_0$  as  $\tilde{\phi}$  is the sum of  $d$ -monotone functions, recalling that  $1 - p_d(0)$  is a non-negative coefficient.

It remains to be shown that  $(-1)^{d-2} \tilde{\phi}^{(d-2)}$  is a non-increasing convex function. From  $d$ -monotonicity, functions  $(-1)^{d-2} p_{d-1}^{(d-2)}$ ,  $(-1)^{d-2} \phi_D^{(d-2)}$  and  $(-1)^{d-2} \phi^{(d-2)}$  are all three non-increasing and convex functions. As  $\tilde{\phi}^{(d-2)}$  is continuous, it follows that  $(-1)^{d-2} \tilde{\phi}^{(d-2)}$  is decreasing. It is also clear that  $(-1)^{d-2} \tilde{\phi}^{(d-2)}$  is convex on  $[0, t_0]$  and convex on  $[t_0, +\infty)$ . At the junction  $t = t_0$  of these convex functions  $(-1)^{d-2} \tilde{\phi}^{(d-2)}(t_0) = (-1)^{d-2} p_{d-1}^{(d-2)}(t_0)$ , and from Proposition 2.1,  $(-1)^{d-2} \tilde{\phi}^{(d-2)}(t) \geq (-1)^{d-2} p_{d-1}^{(d-2)}(t)$  so that necessarily  $(-1)^{d-2} \tilde{\phi}^{(d-2)}(t)$  is convex. Finally, we have shown that  $\tilde{\phi}$  is a continuous  $d$ -monotone function with  $\tilde{\phi}(0) = 1$  and  $\tilde{\phi}(+\infty) = 0$ , hence  $\tilde{\phi}$  is a valid Archimedean generator.  $\square$

**2.2. Upper-tail dependence.** Here, we investigate the upper-tail dependence of a upper-patched generator.

In the following, we denote by  $f \in \mathcal{RV}_r(x_0)$ ,  $r \in \mathbb{R}$ , a measurable function  $f : (0, \infty) \rightarrow (0, \infty)$  which is regularly varying at  $x_0$  with index  $r$ , where typically  $x_0$  stands for 0, 1 or  $+\infty$  (see e.g. Bingham et al. (1989)). In particular, we say that  $f$  is regularly varying at 0 with index  $r$  if

$$f \in \mathcal{RV}_r(0) \Leftrightarrow \lim_{x \rightarrow 0^+} \frac{f(sx)}{f(x)} = s^r, \forall s > 0.$$

Now, it is well known that the upper-tail behaviour of a copula is directly linked with the behavior of  $\phi$  in the neighbourhood of the attachment point  $\phi(0) = 1$  (see e.g., Charpentier and Segers (2009), Di Bernardino and Rullière (2016)). The following proposition shows that the regular variation properties of the proposed patched generator  $\tilde{\phi}$  are directly linked to those of the distortion generator  $\phi_D$ .



**Proposition 2.3** (Regular index for the patched generator). *Consider two generators  $\phi$  and  $\phi_D$  as in Definition 2.1, and let  $\tilde{\phi}$  be a patched distorted generator as in Definition 2.2. If  $1 - \phi_D(t) \in \mathcal{RV}_{1/\rho}(0)$  with  $\rho \in [1, +\infty]$ , then  $1 - \tilde{\phi} \in \mathcal{RV}_{1/\rho}(0)$ . Furthermore, the patched upper tail dependence coefficient is given by*

$$(3) \quad \tilde{\lambda}_U^{(h,d-h)} = \begin{cases} 0, & \text{if } \rho = 1, \\ \frac{\sum_{i=1}^d (-1)^i C_d^i \cdot i^{1/\rho}}{\sum_{i=1}^{d-h} (-1)^i C_{d-h}^i \cdot i^{1/\rho}}, & \text{if } \rho \in (1, +\infty), \\ 1, & \text{if } \rho = +\infty. \end{cases}$$

