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L² WELL-POSEDNESS OF BOUNDARY VALUE PROBLEMS FOR PARABOLIC SYSTEMS WITH MEASURABLE COEFFICIENTS

PASCAL AUSCHER, MORITZ EGERT, AND KAJ NYSTRÖM

In memory of Alan McIntosh

ABSTRACT. We prove the first positive results concerning boundary value problems in the upper half-space of second order parabolic systems only assuming measurability and some transversal regularity in the coefficients of the elliptic part. To do so, we introduce and develop a first order strategy by means of a parabolic Dirac operator at the boundary to obtain, in particular, Green’s representation for solutions in natural classes involving square functions and non-tangential maximal functions, well-posedness results with data in L²-Sobolev spaces together with invertibility of layer potentials, and perturbation results. In the way, we solve the Kato square root problem for parabolic operators with coefficients of the elliptic part depending measurably on all variables. The major new challenge, compared to the earlier results by one of us under time and transversally independence of the coefficients, is to handle non-local half-order derivatives in time which are unavoidable in our situation.

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1. INTRODUCTION

In this paper we study boundary value problems for parabolic equations of type

$$(1.1) \quad \mathcal{L}u := \partial_t u - \text{div}_X A(X, t) \nabla_X u = 0.$$

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Here, $X = (\lambda, x) \in \mathbb{R}_+^{n+1} := (0, \infty) \times \mathbb{R}^n$ and $t \in \mathbb{R}$. We assume that the matrix A is measurable, strongly elliptic, and possibly depends on all variables. Parabolic systems will be included in our considerations as well. The focus on the *upper* parabolic half-space \mathbb{R}_+^{n+2} here is of course arbitrary and we could as well have decided to work in the lower one.

We stress that with mere measurable dependence in time there are no positive results on solvability of boundary value problems in this context up to now. Indeed, most studies concerning second order parabolic boundary value problems have focused on the heat equation in (parabolic) Lipschitz-type domains and some generalisations. For the heat equation, we mention [25, 26] devoted to time-independent Lipschitz cylinders and the important achievements in [38, 39, 49] for time dependent Lipschitz-type domains. As for generalisations, [40] studies the Dirichlet problem for parabolic equations with singular drift terms (containing the ones coming from an elaborate change of variables from the heat equation) with coefficients having some smoothness and smallness in the sense of a Carleson condition. See also [33] and [34].

In a series of recent papers [27, 53, 54], the solvability for Dirichlet, regularity and Neumann problems with L^2 data was established for the class of parabolic equations (1.1) under the assumptions that the elliptic part is independent of both the time variable t and the variable λ transverse to the boundary, and that the elliptic part has either constant (complex) coefficients, real symmetric coefficients, or small perturbations thereof. This is proved by first studying fundamental solutions and proving square function estimates and non-tangential estimates for the single layer potential along the lines of the approach for elliptic equations in [2]. In the case of real symmetric coefficients, solvability is then established using a Rellich-type argument.

An important step in [53] – revisiting an earlier idea of [39, 44] – is the introduction of a reinforced notion of weak solution by means of a sesquilinear form associated with (1.1), showing its coercivity and proving a fundamental square function estimate. In fact, what is implicitly shown in [53] by adapting the proof of the Kato conjecture for elliptic operators [13], is that the operator $\partial_t - \operatorname{div}_x A(x) \nabla_x$ is maximal-accretive in $L^2(\mathbb{R}^{n+1})$ and that the domain of its square root is that of the associated form. Note that this is for one dimension lower than (1.1). With this at hand, [27] proves a number of technical estimates for the single layer potential for \mathcal{L} based on the DeGiorgi-Nash-Moser condition. Finally, [54] addresses solvability and uniqueness issues by proving invertibility of the layer potentials. However, the methods there are limited to time independent coefficients and removing this constraint requires new ideas.

We owe to Alan McIntosh the insight that boundary value problems involving accretive sesquilinear forms could be addressed via a first order system of Cauchy-Riemann type. For elliptic equations this approach was pioneered in [7] and carried out systematically in [8] using a simpler setup. It relies on three main properties of the corresponding perturbed Dirac operator at the boundary: bisectoriality, off-diagonal estimates for its resolvents and a bounded holomorphic functional calculus on L^2 , which in this case has been obtained by a $T(b)$ argument based on a remarkable elaboration on the solution of the Kato conjecture in [22]. Non-tangential maximal estimates to precise the boundary behaviour were also obtained. Uniqueness via a semigroup representation, later identified to layer potential integrals in the case of real elliptic equations in [55], are obtained in [5].

In this paper we implement this strategy for boundary value problems of parabolic equations. The outcome of our efforts is the possibility to address arbitrary parabolic equations (and systems) as in (1.1) with coefficients depending measurably on time and on the transverse variable with additional transversal regularity, the removal of time independence being the substantial step.

In the spirit of a first order approach for parabolic operators, we mention two articles [28, 29] proposing a factorisation of the heat operator via a (non-homogeneous) Dirac operator valued in some Clifford algebra that involves only first and zero order partial derivatives. But this does not seem to meet our needs here: First, we have non-smooth coefficients and, second, half-order derivatives in time turn out to be part of the data for the boundary value problems. Hence, it is natural that our Dirac operator comes with such derivatives, too.

To proceed, we have to overcome serious difficulties compared with the elliptic counterpart since half-order time-derivatives appearing in the Dirac operator are non-local and have poor decay properties. However, we are still able to obtain just enough decay in the off-diagonal estimates of its

resolvents to obtain through a $T(b)$ argument the square function estimate that opens the door to all our further results concerning boundary value problems with L^2 data. Additionally, we prove important non-tangential maximal estimates via some new reverse Hölder inequalities for reinforced weak solutions.

Concerning the Dirichlet problem, the first order approach only comprises uniqueness in the sense of L^2 convergence with square function control. However, in the cases where we can solve, we also prove uniqueness in the classical sense of non-tangential convergence at the boundary with non-tangential maximal control, which is the largest possible class. This includes that the “unique” solution of the Dirichlet problem with non-tangential maximal control in L^2 also enjoys the square function estimates. Proof of uniqueness is done via new arguments, not requiring any form of regularity of solutions such as the DeGiorgi-Nash-Moser estimates, and hence it does apply to parabolic systems in particular.

As a particular outcome of independent interest, we obtain that the parabolic operator $\partial_t - \operatorname{div}_x A(x, t) \nabla_x$ can be defined as a maximal-accretive operator in $L^2(\mathbb{R}^{n+1})$ and that it satisfies a Kato square root estimate – its square root has domain equal to that of the defining form. Note that we allow measurable t -dependence on the coefficients in contrast to [53], which makes a huge difference. This has a worth-mentioning consequence. In [12], we use the solution of the parabolic Kato problem to prove that the L^p Dirichlet problem corresponding to (1.1) is well-posed if $A = A(x, t)$ is real valued but possibly non-symmetric, measurable, and $p \in (1, \infty)$ is sufficiently large, which is the best possible result in general.

In the next section we will give a comprehensive presentation of our results, postponing details for later. It will come with the introduction of some necessary notation, but we try to make the presentation as fluent as possible to give a general overview. Also the progression does not necessarily follow the order in which things are proved. The subsequent Sections 3 - 13 are devoted to the proofs of our results. In the final Section 14 we formulate further generalisations of our results and state additional open problems. Let us mention two of them here. One concerns boundary value problems for (1.1) in time dependent Lipschitz-type domains $\lambda > \varphi(x, t)$. While Lipschitz changes of variables preserve the class of elliptic equations $-\operatorname{div}_X A \nabla_X u = 0$, this is not the case for equations $\partial_t u - \operatorname{div}_X A \nabla_X u = 0$. Already the simple change of variables $(\lambda, x, t) \mapsto (\lambda - \varphi(x, t), x, t)$ requires that φ Lipschitz in both x and t and brings in a drift term. But the natural regularity on φ is some half-order regularity in t and requires more elaborate changes of variables such as the ones used in the works mentioned before. This results in creating drift terms and brings in leading coefficients to which existing results including ours do not apply. Another one would be to allow for data in other spaces which, if one uses the first order approach, requires to develop a Hardy space theory associated with parabolic Dirac operators in the first place.

2. MAIN RESULTS

2.1. The coefficients. For the time being, $A = A(X, t) = \{A_{i,j}(X, t)\}_{i,j=0}^n$ is assumed to be an $(n+1) \times (n+1)$ -dimensional matrix with complex coefficients satisfying the uniform ellipticity condition

$$(2.1) \quad \kappa |\xi|^2 \leq \operatorname{Re}(A(X, t) \xi \cdot \bar{\xi}), \quad |A(X, t) \xi \cdot \zeta| \leq C |\xi| |\zeta|,$$

for some $\kappa, C \in (0, \infty)$, which we refer to as the ellipticity constants of A , and for all $\xi, \zeta \in \mathbb{C}^{n+1}$, $(X, t) \in \mathbb{R}_+^{n+2}$. Here $u \cdot v = u_0 v_0 + \dots + u_n v_n$, \bar{u} denotes the complex conjugate of u and $u \cdot \bar{v}$ is the standard inner product on \mathbb{C}^{n+1} . We use $X = (x_0, x)$ and $x = (x_1, \dots, x_n)$. Most often, we specialize the variable transverse to the boundary by setting $\lambda = x_0$.

2.2. Reinforced weak solutions. If Ω is an open subset of \mathbb{R}^{n+1} , we let $H^1(\Omega) = W^{1,2}(\Omega) = W^{1,2}(\Omega; \mathbb{C})$ be the standard Sobolev space of complex valued functions v defined on Ω , such that v and ∇v are in $L^2(\Omega; \mathbb{C})$ and $L^2(\Omega; \mathbb{C}^n)$, respectively. A subscripted ‘loc’ will indicate that these conditions hold locally. We shall say that u is a *reinforced weak solution* of (1.1) on $\mathbb{R}_+^{n+1} \times \mathbb{R}$ if

$$u \in \dot{E}_{\text{loc}} := \dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}_+^{n+1})) \cap L_{\text{loc}}^2(\mathbb{R}; W_{\text{loc}}^{1,2}(\mathbb{R}_+^{n+1}))$$

and if for all $\phi \in C_0^\infty(\mathbb{R}_+^{n+2})$,

$$\int_0^\infty \iint_{\mathbb{R}^{n+1}} A \nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x} \phi} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} \phi} dx dt d\lambda = 0.$$

Here, $D_t^{1/2}$ is the half-order derivative and H_t the Hilbert transform with respect to the t variable, designed in such a way that $\partial_t = D_t^{1/2} H_t D_t^{1/2}$. The space $\dot{H}^{1/2}(\mathbb{R})$ is the homogeneous Sobolev space of order 1/2. In Section 3 we shall review properties of this space, which we define as a subspace of $L_{\text{loc}}^2(\mathbb{R})$ in order to deal with proper distributions. Still our definition is equivalent to other common ones: it is the completion of $C_0^\infty(\mathbb{R})$ for the norm $\|D_t^{1/2} \cdot\|_2$ and, modulo constants, it embeds into the space $\mathcal{S}'(\mathbb{R})/\mathbb{C}$ of tempered distributions modulo constants.

At this point we remark that for any $u \in \dot{H}^{1/2}(\mathbb{R})$ and $\phi \in C_0^\infty(\mathbb{R})$ the formula

$$\int_{\mathbb{R}} H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} \phi} dt = - \int_{\mathbb{R}} u \cdot \overline{\partial_t \phi} dt$$

holds, where on the right-hand side we use the duality form between $\dot{H}^{1/2}(\mathbb{R})$ and its dual $\dot{H}^{-1/2}(\mathbb{R})$ extending the complex inner product of $L^2(\mathbb{R})$. It follows that a reinforced weak solution is a weak solution in the usual sense on $\mathbb{R}_+^{n+1} \times \mathbb{R}$: it satisfies $u \in L_{\text{loc}}^2(\mathbb{R}; W_{\text{loc}}^{1,2}(\mathbb{R}_+^{n+1}))$ and for all $\phi \in C_0^\infty(\mathbb{R}_+^{n+2})$,

$$\int_{\mathbb{R}} \iint_{\mathbb{R}_+^{n+1}} A \nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x} \phi} dx d\lambda dt - \int_{\mathbb{R}} \iint_{\mathbb{R}_+^{n+1}} u \cdot \overline{\partial_t \phi} dx d\lambda dt = 0.$$

This implies $\partial_t u \in L_{\text{loc}}^2(\mathbb{R}; W_{\text{loc}}^{-1,2}(\mathbb{R}_+^{n+1}))$. Conversely, any weak solution u in $\dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}_+^{n+1}))$ is a reinforced weak solution.

2.3. Energy solutions. As a first illustration of this concept of interpreting (1.1), let us quickly explain how to obtain reinforced weak solutions using the form method. This seems to be surprising at first sight and relies on the fundamental revelation of coercivity for the parabolic operator \mathcal{L} explored first in [44] and later on for instance in [27, 39, 53, 54]. It will also allow us to touch some of the paper's central themes. For the moment, time or even transversal dependency of the coefficients will not be any obstacle.

We are looking for reinforced weak solutions that belong to the *energy class* \dot{E} of all $v \in \dot{E}_{\text{loc}}$ for which

$$\|v\|_{\dot{E}} := \left(\|\nabla_{\lambda,x} v\|_{L^2(\mathbb{R}_+^{n+2})}^2 + \|H_t D_t^{1/2} v\|_{L^2(\mathbb{R}_+^{n+2})}^2 \right)^{1/2} < \infty.$$

Consequently, these are called *energy solutions*. When considered modulo constants, \dot{E} is a Hilbert space and it is in fact the closure of $C_0^\infty(\mathbb{R}_+^{n+2})$ for the homogeneous norm $\|\cdot\|_{\dot{E}}$. Any $v \in \dot{E}$ can be defined as a uniformly continuous function on $[0, \infty)$ with values in the *homogeneous parabolic Sobolev space* $\dot{H}_{\partial_t - \Delta_x}^{1/4}$. Here, $\dot{H}_{\pm \partial_t - \Delta_x}^s$ is defined as the closure of Schwartz functions $v \in \mathcal{S}(\mathbb{R}^{n+1})$ with Fourier support away from the origin for the norm $\|\mathcal{F}^{-1}(|\xi|^2 \pm i\tau)^s \mathcal{F}v\|_2$. This yields a space of tempered distributions if $s < 0$ and of tempered distributions modulo constants in $L_{\text{loc}}^2(\mathbb{R}^{n+1})$ if $0 < s \leq 1/2$. Of course the choice of the sign in front of $i\tau$ defines the same space up to equivalent norms. We implicitly assume the branch cut for z^s on $(-\infty, 0]$ but any other possible choice yields the same space with an isometric norm. Conversely, any $g \in \dot{H}_{\partial_t - \Delta_x}^{1/4}$ can be extended to a function $v \in \dot{E}$ with trace $v|_{\lambda=0} = g$. We will include proofs of these statements in Section 3.2.

By an energy solution to (1.1) with Neumann boundary data $\partial_{\nu_A} u|_{\lambda=0} = f$ we mean a function $u \in \dot{E}$ such that for all $v \in \dot{E}$,

$$(2.2) \quad \iiint_{\mathbb{R}_+^{n+2}} A \nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x} v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v} d\lambda dx dt = -\langle f, v|_{\lambda=0} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing of $\dot{H}_{\partial_t - \Delta_x}^{1/4}$ with its dual $\dot{H}_{-\partial_t - \Delta_x}^{-1/4}$ extending the inner product on $L^2(\mathbb{R}^{n+1})$ and

$$(2.3) \quad \partial_{\nu_A} u(\lambda, x, t) := [1, 0, \dots, 0] \cdot (A \nabla_{\lambda,x} u)(\lambda, x, t)$$

is the *conormal derivative*, inwardly oriented on the upper half-space. In particular, u is a reinforced weak solution to (1.1) and the Neumann boundary data $f \in \dot{H}_{\partial_t - \Delta_x}^{-1/4}$ is attained in the weak sense prescribed by (2.2). Similarly, by an energy solution to (1.1) with Dirichlet boundary datum $u|_{\lambda=0} = f \in \dot{H}_{\partial_t - \Delta_x}^{1/4}$ we mean a function $u \in \dot{E}$ such that for all $v \in \dot{E}_0$, the subspace of \dot{E} with zero boundary trace,

$$\iiint_{\mathbb{R}_+^{n+2}} A \nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x} v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v} \, d\lambda \, dx \, dt = 0.$$

The key to solving these problems is the introduction of the modified sesquilinear form

$$a_\delta(u, v) := \iiint_{\mathbb{R}_+^{n+2}} A \nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x} (1 - \delta H_t) v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} (1 - \delta H_t) v} \, d\lambda \, dx \, dt,$$

where δ is a real number yet to be chosen. The Hilbert transform H_t is a skew-symmetric isometric operator with inverse $-H_t$ on both \dot{E} and $\dot{H}_{\partial_t - \Delta_x}^{1/4}$. Hence, $1 - \delta H_t$ is invertible on these spaces for any $\delta \in \mathbb{R}$. The upshot is that if we fix $\delta > 0$ small enough, then a_δ is coercive on \dot{E} since

$$(2.4) \quad \operatorname{Re} a_\delta(u, u) \geq (\kappa - C\delta) \|\nabla_{\lambda,x} u\|_2^2 + \delta \|H_t D_t^{1/2} u\|_2^2$$

where κ, C are the constants in (2.1). Hence, solving the Neumann problem with data f means finding $u \in \dot{E}$ such that

$$a_\delta(u, v) = -\langle f, (1 - \delta H_t) v|_{\lambda=0} \rangle \quad (v \in \dot{E})$$

and the Lax-Milgram lemma applied to a_δ on \dot{E} yields a unique such u . We remark that this exactly means (2.2) upon replacing $(1 - \delta H_t) v$ by v , so that $\partial_{\nu_A} u|_{\lambda=0} = f$ holds in the respective sense. For the Dirichlet problem we take an extension $w \in \dot{E}$ of the data f and apply the Lax-Milgram lemma to a_δ on \dot{E}_0 to obtain some $u \in \dot{E}_0$ such that

$$a_\delta(u, v) = -a_\delta(w, v) \quad (v \in \dot{E}_0).$$

Hence, $u + w$ is an energy solution with data f . Would there exist another solution v , then $a_\delta(u + w - v, u + w - v) = 0$ and hence $\|u + w - v\|_{\dot{E}} = 0$ by coercivity.

We shall rephrase these observations by saying the the Dirichlet and Neumann problems associated with (1.1) are *well-posed* for the energy class. In the case of λ -independent coefficients we will rediscover the energy solutions in Section 2.8 within a much broader context.

2.4. The correspondence to a first order system. Given a reinforced weak solution u as in Section 2.2, we create a vector with $n + 2$ components, which we call the (*parabolic*) *conormal differential* of u ,

$$D_A u(\lambda, x, t) := \begin{bmatrix} \partial_{\nu_A} u(\lambda, x, t) \\ \nabla_x u(\lambda, x, t) \\ H_t D_t^{1/2} u(\lambda, x, t) \end{bmatrix},$$

where the inwardly oriented conormal derivative was defined in (2.3). Note that the last component of $D_A u$ contains the half-order time derivative of u , which exists by assumption. Whether or not we choose to include the Hilbert transform in the definition of $D_A u$ is not important at this stage: it only affects the algebra of the representation but not its analysis. Note also that in a pointwise fashion,

$$|D_A u|^2 \sim |\nabla_{\lambda,x} u|^2 + |H_t D_t^{1/2} u|^2.$$

Throughout the paper we will represent vectors $\xi \in \mathbb{C}^{n+2}$ as

$$\xi = \begin{bmatrix} \xi_\perp \\ \xi_\parallel \\ \xi_\theta \end{bmatrix},$$

where the *normal* (or *perpendicular*) part ξ_\perp is scalar valued, the *tangential* part ξ_\parallel is valued in \mathbb{C}^n and the *time* part ξ_θ is again scalar valued.

If u is a reinforced weak solution to (1.1), then the vector valued function $F := D_A u$ belongs to the space $L^2_{\text{loc}}(\mathbb{R}; L^2_{\text{loc}}(\mathbb{R}_+^{n+1}; \mathbb{C}^{n+1})) \times L^2(\mathbb{R}; L^2_{\text{loc}}(\mathbb{R}_+^{n+1}; \mathbb{C}))$, the global square integrability for the θ -component being with respect to the t variable, and it has the additional structure

$$(2.5) \quad \text{curl}_x F_{\parallel} = 0, \quad \nabla_x F_{\theta} = H_t D_t^{1/2} F_{\parallel},$$

when computed in the sense of distributions on $\mathbb{R}_+^{n+1} \times \mathbb{R}$. In fact, the first equation is contained in the second one.

To eventually address boundary value problems, we will assume global square integrability with respect to (x, t) for all components of F . Hence, with the help of Fubini's theorem we put $D_A u$ in the space $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2}))$, where the local square integrability now is with respect to the λ variable. Throughout the paper we let \mathbb{L}^2 be $L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$ and \mathcal{H}_{loc} be the subspace of $L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{L}^2)$ defined by the compatibility conditions (2.5).

Next, we split the coefficient matrix A as

$$(2.6) \quad A(\lambda, x, t) = \begin{bmatrix} A_{\perp\perp}(\lambda, x, t) & A_{\perp\parallel}(\lambda, x, t) \\ A_{\parallel\perp}(\lambda, x, t) & A_{\parallel\parallel}(\lambda, x, t) \end{bmatrix}$$

and we recall from [5] that there is a purely algebraic transformation on the bounded, uniformly elliptic matrix functions that will eventually allow us to write (1.1) as system for the unknown $D_A u$.

Proposition 2.1 ([5, Prop. 4.1, p. 68]). *The pointwise transformation*

$$A \mapsto \hat{A} := \begin{bmatrix} 1 & 0 \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix} \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1} A_{\perp\parallel} \\ A_{\parallel\perp} A_{\perp\perp}^{-1} & A_{\parallel\parallel} - A_{\parallel\perp} A_{\perp\perp}^{-1} A_{\perp\parallel} \end{bmatrix}$$

is a self-inverse bijective transformation on the set of bounded matrices which are uniformly elliptic.

We introduce the operators

$$(2.7) \quad P := \begin{bmatrix} 0 & \text{div}_x & -D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{1/2} & 0 & 0 \end{bmatrix}, \quad M := \begin{bmatrix} \hat{A}_{\perp\perp} & \hat{A}_{\perp\parallel} & 0 \\ \hat{A}_{\parallel\perp} & \hat{A}_{\parallel\parallel} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For each fixed $\lambda > 0$, the operator M is a multiplication operator on \mathbb{L}^2 . The operator P is independent of λ and defined as an unbounded operator in \mathbb{L}^2 with maximal domain. The adjoint of P is

$$(2.8) \quad P^* = \begin{bmatrix} 0 & \text{div}_x & H_t D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -D_t^{1/2} & 0 & 0 \end{bmatrix}.$$

In Section 5 we will prove the following theorem, establishing the connection between reinforced weak solutions u to (1.1) and a first order differential equation.

Theorem 2.2. *If u is a reinforced weak solution u to (1.1) and $F := D_A u \in \mathcal{H}_{\text{loc}}$, then*

$$(2.9) \quad \iiint_{\mathbb{R}_+^{n+2}} F \cdot \overline{\partial_\lambda \phi} \, d\lambda \, dx \, dt = \iiint_{\mathbb{R}_+^{n+2}} M F \cdot \overline{P^* \phi} \, d\lambda \, dx \, dt$$

for all $\phi \in C_0^\infty(\mathbb{R}_+^{n+2}; \mathbb{C}^{n+2})$. Conversely, to any $F \in \mathcal{H}_{\text{loc}}$ satisfying (2.9) for all ϕ , there exists a reinforced weak solution u to (1.1), unique up to a constant, such that $F = D_A u$.

In other words, up to additive constants one can construct all reinforced weak solutions u to (1.1) with the property $D_A u \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2}))$ by solving the differential equation

$$(2.10) \quad \partial_\lambda F + P M F = 0$$

in the distributional sense (2.9) in the space \mathcal{H}_{loc} .

The equation (2.10) depends on the operator $P M$, hence on the choice of A to represent \mathcal{L} . One can show that any choice of A leads to the same conclusions as far as Dirichlet problems are concerned: solvability and uniqueness, respectively, hold for all choices or none. On the other hand, Neumann

problems depend on the choice of A in the formulation and in the conclusion. It could well be that one Neumann problem is solvable or well-posed but not all of them.

2.5. The parabolic Dirac operator for transversally independent equations. The first case to study is that of equations with coefficients independent of the transverse variable λ . In this case also the entries of M are independent of λ . Thus, the differential equation (2.10) becomes autonomous and can be solved via semigroup techniques, provided the semigroup is well-defined. This requires that PM has a *bounded holomorphic functional calculus*.

Let us quickly recall that an operator T in a Hilbert space is *bisectorial* of *angle* $\omega \in (0, \pi/2)$ if its spectrum is contained in the closure of the open double sector

$$S_\omega := \{z \in \mathbb{C} : |\arg z| < \omega \text{ or } |\arg z - \pi| < \omega\}$$

and if for each $\mu \in (\omega, \pi/2)$ the map $z \mapsto z(z-T)^{-1}$ is uniformly bounded on $\mathbb{C} \setminus S_\mu$. Throughout this paper we require basic knowledge of such operators and their functional calculus allowing to define $b(T)$ for suitable holomorphic functions b on S_μ . A reader without a background in this field will find all the necessary results and further information in various comprehensive treatments including [35, 37, 51].

As can be seen from (2.7) and (2.8), the operator P contains the non-local operators $D_t^{1/2}$ and $H_t D_t^{1/2}$ and it is *not* self-adjoint. These are the two major differences compared to the corresponding construction of the Dirac operator for elliptic equations. Note that P arises from the heat equation $\partial_t u - \Delta_x u = 0$. The non-selfadjointness of P is reflected in the fact that the adjoint equation is not the heat equation itself but the backward heat equation $-\partial_t u - \Delta_x u = 0$. Hence, we (unfortunately) cannot just quote results from the literature devoted to elliptic equations without investigating the arguments. Nevertheless, we (fortunately) can prove the similar results yielding a harvest of consequences from abstract reasoning regardless of the precise definition of P and M .

We are ready to state a fundamental result of the paper:

Theorem 2.3. *The operator PM is a bisectorial operator on \mathbb{L}^2 with range $\mathbf{R}(PM) = \mathbf{R}(P)$. It satisfies the quadratic estimate*

$$\int_0^\infty \|\lambda PM(1 + \lambda^2 PMPM)^{-1} h\|_2^2 \frac{d\lambda}{\lambda} \sim \|h\|_2^2 \quad (h \in \overline{\mathbf{R}(PM)}).$$

The angle ω of bisectoriality and implicit constants in the quadratic estimate depend upon n and the ellipticity constants of A . In particular, PM has a bounded holomorphic functional calculus on $\overline{\mathbf{R}(PM)} = \overline{\mathbf{R}(P)}$ on open double sectors S_μ for all $\mu \in (\omega, \pi/2)$. The same holds true for MP on $\overline{\mathbf{R}(MP)} = \overline{\mathbf{R}(P)}$.