Equation (3) in the particular bivariate case (i.e.,  $d = 2$  and  $h = 1$ ) is given by

$$(4) \quad \tilde{\lambda}_U^{(1,1)} = \begin{cases} 0, & \text{if } \rho = 1, \\ 2 - 2^{1/\rho}, & \text{if } \rho \in (1, +\infty), \\ 1, & \text{if } \rho = +\infty. \end{cases}$$

*Proof.* Let  $x > 0$  and  $x \leq t_0$ . We prove that  $1 - \tilde{\phi} \circ I \in \mathcal{RV}_{-1/\rho}(+\infty)$ . To this aim, we write:

$$\begin{aligned} 1 - \tilde{\phi}(x) &= 1 - p_{d-1}(x) - (1 - p_{d-1}(0))\phi_D(x) \\ &= 1 - p_{d-1}(x) - (1 - p_{d-1}(0))\phi_D(x) + (1 - p_{d-1}(0)) - (1 - p_{d-1}(0)) \\ &= 1 - p_{d-1}(x) + (1 - p_{d-1}(0))(1 - \phi_D(x)) - (1 - p_{d-1}(0)) \\ &= p_{d-1}(0) - p_{d-1}(x) + (1 - p_{d-1}(0))(1 - \phi_D(x)). \end{aligned}$$

Denote  $\Delta(x) := p_{d-1}(0) - p_{d-1}(x)$ . Remark that  $\Delta(x)$  is a positive function. Furthermore, if  $\phi'(t_0) \neq 0$ ,  $\Delta(x) \in \mathcal{RV}_1(0)$ . Let  $I(x) := \frac{1}{x}$ . Notice that  $1 - \tilde{\phi} \circ I$  can be written as a sum of positive functions, i.e.,  $1 - \tilde{\phi}(\frac{1}{x}) = f_1(x) + f_2(x)$ , with  $f_1(x) = p_{d-1}(0) - p_{d-1}(\frac{1}{x})$  and  $f_2(x) = (1 - p_{d-1}(0))(1 - \phi_D(\frac{1}{x}))$ . Furthermore, as provided before,  $f_1 \in \mathcal{RV}_{-1}(+\infty)$  and by assumption  $f_2 \in \mathcal{RV}_{-1/\rho}(+\infty)$ . Then, by using Proposition B.1.9 in de Haan and Ferreira (2006), we obtain that  $1 - \tilde{\phi} \circ I \in \mathcal{RV}_{\max(-1, -1/\rho)}(+\infty)$ . Hence the first result. As a consequence, by using Charpentier and Segers (2009) and Di Bernardino and Rullière (2016) we get the upper tail dependence coefficient for the patched generator. The bivariate case comes down from the application of Theorem 4.4 in Juri and Wüthrich (2003). Hence the results.  $\square$

**2.3. Likelihood.** We now show, on a given dataset, that it is possible to choose a given upper-tail dependence behaviour, without reducing the likelihood of the copula on pseudo-observations.

The patched generator  $\tilde{\phi}$  is equal to  $\phi$  on a set  $[t_0, +\infty)$ , so that the resulting copula is unchanged on a domain  $D_0$  that following result gives explicitly. As a consequence of previous Proposition 2.3, one can easily choose any upper tail coefficient of a copula  $\tilde{C}$ . This can be done without modifying the generator  $\phi$  on a set  $[t_0, +\infty)$ , i.e. without modifying the initial copula  $C$  on a domain  $D_0 = [0, \phi(t_0)]^d$ . It follows that whatever the pseudo-observations in  $(0, 1)^d$ , one can find  $t_0$  such that all pseudo-observations are included in  $D_0$ . For such a value  $t_0$  the likelihood of the data using copula  $\tilde{C}$  is equal to the likelihood using copula  $C$ . The following result indicates that it is thus possible to transform copula  $C$  into  $\tilde{C}$ , in order to choose its upper tail coefficient, without reducing the likelihood function on a given data-set.

**Proposition 2.4** (Likelihood improvement). *Let  $C$  be the Archimedean copula associated to an initial generator  $\phi$ . Consider a patched generator  $\tilde{\phi}$ , build from the initial generator*

$\phi$  and the distortion one  $\phi_D$  as in Definition 2.2. Denote  $\tilde{\psi}$  the inverse function of  $\tilde{\phi}$  and let  $\tilde{C}(u_1, \dots, u_d) = \tilde{\phi}(\tilde{\psi}(u_1) + \dots + \tilde{\psi}(u_d))$ , then

$$(5) \quad \tilde{C}(u_1, \dots, u_d) = C(u_1, \dots, u_d), \quad \text{for all } (u_1, \dots, u_d) \in [0, \phi(t_0)]^d.$$

Let  $\lambda_U^{(h, d-h)}$  the upper-tail coefficient of  $C$  and  $\mathcal{L}$  the likelihood function under  $C$  on given (pseudo-)observations  $\mathbf{U}_1, \dots, \mathbf{U}_n \in (0, 1)^d$ . Let  $\tilde{\lambda}_U$  and  $\tilde{\mathcal{L}}$  be corresponding quantities using the patched generator  $\tilde{\phi}$ . Let  $\lambda_0 \in [0, 1]$  be a chosen upper-tail coefficient, one can find  $t_0 \in \mathbb{R}^{+*}$  and an associated non-strict generator  $\phi_D$  such that

$$(6) \quad \tilde{\lambda}_U^{(h, d-h)} = \lambda_0 \quad \text{and} \quad \tilde{\mathcal{L}} \geq \mathcal{L}.$$

*Proof.* The generator  $\tilde{\phi}$  is equal to  $\phi$  on a set  $[t_0, +\infty)$ . Equivalently, the inverse generator  $\tilde{\psi}$  is equal to  $\psi$  on  $[0, \phi(t_0)]$ , where  $t_0$  can be close to 0 and  $\phi(t_0)$  close to 1. A sufficient condition for the copula to be unchanged is that all  $u_i \in [0, \phi(t_0)]$ , so that all  $\tilde{\psi}(u_i) = \psi(u_i)$ . As  $\psi$  is decreasing, the minimal value of  $\psi(u_i)$  when  $u_i \leq \phi(t_0)$  is  $t_0$ , so that in all cases  $\psi(u_1) + \dots + \psi(u_d) \geq dt_0 \geq t_0$ , ensuring that  $\tilde{\phi}(\psi(u_1) + \dots + \psi(u_d)) = \phi(\psi(u_1) + \dots + \psi(u_d))$ . One can find a point  $t_0$  such that  $\tilde{\mathcal{L}} = \mathcal{L}$  on a domain  $D_0 = [0, \phi(t_0)]^d$ . Since Equation (3) gives a bijection between  $\rho \in [0, +\infty]$  and  $\tilde{\lambda}_U^{(h, d-h)} \in [0, 1]$ , we can find  $\phi_D$  such that  $\tilde{\lambda}_U^{(h, d-h)} = \lambda_0$ , whatever the choice of  $t_0$ . Furthermore, from Equation (5), we can choose  $t_0$  such that  $\tilde{\mathcal{L}} = \mathcal{L}$ : the likelihood is ensured to be not reduced, but other choices of  $t_0$  can possibly improve the likelihood. Hence the result.  $\square$

### 3. NUMERICAL ILLUSTRATIONS

In the following example we build some valid patched generators  $\tilde{\phi}$  under assumptions of Definition 2.1 (see Proposition 2.2) and we investigate the tail properties of the obtained distorted Archimedean copulas by using the tail concentration functions.

**Example 1** (Tail concentration function for some valid patched generators). We consider the tail concentration function (TCF), for  $u \in [0, 1]$ ,  $h \geq 1$  and  $d - h \geq 1$ ,

$$(7) \quad \lambda_{LU}^{(h, d-h)}(u) = \mathbf{1}_{\{u \leq 1/2\}} \lambda_L^{(h, d-h)}(u) + \mathbf{1}_{\{u > 1/2\}} \lambda_U^{(h, d-h)}(u),$$

where  $\lambda_L^{(h, d-h)}$  and  $\lambda_U^{(h, d-h)}$  are as in Definition 1.3 (see for instance Venter (2001) and Durante et al. (2015)).