The proof is given in Sections 4 and 7. We emphasize that the entries of M are allowed to depend on x and t in a merely measurable fashion. As a consequence of bisectoriality there are topological splittings

$$(2.11) \quad \mathbb{L}^2 = \overline{\mathbf{R}(P)} \oplus \mathbf{N}(PM) = M\overline{\mathbf{R}(P)} \oplus \mathbf{N}(P).$$

Existence of a bounded holomorphic functional calculus for PM on $\overline{\mathbf{R}(P)}$ on sectors S_μ in our case means that for any bounded holomorphic function $b : S_\mu \rightarrow \mathbb{C}$ the functional calculus operator $b(PM)$ on $\overline{\mathbf{R}(P)}$ is bounded by $\|b(PM)\|_{\overline{\mathbf{R}(P)} \rightarrow \overline{\mathbf{R}(P)}} \lesssim \|b\|_{L^\infty(S_\mu)}$ with an implicit constant depending upon μ , n and the ellipticity constants of A . Due to the seminal result of McIntosh [51] this property is in fact equivalent to the quadratic estimate stated above. If b is unambiguously defined at the origin, then $b(PM)$ extends to \mathbb{L}^2 by $b(0)$ on $\mathbf{N}(PM)$. This general theory also applies to MP on the closure of its range.

Based on Theorem 2.3 and the observation that \mathcal{H}_{loc} identifies with $L^2_{\text{loc}}(\mathbb{R}_+; \overline{\mathbf{R}(P)})$, see Lemma 4.1 below, we can construct solutions to the differential equation (2.10), using functional calculus.

We define the characteristic functions $\chi^+(z)$ and $\chi^-(z)$ for the right and left open half planes in \mathbb{C} and the exponential functions $e^{-\lambda|z|}$, $\lambda > 0$. Here $[z] := z \operatorname{sgn}(z)$ and $\operatorname{sgn}(z) := \chi^+(z) - \chi^-(z)$. This gives the generalized *spectral projections* $\chi^\pm(PM)$, which are bounded on $\overline{\mathbf{R}(P)}$ by Theorem 2.3, and

the holomorphic semigroup $(e^{-\lambda[PM]})_{\lambda>0}$ generated by $[PM] = PM \operatorname{sgn}(PM)$. The boundedness of the spectral projections yields a topological splitting into spectral spaces $\mathbf{H}^\pm(PM) := \chi^\pm(PM)\overline{\mathbf{R}(P)}$, which can be called *generalized Hardy spaces*,

$$(2.12) \quad \overline{\mathbf{R}(P)} = \mathbf{H}^+(PM) \oplus \mathbf{H}^-(PM).$$

It is convenient to introduce the following *generalized Cauchy extension*: for $h \in \overline{\mathbf{R}(P)}$ and $\lambda \in \mathbb{R} \setminus \{0\}$,

$$(2.13) \quad (C_0h)_\lambda = (C_0h)(\lambda, \cdot) := \begin{cases} (C_0^+h)(\lambda, \cdot) = e^{-\lambda PM} \chi^+(PM)h & (\text{if } \lambda > 0), \\ (C_0^-h)(\lambda, \cdot) = e^{-\lambda PM} \chi^-(PM)h & (\text{if } \lambda < 0). \end{cases}$$

Let us also define the *semigroup extension* of h for $\lambda > 0$ by

$$(2.14) \quad (Sh)_\lambda = (Sh)(\lambda, \cdot) := e^{-\lambda[PM]}h.$$

Hence,

$$\begin{cases} (C_0^+h)_\lambda = (S\chi^+(PM)h)_\lambda & (\text{if } \lambda > 0), \\ (C_0^-h)_\lambda = (S\chi^-(PM)h)_{-\lambda} & (\text{if } \lambda < 0). \end{cases}$$

The above also applies to MP , which has a bounded holomorphic functional calculus on the closure of its range, too. However, unlike for PM , the range of MP is not independent of M .

Proposition 2.4. *The generalized Cauchy extension $F = C_0^+h$ of $h \in \overline{\mathbf{R}(P)}$ gives a solution to $\partial_\lambda F + PMF = 0$ in the strong sense $F \in C([0, \infty); \overline{\mathbf{R}(P)}) \cap C^\infty((0, \infty); \mathbf{D}(PM))$ with bounds*

$$\sup_{\lambda>0} \|F_\lambda\|_2 \sim \|\chi^+(PM)h\|_2 \sim \sup_{\lambda>0} \int_\lambda^{2\lambda} \|F_\mu\|_2^2 d\mu$$

and \mathbb{L}^2 limits

$$\lim_{\lambda \rightarrow 0} F_\lambda = \chi^+(PM)h, \quad \lim_{\lambda \rightarrow \infty} F_\lambda = 0.$$

In addition, the square function estimate

$$(2.15) \quad \int_0^\infty \|\lambda \partial_\lambda F\|_2^2 \frac{d\lambda}{\lambda} \sim \|\chi^+(PM)h\|_2^2$$

holds. Furthermore, F is a solution to $\partial_\lambda F + PMF = 0$ also in the weak sense (2.9).

If h belongs to the spectral space $\mathbf{H}^+(PM)$, then $h = \chi^+(PM)h$ is the initial value of its generalized Cauchy extension C_0^+h . The generalized Cauchy extension C_0^-h gives a solution of the same equation with the analogous estimates for $\lambda \in \mathbb{R}_-$. Moreover, the counterpart of Theorem 2.2 for the lower half-space (with identical proof) provides the correspondence with parabolic conormal differentials of reinforced weak solutions to (1.1) on \mathbb{R}_-^{n+2} .

All statements in Proposition 2.4 but the last one are well-known properties in semigroup theory and they apply to any bisectorial operator with a bounded holomorphic functional calculus [37, 51]. Note that in (2.15) we have $\partial_\lambda F = \phi(\lambda PM)h$ with holomorphic function $\phi(z) = ze^{-[z]}\chi^+(z)$, and thus this square function estimate follows from the analogous one with $\phi(z) = z(1+z^2)^{-1}$ provided by Theorem 2.3. The verification of the equation in the weak sense (2.9) is special to the operator PM and follows by a simple integration by parts.

The converse to Proposition 2.4 is a verbatim modification of the argument originally designed for elliptic systems [5, Thm. 8.2]. In fact, this relies on an abstract approach that works for any bisectorial operator with a bounded holomorphic functional calculus, see [6]. Hence, we can directly state the following result.

Theorem 2.5. *Let $F \in L_{\text{loc}}^2(\mathbb{R}_+; \overline{\mathbf{R}(P)})$ be a solution of (2.10) in the weak sense such that*

$$(2.16) \quad \sup_{\lambda>0} \int_\lambda^{2\lambda} \|F_\mu\|_2^2 d\mu < \infty.$$

Then F has a \mathbb{L}^2 limit $h \in \mathbf{H}^+(PM)$ at $\lambda = 0$ and F is given by the Cauchy extension of h . The analogous result for weak solutions $F \in L_{\text{loc}}^2(\mathbb{R}_-; \overline{\mathbf{R}(P)})$ of (2.10) also holds upon using the spectral space $\mathbf{H}^-(PM)$.

Finally, we mention that the whole functional calculus for PM becomes analytic in A equipped with the L^∞ -norm (or equivalently, in M for the same norm). In particular, this provides us with Lipschitz estimates for the operator norm on $\overline{\mathbb{R}(P)}$: for all b bounded and holomorphic in a sector S_μ with $\omega < \mu < \pi/2$,

$$(2.17) \quad \|b(PM) - b(PM')\|_{\overline{\mathbb{R}(P)} \rightarrow \overline{\mathbb{R}(P)}} \lesssim \|b\|_\infty \|M - M'\|_\infty,$$

where M' is associated with an elliptic matrix A' in the same class as A and $\|M - M'\|_\infty$ is small. The implicit constant depends on n , μ and the ellipticity constants of A . This is again an abstract property of the functional calculus for operators of type PM once it is known that implicit constants in the quadratic estimate of Theorem 2.3 depend on the coefficients A only through the ellipticity constants. Details are written out for example in [8, p. 269] or [35, Prop. 6.1.25].

2.6. Kato square root estimate. As another important consequence of Theorem 2.3 we obtain the resolution of the *Kato problem for parabolic operators* in full generality.

In order to set the context, we note that similar to Section 2.3 the use of the energy space $\mathbb{V} := H^{1/2}(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}; W^{1,2}(\mathbb{R}^n))$ allows us to define the parabolic operator $L := \partial_t - \operatorname{div}_x A_{\parallel\parallel}(x, t) \nabla_x$ as an operator $\mathbb{V} \rightarrow \mathbb{V}^*$ via a sesquilinear form,

$$\langle Lu, v \rangle := \iint_{\mathbb{R}^{n+1}} A_{\parallel\parallel} \nabla_x u \cdot \overline{\nabla_x v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v} \, dx \, dt \quad (u, v \in \mathbb{V}).$$

Carving out the surprising analogy with elliptic operators even further, [11, Lem. 4] shows that L with maximal domain $\mathbb{D}(L) = \{u \in \mathbb{V} : Lu \in L^2(\mathbb{R}^{n+1})\}$ in $L^2(\mathbb{R}^{n+1})$ is *maximal accretive*: That is to say L is closed, $\operatorname{Re}\langle Lu, u \rangle \geq 0$ holds for all $u \in \mathbb{D}(L)$, and $1 + L$ is onto. We refer to [45] for background on this class of operators. Note that all this is for one dimension lower than in Section 2.3 and that we are using an inhomogeneous energy space since we are studying L as an operator in $L^2(\mathbb{R}^{n+1})$. The reader will notice that the following result reads almost identically to the famous Kato problem for elliptic operators solved in [13]. We shall give its proof in Section 12.1. We emphasize that no assumptions on $A_{\parallel\parallel} = A_{\parallel\parallel}(x, t)$ besides measurability and uniform ellipticity have been imposed.

Theorem 2.6. *The operator $L = \partial_t - \operatorname{div}_x A_{\parallel\parallel}(x, t) \nabla_x$ arises from an accretive form, it is maximal-accretive in $L^2(\mathbb{R}^{n+1})$, the domain of its square root is that of the accretive form, that is, $\mathbb{D}(\sqrt{L}) = \mathbb{V} = H^{1/2}(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}; W^{1,2}(\mathbb{R}^n))$. The two-sided estimate*

$$\|\sqrt{L}u\|_2 \sim \|\nabla_x u\|_2 + \|D_t^{1/2}u\|_2 \quad (u \in \mathbb{V})$$

holds with implicit constants depending only upon n and ellipticity constants of $A_{\parallel\parallel}$.

The reader should remark that not even the case $A_{\parallel\parallel}^* = A_{\parallel\parallel}$ can be treated by abstract functional analysis (while this is the case for the elliptic Kato problem) because L is never self-adjoint.

2.7. Sobolev spaces. The previously obtained characterisation of solutions to the differential equation in (2.10) extends to other topologies.

For $s \in \mathbb{R}$ we can define the fractional powers $[PM]^s$ as closed and injective operators in $\overline{\mathbb{R}(PM)} = \overline{\mathbb{R}(P)}$, using the functional calculus for PM . The homogeneous Sobolev space \dot{H}_{PM}^s based on PM is the completion of $\mathbb{D}([PM]^s) \cap \overline{\mathbb{R}(PM)}$ under the norm $\|[PM]^s \cdot\|_2$. Another option taken in [4] is to use the fractional powers of $(PM)^s$ with branch cut on $i(-\infty, 0]$, which gives the same spaces with isometric norms. This yields an abstract scale of Hilbert spaces that can all be realised within the same linear Hausdorff space and the intersection of any two of them is dense in both of them. Of course, $\dot{H}_{PM}^0 = \overline{\mathbb{R}(P)}$. As a matter of fact, the bounded holomorphic functional calculus extends to \dot{H}_{PM}^s , yielding in particular the spectral spaces $\dot{H}_{PM}^{s,\pm}$, and PM extends to an isomorphism from \dot{H}_{PM}^s onto \dot{H}_{PM}^{s-1} . The same abstract construction can be done with MP starting from the base space $\overline{\mathbb{R}(MP)}$. Further background on these operator-adapted Sobolev spaces is provided in [17] or [37, Ch. 6].

In particular, this construction applies to P and in this case we re-obtain concrete homogeneous parabolic Sobolev spaces as defined in Section 2.3. To this end, we note that if $h \in \mathcal{D}(P^2) \cap \mathcal{R}(P)$, then

$$(2.18) \quad P^2 h = \begin{bmatrix} \partial_t - \Delta_x & 0 & 0 \\ 0 & -\nabla_x \operatorname{div}_x & \nabla_x D_t^{1/2} \\ 0 & -H_t D_t^{1/2} \operatorname{div}_x & \partial_t \end{bmatrix} h = \begin{bmatrix} \partial_t - \Delta_x & 0 & 0 \\ 0 & \partial_t - \Delta_x & 0 \\ 0 & 0 & \partial_t - \Delta_x \end{bmatrix} h.$$

Taking the Fourier transform, we easily see that \dot{H}_P^s is a closed subspace of $(\dot{H}_{\partial_t - \Delta_x}^{s/2})^{n+2}$ and that their norms are equal. Moreover, (2.18) reveals the bounded projection π_P onto $\overline{\mathcal{R}(P)}$ along $\mathcal{N}(P)$, which is well-defined in virtue of the decomposition (2.11) for $M = 1$, as the smooth parabolic singular integral of convolution $\pi_P = P^2(\partial_t - \Delta_x)^{-1}$. Thus, \dot{H}_P^s is precisely the image of $(\dot{H}_{\partial_t - \Delta_x}^{s/2})^{n+2}$ under the bounded extension of π_P .

The following lemma allows one to work more concretely with the Sobolev spaces associated with PM . It is a consequence of the bounded holomorphic functional calculus for PM (and P) and the fact that $\overline{\mathcal{R}(PM)} = \overline{\mathcal{R}(P)}$ holds, see [16, Prop. 4.5] for details.

Lemma 2.7. *If $-1 \leq s \leq 0$, then $\dot{H}_{PM}^s = \dot{H}_P^s$ with equivalent norms. Equivalence constants depend only on n and ellipticity constants of A .*

With this notation set up, we can give the alluded *a priori* characterisations of weak solutions to (2.10) with data in parabolic homogeneous Sobolev spaces. Note that all appearing abstract Sobolev spaces identify with concrete parabolic Sobolev spaces only in the range of the above lemma.

Theorem 2.8. *Let $-1 \leq s < 0$. The generalized Cauchy extension $F = C_0^+ h$ of $h \in \dot{H}_P^s$ gives a solution to $\partial_\lambda F + PMF = 0$, in the strong sense $F \in C([0, \infty); \dot{H}_P^s) \cap C^\infty((0, \infty); \dot{H}_P^s \cap \dot{H}_{PM}^{1+s})$ and in the weak sense (2.9), with \dot{H}_P^s bounds*

$$\sup_{\lambda > 0} \|F_\lambda\|_{\dot{H}_P^s} \sim \|\chi^+(PM)h\|_{\dot{H}_P^s},$$

and \dot{H}_P^s limits

$$\lim_{\lambda \rightarrow 0} F_\lambda = \chi^+(PM)h, \quad \lim_{\lambda \rightarrow \infty} F_\lambda = 0.$$

In addition, the square function estimate

$$(2.19) \quad \int_0^\infty \|\lambda^{-s} F\|_2^2 \frac{d\lambda}{\lambda} \sim \|\chi^+(PM)h\|_{\dot{H}_P^s}^2$$

holds. Furthermore, let $F \in L_{\text{loc}}^2(\mathbb{R}_+; \overline{\mathcal{R}(P)})$ be a solution of (2.10) in the weak sense (2.9) such that

$$\int_0^\infty \|\lambda^{-s} F\|_2^2 \frac{d\lambda}{\lambda} < \infty.$$

Then F has a \dot{H}_P^s limit $h \in \dot{H}_{PM}^{s,+}$ at $\lambda = 0$ and F is given by the Cauchy extension of h . The analogous results for the same equation on \mathbb{R}_- also hold upon replacing the positive spectral spaces and projections with their negative counterparts.

As in the case of Proposition 2.4 and Theorem 2.5, also Theorem 2.8 was first obtained in the context of elliptic equations but by an abstract argument which literally applies to any bisectorial operator with bounded holomorphic functional calculus and to PM in particular. References in the context of elliptic equations are [5, Thm. 9.2] for $s = -1$ and [55, Thm. 1.3] for $s \in (-1, 0)$. These results have been put into a much broader context in [19, Sec. 11]. In particular, in their proof it is not important whether or not P is self-adjoint as long as P itself is bisectorial with bounded holomorphic functional calculus. However, expressions involving B^*D in these references should be replaced with M^*P^* in our context.

2.8. Three special cases. In order to make things more explicit, we single out three special cases of Theorems 2.5 and 2.8. Throughout, we let u be a reinforced weak solution to (1.1) and we write $F = D_A u$, where we recall $|D_A u|^2 \sim |\nabla_{\lambda,x} u|^2 + |H_t D_t^{1/2} u|^2$.

Let us begin by revisiting the case $s = -1/2$, which already appeared in our discussion of energy solutions in Section 2.3. Indeed, in this case (2.19) becomes

$$(2.20) \quad \iiint_{\mathbb{R}_+^{n+2}} |\nabla_{\lambda,x} u|^2 + |H_t D_t^{1/2} u|^2 \, d\lambda \, dx \, dt \sim \|h\|_{\dot{H}_P^{-1/2}}^2 \sim \|\partial_{\nu_A} u|_{\lambda=0}\|_{\dot{H}_{\partial_t-\Delta_x}^{-1/4}}^2 + \|u|_{\lambda=0}\|_{\dot{H}_{\partial_t-\Delta_x}^{1/4}}^2,$$

which is the *energy estimate* for reinforced weak solutions, and finiteness of the left-hand side exactly means that u belongs to the energy class \dot{E} introduced in Section 2.3. The *a priori* information that $F \in C_0([0, \infty); \dot{H}_P^{-1/2}) \cap C^\infty((0, \infty); \dot{H}_P^{-1/2} \cap \dot{H}_{PM}^{1/2})$ satisfies $\partial_\lambda F + PMF = 0$ in the strong sense, allows to integrate by parts in λ for every $v \in \dot{E}$, so to obtain

$$(2.21) \quad \iiint_{\mathbb{R}_+^{n+2}} A \nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x} v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v} \, d\lambda \, dx \, dt = -\langle \partial_{\nu_A} u|_{\lambda=0}, v|_{\lambda=0} \rangle.$$

Here, the trace of $\partial_{\nu_A} u$ is defined in $\dot{H}_{\partial_t-\Delta_x}^{-1/4}$ by continuity of $F = D_A u$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing with $\dot{H}_{\partial_t-\Delta_x}^{1/4}$ extending the inner product on $L^2(\mathbb{R}^{n+1})$. Strictly speaking, we would carry out this integration for v smooth and compactly supported in all variables first and then extend by density. A comparison with (2.2) reveals that this understanding of the Neumann boundary data is the same as in the abstract variational setup but the restriction to transversally independent coefficients also allows for a more concrete understanding in the sense of a L^2 limit. In particular, we may plug in $v = (1 - \delta H_t)u$ with $\delta > 0$ small enough as in Section 2.3 to conclude by coercivity

$$(2.22) \quad \iiint_{\mathbb{R}_+^{n+2}} |\nabla_{\lambda,x} u|^2 + |H_t D_t^{1/2} u|^2 \, d\lambda \, dx \, dt \lesssim \|\partial_{\nu_A} u|_{\lambda=0}\|_{\dot{H}_{\partial_t-\Delta_x}^{-1/4}} \|u|_{\lambda=0}\|_{\dot{H}_{\partial_t-\Delta_x}^{1/4}}.$$

Hence, we obtain the boundedness and invertibility of the *Dirichlet to Neumann map* at the boundary expressed in the comparability

$$(2.23) \quad \|\partial_{\nu_A} u|_{\lambda=0}\|_{\dot{H}_{\partial_t-\Delta_x}^{-1/4}} \sim \|u|_{\lambda=0}\|_{\dot{H}_{\partial_t-\Delta_x}^{1/4}} \sim \|h\|_{\dot{H}_P^{-1/2}}.$$

We shall come back to this point later on in Section 2.13.

When $s = 0$, (2.16) rewrites

$$\sup_{\lambda > 0} \int_{\lambda}^{2\lambda} \iint_{\mathbb{R}^{n+1}} |\nabla_{\lambda,x} u|^2 + |H_t D_t^{1/2} u|^2 \, dx \, dt \, d\mu \sim \|h\|_2^2 \sim \|\partial_{\nu_A} u|_{\lambda=0}\|_2^2 + \|u|_{\lambda=0}\|_{\dot{H}_{\partial_t-\Delta_x}^{1/2}}^2$$

and, as we will see in Section 2.10, the left hand side is also comparable to a non-tangential maximal norm on $D_A u$. The comparability

$$\|\partial_{\nu_A} u|_{\lambda=0}\|_2 \sim \|u|_{\lambda=0}\|_{\dot{H}_{\partial_t-\Delta_x}^{1/2}} \sim \|\nabla_x u|_{\lambda=0}\|_2 + \|H_t D_t^{1/2} u|_{\lambda=0}\|_2$$

for this class of u would be the *Rellich estimates* for proving solvability of regularity problems and Neumann problems with L^2 data.

When $s = -1$, (2.19) becomes

$$\int_0^\infty \iint_{\mathbb{R}^{n+1}} |\lambda \nabla_{\lambda,x} u|^2 + |\lambda H_t D_t^{1/2} u|^2 \, dx \, dt \, \frac{d\lambda}{\lambda} \sim \|h\|_{\dot{H}_P^{-1}}^2 \sim \|\partial_{\nu_A} u|_{\lambda=0}\|_{\dot{H}_{\partial_t-\Delta_x}^{-1/2}}^2 + \|u|_{\lambda=0}\|_2^2.$$

In this case the inequality

$$\|\partial_{\nu_A} u|_{\lambda=0}\|_{\dot{H}_{\partial_t-\Delta_x}^{-1/2}} \lesssim \|u|_{\lambda=0}\|_2$$

would be the *Rellich estimate* that provides one way to solve the Dirichlet problem knowing $u|_{\lambda=0}$ in L^2 . Note that u is only known up to a constant from Theorem 2.2. The semigroup representation for $D_A u$ in Theorem 2.8 immediately implies a representation of u itself by means of the functional calculus for MP . The argument in [5, Thm. 9.3] applies verbatim.

Theorem 2.9. *Let u be a reinforced weak solution of (1.1) with*

$$\int_0^\infty \iint_{\mathbb{R}^{n+1}} |\lambda \nabla_{\lambda, x} u|^2 + |\lambda H_t D_t^{1/2} u|^2 dx dt \frac{d\lambda}{\lambda} < \infty.$$

Then there exist unique $c \in \mathbb{C}$ and $\tilde{h} \in H^+(MP)$ such that $u = c - (e^{-\lambda[MP]}\tilde{h})_\perp$. In particular, $u \in C_0([0, \infty); L^2(\mathbb{R}^{n+1})) + \mathbb{C}$, where the subscripted 0 means that u vanishes at ∞ . Moreover, $P\tilde{h}$ is the element $h \in \dot{H}_{PM}^{-1,+}$ appearing in Theorem 2.8.

2.9. Resolvent estimates. An important ingredient in the proof of Theorem 2.3 are off-diagonal estimates for the resolvents of PM . For anyone who knows how these estimates are obtained in the elliptic case, the non-locality of half-order time derivatives may seem as a serious obstacle. A key observation is the following.

Lemma 2.10. *There exists $\delta_0 > 0$, depending only on dimension and the ellipticity constants of the matrix A , such that for any real p, q with $|\frac{1}{p} - \frac{1}{2}| < \delta_0$ and $|\frac{1}{q} - \frac{1}{2}| < \delta_0$, and any $\lambda \in \mathbb{R}$, the resolvent $(1 + i\lambda PM)^{-1}$ is bounded on $L^p(\mathbb{R}; L^q(\mathbb{R}^n; \mathbb{C}^{n+2}))$ with uniform bounds with respect to λ . The same result holds with MP , P^*M or MP^* in place of PM .*

This result will be proved in Section 6 and it allows us to obtain enough decay for our arguments. We prove the following off-diagonal estimates.

Proposition 2.11. *There exists $\varepsilon_0 > 0$ and $N_0 > 1$ such that if $|\frac{1}{q} - \frac{1}{2}| < \varepsilon_0$, then one can find $\varepsilon = \varepsilon(n, q, \varepsilon_0) > 0$ with the following property. Whenever $N \geq N_0$, then there exists a constant $C = C(\varepsilon, N, q) < \infty$ such that for all balls $Q = B(x, r)$ in \mathbb{R}^n , all intervals $I = (t - r^2, t + r^2)$, and all parameters $j \in \mathbb{N}$, $k \in \mathbb{N}^*$ and $\lambda \in \mathbb{R}$ with $|\lambda| \sim r$:*

$$(2.24) \quad \dashint_{Q \times 4^j I} |(1 + i\lambda PM)^{-1} h|^q dy ds \leq CN^{-q\varepsilon k} \dashint_{C_k(Q \times 4^j I)} |h|^q dy ds,$$

provided $h \in (L^2 \cap L^q)(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$ has support in $C_k(Q \times 4^j I)$. Here,

$$C_k(Q \times J) := (2^{k+1}Q \times N^{k+1}J) \setminus (2^k Q \times N^k J)$$

*and cQ and cJ denote dilates of balls and intervals, respectively, keeping the center fixed and dilating the radius by c . Analogous estimates hold with PM replaced by MP , P^*M or MP^* .*

Some remarks explaining the nature of this result are necessary. First, $Q \times I$ is a standard parabolic cube in $\mathbb{R}^n \times \mathbb{R}$. Some estimates will require that we stretch I leaving Q fixed. Hence the use of $4^j I$. Next, since $|\lambda|$ is on the order of the radius of the parabolic cube there is no power of $r/|\lambda|$ involved nor is there dependence on j . Furthermore, in the definition of $C_k(Q \times 4^j I)$ we take the usual double stretching in the x -direction, but we stretch by a factor N in the t -direction. In principle N will be chosen larger than any power of 2^n that we may need to match the decay in the x -direction. In the end, we still have a small power of N^{-k} in front of the integral on the right hand side but note that we have normalized by taking averages (materialized by the dashed integrals) and observe that the Lebesgue measure of $C_k(Q \times 4^j I)$ is on the order of $2^{kn} N^k$ times that of $Q \times 4^j I$. Hence, this is much more decay than it looks like.

We do not know whether the resolvent has the classical off-diagonal decay as an operator acting on a space of homogeneous type (here, \mathbb{R}^{n+1} with parabolic distance and Lebesgue measure) in the sense of [14]. So, it is kind of a novelty in this topic that a weaker form of off-diagonal estimates apply, using that our space has identified directions, which is not the case in general. In contrast, all decay estimates obtained in [53] for operators with t -independent coefficients have the parabolic homogeneity.