We study both the bivariate ( $d = 2$ ) and the trivariate ( $d = 3$ ) case by choosing distortion generators  $\phi_D$  satisfying assumption of Definition 2.1. Then we obtain valid  $d$ -Archimedean patched generators (see Proposition 2.2).

In both dimensional situations, we start from an initial Clayton generator  $\phi(t) = (1 + \theta t)^{-\frac{1}{\theta}}$ , with  $\theta = 3$ .

- In the case  $d = 2$  and  $h = 1$  we choose the distortion generator  $\phi_D(t) = 1 - t^{\frac{1}{\theta_D}}$  with  $\theta_D = 2$  and end-point  $d_0 = 1$  (see Copula 4.1.2 in Table 1). The obtained tail concentration function is displayed in Figure 2 (first panel).
- In the case  $d = 3$  we take the distortion generator  $\phi_D = (1 - t^{\frac{1}{\theta_D}})^{\theta_D}$  with  $\theta_D = 2$  and end-point  $d_0 = 1$  (see Copula 4.1.15 in Table 1). The obtained tail concentration function for  $d = 3$  and  $h = 1$  (*resp.*  $d = 3$  and  $h = 2$ ) is displayed in the second panel (*resp.* third panel) of Figure 2.

As one can see in Figure 2, the tail concentration function is firstly identical to the one of the initial Clayton generator  $\phi$ , then shifts toward the one of the distortion generator  $\phi_D$ . This is the desired feature where one aims at changing an initial generator in order to choose the final tail dependence behavior. It is also noticeable that the patched generator  $\tilde{\phi}$ , despite its validity, generates a large variety of shapes of the tail concentration function, contrary to most classical copulas where the concentration function is first increasing and then decreasing. For a detailed discussion of this aspect, the interested reader is referred to Section 3 in Di Bernardino and Rullière (2016).