2.10. Non-tangential maximal function estimates. The off-diagonal estimates in Proposition 2.11 are strong enough to prove non-tangential maximal function estimates in Section 8. For $(x, t) \in \mathbb{R}^{n+1}$ we define the non-tangential maximal function

$$\tilde{N}_* F(x, t) = \sup_{\lambda > 0} \left(\dashint_{\Lambda \times Q \times I} |F(\mu, y, s)|^2 d\mu dy ds \right)^{1/2},$$

where $\Lambda = (c_0\lambda, c_1\lambda)$, $Q = B(x, c_2\lambda)$ and $I = (t - c_3\lambda^2, t + c_3\lambda^2)$ with positive constants c_i and $c_0 < c_1$. The numerical values of the constants are irrelevant since by a covering argument any change gives equivalent norms.

Theorem 2.12. *Let $h \in \overline{\mathbb{R}(PM)}$ and let $F = Sh$ be its semigroup extension as in (2.14). Then*

$$\|\tilde{N}_*F\|_2 \sim \|h\|_2,$$

where the implicit constants depend only on dimension and the ellipticity constants of A . Furthermore, there is almost everywhere convergence of Whitney averages

$$\lim_{\lambda \rightarrow 0} \iiint_{\Lambda \times Q \times I} |F(\mu, y, s) - h(x, t)|^2 d\mu dy ds = 0$$

for almost every $(x, t) \in \mathbb{R}^{n+1}$.

Recall that if $h \in \mathbf{H}^+(PM)$, then the semigroup extension is the same as the Cauchy extension of h , so that this theorem provides us with estimates for reinforced weak solutions to (1.1). As the estimate

$$\int_{c_0\lambda}^{c_1\lambda} \|F_\mu\|_2^2 d\mu \lesssim \|\tilde{N}_*F\|_2^2$$

holds for arbitrary functions F , see Lemma 8.11 below for convenience, the non-tangential maximal estimate can be seen as a further regularity estimate for solutions in the uniqueness class of Theorem 2.5. Of course, the class of solutions to (2.10) with $\|\tilde{N}_*F\|_2 < \infty$ is also a uniqueness class by restriction. In addition, the almost everywhere convergence of Whitney averages allows us to give a pointwise meaning to the boundary trace of such solutions. A similar remark applies to data in $\mathbf{H}^-(PM)$ for solutions with $\lambda \in \mathbb{R}_-$.

There is also a companion result that applies to Dirichlet problems with L^2 data.

Theorem 2.13. *Let $\tilde{h} \in \overline{\mathbb{R}(MP)}$ and let $\tilde{F}(\lambda, \cdot) = e^{-\lambda[MP]}\tilde{h}$ be its semigroup extension with respect to MP . Then*

$$\|\tilde{N}_*\tilde{F}\|_2 \sim \|\tilde{h}\|_2,$$

where the implicit constants depend only on dimension and the ellipticity constants of A . Furthermore, there is almost everywhere convergence of Whitney averages,

$$\lim_{\lambda \rightarrow 0} \iiint_{\Lambda \times Q \times I} |\tilde{F}(\mu, y, s) - \tilde{h}(x, t)|^2 d\mu dy ds = 0$$

for almost every $(x, t) \in \mathbb{R}^{n+1}$.

Indeed, we recall from Theorem 2.9 that for \tilde{F} as above and $\tilde{h} \in \mathbf{H}^+(MP)$ the perpendicular part \tilde{F}_\perp is a solution of the Dirichlet problem with data \tilde{h}_\perp .

The non-tangential maximal estimates in Theorem 2.12 and Theorem 2.13 depend on apparently new reverse Hölder estimates for reinforced weak solutions of (1.1) that we also prove in Section 8. These estimates are valid in full generality for λ -dependent equations and also for solutions in the lower half-space up to the obvious changes of notation.

Theorem 2.14. *There is a constant C , depending only on the ellipticity constants of A and the dimension n , such that any reinforced weak solution of (1.1) satisfies the reverse Hölder estimate*

$$(2.25) \quad \left(\iiint_{\Lambda \times Q \times I} |\nabla_{\lambda, x} u|^2 + |H_t D_t^{1/2} u|^2 + |D_t^{1/2} u|^2 d\mu dy ds \right)^{1/2} \leq C \sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \iiint_{8\Lambda \times 8Q \times I_k} |\nabla_{\lambda, x} u| + |H_t D_t^{1/2} u| + |D_t^{1/2} u| d\mu dy ds.$$

Here, $\Lambda = (\lambda - r, \lambda + r)$, $Q = B(x, r)$, $I = (t - r^2, t + r^2)$ define a parabolic cylinder of radius $r < \lambda/8$ and $I_k := k\ell(I) + I$ are the disjoint translates of I covering the real line.

Note that the estimates in (2.25) are non-local in the time variable, reflecting the non-locality of half-order time derivatives. The estimates involves both $H_t D_t^{1/2} u$ and $D_t^{1/2} u$, which is reminiscent of the arbitrary choice that we made when setting up the first order approach. In fact, the proof of Theorem 2.14 will suggest that control for one of these functions on the left requires both of them on the right. The factor of the parabolic enlargement of Λ and Q on the right-hand side and the corresponding relation between r and λ can be changed from 8 to any factor $c > 1$ at the expense of C then depending on c as well. Again, this is best done by a covering argument.

2.11. Characterisation of well-posedness of the BVPs (λ -independent). Let us eventually address boundary value problems for parabolic equations. We write $\mathcal{L}u = 0$ to mean that u is a reinforced weak solution of (1.1). For $-1 \leq s \leq 0$ we let

$$\|F\|_{\mathcal{E}_s} := \begin{cases} \|\tilde{N}_*(F)\|_2 & (\text{if } s = 0), \\ (\int_0^\infty \|\lambda^{-s} F\|_2^2 \frac{d\lambda}{\lambda})^{1/2} & (\text{otherwise}) \end{cases}$$

and define the solution classes

$$\mathcal{E}_s := \{F \in L^2_{\text{loc}}(\mathbb{R}_+^{n+2}; \mathbb{C}^{n+2}); \|F\|_{\mathcal{E}_s} < \infty\}.$$

Given $s \in [-1, 0]$, the regularity of the data, we consider the problems

$$\begin{aligned} (R)_{\mathcal{E}_s}^{\mathcal{L}} : \mathcal{L}u = 0, D_A u \in \mathcal{E}_s, u|_{\lambda=0} = f \in \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}, \\ (N)_{\mathcal{E}_s}^{\mathcal{L}} : \mathcal{L}u = 0, D_A u \in \mathcal{E}_s, \partial_{\nu_A} u|_{\lambda=0} = f \in \dot{H}_{\partial_t - \Delta_x}^{s/2}. \end{aligned}$$

The problem $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ is formulated as a Dirichlet problem with regularity $s/2 + 1/2$ on the data but due to

$$\|f\|_{\dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}} \sim \|[\nabla_x f, H_t D_t^{1/2} f]\|_{(\dot{H}_{\partial_t - \Delta_x}^{s/2})^{n+1}},$$

it is equivalent to a regularity problem with (parabolic) regularity s on the quantities $\nabla_x u$ and $H_t D_t^{1/2} u$ at the boundary. We shall adopt this second point of view because these quantities naturally appear in $D_A u$. On the other hand, $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ is a Neumann problem. Note that we require the full knowledge $D_A u \in \mathcal{E}_s$, whereas in [54] the Neumann problem for $s = 0$ is posed with the non-tangential maximal control of $\nabla_{\lambda, x} u$ only. However, therein additional information, such as DeGiorgi-Nash-Moser estimates and invertibility of layer potentials, are used to prove uniqueness under this weaker assumption.

Solving the problems $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ and $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ means finding a solution with control from the data. For $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ and f the Dirichlet data, we want

$$\|D_A u\|_{\mathcal{E}_s} \lesssim \|f\|_{\dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}}$$

and for $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ and f the Neumann data we want

$$\|D_A u\|_{\mathcal{E}_s} \lesssim \|f\|_{\dot{H}_{\partial_t - \Delta_x}^{s/2}}.$$

The implicit constant must be independent of the data. The behaviour at the boundary is the strong convergence specified by our results above. Existence means that this holds for all data in a chosen space. Uniqueness means that there is at most one solution. Well-posedness is the conjunction of both existence of a solution for all data and uniqueness.

It is instructive to write out the general theory for the three most prominent cases that we considered in Section 2.8: In the case $s = 0$, the boundary value problems are the regularity and Neumann problems with data in $L^2(\mathbb{R}^{n+1})$. For the regularity problem, the data takes the form of an array $[\nabla_x f, H_t D_t^{1/2} f]$ for some $f \in \dot{H}_{\partial_t - \Delta_x}^{1/2}$. Due to Theorem 2.12 the boundary behaviour can also be interpreted in the sense of almost everywhere convergence of Whitney averages. In the case $s = -1$, as said, $(R)_{\mathcal{E}_{-1}}^{\mathcal{L}}$ is nothing but a Dirichlet problem with L^2 data f and thanks to Theorem 2.13 almost everywhere convergence of Whitney averages of u to its boundary data comes again as a bonus. In the case $s = -1/2$ we are back to the class of energy solutions, within which we have already obtained

well-posedness in Section 2.3. In fact, we have seen in Section 2.8 that the class of energy solutions coincides with the class of reinforced weak solutions such that $D_A u \in \mathcal{E}_{-1/2}$.

We will say that the boundary value problem $(BVP)_{\mathcal{E}_s}^{\mathcal{L}}$, where BVP designates either N or R , is *compatibly well-posed* if it is well-posed and if the solution agrees with the energy solution of the problem $(BVP)_{\mathcal{E}_{-1/2}}^{\mathcal{L}}$ when the data belongs to both data spaces.

In order to formulate characterisations of well-posedness and compatible well-posedness, we set some new notation. Recall that the trace h of a conormal differential contains three terms: the first one is called the scalar component and denoted by h_{\perp} , and the two others are the tangential component and the time component, which we concatenate by $h_r = [h_{\parallel}, h_{\theta}]$. Since the trace space of all conormal differentials of reinforced weak solutions has been identified as a generalized Hardy space in Theorems 2.5 and 2.8, the conclusion is that the two maps $N_{\perp} : h \mapsto h_{\perp}$ and $N_r : h \mapsto h_r$ carry the well-posedness. We record this easy but important observation in the following theorem.

Theorem 2.15. *The following assertions hold for $-1 \leq s \leq 0$.*

- (i) $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ is well-posed if and only if $N_r : \dot{H}_{PM}^{s,+} \rightarrow (\dot{H}_P^s)_r$ is an isomorphism.
- (ii) $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ is well-posed if and only if $N_{\perp} : \dot{H}_{PM}^{s,+} \rightarrow (\dot{H}_P^s)_{\perp}$ is an isomorphism.

In each case, onto-ness is equivalent to existence and injectivity to uniqueness, respectively.

As the boundary problems for $s = -1/2$ are well-posed by the method of energy solutions in Section 2.3, compatible well-posedness reduces to another simple statement.

Proposition 2.16. *The following assertions hold for $-1 \leq s \leq 0$.*

- (i) $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ is compatibly well-posed if and only if $N_r : \dot{H}_{PM}^{s,+} \rightarrow (\dot{H}_P^s)_r$ is an isomorphism and its inverse agrees with the one at $s = -1/2$ on $(\dot{H}_P^s)_r \cap (\dot{H}_P^{-1/2})_r$.
- (ii) $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ is compatibly well-posed if and only if $N_{\perp} : \dot{H}_{PM}^{s,+} \rightarrow (\dot{H}_P^s)_{\perp}$ is an isomorphism and its inverse agrees with the one at $s = -1/2$ on $(\dot{H}_P^s)_{\perp} \cap (\dot{H}_P^{-1/2})_{\perp}$.

Let us recall that Theorems 2.5 and 2.8 also provide *a priori* representations for the conormal gradients of solutions to (1.1) on the lower half-space \mathbb{R}^{n+2}_- , leading to similar characterisations of well-posedness upon replacing positive with negative spectral spaces. Given these characterisations, (compatible) well-posedness extrapolates and compatible well-posedness interpolates. The full details for elliptic Dirac operators can be found in [4, Thm. 7.8 & 7.7]. They remain unchanged in our setup.

Theorem 2.17. *The set of $s \in (-1, 0)$ for which $(BVP)_{\mathcal{E}_s}^{\mathcal{L}}$ is well-posed (compatibly well-posed) is open.*

Note that in general we cannot extrapolate from the endpoint cases $s = 0$ and $s = -1$.

Theorem 2.18. *If $(BVP)_{\mathcal{E}_{s_i}}^{\mathcal{L}}$ is compatibly well-posed when $-1 \leq s_0, s_1 \leq 0$, then $(BVP)_{\mathcal{E}_s}^{\mathcal{L}}$ is compatibly well-posed for any s between s_0 and s_1 .*

We mention that all results are analytic with respect to L^{∞} -perturbations of the coefficients $A(x, t)$. Again this is an abstract result proved in [4, Thm. 7.16], compare also with [4, Rem. 7.17 - 7.19]. The assumption of this theorem is that a local Lipschitz estimate holds for the functional calculus of PM on \dot{H}_P^s . This one has been described in (2.17) when $s = 0$ and it extends to any $s \in [-1, 0]$ using Lemma 2.7.

Theorem 2.19. *Let $s \in [-1, 0]$. If $(BVP)_{\mathcal{E}_s}^{\mathcal{L}}$ is (compatibly) well-posed, then so is $(BVP)_{\mathcal{E}_s}^{\mathcal{L}'}$ for any \mathcal{L}' with coefficients A' such that $\|A - A'\|_{\infty}$ is sufficiently small, depending on s and the ellipticity constants of A . Moreover, the operator norm of the inverse of the respective projection N_{\bullet} is Lipschitz continuous as a function of A' in a neighbourhood of A .*

Finally, the Lipschitz estimates (2.17), extended to \dot{H}_P^s as explained above, allow us to use the method of continuity to perturb from any equation to any other within arcwise connected components. Usually, we perturb from the heat equation. We record this observation in the following result to be proved in Section 12.9. We say a bounded operator has *lower bounds* if $\|Tf\| \gtrsim \|f\|$ holds for all f .

Proposition 2.20. *Let $s \in [-1, 0]$. Let \mathbb{A} be a class of matrices $A(x, t)$ with uniform ellipticity bounds that is arcwise connected. If this class contains one matrix A_0 for which the problem $(BVP)_{\mathcal{E}_s}^{\mathcal{L}_0}$ is (compatibly) well-posed and if all the corresponding maps N_\bullet have lower bounds on the trace space $\dot{H}_{PM}^{s,+}$ for each $A \in \mathbb{A}$, then $(BVP)_{\mathcal{E}_s}^{\mathcal{L}}$ is (compatibly) well-posed for all $A \in \mathbb{A}$.*

2.12. The backward equation and duality results. The equation dual to (1.1) is the backward equation

$$(2.26) \quad \mathcal{L}^*v = -\partial_t v - \operatorname{div}_X A^*(X, t) \nabla_X v = 0,$$

where A^* is the (complex) adjoint of A . Note the sign change in front of the time derivative. This equation can also be rephrased as a first order system. However, in this case it will be convenient to use a different representation to simplify the presentation of the duality results.

We use the *backward (parabolic) conormal differential*

$$\tilde{D}_{A^*}v(\lambda, x, t) := \begin{bmatrix} \partial_{\nu_{A^*}}v(\lambda, x, t) \\ \nabla_x v(\lambda, x, t) \\ D_t^{1/2}v(\lambda, x, t) \end{bmatrix},$$

where the difference compared to $D_{A^*}v$ is in the third component. With this choice of conormal differential, the analogue of Theorem 2.2 is that there is, up to constants, a correspondence between reinforced weak solutions v to (2.26) and weak solutions $G \in \tilde{\mathcal{H}}_{\text{loc}} := L_{\text{loc}}^2(\mathbb{R}_+; \overline{\mathbb{R}(P^*)})$ to the first order system

$$(2.27) \quad \partial_\lambda G + P^* \tilde{M} G = 0.$$

Here $\tilde{M} = NM^*$, where M^* is the complex adjoint of M in the standard duality of \mathbb{C}^{n+2} and

$$(2.28) \quad N = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that this is an operator in the same class as M . In the case of λ -independent coefficients, the identity $P^* \tilde{M} = ((\tilde{M})^* P)^*$ shows that bisectoriality of $P^* \tilde{M}$ on L^2 and the bounded holomorphic functional calculus inherit from $(\tilde{M})^* P$. The same analysis as developed above applies to (2.27). It is only when coming back to (2.26) that we have to use a different correspondence.

As the reversal of time maps solutions of the backward equation $\mathcal{L}^*v = 0$ to a solution of a forward equation with coefficients $A^*(x, -t)$, we also have reverse Hölder inequalities for reinforced weak solutions of the backward equation. At this occasion let us recall that the reverse Hölder inequalities of Theorem 2.14 already contain both $H_t D_t^{1/2}v$ and $D_t^{1/2}v$ on the left hand side. All other results for the backward equation, including non-tangential estimates as in Theorem 2.12, are obtained by literally repeating the arguments provided for the forward equation and we shall not give further details.

With the construction outlined above we can formulate duality results for the boundary value problems.

Theorem 2.21. *Let \mathcal{L} and \mathcal{L}^* be as above with λ -independent coefficients and let $-1 \leq s \leq 0$. Then $(BVP)_{\mathcal{E}_s}^{\mathcal{L}}$ is (compatibly) well-posed if and only if $(BVP)_{\mathcal{E}_{-1-s}}^{\mathcal{L}^*}$ is (compatibly) well-posed.*

In a similar spirit we obtain a generalisation of Green's formula, which is of independent interest and worth stating as it avoids integration by parts. Recall that the dual of $\dot{H}_{\partial_t - \Delta_x}^{s/2}$ can be identified with $\dot{H}_{-\partial_t - \Delta_x}^{-s/2}$, which coincides with $\dot{H}_{\partial_t - \Delta_x}^{-s/2}$ up to equivalent norms, and the duality pairing extends the inner product on $L^2(\mathbb{R}^{n+1})$.

Proposition 2.22. *Let $-1 \leq s \leq 0$. For any weak solution u to (1.1) with $D_A u \in \mathcal{E}_s$ and any weak solution w to (2.26) with $\tilde{D}_{A^*} w \in \mathcal{E}_{-s-1}$ it holds*

$$(2.29) \quad \langle \partial_{\nu_A} u|_{\lambda=0}, w|_{\lambda=0} \rangle = \langle u|_{\lambda=0}, \partial_{\nu_{A^*}} w|_{\lambda=0} \rangle.$$

Here, the first pairing is the $\langle \dot{H}_{\partial_t - \Delta_x}^{s/2}, \dot{H}_{-\partial_t - \Delta_x}^{-s/2} \rangle$ sesquilinear duality while the second one is the $\langle \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}, \dot{H}_{-\partial_t - \Delta_x}^{-s/2-1/2} \rangle$ sesquilinear duality.

The proof of these two results will be given in Section 9.

2.13. The role of $\text{sgn}(PM)$. In Section 2.11 we have characterised well-posedness separately for each BVP on each parabolic half-space by means of the spectral projections $\chi^\pm(PM)$ and the coordinate projections N_\perp, N_r . In contrast, most classical methods to solving BVPs, such as the method of layer potentials [27, 54, 58], the closely related Rellich estimates, or the study of Dirichlet-to-Neumann maps, usually work equally well on both half-spaces and yield well-posedness of either all boundary value problems *simultaneously* – or none. We shall see that the general theory presented here comprises all these approaches.

This link becomes apparent by studying the operator $\text{sgn}(PM) = \chi^+(PM) - \chi^-(PM)$, which by the bounded holomorphic functional calculus is an involution on $\dot{H}_{PM}^s = \dot{H}_P^s$ for $-1 \leq s \leq 0$. Gathering again \parallel - and θ -components of vectors in these spaces in the single component r , we represent $\text{sgn}(PM)$ by a (2×2) -matrix

$$(2.30) \quad \text{sgn}(PM) := \begin{bmatrix} s_{\perp\perp}(PM) & s_{\perp r}(PM) \\ s_{r\perp}(PM) & s_{rr}(PM) \end{bmatrix}.$$

Since we have $\chi^\pm = \frac{1}{2}(1 \pm \text{sgn})$, the four components of $\text{sgn}(PM)$ reappear in the representations of the spectral projections as

$$(2.31) \quad \begin{aligned} \chi^+(PM) &= \frac{1}{2} \begin{bmatrix} 1 + s_{\perp\perp}(PM) & s_{\perp r}(PM) \\ s_{r\perp}(PM) & 1 + s_{rr}(PM) \end{bmatrix}, \\ \chi^-(PM) &= \frac{1}{2} \begin{bmatrix} 1 - s_{\perp\perp}(PM) & -s_{\perp r}(PM) \\ -s_{r\perp}(PM) & 1 - s_{rr}(PM) \end{bmatrix} \end{aligned}$$

and in view of Section 2.11, well-posedness of the BVPs on any of the two half-spaces has to be encoded in the six maps $1 \pm s_{\perp\perp}(PM), s_{\perp r}(PM), s_{r\perp}(PM), 1 \pm s_{rr}(PM)$. This can be made precise as follows.

Theorem 2.23. *Let $s \in [-1, 0]$. The following hold true.*

- (i) $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ is well-posed on both half-spaces if and only if $s_{r\perp}(PM) : (\dot{H}_P^s)_\perp \rightarrow (\dot{H}_P^s)_r$ is invertible.
- (ii) $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ is well-posed on both half-spaces if and only if $s_{\perp r}(PM) : (\dot{H}_P^s)_r \rightarrow (\dot{H}_P^s)_\perp$ is invertible.
- (iii) The two operators $s_{r\perp}(PM) : (\dot{H}_P^s)_\perp \rightarrow (\dot{H}_P^s)_r, s_{\perp r}(PM) : (\dot{H}_P^s)_r \rightarrow (\dot{H}_P^s)_\perp$ are invertible if and only if the four operators $1 \pm s_{\perp\perp}(PM) : (\dot{H}_P^s)_\perp \rightarrow (\dot{H}_P^s)_\perp, 1 \pm s_{rr}(PM) : (\dot{H}_P^s)_r \rightarrow (\dot{H}_P^s)_r$ are invertible.
- (iv) In particular, well-posedness of $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ and $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ on both half-spaces is equivalent to simultaneous invertibility of the six operators $1 \pm s_{\perp\perp}(PM), s_{\perp r}(PM), s_{r\perp}(PM), 1 \pm s_{rr}(PM)$ acting between the respective components of \dot{H}_P^s .

The experienced reader in parabolic boundary value problems may wonder about the relation between the spectral projections and the so-called Calderón projections. We shall come back to this in Section 2.15.

The proof of Theorem 2.23 will be provided in Section 10. Let us stress that this theorem *cannot* be improved by purely algebraic reasoning on complementary projections.

First, we could have invertibility of solely $s_{r\perp}(PM)$ and failure of invertibility of one $1 \pm s_{\perp\perp}(PM)$ as well as one of $1 \pm s_{rr}(PM)$. Indeed, using a representation as in (2.31) for general pairs of complementary projections χ^\pm , we already see from the simple example in \mathbb{R}^2 ,

$$\chi^+ := \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \chi^- := \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix},$$

that $s_{r\perp}$ can be invertible, while $1 - s_{\perp\perp}$ and $1 + s_{rr}$ are not. In the opposite direction, let $\ell^2(\mathbb{N})$ be the space of complex square-summable sequences, let $S : (a_n) \mapsto (0, a_1, a_2, \dots)$ be the right shift with $S^* : (a_n) \mapsto (a_2, a_3, a_4, \dots)$ its adjoint and let $z \in \mathbb{C}$ be a root of $z^2 - z + 1$. Taking into account

the relations $S^*S = 1$ and $z + \bar{z} = 1$, we see that on $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$ we have a pair of complementary projections

$$\chi^+ := \begin{bmatrix} zSS^* & S \\ S^* & \bar{z}S^*S \end{bmatrix}, \quad \chi^- := \begin{bmatrix} S^*S - zSS^* & -S \\ -S^* & zS^*S \end{bmatrix}$$

for which both $1 \pm s_{rr}$ are invertible, while $s_{r\perp} = 2S^*$ has one-dimensional nullspace and $s_{\perp r} = 2S$ is not onto.

Upon interchanging \perp - and r -coordinates, the same two examples give the same conclusions with the roles of \perp and r reversed.

In Section 2.3 we have obtained well-posedness of both boundary value problems for the energy class on the upper half-space but the discussion there applies verbatim to the lower one. These are precisely the problems $(R)_{\mathcal{E}_{-1/2}}^{\mathcal{L}}$ and $(N)_{\mathcal{E}_{-1/2}}^{\mathcal{L}}$, see Section 2.8. Hence, we can record

Corollary 2.24. *Invertibility of the six operators in Theorem 2.23 above always holds at regularity $s = -1/2$.*

The inverses for $s = -1/2$ have an illuminating interpretation in terms of boundary operators. Indeed, let us define the *Neumann to Dirichlet operator* on the upper half-space by

$$\Gamma_{ND}^{\mathcal{L},+} f := \left[\begin{array}{c} \nabla_x \\ H_t D_t^{1/2} \end{array} \right] u|_{\lambda=0},$$

where u is the energy solution to the Neumann problem for $\mathcal{L}u = 0$ on \mathbb{R}_+^{n+2} with data f . Owing to well-posedness, this defines an isomorphism $(\dot{H}_P^{-1/2})_{\perp} \rightarrow (\dot{H}_P^{-1/2})_r$. Its inverse is the *Dirichlet to Neumann operator* $\Gamma_{DN}^{\mathcal{L},+} g := \partial_{\nu_A} v|_{\lambda=0}$, where v is the energy solution to the Dirichlet problem for $\mathcal{L}v = 0$ on \mathbb{R}_+^{n+2} with regularity data g . Hence, the parabolic conormal differential of an energy solution to $\mathcal{L}u = 0$ on the upper half-space has a trace in $\dot{H}_P^{-1/2} = \dot{H}_{PM}^{-1/2}$ given by $DAu|_{\lambda=0} = [f, \Gamma_{ND}^{\mathcal{L},+} f]$. However, due to Theorem 2.8 these traces also describe the spectral space $\dot{H}_{PM}^{-1/2,+}$. Consequently,

$$(2.32) \quad \dot{H}_{PM}^{-1/2,+} = \{h \in \dot{H}_P^{-1/2} : h_r = \Gamma_{ND}^{\mathcal{L},+} h_{\perp}\},$$

which can be rephrased as saying that for $h \in \dot{H}_P^{-1/2}$ the conditions $\text{sgn}(PM)h = h$ and $h_r = \Gamma_{ND}^{\mathcal{L},+} h_{\perp}$ are equivalent. Unravelling the latter equivalence gives

$$(2.33) \quad \Gamma_{ND}^{\mathcal{L},+} = s_{\perp r}(PM)^{-1}(1 - s_{\perp\perp}(PM)) = (1 - s_{rr}(PM))^{-1}s_{r\perp}(PM)$$

and

$$(2.34) \quad \Gamma_{DN}^{\mathcal{L},+} = (\Gamma_{ND}^{\mathcal{L},+})^{-1} = s_{r\perp}(PM)^{-1}(1 - s_{rr}(PM)) = (1 - s_{\perp\perp}(PM))^{-1}s_{\perp r}(PM).$$

Similarly, akin operators for the lower half-space can be rediscovered within the functional calculus of PM . In Section 10 we shall use the operators $\Gamma_{ND}^{\mathcal{L},\pm}$, $\Gamma_{DN}^{\mathcal{L},\pm}$ to prove the following.