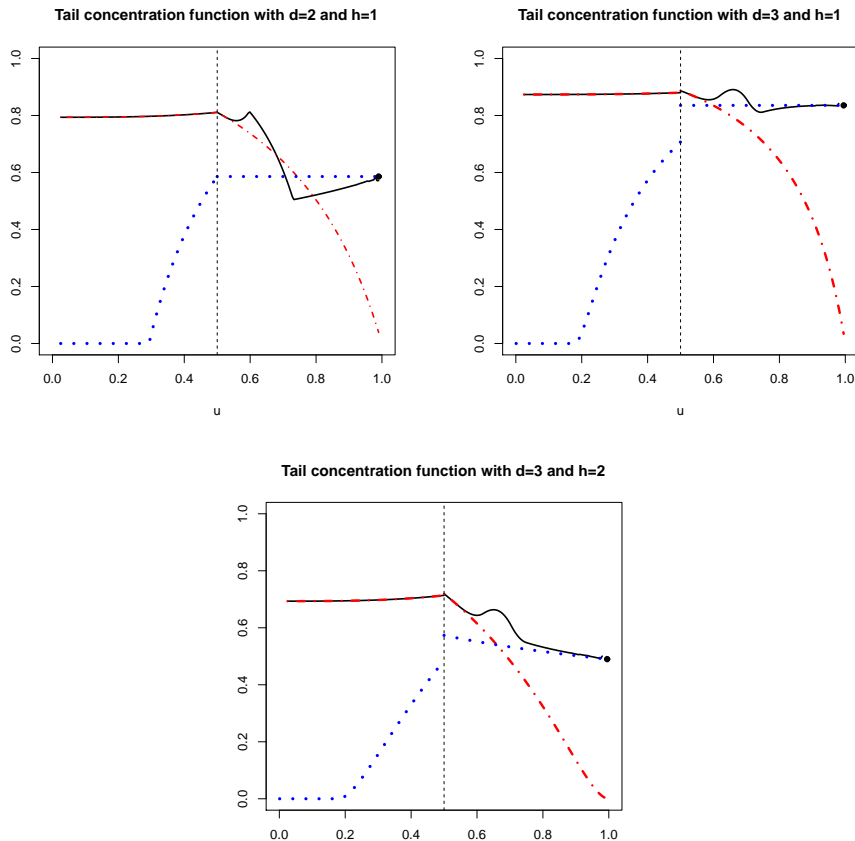


FIGURE 2. The tail concentration<sup>u</sup> functions are represented respectively in full black line for the patched generator  $\tilde{\phi}$ , in blue dotted line for the distortion generator  $\phi_D$  and in red dashed line for initial Clayton generator  $\phi$ . First panel  $d = 2$ ,  $h = 1$ ; second panel  $d = 3$ ,  $h = 1$ ; third panel  $d = 3$ ,  $h = 2$ .

**Example 2** (Tail improvement). In practical applications, the proposed initial model never fits perfectly to the reality, it is thus natural to try improving this initial model by using both the log-likelihood and the estimated tail behavior on the data. This can be particularly useful in the case of model misspecification. In this example we generate a data-set from a Gumbel model and we have deliberately chosen to improve an initial misspecified Frank model. We show hereafter that empirically, one can improve both classical criteria (as log-likelihood, AIC or BIC) and estimated tail dependence on the data. As we will see, it is noticeable that the results obtained by the patched model are

quite good compared to the true (supposed unknown) Gumbel model.

i. Initial context. We present here the initial data and the classical estimation procedure for the chosen initial model. We sample 2000 observations from a bivariate Gumbel copula with parameter  $\theta = 2$ . We fit an initial Frank copula on this data-set based on the ML estimator. We estimate the tail concentration function (TCF) in (7) by using the empirical rank based estimator of  $\lambda_L^{(h,d-h)}(u)$  and  $\lambda_U^{(h,d-h)}(u)$ , with  $d = 2$  and  $h = 1$  (the interested reader is referred to Schmidt and Stadtmüller (2006)).

In Figure 3, one illustrates the empirical estimated TCF based on the data, the theoretical TCF from the initial Frank generator  $\phi_{\hat{\theta}}$  and from the Gumbel copula generator with ML estimated parameters. Furthermore we represent in Figure 3, the Monte Carlo confidence intervals of 300 empirical TCFs from the initial Frank copula with generator  $\phi_{\hat{\theta}}$ . These confidence intervals are denoted  $IC_{[0.05,0.95]}(\text{TCF}_{\phi_{\hat{\theta}}})$ . In particular, we empirically estimate the upper tail dependence coefficient  $\hat{\lambda}_U^{(h,d-h)}$ , with  $d = 2$  and  $h = 1$ .