Proposition 2.25. *Let $s \in [-1, 0]$.*

- (i) *If $1 \pm s_{\perp\perp}(PM)$ is invertible between the respective components of \dot{H}_P^s and the inverse agrees with the one for $s = -1/2$, then $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ is compatibly well-posed on \mathbb{R}_{\mp}^{n+2} .*
- (ii) *If $1 \pm s_{rr}(PM)$ is invertible between the respective components of \dot{H}_P^s and the inverse agrees with the one for $s = -1/2$, then $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ is compatibly well-posed on \mathbb{R}_{\mp}^{n+2} .*

It is instructive to compare Proposition 2.25 with Theorem 2.23 and the succeeding remark. Indeed, the additional topological restriction that inverses should be compatible with the ones on energy solutions allows us to overcome the general algebraic obstructions to proving that invertibility of $1 \pm s_{\perp\perp}(PM)$ (or $1 \pm s_{rr}(PM)$) implies invertibility of $s_{r\perp}(PM)$ (or $s_{\perp r}(PM)$, respectively).

2.14. Layer potentials. We continue to carve out the link with classical theory of BVPs by introducing abstract layer potentials in the spirit of [55], following the presentation in [19, Section 12.3]. We also go further by constructing explicitly the inverse of $\partial_t - \operatorname{div}_{\lambda,x} A(x,t) \nabla_{\lambda,x}$ on \mathbb{R}^{n+2} from the layer potentials and without local DeGiorgi-Nash-Moser regularity assumptions.

In the following we designate by π_{MP} the projection onto $\overline{\mathbb{R}(MP)}$ along $\mathbb{N}(MP) = \mathbb{N}(P)$ in (2.11), noting that the notation π_P is consistent with Section 2.5. We collect two further properties of the functional calculi that are required for defining the layer potentials rigorously. The first one follows since π_{MP} and π_P share the same nullspace and the second one comes from general functional calculus of these operators, see [19, Sec. 11].

Lemma 2.26. *The restrictions $\pi_{MP}: \overline{\mathbb{R}(P)} \rightarrow \overline{\mathbb{R}(MP)}$ and $\pi_P: \overline{\mathbb{R}(MP)} \rightarrow \overline{\mathbb{R}(P)}$ are isomorphisms and mutual inverses of one another. For every $-1 \leq s \leq 0$, they extend to isomorphisms $\pi_{MP}: \dot{H}_P^{s+1} \rightarrow \dot{H}_{MP}^{s+1}$ and $\pi_P: \dot{H}_{MP}^{s+1} \rightarrow \dot{H}_P^{s+1}$, respectively. Moreover, the intertwining property*

$$P\pi_P b(MP)\pi_{MP}h = b(PM)Ph$$

is valid for all $h \in \dot{H}_P^{s+1}$ and all bounded holomorphic functions $b: S_\mu \rightarrow \mathbb{C}$.

Note that the holomorphic functions $z \mapsto e^{-\lambda z} \chi^{\operatorname{sgn} \lambda}(z)$ are bounded on $\operatorname{Re} z \neq 0$ when $\lambda \neq 0$. Thus, Lemma 2.7 shows that if $-1 \leq s \leq 0$, then $e^{-\lambda PM} \chi^{\operatorname{sgn} \lambda}(PM)$ is bounded on \dot{H}_P^s , while P is an isomorphism from \dot{H}_P^{s+1} onto \dot{H}_P^s .

We regroup again \parallel - and θ -components of vectors as a single component denoted r . For instance, for a scalar function $f \in \dot{H}_{\partial_t - \Delta_x}^{s/2}$, we let $h = [f, 0]$ be the vector function in \dot{H}_P^s with $h_\perp = f$ and $h_r = 0$. Throughout, we assume $-1 \leq s \leq 0$. For $\lambda \in \mathbb{R} \setminus \{0\}$ and $f \in \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}$, we define the single layer operator by

$$(2.35) \quad \mathcal{S}_\lambda f := \begin{cases} - \left(P^{-1} e^{-\lambda PM} \chi^+(PM) \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp & (\text{if } \lambda > 0), \\ + \left(P^{-1} e^{-\lambda PM} \chi^-(PM) \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp & (\text{if } \lambda < 0) \end{cases}$$

and for $f \in \dot{H}_{\partial_t - \Delta_x}^{s/2}$ we define the double layer operator

$$(2.36) \quad \mathcal{D}_\lambda f := \begin{cases} - \left(\pi_P e^{-\lambda MP} \chi^+(MP) \pi_{MP} \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp & (\text{if } \lambda > 0), \\ + \left(\pi_P e^{-\lambda MP} \chi^-(MP) \pi_{MP} \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp & (\text{if } \lambda < 0). \end{cases}$$

Note that these definitions differ from the ones in [19] in that we have introduced some additional projectors in (2.36) to yield rigorous formulæ for all Sobolev functions while usage of the projectors is not necessary for f in dense spaces, see [4, Sec. 7.4] for details. To avoid further confusion, let us notice an unfortunate typo in [19]: the four formulæ (81) - (85) come with $e^{+t\dots}$ instead of $e^{-t\dots}$ when $t < 0$.

It follows from Lemma 2.26, that \mathcal{S}_λ is a bounded operator $\dot{H}_{\partial_t - \Delta_x}^{s/2} \rightarrow \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}$, while \mathcal{D}_λ is a bounded operator $\dot{H}_{\partial_t - \Delta_x}^{s/2+1/2} \rightarrow \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}$ with uniform norm with respect to λ . Moreover, they are all compatible on common dense subspaces.

We can relate the conormal differentials of the layer operators to the Cauchy extension operators of (2.13). By a direct calculation from the intertwining property in Lemma 2.26,

$$D_A \mathcal{S}_\lambda f = \begin{cases} + e^{-\lambda PM} \chi^+(PM) \begin{bmatrix} f \\ 0 \end{bmatrix} & (\text{if } \lambda > 0), \\ - e^{-\lambda PM} \chi^-(PM) \begin{bmatrix} f \\ 0 \end{bmatrix} & (\text{if } \lambda < 0) \end{cases}$$

and

$$D_A \mathcal{D}_\lambda f = \begin{cases} +e^{-\lambda PM} \chi^+(PM) P \begin{bmatrix} f \\ 0 \end{bmatrix} & (\text{if } \lambda > 0), \\ -e^{-\lambda PM} \chi^-(PM) P \begin{bmatrix} f \\ 0 \end{bmatrix} & (\text{if } \lambda < 0). \end{cases}$$

Note that $P \begin{bmatrix} f \\ 0 \end{bmatrix}$ has zero \perp -component and r -component equal to $\begin{bmatrix} -\nabla_x f \\ -H_t D_t^{1/2} f \end{bmatrix}$. In other words,

$$(2.37) \quad D_A \mathcal{S}_\lambda f = \operatorname{sgn}(\lambda) \left(C_0 \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\lambda, \quad D_A \mathcal{D}_\lambda f = \operatorname{sgn}(\lambda) \left(C_0 P \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\lambda, \quad (\lambda \neq 0),$$

where, as we recall, C_0 is the generalized Cauchy extension operator. These expressions, together with the correspondence stated in Theorem 2.2 and the boundedness of the Cauchy extensions, show that $\mathcal{D}_\lambda f$ and $\mathcal{S}_\lambda f$ are reinforced weak solutions to (1.1). Moreover, the link with the functional calculus of PM provides us with existence of limits at $\lambda = 0^\pm$ including the jump relations and the duality relations between layer potentials.

Proposition 2.27. *Let $-1 \leq s \leq 0$. Given $f \in \dot{H}_{\partial_t - \Delta_x}^{s/2}$, the limits*

$$D_A \mathcal{S}_{0^+} f = \chi^+(PM) \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad D_A \mathcal{S}_{0^-} f = -\chi^-(PM) \begin{bmatrix} f \\ 0 \end{bmatrix}$$

exist in the strong topology of \dot{H}_P^s and hence the jump relation for the conormal derivatives and the no-jump relation for tangential derivatives are encoded as

$$D_A \mathcal{S}_{0^+} f - D_A \mathcal{S}_{0^-} f = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

Given $f \in \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}$, the limits

$$\mathcal{D}_{0^+} f = - \left(\pi_P \chi^+(MP) \pi_{MP} \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp, \quad \mathcal{D}_{0^-} f = + \left(\pi_P \chi^-(MP) \pi_{MP} \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp$$

exist in the strong topology and hence

$$\mathcal{D}_{0^+} f - \mathcal{D}_{0^-} f = -f, \quad \mathcal{D}_{0^+} f + \mathcal{D}_{0^-} f = - \left(\pi_P \operatorname{sgn}(MP) \pi_{MP} \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp =: 2Kf,$$

with K being the double layer potential at the boundary.

In addition, there are the duality formulæ for $f \in \dot{H}_{\partial_t - \Delta_x}^{s/2}$ and $g \in \dot{H}_{-\partial_t - \Delta_x}^{-s/2-1/2}$ given by

$$\langle g, \mathcal{S}_\lambda^\mathcal{L} f \rangle = \langle \mathcal{S}_{-\lambda}^{\mathcal{L}*} g, f \rangle,$$

and for $f \in \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}$ and $g \in \dot{H}_{-\partial_t - \Delta_x}^{-s/2-1/2}$ given by

$$\langle g, \mathcal{D}_\lambda^\mathcal{L} f \rangle = \langle \partial_{\nu_A^*} \mathcal{S}_{-\lambda}^{\mathcal{L}*} g, f \rangle.$$

Here, $\mathcal{S}_\lambda^\mathcal{L}$, $\mathcal{D}_\lambda^\mathcal{L}$ are the layer potential operators defined above and associated with the parabolic equation (1.1), while $\mathcal{S}_\lambda^{\mathcal{L}}$ is the single layer operator associated with the backward parabolic equation (2.26) defined using the correspondence with the operator $P^* \widetilde{M}$ of Section 2.12.*

Due to the relation (2.37) between layer potentials and the Cauchy extension, there is always a representation by (abstract) layer potential operators,

$$D_A u(\lambda, \cdot) = D_A \mathcal{S}_\lambda(\partial_{\nu_A} u|_{\lambda=0}) - D_A \mathcal{D}_\lambda(u|_{\lambda=0})$$

provided all terms are defined. More precisely, the integrated version of this is as follows.

Theorem 2.28 (Green's representation). *Let $-1 \leq s \leq 0$. Any solution u of (1.1) with λ -independent coefficients such that $D_A u$ belongs to the class \mathcal{E}_s can be represented as*

$$u(\lambda, \cdot) = \mathcal{S}_\lambda(\partial_{\nu_A} u|_{\lambda=0}) - \mathcal{D}_\lambda(u|_{\lambda=0}) + c, \quad (\lambda > 0),$$

where c is a constant. The representation holds in $C([0, \infty); L^2_{\text{loc}}(\mathbb{R}^{n+1}))$ and, moding out the constant, in $C([0, \infty); \dot{H}^{s/2+1/2}_{\partial_t-\Delta_x})$. In the lower half-space, with our convention of the conormal derivative, the formula reads

$$u(\lambda, \cdot) = -\mathcal{S}_\lambda(\partial_{\nu_A} u|_{\lambda=0}) + \mathcal{D}_\lambda(u|_{\lambda=0}) + c, \quad (\lambda < 0).$$

Again, Proposition 2.27 and Theorem 2.28 hold as abstract results for operators of type PM with a bounded holomorphic functional calculus, see [19, Sec. 12.3] for the proof in the context of elliptic systems and also [4, Section 7.4] for a cleaner presentation.

The statement in Theorem 2.28 looks like the representation by the familiar Green's formula although we neither have defined a fundamental solution nor can perform integration by parts. In fact, at this level of generality, one can only construct an inverse, whose kernel is to become the fundamental solution when one has regularity assumptions on solutions. Nevertheless, our method says that *all solutions of the BVPs must be obtained by Green's representation*, once the abstract operators have integral representations.

We present two different constructions of the inverse. Both inversion formulæ will be proved in Section 11. For the first one, we consider the operator \mathcal{L} of (1.1) with all variables $(\lambda, x, t) \in \mathbb{R}^{n+2}$ and coefficients independent of λ . The discussion of Section 2.3 adapts and in virtue of the Lax-Milgram lemma \mathcal{L} becomes an isomorphism from $\dot{H}^{1/2}_{\partial_t-\Delta_{\lambda,x}}$ onto $\dot{H}^{-1/2}_{\partial_t-\Delta_{\lambda,x}}$.

Theorem 2.29. *Assume that the coefficients do not depend on λ . The inverse of \mathcal{L} is formally given by the convolution with the operator valued kernel \mathcal{S}_λ . More precisely, let $E = L^2(\mathbb{R}; \dot{H}^{-1/2}_{\partial_t-\Delta_x})$. Given $f \in \dot{H}^{-1/2}_{\partial_t-\Delta_{\lambda,x}}$, the integrals*

$$(\Gamma_{\varepsilon,R} f)_\lambda = \int_{\varepsilon < |\lambda-\mu| < R} \mathcal{S}_{\lambda-\mu} f_\mu \, d\mu,$$

where $f_\mu := f(\mu, \cdot)$, exists for each $0 < \varepsilon < R < \infty$ in E' . They define uniformly bounded operators $\Gamma_{\varepsilon,R} : \dot{H}^{-1/2}_{\partial_t-\Delta_{\lambda,x}} \rightarrow E'$ and

$$\lim_{\varepsilon \rightarrow 0, R \rightarrow \infty} \Gamma_{\varepsilon,R} f = \mathcal{L}^{-1} f$$

holds in the sense of strong convergence in E' .

Note that $\dot{H}^{1/2}_{\partial_t-\Delta_{\lambda,x}}$ continuously embeds into E' and so the last equality makes sense. In other words, we have approximated $\mathcal{L}^{-1} f$ not within $\dot{H}^{1/2}_{\partial_t-\Delta_{\lambda,x}}$ but within the larger space E' . It does not seem as if $\Gamma_{\varepsilon,R}$ or any modification of this convolution maps into $\dot{H}^{1/2}_{\partial_t-\Delta_{\lambda,x}}$ except if $f \in E$. Nevertheless, this representation is in agreement with the natural interpretation of the single layer potential being the fundamental solution computed with pole at $\mu = 0$.

One can also consider a parabolic operator with coefficients depending on all variables and construct its inverse with our methods. This gives us a definition in much greater generality than in the literature. Let us do that for $L = \partial_t - \text{div}_x A_{\parallel\parallel}(x, t) \nabla_x$ with $(x, t) \in \mathbb{R}^{n+1}$. Consider $\mathcal{L} = \partial_t - \text{div}_x A_{\parallel\parallel}(x, t) \nabla_x - \partial_\lambda^2$ in $n+2$ variables and let \mathcal{S}_λ be the single layer potential operator associated with \mathcal{L} .

Theorem 2.30. *The inverse of $L : \dot{H}^{1/2}_{\partial_t-\Delta_x} \rightarrow \dot{H}^{-1/2}_{\partial_t-\Delta_x}$ is given by $L^{-1} = -\int_{-\infty}^{\infty} \mathcal{S}_\lambda \, d\lambda$. More precisely, given $f \in \dot{H}^{-1/2}_{\partial_t-\Delta_x}$, the integrals*

$$\int_{\varepsilon \leq |\lambda| \leq R} \mathcal{S}_\lambda f \, d\lambda \quad (0 < \varepsilon < R < \infty)$$

define uniformly bounded operators from $\dot{H}_{\partial_t - \Delta_x}^{-1/2}$ to $\dot{H}_{\partial_t - \Delta_x}^{1/2}$ and as a strong limit in $\dot{H}_{\partial_t - \Delta_x}^{1/2}$,

$$\lim_{\varepsilon \rightarrow 0, R \rightarrow \infty} \int_{\varepsilon \leq |\lambda| \leq R} \mathcal{S}_\lambda f \, d\lambda = -L^{-1}f.$$

In this result, the approximation is within $\dot{H}_{\partial_t - \Delta_x}^{1/2}$, but is not directly related to the layer potentials of L .

2.15. Invertibility of layer potentials. Let us denote the (abstract) layer potentials from the previous section with superscripts \mathcal{L} or \mathcal{L}^* to distinguish them for the forward and backward equation. So far, we have obtained their boundedness. Provided that the boundary layers are invertible, we can solve the Dirichlet/regularity problem for $\mathcal{L}u = 0$ with boundary data f (recall that these can be thought as the same problem in different topologies) by

$$(2.38) \quad u = \mathcal{S}_\lambda^\mathcal{L} (\mathcal{S}_0^\mathcal{L})^{-1} f$$

or

$$(2.39) \quad u = \mathcal{D}_\lambda^\mathcal{L} (\mathcal{D}_{0^+}^\mathcal{L})^{-1} f,$$

while the Neumann problem $\mathcal{L}u = 0$ with Neumann data g can be solved via

$$(2.40) \quad u = \mathcal{D}_\lambda^\mathcal{L} (\partial_{\nu_A} \mathcal{D}_0^\mathcal{L})^{-1} g$$

or

$$(2.41) \quad u = \mathcal{S}_\lambda^\mathcal{L} (\partial_{\nu_A} \mathcal{S}_{0^+}^\mathcal{L})^{-1} g = \mathcal{S}_\lambda^\mathcal{L} (\mathcal{D}_{0^-}^{\mathcal{L}^*})^{*-1} g.$$

Here we consider $\lambda > 0$. Since $\mathcal{S}_\lambda^\mathcal{L}$ and $\partial_{\nu_A} \mathcal{D}_\lambda^\mathcal{L}$ do not jump across the boundary $\lambda = 0$, we do not need to use the 0^\pm symbol. On the other hand, the double layer potentials $\mathcal{D}_\lambda^\mathcal{L}$ and $\mathcal{D}_\lambda^{\mathcal{L}^*} = (\partial_{\nu_A} \mathcal{S}_{-\lambda}^\mathcal{L})^*$ jump. In the lower half-space, we would assume $\lambda < 0$ and change 0^+ to 0^- and vice versa. Hence, including the lower half-space in the discussion, we have eight possible formulæ, depending on the invertibility of six boundary operators, which is reminiscent of the connection between simultaneous well-posedness of BVPs in both half-spaces and invertibility of the entries of $\text{sgn}(PM)$ discussed in Section 2.13.

This is of course no coincidence – there is a direct translation of the results on $\text{sgn}(PM)$ into the language of layer potentials. Indeed, setting $D_r f = [\nabla_x f, H_t D_t^{1/2} f]$ and using the formulæ for $D_A \mathcal{S}_\lambda^\mathcal{L}$ and $D_A \mathcal{D}_\lambda^\mathcal{L}$ as well as the jump relations from Proposition 2.27, the reader may readily check that

$$(2.42) \quad \begin{aligned} \chi^+(PM) &= \begin{bmatrix} \partial_{\nu_A} \mathcal{S}_{0^+}^\mathcal{L} & -\partial_{\nu_A} \mathcal{D}_0^\mathcal{L} D_r^{-1} \\ D_r \mathcal{S}_0^\mathcal{L} & -D_r \mathcal{D}_{0^+}^\mathcal{L} D_r^{-1} \end{bmatrix}, \\ \chi^-(PM) &= \begin{bmatrix} -\partial_{\nu_A} \mathcal{S}_{0^-}^\mathcal{L} & +\partial_{\nu_A} \mathcal{D}_0^\mathcal{L} D_r^{-1} \\ -D_r \mathcal{S}_0^\mathcal{L} & D_r \mathcal{D}_{0^-}^\mathcal{L} D_r^{-1} \end{bmatrix}, \\ \text{sgn}(PM) &= \begin{bmatrix} \partial_{\nu_A} \mathcal{S}_{0^+}^\mathcal{L} + \partial_{\nu_A} \mathcal{S}_{0^-}^\mathcal{L} & -2\partial_{\nu_A} \mathcal{D}_0^\mathcal{L} D_r^{-1} \\ 2D_r \mathcal{S}_0^\mathcal{L} & -D_r (\mathcal{D}_{0^+}^\mathcal{L} + \mathcal{D}_{0^-}^\mathcal{L}) D_r^{-1} \end{bmatrix}. \end{aligned}$$

Comparison with (2.31) reveals $\pm 2\partial_{\nu_A} \mathcal{S}_{0^\pm}^\mathcal{L} = 1 \pm s_{\perp\perp}(PM)$, $-2\partial_{\nu_A} \mathcal{D}_0^\mathcal{L} = s_{\perp r}(PM)$, $2D_r \mathcal{S}_0^\mathcal{L} = s_{r\perp}(PM)$ and $\mp 2D_r \mathcal{D}_{0^\pm}^\mathcal{L} D_r^{-1} = 1 \pm s_{rr}(PM)$. Hence, Theorem 2.23 and Corollary 2.24 in the language of layer potentials become

Theorem 2.31. *Let $s \in [-1, 0]$. The following hold true.*

- (i) $(R)_{\mathcal{E}_s}^\mathcal{L}$ is well-posed on both half-spaces if and only if $\mathcal{S}_0^\mathcal{L} : \dot{H}_{\partial_t - \Delta_x}^{s/2} \rightarrow \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}$ is invertible.
- (ii) $(N)_{\mathcal{E}_s}^\mathcal{L}$ is well-posed on both half-spaces if and only if $\partial_{\nu_A} \mathcal{D}_0^\mathcal{L} : \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2} \rightarrow \dot{H}_{\partial_t - \Delta_x}^{s/2}$ is invertible.
- (iii) The two operators

$$\mathcal{S}_0^\mathcal{L} : \dot{H}_{\partial_t - \Delta_x}^{s/2} \rightarrow \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}, \quad \partial_{\nu_A} \mathcal{D}_0^\mathcal{L} : \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2} \rightarrow \dot{H}_{\partial_t - \Delta_x}^{s/2}$$

are invertible if and only if the four operators

$$\partial_{\nu_A} \mathcal{S}_{0^\pm}^{\mathcal{L}} : \dot{H}_{\partial_t - \Delta_x}^{s/2} \rightarrow \dot{H}_{\partial_t - \Delta_x}^{s/2}, \quad \mathcal{D}_{0^\pm}^{\mathcal{L}} : \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2} \rightarrow \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2},$$

are invertible.

- (iv) In particular, well-posedness of $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ and $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ on both half-spaces is equivalent to simultaneous invertibility of the six boundary layer operators appearing in (iii).
- (v) The six operators are always invertible when $s = -1/2$.

The direction from invertibility to *solvability* is well-known in this field. Our characterisations of well-posedness allowed us to include *uniqueness* and the full converse. This statement was implicit in the elliptic counterpart in [18].

The conormal derivative of the double layer potential appearing in (ii) is hardly used in the literature and indeed, (iii) makes clear that it needs not be considered for a simultaneous treatment of regularity and Neumann problems. This is in accordance with the case of a real, symmetric and t -independent matrix A , where invertibility of the boundary layers for $s = 0$ can be achieved through a Rellich-type estimate

$$\|\partial_{\nu_A} u|_{\lambda=0}\|_2 \sim \|\nabla_x u|_{\lambda=0}\|_2 + \|H_t D_t^{1/2} u|_{\lambda=0}\|_2,$$

at the boundary for $u = \mathcal{S}_\lambda f$ or $u = \mathcal{D}_\lambda f$. Its proof is typically carried out by integration by parts and thus applies equally well in both half-spaces. As a consequence of this, there are lower bounds for all six operators in Theorem 2.31 and invertibility as well as compatibility with the inverses for the energy class can be obtained from the method of continuity, perturbing from the heat equation to \mathcal{L} as in Proposition 2.20. This method has been pioneered for elliptic equations in [58]. The parabolic version was recently obtained in [54, Sec. 11.4].

In the general situation, however, Neumann and regularity problems on both half-spaces might not be simultaneously well-posed and then invertibility of one boundary layer operator needs not be necessary for solving the corresponding problem. Indeed, invertibility of $\mathcal{D}_{0^\pm}^{\mathcal{L}} : \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2} \rightarrow \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}$ is sufficient for solving $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ on \mathbb{R}_\pm^{n+2} via (2.39) but it does *not* follow from a purely algebraic reasoning that invertibility, or even lower bounds, are also necessary for having well-posedness of $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ solely in both half-spaces, see the remarks after Theorem 2.23. The same discussion applies to $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ and $\mathcal{D}_{0^\mp}^{\mathcal{L}*}$.

To finish this section, we comment on the connection between the spectral projections and the Calderón projections. We have seen that the entries of $\chi^+(PM)$ are given in terms of layer potentials but using the representation at the level of \dot{H}_P^s , $-1 \leq s \leq 0$. However, if we use the boundary space $\mathcal{H}^s := \dot{H}_{\partial_t - \Delta_x}^{s/2} \times \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}$, then through the isometry $[\psi, \omega] \mapsto [\psi, -D_r \omega]$ onto \dot{H}_P^s , the projection $\chi^+(PM)$ is similar to

$$\begin{bmatrix} \partial_{\nu_A} \mathcal{S}_{0^+}^{\mathcal{L}} & \partial_{\nu_A} \mathcal{D}_0^{\mathcal{L}} \\ -\mathcal{S}_0^{\mathcal{L}} & -\mathcal{D}_{0^+}^{\mathcal{L}} \end{bmatrix},$$

defined on \mathcal{H}^s , which is the celebrated *Calderón projection* for our parabolic equation on the upper half-space. The same can be done for the lower half-space. As for $\text{sgn}(PM)$, it is similar to the involution on \mathcal{H}^s given by

$$2\mathcal{A}_{\mathcal{L}} := \begin{bmatrix} \partial_{\nu_A} \mathcal{S}_{0^+}^{\mathcal{L}} + \partial_{\nu_A} \mathcal{S}_{0^-}^{\mathcal{L}} & 2\partial_{\nu_A} \mathcal{D}_0^{\mathcal{L}} \\ -2\mathcal{S}_0^{\mathcal{L}} & -\mathcal{D}_{0^+}^{\mathcal{L}} - \mathcal{D}_{0^-}^{\mathcal{L}} \end{bmatrix}.$$

In [32, Section 3], these calculations are carried out for the layer potentials in the case of the heat equation on a bounded Lipschitz cylinder when $s = -1/2$. The built-in algebra provided by the Dirac operator PM hence gives us immediate access to this technology. This reference shows also a certain coercivity of the involution, which is not available here since t lives in a non-compact interval. In fact, using the natural duality between $\mathcal{H}^{-1/2}$ and its dual, one can only show that

$$\text{Re} \left\langle \mathcal{A}_{\mathcal{L}} \begin{bmatrix} \psi \\ -\omega \end{bmatrix}, \begin{bmatrix} -\omega \\ \psi \end{bmatrix} \right\rangle = \text{Re} \iiint_{\mathbb{R}_+^{n+2}} A \nabla u^+ \cdot \overline{\nabla u^+} + \text{Re} \iiint_{\mathbb{R}_-^{n+2}} A \nabla u^- \cdot \overline{\nabla u^-},$$

where u^\pm are the energy solutions to (1.1) for λ -independent coefficients in \mathbb{R}_\pm^{n+2} with data corresponding to the splitting of $[\psi, D_r\omega] \in \dot{H}_P^{-1/2}$ in virtue of the spectral spaces $\dot{H}_{PM}^{-1/2, \pm}$. So, we are missing the terms with half-order derivatives in time and there does not seem to be a hidden coercivity either, as already the example of the heat equation and calculations via Fourier transform show. We leave further details to the reader. [For the interested reader, the very same calculations work with the DB formalism of [19] for the elliptic equation $-\operatorname{div}_{\lambda, x} A(x) \nabla_{\lambda, x} u = 0$ on both half-spaces showing that the coercivity holds because the right hand side (integrals on \mathbb{R}_+^{n+1}) would be equivalent to the square of the norm in $\dot{H}_{\Delta_x}^{-1/4} \times \dot{H}_{\Delta_x}^{1/4} = \dot{H}^{-1/2}(\mathbb{R}^n) \times \dot{H}^{1/2}(\mathbb{R}^n)$ of the data using the spectral splitting.]