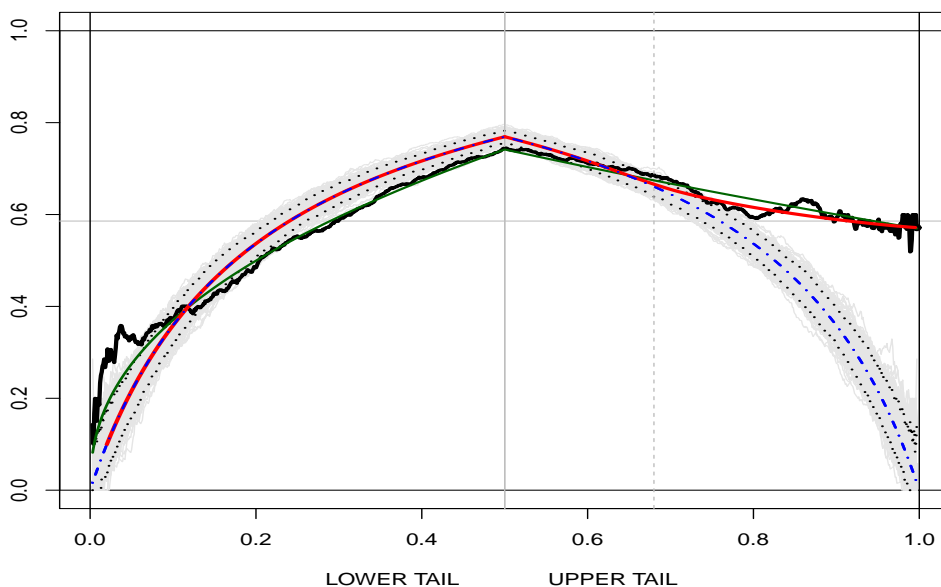


FIGURE 3. **Bold black line:** Estimated empirical TCF based on bivariate data with sample size 2000 from Gumbel copula with parameter  $\theta = 2$ . **Dashed blue line:** Theoretical TCF from the initial Frank generator  $\phi_{\hat{\theta}}$  with ML estimated parameter. **Full dark green line:** Theoretical TCF from Gumbel copula generator with ML estimated parameter. **Full red line:** Theoretical TCF from the patched generator  $\tilde{\phi}$ .

ii. Patched generator. We now try to improve the initial Frank copula fit proposed in the previous paragraph. As one can see in Figure 3 (bold black and dashed blue lines) the fit is roughly suited to the central part of the copula, but not to the upper tail of this data-set. Creating a patched generator is reduced to the choice of the break-point  $t_0$  and the choice of the distortion generator  $\phi_D$ . The break-point is chosen large enough in order to change

mainly the tail dependence of the fitted copula (see Proposition 2.4), and the distortion generator  $\phi_D$  will be chosen in order to ensure that the patched copula is a valid copula which exhibits the target upper tail dependence coefficient, i.e.,  $\widehat{\lambda}_U^{(h,d-h)}$ . More precisely, here is the detailed procedure to patch the initial Frank generator  $\phi_{\widehat{\theta}}$ :

1. **Cutting point choice:** We choose the cutting point  $u^*$  as the largest value  $u \in (0.5, 1)$  such that  $\widehat{\lambda}_U^{(h,d-h)}(u^*) \in \text{IC}_{[0.05,0.95]}(\text{TCF}_{\phi_{\widehat{\theta}}})$  and set  $t_0 = \phi_{\widehat{\theta}}^{-1}(u^*)$ .
2. **Tail improvement:** We choose a distortion generator  $\phi_D(t)$  satisfying admissibility conditions in Definition 2.1 and exhibiting desired tail behavior, i.e. such that  $1 - \phi_D \in \mathcal{RV}_{1/\widehat{\rho}}(0)$  with  $\widehat{\rho} = \ln(2)/\ln(-\widehat{\lambda}_U^{(h,d-h)} + 2)$ , see Equation (4) when  $d = 2$  and  $h = 1$ . Here, we have chosen  $\phi_D(t) = (1 - t^{\frac{1}{\widehat{\rho}}})^{\widehat{\rho}}$  (i.e., generator 4.1.15 in Table 1).
3. The patched generator is then directly given by Definition 2.2.

The TCF for the obtained patched generator is displayed in Figure 3 (full red line). In this figure, the obtained cutting point  $u^*$  is displayed by using the vertical dashed grey line. One can see on this figure that the patched generator fits nicely the empirical tail concentration function, and that it fits quite well the theoretical concentration function for large values of  $u$ . This fit is however obtained on one only sample. In the following, we check the statistical procedure described in paragraph *i.* and *ii.* on  $M = 100$  Monte Carlo simulations.

In Figure 4, we represent the obtained boxplots for the log-likelihood, AIC and BIC criteria for the 3 considered models (true Gumbel model, initial Frank model and patched distorted model).

To avoid the unnecessary calculation of the density of the patched copula, the log-likelihood has been computed numerically in all three models, using finite differences for the second order partial derivatives. We have checked that it was giving exactly the same results when using theoretical densities for Gumbel and Frank models.