2.16. Well-posedness results for equations with λ -independent coefficients. Finally, we shall address well-posedness of the BVPs. All structural properties of the coefficients A refer to the schematic representation (2.6).

- Theorem 2.32.** (i) $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ and $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ are compatibly well-posed when $-1 \leq s \leq 0$ and $A(x, t)$ has block structure.
- (ii) $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ is compatibly well-posed when $-1/2 \leq s \leq 0$ and $A(x, t)$ is upper triangular.
 - (iii) $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ is compatibly well-posed when $-1 \leq s \leq -1/2$ and $A(x, t)$ is lower triangular.
 - (iv) $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ is compatibly well-posed when $-1/2 \leq s \leq 0$ and $A(x, t)$ is lower triangular.
 - (v) $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ is compatibly well-posed when $-1 \leq s \leq -1/2$ and $A(x, t)$ is upper triangular.
 - (vi) $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ and $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ are compatibly well-posed when $-1 \leq s \leq 0$ and $A(x, t) = A(x)$ is Hermitian.
 - (vii) $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ and $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ are compatibly well-posed when $-1 \leq s \leq 0$ and $A(x, t) = A$ is constant.

We remark that additionally all ranges for compatible well-posedness are relatively open in $(-1, 0)$ by Theorem 2.17 and all results are stable under L^∞ -perturbations of the coefficients as in Theorem 2.19. The proof of this theorem will be given in Section 12.

Let us mention that (vi) was proved in [54] in the case when \mathcal{L} is a real equation and $s = -1, 0$. Here, we may even allow systems of complex parabolic equations, see the discussion in Section 14.1 at the end of this paper. Similarly, (vii) is done in [54] when $s = -1, 0$. Item (i) is also proved as a corollary of the estimates in [53] when \mathcal{L} has additionally t -independent coefficients. In (ii) and (iii) the triangular condition can be improved to the effect that the lower left and upper right coefficient of A can be real-valued with vanishing x -divergence, respectively, since such a change of the representing matrix results in a new elliptic operator that could also be represented by a triangular matrix [8, Rem. 6.7].

The results on well-posedness are identical for the backward equation with coefficients $A(x, t)$: In fact, it suffices to revert time and to observe that $A(x, -t)$ satisfies the same requirements as $A(x, t)$ in the list above. Likewise, the results remain valid for BVPs on the lower half-space: Here, it suffices to observe that by even reflection at the boundary we obtain a one-to-one correspondence with weak solutions on the upper half-space for an elliptic equation with coefficients

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

sharing the same structural properties.

In combination with Theorem 2.31 we obtain *a posteriori* all possible representation formulæ for the unique solution proposed by layer potential theory.

Theorem 2.33. *Let $s \in [-1, 0]$ and u be a solution to $\mathcal{L}u = 0$ with $D_{Au} \in \mathcal{E}_s$ (on either the upper or the lower half-space). All eight representations in (2.38) - (2.41) are valid if A is either of block form, or constant, or Hermitian and independent of t , or if A is arbitrary and s is sufficiently close to $-1/2$ (depending on A). Moreover, (2.38) remains valid for the setup of Theorem 2.32(ii)-(iii) and (2.40) for that of Theorem 2.32(iv)-(v).*

The Hermitian case (vi) will be treated in Section 12 by a Rellich-type argument. It will then become clear that the assumption of t -independence is necessary in order to compensate some lack of

compactness, stemming from the fact that t varies over the whole real line. If we drop this additional assumption, then the natural solution class changes. The following result, also proved in Section 12, captures this effect.

Proposition 2.34. *Assume $A(x, t)$ is Hermitian with $D_t^{1/2}A(x, t) \in L^\infty(\mathbb{R}^n; \text{BMO}(\mathbb{R}))$.*

- (i) *Let $f \in \dot{H}_{\partial_t - \Delta_x}^{1/2} \cap \dot{H}_{\partial_t - \Delta_x}^{1/4}$. Then there is a unique reinforced weak solution u to (1.1) such that $u|_{\lambda=0} = f$ and $D_A u \in \mathcal{E}_0 \cap \mathcal{E}_{-1/2}$.*
- (ii) *Let $f \in L^2(\mathbb{R}^{n+1}) + \dot{H}_{\partial_t - \Delta_x}^{1/4}$. Then there is a unique reinforced weak solution u to (1.1) with $u|_{\lambda=0} = f$ and the additional property that u can be written as the sum of two solutions $u_1 + u_2$ with $D_A u_1 \in \mathcal{E}_{-1}$ and $D_A u_2 \in \mathcal{E}_{-1/2}$.*
- (iii) *Well-posedness in the same classes holds for the Neumann problem: For (i), $\partial_{\nu_A} u|_{\lambda=0}$ is taken in $L^2(\mathbb{R}^{n+1}) \cap \dot{H}_{\partial_t - \Delta_x}^{-1/4}$ and for (ii) it is taken in $\dot{H}_{\partial_t - \Delta_x}^{-1/2} + \dot{H}_{\partial_t - \Delta_x}^{-1/4}$.*

Note that (i) is a regularity problem with restricted data. The dual problem (ii) looks like a curious Dirichlet problem but if the data is restricted to $f \in L^2(\mathbb{R}^{n+1})$, it will have a strong implication on the (classical) L^2 Dirichlet problem discussed in the next section.

As before, also Proposition 2.34 remains valid for BVPs on the lower half space. The interested reader may verify that the transposition into the language of layer potentials yields the following.

Proposition 2.35. *Assume $A(x, t)$ is Hermitian with $D_t^{1/2}A(x, t) \in L^\infty(\mathbb{R}^n; \text{BMO}(\mathbb{R}))$.*

- (i) $\mathcal{S}_0^\mathcal{L} : \dot{H}_{\partial_t - \Delta_x}^0 \cap \dot{H}_{\partial_t - \Delta_x}^{-1/4} \rightarrow \dot{H}_{\partial_t - \Delta_x}^{1/2} \cap \dot{H}_{\partial_t - \Delta_x}^{1/4}$ is invertible and (2.38) holds for the solution (modulo constants) of $\mathcal{L}u = 0$ with $D_A u \in \mathcal{E}_0 \cap \mathcal{E}_{-1/2}$ and boundary data $f \in \dot{H}_{\partial_t - \Delta_x}^{1/2} \cap \dot{H}_{\partial_t - \Delta_x}^{1/4}$.
- (ii) $\mathcal{D}_{0+}^\mathcal{L} : L^2(\mathbb{R}^{n+1}) + \dot{H}_{\partial_t - \Delta_x}^{1/4} \rightarrow L^2(\mathbb{R}^{n+1}) + \dot{H}_{\partial_t - \Delta_x}^{1/4}$ is invertible and (2.39) holds for the solution of $\mathcal{L}u = 0$ with $D_A u \in \mathcal{E}_{-1} + \mathcal{E}_{-1/2}$ and boundary data $f \in L^2(\mathbb{R}^{n+1}) + \dot{H}_{\partial_t - \Delta_x}^{1/2}$.

2.17. Uniqueness for Dirichlet problems. So far, the Dirichlet problem with L^2 data is only formulated as $(R)_{\mathcal{E}_{-1}}^\mathcal{L}$, that is, within the class of reinforced weak solutions having controlled square function norm

$$\|D_A u\|_{\mathcal{E}_{-1}}^2 = \iiint_{\mathbb{R}_+^{n+2}} |\lambda \nabla_{\lambda, x} u|^2 + |\lambda H_t D_t^{1/2} u|^2 \frac{d\lambda dx dt}{\lambda} < \infty$$

and the boundary behaviour can be understood in the sense of L^2 convergence of $u(\lambda, \cdot)$ when $\lambda \rightarrow 0$ since u is continuous with values in $L^2(\mathbb{R}^{n+1})$ as a consequence of Theorem 2.9. As we have seen in Theorem 2.13, such solutions also satisfy the non-tangential maximal bound $\|\tilde{N}_*(u)\|_2 \lesssim \|D_A u\|_{\mathcal{E}_{-1}}$ and enjoy a form of non-tangential convergence for Whitney averages. Thus it makes sense to pose the question of uniqueness in the larger and more classical class involving non-tangential approach in the spirit of earlier references such as [39, 40, 54].

Definition 2.36. We say that $(D)_{\mathcal{E}_{-1}}^\mathcal{L}$ is well-posed if given any $f \in L^2(\mathbb{R}^{n+1})$ there is a unique weak solution $\mathcal{L}u = 0$ which satisfies $\|\tilde{N}_*(u)\|_2 < \infty$ and

$$\iiint_{\Lambda \times Q \times I} |u(\mu, y, s) - f(x, t)| d\mu dy ds \rightarrow 0$$

almost everywhere when $\lambda \rightarrow 0$, where $\Lambda \times Q \times I$ and (x, t) are related as in the definition of \tilde{N}_* . Uniqueness of $(D)_{\mathcal{E}_{-1}}^\mathcal{L}$ means that if a weak solution $\mathcal{L}u = 0$ satisfies $\|\tilde{N}_*(u)\|_2 < \infty$ and tends to zero in the sense above when $\lambda \rightarrow 0$, then $u = 0$ almost everywhere.

Note that this problem is concerned with general weak solutions but we will see in Lemma 13.7 below that the non-tangential control *a priori* forces u to be reinforced. By reverse Hölder estimates on weak solutions (see Lemma 8.2 below), we may replace L^2 -averages of u by L^1 -averages up to taking slightly bigger Whitney regions. Hence, this is the largest possible class in which uniqueness may be asked for. When DeGiorgi-Nash-Moser regularity on solutions is assumed, the conditions in our definition amount to the usual pointwise non-tangential control and non-tangential convergence

to the boundary almost everywhere. As our main result on $(D)_{\frac{c}{2}}$ we will prove in Section 13 the following

Theorem 2.37. *Existence for the adjoint regularity problem $(R)_{\mathcal{E}_0}^{\mathcal{L}^*}$ (that is, with regularity data in $L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+1})$) implies uniqueness for $(D)_{\frac{c}{2}}$.*

This result is in spirit of the earlier results in [46] on elliptic equations, see also [2, Lem. 4.31]. For parabolic equations, compare with [54, Lem. 6.1], where much more is assumed to get the same conclusion, including in particular DeGiorgi-Nash-Moser regularity as in all earlier references.

Corollary 2.38. *Assume the adjoint regularity problem $(R)_{\mathcal{E}_0}^{\mathcal{L}^*}$ is (compatibly) well-posed. Then so is $(D)_{\frac{c}{2}}$. Moreover, given $f \in L^2(\mathbb{R}^{n+1})$, the weak solution u with data f is reinforced, continuous in λ valued in $L^2(\mathbb{R}^{n+1})$ and satisfies*

$$\|\tilde{N}_*(u)\|_2 \sim \|D_A u\|_{\mathcal{E}_{-1}} \sim \sup_{\lambda > 0} \|u(\lambda, \cdot)\|_2 \sim \|f\|_2.$$

Indeed, uniqueness in the corollary above follows from the previous theorem, while existence follows from the previous discussion and Theorem 2.21 on duality of boundary value problems. The last two equivalences are consequences of Theorem 2.2 and Theorem 2.8 (or Theorem 2.9). For the first equivalence, the comparison “ \lesssim ” was already mentioned in Theorem 2.13 while the inequality $\|f\|_2 \lesssim \|\tilde{N}_*(u)\|_2$ is an easy consequence of the non-tangential convergence at the boundary. Compatible well-posedness was already stated in Theorem 2.21 and is preserved as well. Comparing with Theorem 2.32, we also obtain

Corollary 2.39. *$(D)_{\frac{c}{2}}$ is compatibly well-posed if A is either of block form, lower triangular, Hermitian and time independent, or constant.*

Finally, it turns out that there also is a natural solution class defined by the non-tangential maximal function within which we have uniqueness in the setup of Proposition 2.34. By this means, we can complete the discussion of the previous section by adding the following result for the Dirichlet problem with Hermitian time dependent coefficients.

Theorem 2.40. *Assume $A(x, t)$ is Hermitian with $D_t^{1/2} A(x, t) \in L^\infty(\mathbb{R}^n; \text{BMO}(\mathbb{R}))$. Let $f \in L^2(\mathbb{R}^{n+1})$. There is a unique weak solution $\mathcal{L}u = 0$ which satisfies $\|\tilde{N}_*(\frac{u}{\max\{1, \sqrt{\lambda}\}})\|_2 < \infty$ and*

$$\iiint_{\Lambda \times Q \times I} |u(\mu, y, s) - f(x, t)| \, d\mu \, dy \, ds \rightarrow 0$$

almost everywhere when $\lambda \rightarrow 0$. This solution is the same as in Proposition 2.34(ii).

We note that the growth in λ in the non-tangential control is reflecting the regularity of A in the assumption. The proof will also be given in Section 13.

2.18. Consequences for λ -dependent coefficients. There is an automatic improvement of our results to a situation where A depends on the transversal variable λ by treating A in (1.1) as a perturbation of λ -independent coefficients A_0 . This is done at the level of the first order equation, reformulating $\partial_\lambda F + PMF = 0$ as

$$(2.43) \quad \partial_\lambda F + PM_0 F = P(M_0 - M)F.$$

This inhomogeneous problem can then be treated by maximal regularity results obtained in [5] in the cases $s = 0$ and $s = -1$ and in [55] for the intermediate cases $-1 < s < 0$ but changing DB to PM therein. We could have just stopped here but we felt that it is worth helping the reader making it through the transposition to our setup.

Generally, the estimates for $-1 < s < 0$ transpose immediately since they only use semigroup theory. The estimates for $s = -1$ and $s = 0$ require more harmonic analysis. They are obtained using an abstract framework that consists of the following:

- (i) A Hilbert space \mathcal{H} , vector valued Lebesgue spaces $\mathcal{Y} := L^2(\mathbb{R}_+, \lambda \, d\lambda; \mathcal{H})$, $\mathcal{Y}^* := L^2(\mathbb{R}_+, d\lambda/\lambda; \mathcal{H})$ and a space \mathcal{X} with continuous embedding $\mathcal{Y}^* \subset \mathcal{X} \subset L^2_{\text{loc}}(\mathbb{R}_+, d\lambda; \mathcal{H})$.

- (ii) A bisectorial operator DB_0 with a bounded holomorphic functional calculus on \mathcal{H} such that $h \mapsto (e^{-\lambda[DB_0]}h)_{\lambda>0}$ and $h \mapsto (e^{-\lambda[D^*B_0]}h)_{\lambda>0}$ are bounded from \mathcal{H} into \mathcal{X} .

As for our setup, we let $\mathcal{H} = \overline{\mathbf{R}(\overline{P})}$, $\mathcal{X} = \{F \in L^2_{\text{loc}}(\mathbb{R}_+, d\lambda; \mathcal{H}) : \|\tilde{N}_*(F)\|_2 < \infty\}$ and use PM_0 instead of DB_0 . Then the embedding $\mathcal{Y}^* \subset \mathcal{X}$ is classical, see also Lemma 8.11 below. The $\mathcal{H} \rightarrow \mathcal{X}$ boundedness of the PM -semigroup follows from the non-tangential estimate stated in Theorem 2.12 and for the P^*M -semigroup we use the analogue for the backward equation, see also Section 2.12. The fact that [5] works within Euclidean space and Lebesgue measure is of no harm either since only the doubling character of the measure is used. This is also the case for \mathbb{R}^{n+1} equipped with parabolic distance and Lebesgue measure. The needed theory on tent spaces above such spaces of homogeneous type has been written in detail in [3]. The needed estimates for tent spaces with Whitney averages, proved in [5] and next in [41, 42], transpose to such a setting, see the final remark in [41]. Below, we will summarize the main results informally and give an outline of the general strategy, leaving the care of checking details to the interested readers.

The condition imposed on $A(\lambda, x, t)$ is that there exists some λ -independent coefficients $A_0(x, t)$, uniformly elliptic and bounded, such that the *discrepancy*

$$\|A - A_0\|_s = \begin{cases} \|A - A_0\|_\infty & (\text{if } -1 < s < 0), \\ \|A - A_0\|_* & (\text{if } s = -1, 0) \end{cases}$$

is finite. Here, $\|\cdot\|_\infty$ is the $L^\infty(\mathbb{R}_+^{n+2})$ -norm and $\|\cdot\|_*$ is the Carleson-Dahlberg measure norm defined by $\|\cdot\|_* := \|C_2(W_\infty(\cdot))\|_\infty$ with

$$C_2(g)(x, t) := \sup_{R \ni (x, t)} \left(\frac{1}{|R|} \int_0^{\ell(R)} \iint_R |g(\lambda, y, s)|^2 dy ds \frac{d\lambda}{\lambda} \right)^{1/2},$$

where the supremum is taken over all parabolic cubes R with sidelength $\ell(R)$ in \mathbb{R}^{n+1} containing (x, t) and $W_\infty(\cdot)(\lambda, y, s)$ is the essential supremum over the Whitney region given by $\Lambda = (\lambda/2, 2\lambda)$, $Q = B(y, \lambda)$ and $I = (s - \lambda^2, s + \lambda^2)$. The condition $\|A - A_0\|_* < \infty$ is strictly stronger than $\|A - A_0\|_\infty < \infty$. Note that $\|A - A_0\|_s \sim \|M - M_0\|_s$ where M_0 corresponds to A_0 just as M corresponds to A . We also remark that A_0 is uniquely defined when using the $\|\cdot\|_*$ -norm and the ellipticity constants relate to those of A . On the other hand, when using only the L^∞ -norm there are many different possible A_0 . Nevertheless we implicitly assume that the ellipticity constants of A and A_0 are comparable.

The operator related to solving (2.43) with initial data equal to zero writes formally in vector-valued form as

$$(S_A F)_\lambda := \int_0^\lambda e^{-(\lambda-\mu)PM_0} E_0^+ P(M_0 - M_\mu) F_\mu d\mu - \int_\lambda^\infty e^{(\mu-\lambda)PM_0} E_0^- P(M_0 - M_\mu) F_\mu d\mu.$$

Here, $E_0^\pm = \chi^\pm(PM_0)$ are the projections defined in Section 2.5. We also use the notation $F_\lambda = F(\lambda, \cdot)$. The integral can be appropriately defined when $-1 \leq s \leq 0$ and enjoys the norm estimate

$$\|S_A F\|_{\mathcal{E}_s} \leq C \|A - A_0\|_s \|F\|_{\mathcal{E}_s},$$

where the \mathcal{E}_s -norms were defined in Section 2.11 and C depends on ellipticity, s and dimension. Also S_A maps into $L^2_{\text{loc}}(\mathbb{R}_+; \overline{\mathbf{R}(\overline{P})})$. Then it is indeed true that for any solution of (1.1) with $D_A u \in \mathcal{E}_s$ there exists a unique $h^+ \in \dot{H}_{PM_0}^{s,+}$ such that we have an implicit representation

$$(2.44) \quad (D_A u)_\lambda = e^{-\lambda PM_0} h^+ + (S_A D_A u)_\lambda$$

with equality in \mathcal{E}_s and $\|h^+\|_{\dot{H}_P^s} \sim \|D_A u\|_{\mathcal{E}_s}$ with implicit constants depending on dimension, s and ellipticity. The boundary behaviour of $D_A u$ at $\lambda = 0$ is captured by h^+ and the complementary data

$$h^- := - \int_0^\infty e^{\mu PM_0} E_0^- P(M_0 - M_\mu) (D_A u)_\mu d\mu \in \dot{H}_{PM_0}^{s,-}$$

satisfies for implicit constant depending on dimension, s and ellipticity,

$$(2.45) \quad \|h^-\|_{\dot{H}_P^s} \lesssim \|A - A_0\|_s \|D_A u\|_{\mathcal{E}_s}.$$

Then, writing $h := h^+ + h^-$, there are limits

$$\lim_{\lambda \rightarrow 0} \lambda^{-1} \int_{\lambda}^{2\lambda} \|D_A u - h\|_{\dot{H}_P^s}^2 d\mu = 0 = \lim_{\lambda \rightarrow \infty} \lambda^{-1} \int_{\lambda}^{2\lambda} \|D_A u\|_{\dot{H}_P^s}^2 d\mu.$$

In the case $s = -1$, there is a direct representation of u by means of another operator [5, Thm. 9.2], showing that $u - c \in C_0([0, \infty); L^2(\mathbb{R}^{n+1}))$ for some constant c . Moreover, [5, Thm. 10.1] can be adapted as well in order to show the control of the non-tangential maximal function by the square function

$$\|\tilde{N}_*(u - c)\|_2 \lesssim \|D_A u\|_{\mathcal{E}_{-1}}$$

for arbitrary reinforced weak solutions. This uses the dual estimate to (2.24) with M^*P^* in place of PM when it comes to off-diagonal decay.

When $s = 0$, the Whitney averages of $D_A u$ converge to h almost everywhere on \mathbb{R}^{n+1} . This is best done by showing $\int_0^\lambda \|D_A u - Sh\|_2^2 \frac{d\lambda}{\lambda} < \infty$ with Sh the semigroup extension of h and using Theorem 2.12 for Sh . If $-1 < s \leq 0$, then one can adapt the proof of Lemma 13.6 below to show that Whitney averages of u (which can be chosen in $L_{\text{loc}}^2(\mathbb{R}_+^{n+2})$) converge almost everywhere on \mathbb{R}^{n+1} and the limit turns out to be an element $u_0 \in \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2} \cap L_{\text{loc}}^2(\mathbb{R}^{n+1})$. Thus, the boundary equation can be understood in $L_{\text{loc}}^2(\mathbb{R}^{n+1})$ with $\nabla_x u_0 = h_{\parallel}$ and $H_t D_t^{1/2} u_0 = h_{\theta}$.

So far, we have seen that all conormal differentials $D_A u \in \mathcal{E}_s$ of reinforced weak solutions to (1.1) have a trace $h = D_A u|_{\lambda=0}$ and that this trace maps \mathcal{E}_s boundedly into \dot{H}_P^s . If now $\|A - A_0\|_s$ is small enough, then $\|h\|_{\dot{H}_P^s} \sim \|h^+\|_{\dot{H}_P^s}$ as a consequence of (2.45). Hence,

$$\|D_A u\|_{\mathcal{E}_s} \sim \|D_A u|_{\lambda=0}\|_{\dot{H}_P^s}.$$

In this case, the trace map is an isomorphism onto its range. Let us call $\mathbf{H}_A^{s,+}$ the range of this map just for the discussion. With this notation, $\mathbf{H}_{A_0}^{s,+} = \mathbf{H}_{PM_0}^{s,+}$. The conclusion is that if $\|A - A_0\|_s$ is small enough, then the equation (2.44) can be used to construct solutions as the operator $1 - S_A$ becomes invertible on \mathcal{E}_s : given any $g \in \dot{H}_P^s$, we can construct a reinforced weak solution to (1.1) with $D_A u \in \mathcal{E}_s$ by

$$D_A u = (1 - S_A)^{-1}(C_0^+ g)$$

with $C_0^+ g$ the Cauchy extension of g using the operator PM_0 . Set $E_A^+ g = h$ to be the trace of $D_A u$. Taking limits, we find $h^+ = E_0^+ g$, $\|h^-\|_{\dot{H}_P^s} \lesssim \|A - A_0\|_s \|g\|_{\dot{H}_P^s}$ and so we obtain that

$$E_A^+ g = E_0^+ g - \int_0^\infty e^{\mu PM_0} E_0^- P(M_0 - M_\mu)(1 - S_A)^{-1}(C_0^+ g)_\mu d\mu$$

is a bounded projection of \dot{H}_P^s onto $\mathbf{H}_A^{s,+}$ along $\dot{H}_{PM_0}^{s,-}$. Moreover, there is a Lipschitz estimate

$$\|E_A^+ - E_0^+\| \lesssim \|A - A_0\|_s.$$

Turning eventually to boundary value problems, we can use h^+ to construct u from a boundary data and obtain the following theorem.

Theorem 2.41. *Let $-1 \leq s \leq 0$. If $\|A - A_0\|_s$ is small enough, then the following assertions hold.*

- (i) $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ is well-posed if and only if $N_r : \dot{H}_{PM_0}^{s,+} \rightarrow (\dot{H}_P^s)_r$ is an isomorphism.
- (ii) $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ is well-posed if and only if $N_{\perp} : \dot{H}_{PM_0}^{s,+} \rightarrow (\dot{H}_P^s)_{\perp}$ is an isomorphism.

In fact, as we have characterized the traces of conormal derivatives of solutions in \mathcal{E}_s provided that $\|A - A_0\|_s$ is small enough, well-posedness of $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ is equivalent to invertibility of $N_r : \mathbf{H}_A^{s,+} \rightarrow (\dot{H}_P^s)_r$. By the Lipschitz estimate on the difference of projectors, we may equivalently replace $\mathbf{H}_A^{s,+}$ by $\mathbf{H}_{A_0}^{s,+} = \mathbf{H}_{PM_0}^{s,+}$. The proof for $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ is the same.