On Figure 4, one can check that in average the patched model performs as well as the true Gumbel model, even on penalized criterions like AIC or BIC. Due to the estimation of the empirical upper tail dependence coefficient  $\widehat{\lambda}_U^{(h,d-h)}$ , the log-likelihood is more widespread. We clearly check here on Figures 3 and 4, that empirically, the patched generator improve both the likelihood and the tail dependence compared to the initial unsuited Frank model. This is consistent with theoretical results of Proposition 2.4 which state that one can choose any target upper tail dependence coefficient for the patched copula without lowering the likelihood.

To conclude, we represent in Figure 5 the boxplots on the  $M = 100$  Monte Carlo simulations of the difference between the theoretical upper tail dependence coefficient for a Gumbel copula with parameter 2, i.e.,  $\lambda_U = 0.58$ , and the estimated values under the 3 considered models (Gumbel, Frank, Patched). Trivially the Frank model is not able to capture the considered tail dependency (Figure 5, second panel). Conversely, as remarked in Figures 3 and 4, in average the patched model performs as well as the true Gumbel model (see Figure 5, first and third panels). However, as before, the patched case is more dispersed than the (true) Gumbel one, due to the supplementary empirical estimation of the upper tail dependence coefficient.

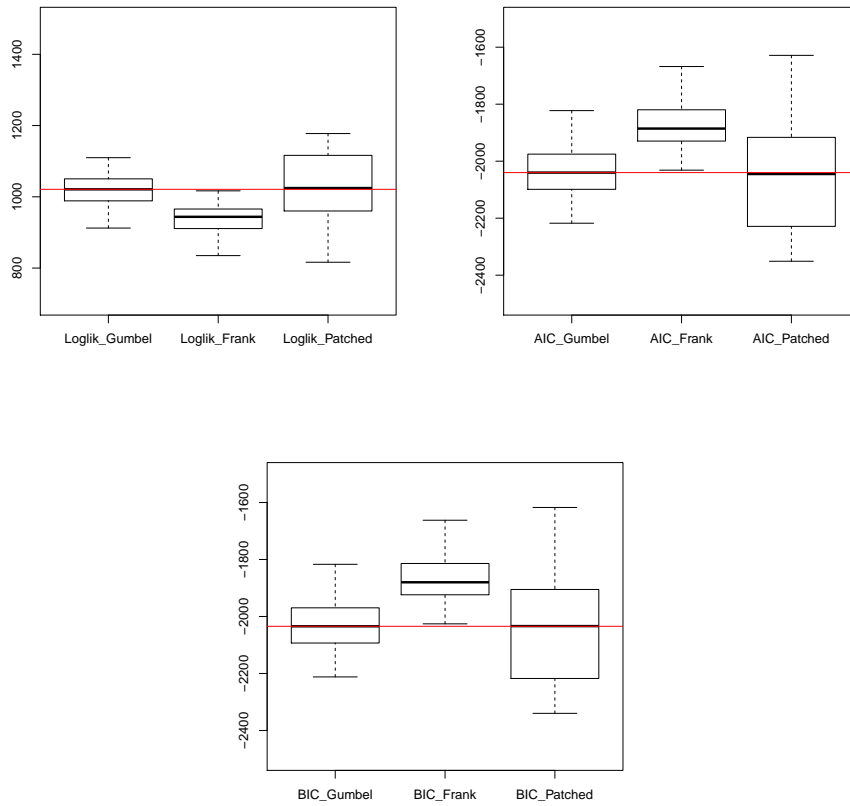


FIGURE 4. Boxplots for the log-likelihood, AIC and BIC criteria for the 3 considered models (true Gumbel model, initial Frank model and patched distorted model). In the red horizontal line we display the median values for the Gumbel model. We consider  $M = 100$  Monte Carlo simulations.