This result is not formulated like this in [5] or [55] but it is an illuminating formulation. Indeed, note that the right hand conditions mean well-posedness for the λ -independent operator \mathcal{L}_0 with coefficients A_0 by Theorem 2.15. Hence, this is also a perturbation result in that well-posedness

for A_0 implies well-posedness for nearby A in the norm $\|\cdot\|_s$. We remark that compatible well-posedness is also stable in this process. (This is a little tedious but not difficult using a Neumann series arguments when computing inverses.) In the same spirit we can use that the right hand sides in Theorem 2.41 are also stable under the change (\mathcal{L}_0, s) to $(\mathcal{L}_0^*, -1-s)$ by Theorem 2.21, to obtain

Corollary 2.42. *Let $-1 \leq s \leq 0$ and assume that $\|A - A_0\|_s$ is small enough. Let \mathcal{L} and \mathcal{L}^* be as above with coefficients A and A^* , respectively. Then $(BVP)_{\mathcal{E}_s}^{\mathcal{L}}$ is (compatibly) well-posed if and only if $(BVP)_{\mathcal{E}_{-1-s}}^{\mathcal{L}^*}$ is (compatibly) well-posed, where BVP designates either N or R .*

3. HOMOGENEOUS FRACTIONAL SOBOLEV SPACES

We introduce the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R})$ and related spaces in several variables. It will be necessary to have them realised not only within the tempered distributions modulo polynomials but, equivalently and more concretely, within $L_{\text{loc}}^2(\mathbb{R})$ by means of a Riesz potential. In doing so we have to deviate from common literature [1, 23], see also [24], since $1/2$ is a critical exponent for homogeneous L^2 -Sobolev space on the real line. For the reader's convenience we shall provide all necessary details.

In the following $D_t^{1/2}$ denotes the *half-order time derivative*, defined $L^2(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$ by the Fourier multiplier $\tau \mapsto |\tau|^{1/2}$, and H_t is the *Hilbert transform* corresponding to $\tau \mapsto i \operatorname{sgn}(\tau)$. The sign convention for the Hilbert transform is in accordance with the textbook [47].

3.1. Spaces on the real line. For $v \in \mathcal{S}(\mathbb{R})$ there are well-known kernel representations

$$(3.1) \quad D_t^{1/2}v(t) = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{|t-s|^{3/2}} (v(t) - v(s)) ds,$$

and

$$(3.2) \quad H_t D_t^{1/2}v(t) = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\operatorname{sgn}(t-s)}{|t-s|^{3/2}} (v(t) - v(s)) ds,$$

valid for a.e. $t \in \mathbb{R}$, see [47, §12.1]. These formulæ can be used to prove that $D_t^{1/2}v$ and $H_t D_t^{1/2}v$ are integrable functions satisfying

$$(3.3) \quad |D_t^{1/2}v(t)| + |H_t D_t^{1/2}v(t)| \leq C \min\{1, |t|^{-3/2}\} \quad (t \in \mathbb{R}),$$

where C depends only on the size of v in the topology of $\mathcal{S}(\mathbb{R})$. In fact, for boundedness we would split the integral at height $|t-s|=1$ and treat both terms separately and in order to reveal the decay we would split at $|t-s|=|t|/2$. We will freely use these facts in the following without referring to them at each occurrence.

Definition 3.1. The space $\dot{H}^{1/2}(\mathbb{R})$ consists of all $v \in L_{\text{loc}}^2(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$ such that firstly for every $\phi \in \mathcal{S}(\mathbb{R})$ the product $v D_t^{1/2}\phi \in L^1(\mathbb{R})$ and secondly there is $g \in L^2(\mathbb{R})$ with

$$\int_{\mathbb{R}} v \cdot \overline{D_t^{1/2}\phi} dt = \int_{\mathbb{R}} g \cdot \overline{\phi} dt \quad (\phi \in \mathcal{S}(\mathbb{R})).$$

In this case we define $D_t^{1/2}v := g$ and $\|v\|_{\dot{H}^{1/2}(\mathbb{R})} := \|D_t^{1/2}v\|_{L^2(\mathbb{R})}$.

Remark 3.2. For $v \in L^2(\mathbb{R})$ we have $v \in \dot{H}^{1/2}(\mathbb{R})$ if and only if $\hat{g} := |\tau|^{1/2}\hat{v} \in L^2(\mathbb{R})$. Hence, $L^2(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) = H^{1/2}(\mathbb{R})$ is the usual inhomogeneous fractional Sobolev space.

There is an important growth estimate implicit in Definition 3.1.

Lemma 3.3. *Any $v \in \dot{H}^{1/2}(\mathbb{R})$ satisfies $\int_{\mathbb{R}} |v(t)| \frac{dt}{1+|t|^{3/2}} < \infty$ and $D_t^{1/2}v = 0$ holds if and only if v is constant.*

Proof. The estimate follows from $v \in L_{\text{loc}}^1(\mathbb{R})$ and since for any $\phi \in C_0^\infty(\mathbb{R})$ positive and non-zero at some point, (3.1) yields $|D_t^{1/2}\phi(t)| \sim |t|^{-3/2}$ for $|t|$ large. Clearly the constant functions are in $\dot{H}^{1/2}(\mathbb{R})$ and satisfy $D_t^{1/2}v = 0$. Conversely, let $v \in \dot{H}^{1/2}(\mathbb{R})$ be of that kind. We can write any $\psi \in \mathcal{S}(\mathbb{R})$ with Fourier transform supported away from 0 as $\psi = D_t^{1/2}\phi$ for some $\phi \in \mathcal{S}(\mathbb{R})$. Hence

$\langle v, \psi \rangle = 0$, showing that \widehat{v} is supported at the origin. Thus, v is a polynomial and according to our estimate it must be constant. \square

Further properties of $\dot{H}^{1/2}(\mathbb{R})$ follow from a representation in terms of the Riesz-type potential

$$I_t^{1/2}g(t) = \frac{1}{\sqrt{2\pi}} \left(\int_{(-1,1)} \frac{1}{|t-s|^{1/2}} g(s) ds + \int_{c(-1,1)} \left(\frac{1}{|t-s|^{1/2}} - \frac{1}{|s|^{1/2}} \right) g(s) ds \right),$$

where typically $g \in L^2(\mathbb{R})$. A similar modification of the classical Riesz potential was considered in [48]. A few words concerning well-definedness: the Hardy-Littlewood-Sobolev inequality [57, Sec. V.1.2] ensures that the integral over $(-1, 1)$ converges absolutely for a.e. $t \in \mathbb{R}$ and the same argument applies to the integral over ${}^c(-1, 1)$ near the singularity of the kernel. On the remaining part this kernel is controlled by $(1 + |s|)^{-3/2}$ and the Cauchy-Schwarz inequality applies. This argument reveals that $I_t^{1/2} : L^2(\mathbb{R}) \rightarrow L_{\text{loc}}^2(\mathbb{R})$ is bounded.

Proposition 3.4. *Let $g \in L^2(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$. If T denotes any of the operators $1, H_t$, or $H_t D_t^{1/2}$, then $I_t^{1/2}g \cdot D_t^{1/2}T\phi \in L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} I_t^{1/2}g \cdot \overline{D_t^{1/2}T\phi} dt = \int_{\mathbb{R}} g \cdot \overline{T\phi} dt.$$

Moreover, $I_t^{1/2}H_tg$ agrees up to a constant with the potential

$$-\frac{1}{\sqrt{2\pi}} \left(\int_{(-1,1)} \frac{\text{sgn}(t-s)}{|t-s|^{1/2}} g(s) ds + \int_{c(-1,1)} \left(\frac{\text{sgn}(t-s)}{|t-s|^{1/2}} - \frac{\text{sgn}(s)}{|s|^{1/2}} \right) g(s) ds \right).$$

Proof. For $g \in \mathcal{S}(\mathbb{R})$ a calculation, using the Riemann integral $\int_{-\infty}^{\infty} e^{is} |s|^{-1/2} ds = \sqrt{2\pi}$, reveals

$$\int_{\mathbb{R}} \frac{e^{it\tau}}{|\tau|^{1/2}} \widehat{g}(\tau) d\tau = \sqrt{2\pi} \int_{\mathbb{R}} \frac{1}{|t-s|^{1/2}} g(s) ds.$$

Hence, in this case $I_t^{1/2}g$ agrees with the following function of t up to a constant:

$$(3.4) \quad \frac{1}{2\pi} \int_{(-1,1)} \frac{e^{it\tau} - 1}{|\tau|^{1/2}} \widehat{g}(\tau) d\tau + \mathcal{F}^{-1} \left(1_{c(-1,1)}(\tau) \frac{\widehat{g}(\tau)}{|\tau|^{1/2}} \right) (t).$$

This representation of $I_t^{1/2}g$ is better suited for the extension to general $g \in L^2(\mathbb{R})$. In fact, define v_1 to be the first function in (3.4) and v_2 the second. Due to Plancherel's theorem $\|v_2\|_2 \leq \|g\|_2$ and, using $|e^{it\tau} - 1| \leq 2|t\tau|^{1/4}$ for $t, \tau \in \mathbb{R}$,

$$(3.5) \quad |v_1(t)| \leq \frac{1}{2\pi} \int_{(-1,1)} \frac{|e^{it\tau} - 1|}{|\tau|^{1/2}} |\widehat{g}(\tau)| d\tau \leq 4|t|^{1/4} \|g\|_2 \quad (t \in \mathbb{R}).$$

This proves that the maps $g \mapsto v_j$ are bounded $L^2(\mathbb{R}) \rightarrow L_{\text{loc}}^2(\mathbb{R})$, just as is $I_t^{1/2}$. Thus, it remains true for all $g \in L^2(\mathbb{R})$ that the function defined in (3.4) coincides with $I_t^{1/2}g$ up to a constant. The potential representation for $I_t^{1/2}H_tg$ follows by a similar reasoning with H_tg in place of g using the Riemann integral $\int_{-\infty}^{\infty} e^{is} |s|^{-1/2} \text{sgn}(s) ds = i\sqrt{2\pi}$.

To prove the actual claim, we let $g \in L^2(\mathbb{R})$ and keep on denoting the two summands in (3.4) by v_1 and v_2 . In the cases $T = 1$ and $T = H_t$ we recall from (3.3) the bound

$$|D_t^{1/2}T\phi(t)| \lesssim \min\{1, |t|^{-3/2}\} \quad (t \in \mathbb{R}),$$

with implicit constants depending only on the size of ϕ in $\mathcal{S}(\mathbb{R})$. In the case $T = H_t D_t^{1/2}$ we even have $D_t^{1/2}T\phi = \partial_t \phi \in \mathcal{S}(\mathbb{R})$ and so this bound holds all the more. After the preceding discussion we may directly argue on $v_1 + v_2$ instead of $I_t^{1/2}g$. Due to (3.4),

$$\int_{\mathbb{R}} |v_1(t)| |D_t^{1/2}T\phi(t)| dt \leq 4\|g\|_2 \int_{\mathbb{R}} |t|^{1/4} |D_t^{1/2}T\phi(t)| dt < \infty,$$

thereby justifying the use of Fubini's theorem when computing

$$\begin{aligned} \int_{\mathbb{R}} v_1(t) \cdot \overline{D_t^{1/2} T \phi(t)} dt &= \int_{(-1,1)} \frac{\widehat{g}(\tau)}{|\tau|^{1/2}} \cdot \overline{\int_{\mathbb{R}} D_t^{1/2} T \phi(t) e^{-it\tau} dt} d\tau \\ &= 2\pi \int_{(-1,1)} \widehat{g}(\tau) \cdot \overline{\widehat{T\phi}(\tau)} d\tau, \end{aligned}$$

where in the first step we have used that $D_t^{1/2} T \phi$ has integral zero since its Fourier transform vanishes at the origin. Concerning v_2 , Plancherel's theorem yields

$$\int_{\mathbb{R}} |v_2(t)| |D_t^{1/2} T \phi(t)| dt \leq \|g\|_2 \|D_t^{1/2} T \phi\|_2 < \infty$$

and

$$\int_{\mathbb{R}} v_2(t) \cdot \overline{D_t^{1/2} T \phi(t)} dt = 2\pi \int_{c(-1,1)} \widehat{g}(\tau) \cdot \overline{\widehat{T\phi}(\tau)} d\tau.$$

Adding up the corresponding estimates and identities together with a final application of Plancherel's theorem leads us to the claim. \square

The first consequence of Proposition 3.4 is a Riesz potential representation for $\dot{H}^{1/2}(\mathbb{R})$. It also allows to improve on Lemma 3.3 obtaining a continuous embedding, which is apparently new.

Corollary 3.5. *If $g \in L^2(\mathbb{R})$, then $I_t^{1/2} g \in \dot{H}^{1/2}(\mathbb{R})$ and $D_t^{1/2} I_t^{1/2} g = g$. In particular, for any $v \in \dot{H}^{1/2}(\mathbb{R})$ there exist $c, d \in \mathbb{C}$ such that $v = c + I_t^{1/2} D_t^{1/2} v$ and*

$$\int_{\mathbb{R}} |v(t) - d| \frac{dt}{1 + |t|^{3/2}} \leq 42 \|v\|_{\dot{H}^{1/2}(\mathbb{R})}.$$

Proof. The first statement follows from Proposition 3.4 applied with $T = 1$. This implies the representation of v by taking $g = D_t^{1/2} v$ and then using Lemma 3.3. From the proof of Proposition 3.4 we know that (3.4) with $g = D_t^{1/2} v$ reproduces v up to a constant $d \in \mathbb{C}$ and therefore the final estimate follows from the bounds for v_1 and v_2 obtained therein, see (3.5) and the lines above. \square

Next, Proposition 3.4 with $T = H_t$ and $T = H_t D_t^{1/2}$ yields the fractional integration by parts formulæ that justify to speak of reinforced weak solutions in the first place.

Corollary 3.6. *Let $v \in \dot{H}^{1/2}(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$. Then $v H_t D_t^{1/2} \phi \in L^1(\mathbb{R})$ and there are fractional integration by parts formulæ*

$$\int_{\mathbb{R}} v \cdot \overline{H_t D_t^{1/2} \phi} dt = - \int_{\mathbb{R}} H_t D_t^{1/2} v \cdot \overline{\phi} dt$$

and

$$\int_{\mathbb{R}} v \cdot \overline{\partial_t \phi} dt = - \int_{\mathbb{R}} H_t D_t^{1/2} v \cdot \overline{D_t^{1/2} \phi} dt.$$

The kernel representations (3.1) and (3.2) remain valid for $v \in \dot{H}^{1/2}(\mathbb{R})$ in a certain sense.

Corollary 3.7. *If $v \in \dot{H}^{1/2}(\mathbb{R})$, then for almost every $t \in {}^c(\text{supp } v)$,*

$$D_t^{1/2} v(t) = -\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \frac{v(s)}{|t-s|^{3/2}} ds \quad H_t D_t^{1/2} v(t) = -\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\text{sgn}(t-s)v(s)}{|t-s|^{3/2}} ds.$$

Proof. Let $\phi \in C_0^\infty(\mathbb{R})$ be supported in ${}^c(\text{supp } v)$. For the first formula we start out with

$$\int_{\mathbb{R}} D_t^{1/2} v(t) \cdot \overline{\phi(t)} dt = \int_{\mathbb{R}} v(t) \cdot \overline{D_t^{1/2} \phi(t)} dt,$$

represent $D_t^{1/2} \phi$ via (3.1), and apply Fubini's theorem (justified by Lemma 3.3), to obtain

$$\int_{\mathbb{R}} D_t^{1/2} v(t) \cdot \overline{\phi(t)} dt = \int_{\mathbb{R}} \left(\frac{-1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \frac{v(s)}{|t-s|^{3/2}} ds \right) \overline{\phi(t)} dt.$$

For the second formula we start out with the first identity in Corollary 3.6 instead. \square

We stress that our, seemingly new, definition of $\dot{H}^{1/2}(\mathbb{R})$ is in accordance with the relevant literature: It realises the closure of $C_0^\infty(\mathbb{R})$ for the homogeneous norm $\|D_t^{1/2} \cdot\|_{L^2(\mathbb{R})}$.

Corollary 3.8. *The space $\dot{H}^{1/2}(\mathbb{R})/\mathbb{C}$ is a Hilbert space for the norm $\|\cdot\|_{\dot{H}^{1/2}(\mathbb{R})}$ and it contains $C_0^\infty(\mathbb{R})$ as a dense subspace.*

Proof. In view of Lemma 3.3, $\|\cdot\|_{\dot{H}^{1/2}(\mathbb{R})}$ is a Hilbertian norm on $\dot{H}^{1/2}(\mathbb{R})/\mathbb{C}$. Due to Corollary 3.5 we have an isomorphism

$$I_t^{1/2} : L^2(\mathbb{R}) \rightarrow \dot{H}^{1/2}(\mathbb{R})/\mathbb{C}.$$

Thus, $\dot{H}^{1/2}(\mathbb{R})/\mathbb{C}$ is a Hilbert space. L^2 -functions with Fourier transform supported away from 0 are dense in $L^2(\mathbb{R})$, hence their image under $I_t^{1/2}$ is dense in $\dot{H}^{1/2}(\mathbb{R})/\mathbb{C}$. For such g , however, (3.4) defines the $L^2(\mathbb{R})$ -function $\mathcal{F}^{-1}(|\tau|^{-1/2}\widehat{g}(\tau))$ up to a constant. This proves that the inhomogeneous space $\dot{H}^{1/2}(\mathbb{R}) = L^2(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})$ is dense in $\dot{H}^{1/2}(\mathbb{R})/\mathbb{C}$. The density of $C_0^\infty(\mathbb{R})$ in $\dot{H}^{1/2}(\mathbb{R})$ is classical and can be proved by truncation and convolution with smooth kernels. \square

3.2. Spaces of several variables. The notions introduced so far can be extended straightforwardly to functions of several variables.

Definition 3.9. The space $\dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}_+^{n+1}))$ consists of all $v \in L_{\text{loc}}^2(\mathbb{R}_+^{n+2})$ such that $v(\lambda, x, \cdot) \in \dot{H}^{1/2}(\mathbb{R})$ for almost every $(\lambda, x) \in \mathbb{R}_+^{n+1}$ and such that

$$\iint_K \int_{\mathbb{R}} |D_t^{1/2} v(\lambda, x, t)|^2 dt dx d\lambda < \infty$$

for every compact set $K \subset \mathbb{R}_+^{n+1}$. Similarly, $\dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}^n))$ is defined as a subspace of $L_{\text{loc}}^2(\mathbb{R}^{n+1})$.

There is an analogue of the embedding stated in Corollary 3.5 for the spaces in several variables.

Lemma 3.10. *Every $v \in \dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}_+^{n+1}))$ satisfies $vD_t^{1/2}\phi$, $vH_tD_t^{1/2}\phi \in L^1(\mathbb{R}_+^{n+2})$ if $\phi \in \mathcal{S}(\mathbb{R}_+^{n+2})$ is compactly supported in λ and x . A similar statement holds for $\dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}^n))$.*

Proof. Corollary 3.5 and the growth estimate (3.3) yield the global integrability conditions with $v-d$ in place of v , where $d \in L_{\text{loc}}^1(\mathbb{R}_+^{n+2})$ does not depend on t . However, $dD_t^{1/2}\phi$ and $dH_tD_t^{1/2}\phi$ are in $L^1(\mathbb{R}_+^{n+2})$ by the support assumption on ϕ . \square

In view of Definition 3.9 the ground space \dot{E}_{loc} for reinforced weak solutions,

$$\dot{E}_{\text{loc}}(\mathbb{R}_+^{n+2}) = \dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}_+^{n+1})) \cap L_{\text{loc}}^2(\mathbb{R}; W_{\text{loc}}^{1,2}(\mathbb{R}_+^{n+1})),$$

is unambiguously defined as a subspace of $L_{\text{loc}}^2(\mathbb{R}_+^{n+2})$ and within there is the energy space $\dot{E}(\mathbb{R}_+^{n+2})$ of those $v \in \dot{E}_{\text{loc}}(\mathbb{R}_+^{n+2})$ for which

$$\|v\|_{\dot{E}(\mathbb{R}_+^{n+2})} := \left(\|D_t^{1/2} v\|_{L^2(\mathbb{R}_+^{n+2})}^2 + \|\nabla_{\lambda, x} v\|_{L^2(\mathbb{R}_+^{n+2})}^2 \right)^{1/2} < \infty.$$

Note that in Section 2 we simply wrote \dot{E}_{loc} and \dot{E} . We stress the dependence of the domain since for the time being we shall concentrate on the analogous spaces of functions in the variables x and t only. Similar to the situation on the real line, there is a corresponding parabolic Riesz potential given by

$$I_{t,x}^{1/2} g(x, t) = \iint_{|\xi| \vee |\tau| < 1} \frac{e^{ix \cdot \xi} e^{it\tau} \widehat{g}(\xi, \tau)}{|\xi| + |\tau|^{1/2}} d\xi d\tau + \mathcal{F}^{-1} \left(\frac{1_{\{|\xi| \vee |\tau| \geq 1\}} \widehat{g}(\xi, \tau)}{|\xi| + |\tau|^{1/2}} \right).$$

Here, $g \in L^2(\mathbb{R}^{n+1})$, ξ, τ are the Fourier variables corresponding to x, t and $a \vee b = \max(a, b)$.

Lemma 3.11. *Let $g \in L^2(\mathbb{R}^{n+1})$ and let $v = I_{t,x}^{1/2} g$. Then $v \in \dot{E}(\mathbb{R}^{n+1})$ with derivatives*

$$\begin{bmatrix} \nabla_x v \\ D_t^{1/2} v \end{bmatrix} = \mathcal{F}^{-1} \left(\frac{1}{|\xi| + |\tau|^{1/2}} \begin{bmatrix} i\xi \\ |\tau|^{1/2} \end{bmatrix} \widehat{g} \right)$$

and estimates $\|v\|_{L^2+L^\infty} \lesssim \|g\|_{L^2}$ and $\|v\|_{\dot{E}(\mathbb{R}^{n+1})} \sim \|g\|_{L^2}$. Conversely, each $v \in \dot{E}(\mathbb{R}^{n+1})$ can be represented as $v = c + I_{t,x}^{1/2} g$ for unique $g \in L^2(\mathbb{R}^{n+1})$ and $c \in \mathbb{C}$.

Proof. Let us denote the two summands in the definition of $v = I_{t,x}^{1/2} g$ by v_1 and v_2 . Thanks to Plancherel's theorem we have $\|v_2\|_2 \leq \|g\|_2$ and

$$|v_1(t)| \leq \iint_{|\xi| \vee |\tau| < 1} \frac{|\widehat{g}(\xi, \tau)|}{|\xi| + |\tau|^{1/2}} d\xi d\tau \leq \left(\iint_{|\xi| \vee |\tau| < 1} \frac{1}{|\xi|^2 + |\tau|} d\xi d\tau \right) \|g\|_2 \quad (t \in \mathbb{R}),$$

showing $v_1 \in L^\infty(\mathbb{R}^{n+1})$. This already proves the first of the two estimates for v . For any $\phi \in \mathcal{S}(\mathbb{R}^{n+1})$ the functions $v_1 D_t^{1/2} \phi$ and $v_2 D_t^{1/2} \phi$ are in $L^1(\mathbb{R}^{n+1})$ thanks to (3.3). In turn, this allows us to use the theorems of Fubini and Plancherel just as in the proof of Proposition 3.4 to conclude

$$\iint_{\mathbb{R}^{n+1}} v \cdot \overline{\left[\begin{array}{c} \nabla_x \phi \\ D_t^{1/2} \phi \end{array} \right]} d\xi d\tau = \iint_{\mathbb{R}^{n+1}} \mathcal{F}^{-1} \left(\frac{1}{|\xi| + |\tau|^{1/2}} \left[\begin{array}{c} i\xi \\ |\tau|^{1/2} \end{array} \right] \widehat{g} \right) \cdot \overline{\phi(\xi, \tau)} d\xi d\tau.$$

Note that the Fourier multiplication operator appearing on the right-hand side is bounded on $L^2(\mathbb{R}^{n+1})$. Taking ϕ in tensor form $\phi(x, t) = \phi_x(x) \phi_t(t)$ reveals $v \in \dot{E}_{\text{loc}}(\mathbb{R}^{n+1})$ and

$$(3.6) \quad \left[\begin{array}{c} \nabla_x v \\ D_t^{1/2} v \end{array} \right] = \mathcal{F}^{-1} \left(\frac{1}{|\xi| + |\tau|^{1/2}} \left[\begin{array}{c} i\xi \\ |\tau|^{1/2} \end{array} \right] \widehat{g} \right) \in L^2(\mathbb{R}^{n+1}),$$

that is, $v \in \dot{E}(\mathbb{R}^{n+1})$. Finally, the estimate $\|v\|_{\dot{E}} \leq \|g\|_2$ is immediate from Plancherel's theorem and to complete the first part of the lemma we record the reverse estimate

$$\|g\|_2 \leq \left\| \frac{|\xi|^{1/2}}{|\xi| + |\tau|^{1/2}} \widehat{g} \right\|_2 + \left\| \frac{|\tau|^{1/2}}{|\xi| + |\tau|^{1/2}} \widehat{g} \right\|_2 = \|\nabla_x v\|_2 + \|D_t^{1/2} v\|_2 = \|v\|_{\dot{E}}.$$

We turn to the second part where now $v \in \dot{E}(\mathbb{R}_+^{n+2})$ is given. Let $\phi \in C_0^\infty(\mathbb{R}^{n+1}; \mathbb{C}^n)$. It follows from Remark 3.10 that $v D_t^{1/2} \text{div}_x \phi \in L^1(\mathbb{R}^{n+1})$ and this justifies the use of Fubini's theorem when computing

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \nabla_x v \cdot \overline{D_t^{1/2} \phi} dx dt &= - \int_{\mathbb{R}} \int_{\mathbb{R}^n} v \cdot \overline{D_t^{1/2} \text{div}_x \phi} dx dt \\ &= - \int_{\mathbb{R}^n} \int_{\mathbb{R}} v \cdot \overline{D_t^{1/2} \text{div}_x \phi} dt dx = - \int_{\mathbb{R}^n} \int_{\mathbb{R}} D_t^{1/2} v \cdot \overline{\text{div}_x \phi} dt dx. \end{aligned}$$

With $h_{\parallel} := \nabla_x v$ and $h_{\theta} := D_t^{1/2} v$, Plancherel's theorem lets discover $|\tau|^{1/2} \widehat{h_{\parallel}} = i \widehat{h_{\theta}} \xi$. So,

$$g := \mathcal{F}^{-1} \left(\frac{|\xi| + |\tau|^{1/2}}{|\tau|^{1/2}} \widehat{h_{\theta}}(\xi, \tau) \right) \in L^2(\mathbb{R}^{n+1})$$

solves (3.6) and from the first part we deduce $\nabla_x v = \nabla_x I_{t,x}^{1/2} g$ and $D_t^{1/2} v = D_t^{1/2} I_{t,x}^{1/2} g$. Taking into account Lemma 3.3, this implies that $c = v - I_{t,x}^{1/2} g$ is constant. Finally, g is unique due to the estimate in the first part of the lemma. \square

As a consequence we obtain that our definition of the energy spaces \dot{E} gives a realisation of the closure of test functions with respect to their homogeneous norm. In particular, they are the same spaces that have been introduced by the third author and collaborators in this context, see [27, 53, 54].

Corollary 3.12. *The space $\dot{E}(\mathbb{R}^{n+1})/\mathbb{C}$ is a Hilbert space for the norm $\|\cdot\|_{\dot{E}(\mathbb{R}^{n+1})}$ and contains $C_0^\infty(\mathbb{R}^{n+1})$ as a dense subset. The analogous statement holds for $\dot{E}(\mathbb{R}_+^{n+2})/\mathbb{C}$ using $C_0^\infty(\overline{\mathbb{R}_+^{n+2}})$ as dense subset.*

Proof. The statement for $\dot{E}(\mathbb{R}^{n+1})/\mathbb{C}$ can be obtained by a literal repetition of the proof of Corollary 3.8, replacing I_t by $I_{t,x}$ and the multiplier $|\tau|^{-1/2}$ by $(|\xi| + |\tau|)^{-1/2}$ therein. Of course, the same result holds in dimension $n+2$. Since even reflection around $\lambda=0$ allows to extend any function in $\dot{E}(\mathbb{R}_+^{n+2})$ to a function in $\dot{E}(\mathbb{R}^{n+2})$, we readily obtain the second claim. \square

Remark 3.13. (i) Corollary 3.12 can be rephrased as saying that $\dot{E}(\mathbb{R}^{n+1})$ is a realisation of the parabolic Sobolev space $\dot{H}_{\partial_t - \Delta_x}^{1/2}$ within the tempered distributions modulo constants. The same kind of argument can be used to obtain such realisations for $\dot{H}_{\partial_t - \Delta_x}^s$, $0 < s < 1/2$.
(ii) Sobolev embeddings in parabolic space show that $\dot{E}(\mathbb{R}^{n+1})/\mathbb{C}$ has a realisation in the Lebesgue space $L^q(\mathbb{R}^{n+1})$, where $q = \frac{2(n+2)}{n}$. We shall not use this fact.

Finally, we supply a proof of the trace theorem for $\dot{E}(\mathbb{R}_+^{n+2})$ that played an important role for the study of energy solutions in Section 2.3. The reader may recall the definition of the parabolic Sobolev space $\dot{H}_{\partial_t - \Delta_x}^{1/4}$ from Section 2.7.

Lemma 3.14. *The space $\dot{E}(\mathbb{R}_+^{n+2})/\mathbb{C}$ continuously embeds into $C([0, \infty); \dot{H}_{\partial_t - \Delta_x}^{1/4})$. Conversely, any $f \in \dot{H}_{\partial_t - \Delta_x}^{1/4}$ has an extension $v \in \dot{E}(\mathbb{R}_+^{n+2})$ such that $v|_{\lambda=0} = f$.*

Proof. Let $v \in C_0^\infty(\overline{\mathbb{R}_+^{n+2}})$. For any $\lambda_0 > 0$ the fundamental theorem of calculus yields

$$\begin{aligned} \|\tau|^{1/4}\hat{v}|_{\lambda=\lambda_0}\|_2^2 + \|\xi|^{1/2}\hat{v}|_{\lambda=\lambda_0}\|_2^2 &= 2 \operatorname{Re} \int_{\lambda_0}^{\infty} (|\tau|^{1/2}\hat{v}, \partial_\lambda \hat{v}) + (|\xi|\hat{v}, \partial_\lambda \hat{v}) \, d\lambda \\ &\leq \int_0^{\infty} \|\tau|^{1/2}\hat{v}\|_2^2 + \|\xi|\hat{v}\|_2^2 + 2\|\partial_\lambda \hat{v}\|_2^2 \, d\lambda, \end{aligned}$$

where norms, scalar products and the Fourier transform are in $L^2(\mathbb{R}^{n+1})$. By Plancherel's theorem the left-hand side compares to the norm of $v|_{\lambda=\lambda_0}$ in $\dot{H}_{\partial_t - \Delta_x}^{1/4}$ and the right-hand side to the norm of v in $\dot{E}(\mathbb{R}^{n+2})$. This proves uniform continuity of $\lambda_0 \mapsto v|_{\lambda=\lambda_0}$ into $\dot{H}_{\partial_t - \Delta_x}^{1/4}$ and thus also the limit at 0. Due to Corollary 3.12 we obtain the required embedding by density. Conversely, given $f \in C_0^\infty(\mathbb{R}^{n+1})$, we can define an extension to \mathbb{R}_+^{n+2} by

$$v(\lambda, x, t) = \mathcal{F}^{-1}(e^{-\lambda(|\xi|^2 + i\tau)^{1/2}} \hat{f})(x, t).$$

From Plancherel's theorem we can infer $\|v\|_{\dot{E}(\mathbb{R}^{n+2})} \lesssim \|f\|_{\dot{H}_{\partial_t - \Delta_x}^{1/4}}$ and so this extension operator extends to $\dot{H}_{\partial_t - \Delta_x}^{1/4}$ by density. \square

4. THE PARABOLIC DIRAC OPERATOR

In this section we develop the basic operator theoretic properties of the parabolic Dirac operator

$$P = \begin{bmatrix} 0 & \operatorname{div}_x & -D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{1/2} & 0 & 0 \end{bmatrix}$$

with maximal domain

$$D(P) = \left\{ f \in \mathbb{L}^2 : \nabla_x f_\perp \in \mathbb{L}^2, H_t D_t^{1/2} f_\perp \in \mathbb{L}^2, \operatorname{div}_x f_\parallel - D_t^{1/2} f_\theta \in \mathbb{L}^2 \right\}$$

in $\mathbb{L}^2 = L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$. The reader may recall relevant notation from Sections 2.4 and 2.5. Clearly P is closed and its null space is

$$N(P) = \left\{ f \in \mathbb{L}^2 : f_\perp = 0, \operatorname{div}_x f_\parallel = D_t^{1/2} f_\theta \text{ in } \mathcal{S}'(\mathbb{R}^{n+1}) \right\}.$$

Here is an explicit description of $\overline{R(P)}$ as a space of distributions.

Lemma 4.1. *It holds*

$$\overline{R(P)} = \left\{ f \in \mathbb{L}^2 : \nabla_x f_\theta = H_t D_t^{1/2} f_\parallel \text{ in } \mathcal{S}'(\mathbb{R}^{n+1}; \mathbb{C}^n) \right\}.$$

Proof. The inclusion ‘ \subset ’ follows from the definition of P . Conversely assume that f is contained in the right-hand space. Let $g \in \mathbf{N}(P^*)$, that is, $g_\perp = 0$ and $\operatorname{div}_x g_\parallel = -H_t D_t^{1/2} g_\theta$. On the Fourier side the relation for \parallel - and θ -components can be solved by

$$\widehat{f}_\parallel = \frac{\widehat{f}_\theta}{\operatorname{sgn}(\tau)|\tau|^{1/2}} \xi, \quad \widehat{g}_\theta = -\xi \cdot \frac{\widehat{g}_\parallel}{\operatorname{sgn}(\tau)|\tau|^{1/2}} \quad (\text{a.e. on } \mathbb{R}^{n+1}),$$

where τ and ξ are the Fourier variables corresponding to t and x . Hence, $(\widehat{f}, \widehat{g}) = 0$ and thus $(f, g) = 0$ by Plancherel’s theorem. This proves that $f \in \mathbf{N}(P^*)^\perp$ and hence $f \in \overline{\mathbf{R}(P)}$. \square

Bisectoriality of PM and MP can be obtained by implementing the ‘hidden coercivity’ of the parabolic operator \mathcal{L} discussed in Section 2.3 within the first order framework.

Lemma 4.2. *The operators PM and MP are closed and bisectorial on \mathbb{L}^2 with $\mathbf{R}(PM) = \mathbf{R}(P)$ and $\overline{\mathbf{R}(MP)} = \overline{MR(P)}$.*

Proof. Closedness will follow from bisectoriality since the latter implies that the operator’s resolvent set is non-empty. Consider for $\delta \in \mathbb{R}$ the transformation

$$U_\delta = \frac{1}{1 + \delta^2} \begin{bmatrix} 1 - \delta H_t & 0 & 0 \\ 0 & 1 + \delta H_t & 0 \\ 0 & 0 & \delta - H_t \end{bmatrix}.$$

Then, as $H_t^2 = -1$,

$$PU_\delta = \frac{1}{1 + \delta^2} \begin{bmatrix} 0 & \operatorname{div}_x(1 + \delta H_t) & -\delta D_t^{1/2} + H_t D_t^{1/2} \\ -\nabla_x(1 - \delta H_t) & 0 & 0 \\ -H_t D_t^{1/2} - \delta D_t^{1/2} & 0 & 0 \end{bmatrix}.$$

Using that $H_t^* = -H_t$ and that H_t commutes with derivatives in x , we see that PU_δ is self-adjoint. Next, we claim that $U_\delta^{-1}M$ is accretive for $\delta > 0$ small enough. Indeed, on recalling the definition of M from (2.7), we can write $U_\delta^{-1}M$ as

$$\begin{bmatrix} 1 + \delta H_t & 0 & 0 \\ 0 & 1 - \delta H_t & 0 \\ 0 & 0 & \delta + H_t \end{bmatrix} \begin{bmatrix} \hat{A}_{\perp\perp} & \hat{A}_{\perp\parallel} & 0 \\ \hat{A}_{\parallel\perp} & \hat{A}_{\parallel\parallel} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (1 + \delta H_t)\hat{A}_{\perp\perp} & (1 + \delta H_t)\hat{A}_{\perp\parallel} & 0 \\ (1 - \delta H_t)\hat{A}_{\parallel\perp} & (1 - \delta H_t)\hat{A}_{\parallel\parallel} & 0 \\ 0 & 0 & \delta + H_t \end{bmatrix}.$$

Since $\operatorname{Re}(H_t g, g) = 0$, we see that the lower block of the matrix on the right is accretive for all $\delta > 0$. Since the upper block of the product rewrites as \hat{A} plus a bounded perturbation of size δ , it remains accretive if δ is small enough.

Based on the above we can conclude that $PM = (PU_\delta)(U_\delta^{-1}M)$ is a product of a self-adjoint and a bounded accretive operator. Such operators have exhaustively been studied in the literature and it is well-known that they are bisectorial and satisfy

$$\mathbf{R}(PM) = \mathbf{R}((PU_\delta)(U_\delta^{-1}M)) = \mathbf{R}(PU_\delta) = \mathbf{R}(P),$$

see for instance [9] or [35, Prop. 6.1.17]. Next, $MP = M(PM)M^{-1}$ is bisectorial by similarity and $\overline{\mathbf{R}(MP)} = \overline{MR(P)}$ holds by accretivity of M . \square

Remark 4.3. For parabolic systems M is only supposed to be accretive on the range of P , see Section 14. This property is also satisfied by $U_\delta^{-1}M$ if δ is small. In that case, M^{-1} should be interpreted as the inverse of the restriction $M|_{\overline{\mathbf{R}(P)}}$, so that MP and PM are similar on the closures of their ranges and they are null on their null spaces. None of our proofs in the following sections will use the invertibility of M in $L^\infty(\mathbb{R}_+^{n+2}; \mathcal{L}(\mathbb{C}^{n+2}))$.

5. THE CORRESPONDENCE TO A FIRST ORDER SYSTEM: PROOF OF THEOREM 2.2

In this section we prove the correspondence between the parabolic equation (1.1) and the first order system $\partial_\lambda F + PMF = 0$. Whenever convenient, the reader may look up the definitions of P and M in (2.7). We always consider P as a closed operator in \mathbb{L}^2 with maximal domain $D(P)$ and we write (\cdot, \cdot) for the inner product on this space. Let us stress that in the following we do not use the bisectoriality of PM and that M may depend on λ .

Definition 5.1. A weak solution F to the first order system $\partial_\lambda F + PMF = 0$ is a function $F \in L^2_{\text{loc}}(\mathbb{R}_+; \overline{R(P)})$ such that

$$(5.1) \quad \int_0^\infty (F(\lambda), \partial_\lambda \phi(\lambda)) d\lambda = \int_0^\infty (MF(\lambda), P^* \phi(\lambda)) d\lambda \quad (\phi \in C_0^\infty(\mathbb{R}_+^{n+2}; \mathbb{C}^{n+2})).$$

This notion of weak solutions is in accordance with Section 2 since Lemma 4.1 asserts that $L^2_{\text{loc}}(\mathbb{R}_+; \overline{R(P)})$ is exactly the space \mathcal{H}_{loc} introduced beforehand in Section 2.4.

5.1. Conormal differentials are weak solutions. Let u be a reinforced weak solution to (1.1) with the additional property

$$D_A u = \begin{bmatrix} \partial_{\nu_A} u \\ \nabla_x u \\ H_t D_t^{1/2} u \end{bmatrix} \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{L}^2),$$

where $D_t^{1/2}$ is in the sense of Definition 3.9. We put $F := D_A u$ and need to demonstrate that F is a weak solution to the first order system.

To verify the compatibility condition $F \in L^2_{\text{loc}}(\mathbb{R}_+; \overline{R(P)})$, let $\phi \in C_0^\infty(\mathbb{R}^{n+1}; \mathbb{C}^n)$ and note that in view of Lemma 3.10 we have $u(\lambda) \cdot H_t D_t^{1/2} \text{div}_x \phi \in L^1(\mathbb{R}^{n+1})$ for a.e. $\lambda > 0$. Hence, we can compute by means of Fubini's theorem

$$\begin{aligned} (F(\lambda)_\theta, \text{div}_x \phi) &= - \int_{\mathbb{R}^n} \int_{\mathbb{R}} u(\lambda) \cdot \overline{H_t D_t^{1/2} \text{div}_x \phi} dt dx \\ &= - \int_{\mathbb{R}} \int_{\mathbb{R}^n} u(\lambda) \cdot \overline{\text{div}_x H_t D_t^{1/2} \phi} dx dt = (F(\lambda)_\parallel, H_t D_t^{1/2} \phi). \end{aligned}$$

This extends to all $\phi \in \mathcal{S}(\mathbb{R}^{n+1})$ by approximation, showing $\nabla_x F(\lambda)_\theta = H_t D_t^{1/2} F(\lambda)_\parallel$ in the sense of $\mathcal{S}'(\mathbb{R}^{n+1})$. Thus, $F(\lambda) \in \overline{R(P)}$ thanks to Lemma 4.1.

In order to verify (5.1), let $\phi \in C_0^\infty(\mathbb{R}_+^{n+2}; \mathbb{C}^{n+2})$. Taking ϕ_\perp as a test function in the definition of reinforced weak solutions to the parabolic problem, we find

$$\begin{aligned} \int_0^\infty (F_\perp, \partial_\lambda \phi_\perp) d\lambda &= \iiint_{\mathbb{R}_+^{n+2}} (A \nabla_{\lambda, x} u)_\perp \cdot \overline{\partial_\lambda \phi_\perp} dx dt d\lambda \\ &= \iiint_{\mathbb{R}_+^{n+2}} -(A \nabla_{\lambda, x} u)_\parallel \cdot \overline{\nabla_x \phi_\perp} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} \phi_\perp} dx dt d\lambda, \end{aligned}$$

which in the language of P and M reads

$$\int_0^\infty (F_\perp, \partial_\lambda \phi_\perp) d\lambda = \int_0^\infty ((MF)_\parallel, (P^* \phi)_\parallel) + ((MF)_\theta, (P^* \phi)_\theta) d\lambda.$$

For the \parallel -components we deduce, using integration by parts and the definitions of P and M ,

$$\begin{aligned} \int_0^\infty (F_\parallel, \partial_\lambda \phi_\parallel) d\lambda &= \iiint_{\mathbb{R}_+^{n+2}} \nabla_x u \cdot \overline{\partial_\lambda \phi_\parallel} d\lambda = \iiint_{\mathbb{R}_+^{n+2}} \partial_\lambda u \cdot \overline{\text{div}_x \phi_\parallel} d\lambda \\ &= \int_0^\infty ((MF)_\perp, (P^* \phi)_\perp) - (\partial_\lambda u, H_t D_t^{1/2} \phi_\theta) d\lambda. \end{aligned}$$

As for the θ -component we first note that $u \cdot H_t D_t^{1/2}(\partial_\lambda \phi_\theta) \in L^1(\mathbb{R}_+^{n+2})$ holds since u is a reinforced weak solution, see Lemma 3.10. Thus, by definition of $D_t^{1/2}$ and Fubini's theorem

$$\begin{aligned} \int_0^\infty (F_\theta, \partial_\lambda \phi_\theta) \, d\lambda &= - \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}} u \cdot \overline{H_t D_t^{1/2} \partial_\lambda \phi_\theta} \, dt \, dx \, d\lambda \\ &= - \int_{\mathbb{R}^n} \int_{\mathbb{R}} \int_0^\infty u \cdot \overline{\partial_\lambda H_t D_t^{1/2} \phi_\theta} \, d\lambda \, dt \, dx = \int_0^\infty (\partial_\lambda u, H_t D_t^{1/2} \phi_\theta) \, d\lambda, \end{aligned}$$

where we used $H_t D_t^{1/2} \phi(\cdot, x, t) \in C_0^\infty(\mathbb{R}_+)$ for $(x, t) \in \mathbb{R}^{n+1}$ fixed in the last step. Adding up the previous three identities gives (5.1).

5.2. Every weak solution is a conormal differential. Conversely, we are now given a weak solution $F \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}(P))$ to the first order system and we have to construct a potential u such that u is a reinforced weak solution to the parabolic equation in (1.1) and $D_A u = F$. Such u is necessarily unique up to a constant since if $D_A u = 0$, then $\nabla_{\lambda, x} u = 0$ showing that u only depends on t and then $H_t D_t^{1/2} u = 0$ implies that u is constant, see Lemma 3.3. The construction of a specific u is in two steps.

Step 1: Adjusting the (x, t) -direction. By definition of F we have $F(\lambda) \in \overline{\mathbb{R}(P)}$ for almost every $\lambda > 0$. In this case $\nabla_x F(\lambda)_\theta = H_t D_t^{1/2} F(\lambda)_\parallel$ in the sense of tempered distributions due to Lemma 4.1 or, equivalently, $i\xi \widehat{F(\lambda)_\theta} = i \operatorname{sgn}(\tau) |\tau|^{1/2} \widehat{F(\lambda)_\parallel}$ by taking the Fourier transform in (x, t) . As usual, (ξ, τ) is Fourier variable corresponding to (x, t) . With these compatibility conditions at hand, we get for almost every $\lambda > 0$ that the measurable function

$$g(\lambda) := \frac{|\xi| + |\tau|^{1/2}}{i \operatorname{sgn}(\tau) |\tau|^{1/2}} \widehat{F(\lambda)_\theta}$$

belongs to $L^2(\mathbb{R}^{n+1})$ as $|g(\lambda)| \leq |\widehat{F(\lambda)_\theta}| + |\widehat{F(\lambda)_\parallel}|$ almost everywhere and the so-defined function g is in $L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}))$. Now, Lemma 3.11 furnishes a parabolic potential $v(\lambda, x, t) := I_{t,x}^{1/2} g(\lambda, x, t) \in L_{\text{loc}}^2(\mathbb{R}_+^{n+2})$ that satisfies $v(\lambda) \in \dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}^n)) \cap L_{\text{loc}}^2(\mathbb{R}; W_{\text{loc}}^{1,2}(\mathbb{R}^n))$ for almost every $\lambda > 0$ with derivatives

$$\begin{bmatrix} \nabla_x v(\lambda) \\ D_t^{1/2} v(\lambda) \end{bmatrix} = \mathcal{F}^{-1} \left(\frac{1}{|\xi| + |\tau|^{1/2}} \begin{bmatrix} i\xi \\ |\tau|^{1/2} \end{bmatrix} g(\lambda) \right).$$

Taking into account the relation between $\widehat{F(\lambda)_\parallel}$ and $\widehat{F(\lambda)_\theta}$, we obtain

$$(5.2) \quad \begin{bmatrix} \nabla_x v \\ H_t D_t^{1/2} v \end{bmatrix} = \begin{bmatrix} F(\lambda)_\parallel \\ F(\lambda)_\theta \end{bmatrix},$$

so that v already behaves as desired when it comes to derivatives in x - and t -direction. We may also assume that there exists some bounded set $Q \times I \subseteq \mathbb{R}^{n+1}$ with positive Lebesgue measure such that

$$(5.3) \quad \iint_{Q \times I} v(\lambda, x, t) \, dx \, dt = 0 \quad (\text{a.e. } \lambda > 0)$$

since otherwise we could subtract this very average from $v(\lambda)$ for each $\lambda > 0$ without affecting any of the other properties of v discussed above.

To compute the transversal derivative $\partial_\lambda v$, we let $\varphi \in C_0^\infty(\mathbb{R}_+)$, $\psi \in C_0^\infty(\mathbb{R}^n; \mathbb{C}^n)$, $\eta \in C_0^\infty(\mathbb{R})$ and test (5.1) with

$$\phi(\lambda, x, t) = \begin{bmatrix} 0 \\ \varphi(\lambda) \psi(x) \eta(t) \\ 0 \end{bmatrix}.$$

Spelling out the identity

$$\int_0^\infty (F, \partial_\lambda \phi) \, d\lambda = \int_0^\infty (MF, P^* \phi) \, d\lambda$$

and taking into account that for the terms involving φ only the \parallel -component of ϕ and the \perp -component of $P^*\phi$ do not vanish, we readily find

$$-\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(\int_0^\infty v \cdot \overline{\partial_\lambda \varphi} \, d\lambda \right) \overline{\eta \operatorname{div}_x \psi} \, dx \, dt = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(\int_0^\infty (MF)_\perp \cdot \overline{\varphi} \, d\lambda \right) \overline{\eta \operatorname{div}_x \psi} \, dx \, dt.$$

Since ψ and η were arbitrary, the inner integrals over $(0, \infty)$ differ only by a constant, which we can compute by taking an average over $Q \times I \subset \mathbb{R}^{n+1}$ and using (5.3). Thus,

$$\int_0^\infty v \cdot \overline{\partial_\lambda \varphi} + (MF)_\perp \cdot \overline{\varphi} \, d\lambda = \int_0^\infty \left(\iint_{Q \times I} (MF)_\perp \, dx \, dt \right) \cdot \overline{\varphi} \, d\lambda$$

a.e. as functions defined on \mathbb{R}^{n+1} . This means that in the sense of distributions

$$(5.4) \quad \partial_\lambda v = (MF)_\perp - \iint_{Q \times I} (MF)_\perp \, dx \, dt \in L^2_{\text{loc}}(\mathbb{R}_+^{n+2}).$$

In particular, we have now found that $v \in \dot{H}^{1/2}(\mathbb{R}; L^2_{\text{loc}}(\mathbb{R}_+^{n+1})) \cap L^2_{\text{loc}}(\mathbb{R}; W^{1,2}_{\text{loc}}(\mathbb{R}_+^{n+1}))$ has the required regularity of a reinforced weak solution.

Step 2: Adjusting u in λ -direction. From (5.2) and (5.4) we obtain

$$D_A v = \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \partial_\lambda v \\ \nabla_x v \\ H_t D_t^{1/2} v \end{bmatrix} = \begin{bmatrix} F_\perp \\ F_\parallel \\ F_\theta \end{bmatrix} + \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \iint_{Q \times I} (MF)_\perp \, dx \, dt \\ 0 \\ 0 \end{bmatrix}.$$

Let now w be an anti-derivative of the L^2_{loc} -function $\lambda \mapsto \iint_{Q \times I} (MF)_\perp \, dx \, dt$, that is, $w \in W^{1,2}_{\text{loc}}(\mathbb{R}_+)$ and $\partial_\lambda w = \iint_{Q \times I} (MF)_\perp \, dx \, dt$. Then the modified potential $u := v - w$ still has the regularity of a reinforced weak solution since w depends only on λ and from above we can read off $D_A u = F$. Hence, u has the required property. Finally, we let $\varphi \in C_0^\infty(\mathbb{R}_+^{n+2})$ and take $\phi = [\varphi, 0, 0]$ as a test function in (5.1). Unravelling the resulting identity gives

$$\int_0^\infty \int_{\mathbb{R}_+^{n+1}} A \nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x} \varphi} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} \varphi} \, dx \, dt \, d\lambda = 0.$$

This confirms u as a reinforced weak solution to (1.1) and the proof is complete.

Remark 5.2. Independently of the equation (1.1), the argument above shows in fact that any $G \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2}))$ with $\partial_\lambda G_\parallel - \nabla_x G_\perp = 0$ and $H_t D_t^{1/2} G_\parallel = \nabla_x G_\theta$ has a potential $v \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}))$, unique up to constants, such that $[\partial_\lambda v, \nabla_x v, H_t D_t^{1/2} v] = G$.

6. RESOLVENT ESTIMATES

In this section we prove the resolvent estimates stated in Lemma 2.10 and Proposition 2.11.

6.1. Proof of Lemma 2.10. As there is nothing to prove for $\lambda = 0$, we can in the following, by scaling $(x, t) \mapsto (|\lambda|x, |\lambda|^{1/2}t)$, assume $\lambda = \pm 1$ as long as implicit constants in all of our estimates only depend on dimension and the ellipticity constants of A . We shall only argue for $\lambda = 1$ since the proof for $\lambda = -1$ is the same.

For $1 < p, q < \infty$, we define inhomogeneous parabolic Sobolev spaces $H_{p,q}(\mathbb{R}^{n+1})$ by

$$H_{p,q}(\mathbb{R}^{n+1}) := L^p(\mathbb{R}; W^{1,q}(\mathbb{R}^n)) \cap H^{1/2,p}(\mathbb{R}; L^q(\mathbb{R}^n))$$

equipped with the mixed norm

$$\|u\|_{H_{p,q}} := \left\| \| |u| + |\nabla_x u| + |D_t^{1/2} u| \|_{L^q(\mathbb{R}^n)} \right\|_{L^p(\mathbb{R})}.$$

We let $H_{p,q}^*$ denote the space dual to $H_{p,q}$ with respect to the $L^2(\mathbb{R}^{n+1})$ inner product. For any $f \in L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$, there exists a unique $\tilde{f} \in L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$, such that

$$f = \begin{bmatrix} (A\tilde{f})_{\perp} \\ \tilde{f}_{\parallel} \\ \tilde{f}_{\theta} \end{bmatrix},$$

compare with Proposition 2.1. Spelling out the definitions of P and M reveals that the equation $(1 + iPM)^{-1}f = g$ is equivalent to $\tilde{g}_{\perp} \in H_{2,2}$ with

$$\begin{cases} (A\tilde{g})_{\perp} + i \operatorname{div}_x(A\tilde{g})_{\parallel} - iD_t^{1/2}\tilde{g}_{\theta} = f_{\perp}, \\ \tilde{g}_{\parallel} - i\nabla_x\tilde{g}_{\perp} = f_{\parallel}, \\ \tilde{g}_{\theta} - iH_tD_t^{1/2}\tilde{g}_{\perp} = f_{\theta}. \end{cases}$$

Using the second and third equations to eliminate \tilde{g}_{\parallel} and \tilde{g}_{θ} in the first one, this is equivalent to the system

$$(6.1) \quad \begin{cases} (\partial_t + L_A)\tilde{g}_{\perp} = f_{\perp} - A_{\perp\parallel}f_{\parallel} - i \operatorname{div}_x(A_{\parallel\parallel}f_{\parallel}) + iD_t^{1/2}f_{\theta}, \\ \tilde{g}_{\parallel} - i\nabla_x\tilde{g}_{\perp} = f_{\parallel}, \\ \tilde{g}_{\theta} - iH_tD_t^{1/2}\tilde{g}_{\perp} = f_{\theta}, \end{cases}$$

where

$$L_A := [1 \quad i \operatorname{div}_x] A \begin{bmatrix} 1 \\ i\nabla_x \end{bmatrix}.$$

The parabolic operator $\partial_t + L_A$ admits hidden coercivity in the same spirit as discussed for \mathcal{L} in Section 2.3, that is, it can be associated with a bounded and coercive sesquilinear form on $H_{2,2}$. As a consequence of the Lax-Milgram lemma we find that $\partial_t + L_A$ is a bounded and invertible operator from $H_{2,2}$ to its dual $H_{2,2}^*$. Moreover, $\partial_t + L_A$ clearly extends to a bounded operator from $H_{p,q}$ to $H_{p',q'}^*$ for any $1 < p, q < \infty$. All of these bounds depend only upon the ellipticity constants of A . We intend to apply Šneĭberg's lemma [56] in order to extend invertibility to a neighbourhood of $(2, 2)$ in the (p, q) -plane. This is possible provided that $H_{p,q}$ and $H_{p,q}^*$ form complex interpolation scales.