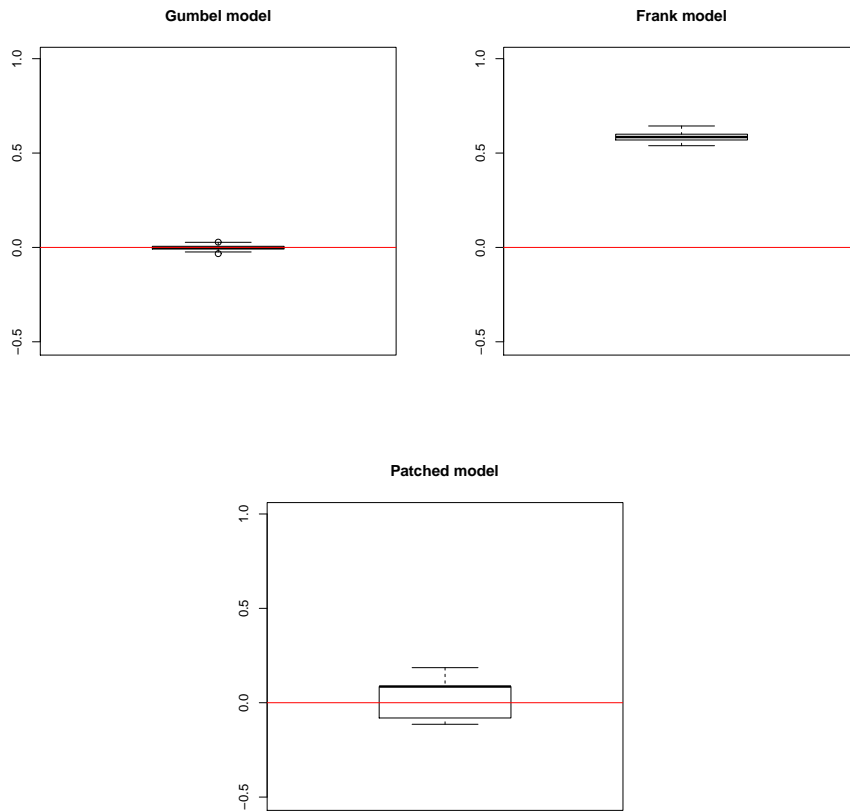


FIGURE 5. Boxplot of the difference between the theoretical  $\lambda_U = 0.58$  and the estimated upper tail dependence coefficients under the 3 considered models (true Gumbel model, initial Frank model and patched distorted model).

#### 4. CONCLUSION

We have presented a specific transformation of a given Archimedean generator, creating what we called a upper-patched generator. One advantage of such a construction is the simplicity of the method and its straightforward applicability. Compared to existing methods, another important advantage is the fact that the patched generator is necessarily valid in the chosen dimension  $d \geq 2$ . At last, another important advantage is that the upper tail dependence coefficient of the patched generator can be easily chosen, without likelihood reduction on a given dataset.

One drawback of the method is that the inverse function of a patched generator is obtained in practice by numerical inversion: it is not given by a direct explicit closed-form formula and depends on the choice of both the initial generator and the distortion generator. Another drawback of the proposed methodology is that it remains suited only to Archimedean dependence structures, as the patched copula remains within the Archimedean class.

Concerning perspectives, we have changed the upper tail dependence of an Archimedean copula by modifying its generator  $\phi$  on an interval  $[0, t_0]$ ; a very natural perspective for modifying the lower tail dependence of an Archimedean copula is to change its generator  $\phi$  on an interval  $[t_0, +\infty)$ . More generally, one can imagine creating an interval-patched generator, where an initial generator is modified on an interval  $I$ . Among difficulties on the interval  $[t_0, +\infty)$ , Taylor expansions of  $\phi$  at different orders are alternatively greater or lower than  $\phi$ , and both diverging toward  $-\infty$  or  $+\infty$ . Creating a upper patched inverse generator from  $\psi = \phi^{-1}$  is also possible, but it is then not straightforward to get the inverse function of this patched inverse, in order to check the  $d$ -monotonicity. Thus, it seems to us that the techniques involved for patching a generator on the other (lower) side are probably different from the one treated in this paper; they are let to future work.

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