Lemma 6.1. *Let $\varepsilon > 0$. For $1 + \varepsilon < p_0, p_1, q_0, q_1 < 1 + \varepsilon^{-1}$ and $0 < \theta < 1$ the complex interpolation identities*

$$[H_{p_0, q_0}, H_{p_1, q_1}]_{\theta} = H_{p, q} \quad \text{and} \quad [H_{p_0, q_0}^*, H_{p_1, q_1}^*]_{\theta} = H_{p, q}^*$$

up to equivalent norms, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Equivalence constants can be chosen only in dependence of ε and n .

Proof. For $1 < p, q < \infty$ we abbreviate $L^p(L^q) = L^p(\mathbb{R}; L^q(\mathbb{R}^n))$. It is well-known that these spaces interpolate according to the rule

$$[L^{p_0}(L^{q_0}), L^{p_1}(L^{q_1})]_{\theta} = L^p(L^q)$$

with *equal* norms, see, for example, [23, Thm. 5.1.1/2]. Let $T : \mathcal{S}(\mathbb{R}^{n+1}) \rightarrow \mathcal{S}'(\mathbb{R}^{n+1})$ be defined on the Fourier side via multiplication by the symbol

$$m(\tau, \xi) = \frac{1}{1 + |\tau|^{1/2} + |\xi_1| + \dots + |\xi_n|},$$

where $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Then m satisfies Mihlin's condition

$$\left| \tau^{\alpha_0} \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} \frac{\partial^{\alpha} m}{\partial \tau^{\alpha_0} \partial \xi_1^{\alpha_1} \dots \partial \xi_n^{\alpha_n}} \right| \leq C(\alpha) < \infty \quad (\tau, \xi_1, \dots, \xi_n \neq 0)$$

for all multi-indices $\alpha \in \mathbb{N}_0^{n+1}$. The Marcinkiewicz multiplier theorem on $L^p(L^q)$, see [50, Cor. 1], yields that T extends to a bounded operator on $L^p(L^q)$ for every pair $(p, q) \in (1, \infty) \times (1, \infty)$. Complex interpolation, starting from the four extremal points with $p, q \in \{1 + \varepsilon, 1 + \varepsilon^{-1}\}$, yields the required uniform bound on the operator norm within their convex hull in the (p, q) -plane. By the

same argument applied to the multipliers $|\tau|^{1/2}m(\tau, \xi)$, $|\xi|m(\tau, \xi)$ and $i\xi_j/|\xi_j|$, we see that in fact the extension of T is bounded $L^p(L^q) \rightarrow H_{p,q}$ and invertible. Thus, the required interpolation rules for the $H_{p,q}$ -spaces including uniformity of the equivalence constants follow from those for $L^p(L^q)$.

As another consequence every space $H_{p,q}$ contains $T(\mathcal{S}(\mathbb{R}^{n+1}))$ as a dense set since $\mathcal{S}(\mathbb{R}^{n+1})$ is dense in $L^p(L^q)$. Thus, the interpolation rules for the dual spaces $H_{p,q}^*$ follow from the duality theorem for the complex interpolation functor [23, Thm. 4.5.1]. \square

Lemma 6.1 allows to apply Šneĭberg's result [56], see also [10, App. A] for a quantitative version, to the effect that $\partial_t + L_A$ remains invertible for (p, q) in a neighbourhood of $(2, 2)$. Technically speaking, this only applies to complex interpolation scales with one parameter: Seeing $(\frac{1}{p}, \frac{1}{q})$ on a line passing through $(\frac{1}{2}, \frac{1}{2})$, it yields an interval for invertibility around $(2, 2)$ on that line. However, the result is quantitative in that the length of that interval depends only upon upper/lower bounds for $\partial_t + L_A : H_{2,2} \rightarrow H_{2,2}^*$ and the equivalence constants in the interpolation from Lemma 6.1. Therefore we can obtain the same interval on each line, eventually summing up to a two-dimensional neighbourhood in the (p, q) -plane. Let us also remark that all inverses are compatible on the intersection of any two such $H_{p,q}^*$ spaces by abstract interpolation theory [43, Thm. 8.1].

Returning to the equation $(1 + iP)M^{-1}f = g$, we obtain from (6.1) that if (p, q) is as above, then given $f \in L^2(L^2) \cap L^p(L^q)$ we have $\|g\|_{L^p(L^q)} \lesssim \|f\|_{L^p(L^q)}$ with implicit constants depending only on dimension and ellipticity. We conclude by density.

Next, we prove the analogous result for MP . If M^{-1} is in L^∞ , which is the case for equations, then the result is immediate by similarity. For systems, however, we only have that $M : \overline{R(P)} \rightarrow \overline{R(MP)}$ is invertible. In this case, we first use the argument above to find that $(1 + iP)^{-1}$ is bounded on $L^p(L^q)$ if $1 < p, q < \infty$. Therefore the domains $D_{p,q}$ of the maximal realisation of P in $L^p(L^q)$, equipped with the graph norm, form a complex interpolation scale. Since MP is bisectorial on $L^2(L^2)$, we have that $1 + iMP$ is an isomorphism $D_{2,2} \rightarrow L^2(L^2)$. For all other p, q it is at least bounded $D_{p,q} \rightarrow L^p(L^q)$. As above, $1 + iMP$ remains an isomorphism $D_{p,q} \rightarrow L^p(L^q)$ for (p, q) in a neighbourhood of $(2, 2)$ with compatible inverses. Consequently, the resolvent estimate holds in $L^p(L^q)$ for those (p, q) .

Finally, M^* is an operator in the same class as M . Thus, we already have resolvent bounds on $L^p(L^q)$ for PM^* and M^*P and those for MP^* and P^*M follow by duality.

6.2. Proof of Proposition 2.11. It suffices to prove (2.24) for $q = 2$ and $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 = \delta_0$ of Lemma 2.10 and N_0 to be determined. Then the proposition follows by interpolation with the global bound provided by Lemma 2.10 as long as q is close enough to 2 to guarantee the announced decay. We shall only argue for PM as the proof for the other operators is literally the same.

Let $h \in L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2}) = L^2(L^2)$. Let Q, I, j, k, λ as in the statement and let $N \geq 4$. We set $J = 4^j I$ and split $C_k(Q \times J) = C_k^1 \cup C_k^2$ with

$$C_k^1 := (2^{k+1}Q \setminus 2^kQ) \times N^k J \quad \text{and} \quad C_k^2 := 2^kQ \times (N^{k+1}J \setminus N^k J).$$

We assume that h is supported in C_k and hence we can write $h = h_1 + h_2$ with h_i supported in C_k^i .

To estimate $\iint_{Q \times J} |(1 + i\lambda PM)^{-1}h_1|^2 dy ds$, we can rely on the same argument as in the elliptic situation since the x support of h_1 is far from Q : More precisely, we have

$$\iint_{Q \times J} |(1 + i\lambda PM)^{-1}h_1|^2 dy ds \leq \frac{1}{\ell(J)|Q|} \int_{\mathbb{R}} \int_Q |(1 + i\lambda PM)^{-1}h_1|^2 dy ds$$

and by the argument in [9, Prop. 5.1], using a commutator with a cut-off function in x -direction only, we obtain for any $m \in \mathbb{N}$,

$$(6.2) \quad \begin{aligned} \iint_{Q \times J} |(1 + i\lambda PM)^{-1}h_1|^2 dy ds &\lesssim \frac{2^{-km}}{\ell(J)|Q|} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |h_1|^2 dy ds \\ &= 2^{-km} N^{k+1} 2^{kn} \iint_{C_k(Q \times J)} |h|^2 dy ds. \end{aligned}$$

In order to estimate $\iint_{Q \times J} |(1 + i\lambda PM)^{-1}h_2|^2 dy ds$, we pick a smooth cut-off function $\eta \in C_0^\infty(N^{k-1}J)$, which is valued in $[0, 1]$, equal to 1 on $N^{k-2}J$ and satisfies $(N^k \ell(J)) \|\partial_t \eta\|_\infty \lesssim 1$ uniformly in k, N

and the size of J . With $p > 2$ in the range of Lemma 2.10 we have

$$\begin{aligned} \iint_{Q \times J} |(1 + i\lambda PM)^{-1} h_2|^2 dy ds &\leq \frac{1}{|Q|} \iint_J \int_{\mathbb{R}^n} |(1 + i\lambda PM)^{-1} h_2|^2 dy ds \\ &\leq \frac{1}{|Q|} \left(\iint_J \left(\int_{\mathbb{R}^n} |(1 + i\lambda PM)^{-1} h_2|^2 dy \right)^{p/2} ds \right)^{2/p} \\ &\leq \frac{1}{|Q| \ell(J)^{2/p}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |\eta(1 + i\lambda PM)^{-1} h_2|^2 dy \right)^{p/2} ds \right)^{2/p}. \end{aligned}$$

Since $\eta(t)h_2(x, t) = 0$, we can re-express $\eta(1 + i\lambda PM)^{-1}h_2$ using a commutator

$$\begin{aligned} \eta(1 + i\lambda PM)^{-1}h_2 &= [\eta, (1 + i\lambda PM)^{-1}]h_2 \\ (6.3) \quad &= (1 + i\lambda PM)^{-1}[\eta, i\lambda PM](1 + i\lambda PM)^{-1}h_2 \\ &= (1 + i\lambda PM)^{-1}i\lambda[\eta, P]M(1 + i\lambda PM)^{-1}h_2, \end{aligned}$$

where

$$[\eta, P] = \begin{bmatrix} 0 & 0 & -[\eta, D_t^{1/2}] \\ 0 & 0 & 0 \\ -[\eta, H_t D_t^{1/2}] & 0 & 0 \end{bmatrix}.$$

The commutators in $[\eta, P]$ depend only on the t variable and due to (3.1) and (3.2) they have kernels $K(t, s)$ bounded by

$$(6.4) \quad |K(t, s)| \leq C \frac{|\eta(t) - \eta(s)|}{|t - s|^{3/2}} \lesssim \tilde{h}^{-1/2} \tilde{h}^{-1} \phi\left(\frac{t - s}{\tilde{h}}\right)$$

with $\phi(t) = \min\{|t|^{-1/2}, |t|^{-3/2}\}$ and $\tilde{h} := N^k \ell(J)$. Thus, by Young's inequality,

$$\|[\eta, D_t^{1/2}]\|_{L^2(\mathbb{R}) \rightarrow L^p(\mathbb{R})} + \|[\eta, H_t D_t^{1/2}]\|_{L^2(\mathbb{R}) \rightarrow L^p(\mathbb{R})} \lesssim \tilde{h}^{-1/2 - ((1/2) - (1/p))}.$$

Using usual vector-valued extension [57, Sec. II.5], it follows that $[\eta, P] : L^2(L^2) \rightarrow L^p(L^2)$ with norm as above. This is the key point. Now, inserting (6.3) into the ongoing estimate, and applying the $L^p(L^2)$ boundedness of the resolvent and the just obtained commutator bound, we find

$$\begin{aligned} &\frac{1}{|Q| \ell(J)^{2/p}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |\eta(1 + i\lambda PM)^{-1} h_2|^2 dy \right)^{p/2} ds \right)^{2/p} \\ &\lesssim \frac{1}{|Q| \ell(J)^{2/p}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |i\lambda[\eta, P]M(1 + i\lambda PM)^{-1} h_2|^2 dy \right)^{p/2} ds \right)^{2/p} \\ &\lesssim \frac{|\lambda|^2}{|Q| \ell(J)^{2/p}} \cdot \frac{1}{(N^k \ell(J))^{1 + (1 - 2/p)}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |M(1 + i\lambda PM)^{-1} h_2|^2 dy ds. \end{aligned}$$

Since the resolvents of PM are $L^2(L^2)$ -bounded, we obtain furthermore

$$\begin{aligned} &\lesssim \frac{|\lambda|^2}{|Q| \ell(J)^2 (N^k)^{2 - 2/p}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |h_2|^2 dy ds \\ &\lesssim \frac{2^{kn}}{(N^k)^{1 - 2/p} \ell(J)} \iint_{C_k(Q \times J)} |h|^2 dy ds. \end{aligned}$$

We remark that $\ell(J) = 4^j \ell(I)$ and $\lambda^2 \sim \ell(I)$ by assumption. Hence, we have proved

$$\iint_{Q \times J} |(1 + i\lambda PM)^{-1} h_2|^2 dy ds \lesssim \frac{2^{kn}}{(N^k)^{1 - 2/p}} \iint_{C_k(Q \times J)} |h|^2 dy ds.$$

To conclude the proof, we only have to compare with (6.2) and choose the parameters at our disposal appropriately: First we pick $0 < \varepsilon < \varepsilon_0$ and then p with $\varepsilon < \frac{1}{2} - \frac{1}{p} < \varepsilon_0$. For N large enough $2^{kn} \lesssim$

$N^{-2k\varepsilon}(N^k)^{1-2/p}$ and given any such choice of N , there is a choice of m large verifying $2^{-km}N^{k+1}2^{kn} \lesssim N^{-2k\varepsilon}$.

7. QUADRATIC ESTIMATES FOR PM : PROOF OF THEOREM 2.3

Based on Lemma 4.2 we see that in order to prove Theorem 2.3 it suffices to prove the quadratic estimate

$$(7.1) \quad \int_0^\infty \|\lambda MP(1+\lambda^2 MPMP)^{-1}h\|_2^2 \frac{d\lambda}{\lambda} \lesssim \|h\|_2^2 \quad (h \in \mathbb{L}^2)$$

and the analogous estimate obtained upon replacing MP by M^*P^* , where $\mathbb{L}^2 = \mathbb{L}^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$ as before: In fact, (7.1) for MP implies the analogous estimate for PM since these two operators are similar on the closure of their ranges by Remark 4.3. By the same argument, the estimate for M^*P^* implies the one for P^*M^* . Once we have (7.1) for all four operators, an abstract duality argument gives the reverse inequalities for all four operators and hence the claim, see [51] or [35, Lem. 3.4.10].

Below, we shall only give the proof for MP since the argument for M^*P^* is identical upon minor changes of the algebra. Note that when $M = 1$, then these estimates with P or P^* can simply be proved using a Fourier transform argument in the (x, t) variables.

In proving (7.1), we shall follow the algorithm developed in [9]. We set $R_\lambda = (1+i\lambda MP)^{-1}$ for $\lambda \in \mathbb{R}$. Then

$$Q_\lambda = \frac{1}{2i}(R_{-\lambda} - R_\lambda) = \lambda MP(1+\lambda^2 MPMP)^{-1}$$

and also $\frac{1}{2}(R_{-\lambda} + R_\lambda) = (1+\lambda^2 MPMP)^{-1}$. Since MP is bisectorial, see Lemma 4.2, the operators R_λ , Q_λ and $(1+\lambda^2 MPMP)^{-1}$ are uniformly bounded on \mathbb{L}^2 .

7.1. Reduction to a Carleson measure. As Q_λ vanishes on $\mathbf{N}(MP) = \mathbf{N}(P)$, it is enough to prove the quadratic estimate (7.1) for $h \in \overline{\mathbf{R}(MP)}$. By density, $h \in \mathbf{R}(MP)$ suffices. So, setting $\Theta_\lambda = Q_\lambda M$, it amounts to showing

$$(7.2) \quad \int_0^\infty \|\Theta_\lambda P v\|_2^2 \frac{d\lambda}{\lambda} \lesssim \|P v\|_2^2 \quad (v \in \mathbf{D}(P)).$$

We define the parabolic approximation of the identity $P_\lambda = (1+\lambda^2(\partial_t - \Delta_x))^{-1}$ for $\lambda \in \mathbb{R}$ and let it act componentwise on \mathbb{C}^{n+2} valued functions. By Plancherel's theorem the operators P_λ are uniformly bounded on \mathbb{L}^2 .

Proposition 7.1. *For all $v \in \mathbf{D}(P)$ the estimate*

$$\int_0^\infty \|\Theta_\lambda(1-P_\lambda)P v\|_2^2 \frac{d\lambda}{\lambda} \lesssim \|P v\|_2^2.$$

Proof. From (2.18) we know that P^2 agrees with $\partial_t - \Delta_x$ on $\mathbf{D}(P^2) \cap \mathbf{R}(P)$. Hence, P_λ agrees with $(1+\lambda^2 P^2)^{-1}$ on $\mathbf{R}(P)$ and to ease the computations we may think of $P_\lambda = (1+\lambda^2 P^2)^{-1}$ just for the proof. Since P_λ and P commute and $(1-P_\lambda)v \in \mathbf{D}(P)$, we have

$$\Theta_\lambda(1-P_\lambda)P v = (\Theta_\lambda P)(1-P_\lambda)v = \lambda(MP)^2(1+(\lambda MP)^2)^{-1}(1-P_\lambda)v.$$

Now, $(\lambda MP)^2(1+(\lambda MP)^2)^{-1} = 1 - (1+(\lambda MP)^2)^{-1}$ is uniformly bounded, hence

$$\|\Theta_\lambda(1-P_\lambda)P v\|_2 \lesssim \frac{1}{\lambda} \|(1-P_\lambda)v\|_2.$$

Using that

$$\frac{1}{\lambda}(1-P_\lambda)v = \lambda P(1+\lambda^2 P^2)^{-1}(P v)$$

we see by the square function estimate for P that

$$\int_0^\infty \|\lambda P(1+\lambda^2 P^2)^{-1}(P v)\|_2^2 \frac{d\lambda}{\lambda} \lesssim \|P v\|_2^2. \quad \square$$

Next, we perform the *principal part approximation*. We use the following dyadic decomposition of \mathbb{R}^{n+1} in parabolic dyadic cubes. Let $\square = \bigcup_{j=-\infty}^{\infty} \square_{2^j}$ where $\square_{2^j} := \{\delta_j(k + (0, 1]^{n+1}) : k \in \mathbb{Z}^{n+1}\}$ with $\delta_j(x, t) = (2^j x, 4^j t)$. For a parabolic dyadic cube $R \in \square_{2^j}$ we denote by $\ell(R) = 2^j$ its *sidelength* and by $|R| = 2^{(n+2)j}$ its *volume*. We set $\square_\lambda = \square_{2^j}$ if $2^{j-1} < \lambda \leq 2^j$. Let the *parabolic dyadic averaging operator* $S_\lambda : \mathbb{L}^2 \rightarrow \mathbb{L}^2$ be given by

$$S_\lambda u(x, t) := u_R := \iint_R u(y, s) \, dy \, ds = \frac{1}{|R|} \iint_R u(y, s) \, dy \, ds$$

for every $(x, t) \in \mathbb{R}^{n+1}$ and $\lambda > 0$, where R is the unique parabolic dyadic cube in \square_λ that contains (x, t) . We remark that $S_\lambda^2 = S_\lambda$.

Definition 7.2. By the *principal part* of $(\Theta_\lambda)_{\lambda>0}$ we mean the multiplication operators γ_λ defined by

$$\gamma_\lambda(x, t)\zeta := (\Theta_\lambda \zeta)(x, t)$$

for every $\zeta \in \mathbb{C}^{n+2}$. We view ζ on the right-hand side of the equation above as the constant function valued in \mathbb{C}^{n+2} defined on \mathbb{R}^{n+1} by $\zeta(x, t) := \zeta$. We identify $\gamma_\lambda(x, t)$ with the (possibly unbounded) multiplication operator $\gamma_\lambda : f(x, t) \mapsto \gamma_\lambda(x, t)f(x, t)$.

Of course $\zeta \notin \mathbb{L}^2$ but still this definition is justified in virtue of the next lemma.

Lemma 7.3. *The operator Θ_λ extends to a bounded operator from L^∞ into L^2_{loc} . In particular we have well defined functions $\gamma_\lambda \in L^2_{\text{loc}}(\mathbb{R}^{n+1}; \mathcal{L}(\mathbb{C}^{n+2}))$ with bounds*

$$\iint_R |\gamma_\lambda(x, t)|^2 \, dx \, dt \lesssim 1$$

for all $R \in \square_\lambda$. Here, $|\cdot|$ is the operator norm of $\mathcal{L}(\mathbb{C}^{n+2})$. Moreover, $\gamma_\lambda S_\lambda$ is bounded on \mathbb{L}^2 with $\|\gamma_\lambda S_\lambda\| \lesssim 1$ uniformly for all $\lambda > 0$.

Proof. Fix a parabolic cube $R \in \square_\lambda$ and $f \in L^\infty(\mathbb{R}^n; \mathbb{C}^{n+2})$ with $\|f\|_\infty = 1$. Then write $f = f_1 + f_2 + \dots$ where $f_1 = f$ on $2R$ and 0 elsewhere and if $j \geq 2$, $f_j = f$ on $C_j(R)$ with the notation of Proposition 2.11 and 0 elsewhere. Then apply $\Theta_\lambda = \frac{1}{2^i}(R_{-\lambda} - R_\lambda)M$, use the off-diagonal estimates of Proposition 2.11 for each term $\Theta_\lambda f_j$, with N large enough, and sum to obtain for $\Theta_\lambda f := \sum_{j=1}^{\infty} \Theta_\lambda f_j$ the estimate

$$\iint_R |(\Theta_\lambda f)(x, t)|^2 \, dx \, dt \leq C.$$

If we do this for the constant functions with values describing an orthonormal basis of \mathbb{C}^{n+2} and sum, we obtain an upper bound for the desired average of γ_λ . Next, for a function $f \in \mathbb{L}^2$,

$$\|\gamma_\lambda S_\lambda f\|_2^2 = \sum_{R \in \square_\lambda} \iint_R \left| \gamma_\lambda(x, t) \left(\iint_R f \right) \right|^2 \, dx \, dt \lesssim \sum_{R \in \square_\lambda} |R| \left| \iint_R f \right|^2 \leq \|f\|_2^2.$$

Here, $|\cdot|$ is the usual Hermitian norm on \mathbb{C}^{n+2} . □

We have the following principal part approximation of Θ_λ by $\gamma_\lambda S_\lambda$.

Lemma 7.4. *It holds*

$$\int_0^\infty \|\Theta_\lambda P_\lambda f - \gamma_\lambda S_\lambda f\|_2^2 \frac{d\lambda}{\lambda} \lesssim \|f\|_2^2 \quad (f \in \mathbb{L}^2).$$

Proof. Let Φ_λ be a nice parabolic approximation of the identity on \mathbb{L}^2 , that is, the convolution operator by a real valued function $\lambda^{-n-2}\phi(\lambda^{-1}x, \lambda^{-2}t)$ with $\phi \in C_0^\infty(B(0, 1) \times (-1, 1))$ satisfying $\iint_{\mathbb{R}^{n+1}} \phi(x, t) \, dx \, dt = 1$. Write

$$(7.3) \quad \Theta_\lambda P_\lambda - \gamma_\lambda S_\lambda = (\Theta_\lambda P_\lambda - \gamma_\lambda S_\lambda P_\lambda) + (\gamma_\lambda S_\lambda (P_\lambda - \Phi_\lambda)) + (\gamma_\lambda S_\lambda (\Phi_\lambda - S_\lambda)) + (\gamma_\lambda S_\lambda^2 - \gamma_\lambda S_\lambda).$$

Because of $S_\lambda^2 = S_\lambda$, the last term vanishes. Next, as the $\gamma_\lambda S_\lambda$ are uniformly bounded as operators on \mathbb{L}^2 ,

$$\int_0^\infty \|\gamma_\lambda S_\lambda(P_\lambda - \Phi_\lambda)f\|_2^2 + \|\gamma_\lambda S_\lambda(\Phi_\lambda - S_\lambda)f\|_2^2 \frac{d\lambda}{\lambda} \lesssim \int_0^\infty \|(P_\lambda - \Phi_\lambda)f\|_2^2 + \|(\Phi_\lambda - S_\lambda)f\|_2^2 \frac{d\lambda}{\lambda}.$$

A straightforward Fourier transform argument reveals control by $\|f\|_2^2$ for the square function of $P_\lambda - \Phi_\lambda$. The analogous bound for the square function of $\Phi_\lambda - S_\lambda$ is done componentwise and for each component it is precisely the statement of [27, Lem. 2.5]. As for the first term in (7.3), we will show that for $\alpha \in (0, 1)$ small enough there is a bound

$$(7.4) \quad \|(\Theta_\lambda P_\lambda - \gamma_\lambda S_\lambda P_\lambda)f\|_2^2 \lesssim \lambda^2 \|\nabla_x P_\lambda f\|_2^2 + \lambda^{4\alpha} \|D_t^\alpha P_\lambda f\|_2^2,$$

where D_t^α is the fractional derivative of order α on \mathbb{R} defined via the Fourier symbol $|\tau|^\alpha$. We postpone the proof of this claim until Section 7.3. Hence, we obtain

$$\int_0^\infty \|\Theta_\lambda P_\lambda f - \gamma_\lambda S_\lambda P_\lambda f\|_2^2 \frac{d\lambda}{\lambda} \lesssim \int_0^\infty \|\lambda \nabla_x P_\lambda f\|_2^2 \frac{d\lambda}{\lambda} + \int_0^\infty \|\lambda^{2\alpha} D_t^\alpha P_\lambda f\|_2^2 \frac{d\lambda}{\lambda} \lesssim \|f\|_2^2$$

from a Fourier transform argument on each component of f , given the definition of P_λ . \square

Combining Lemma 7.4 with Proposition 7.1, we obtain the principal part approximation

$$(7.5) \quad \int_0^\infty \|\Theta_\lambda P v - \gamma_\lambda S_\lambda P v\|_2^2 \frac{d\lambda}{\lambda} \lesssim \|P v\|_2^2 \quad (v \in \mathbf{D}(P)).$$

Hence, to complete the proof of (7.2) we want to apply Carleson's theorem to conclude that

$$\int_0^\infty \|\gamma_\lambda S_\lambda P v\|_2^2 \frac{d\lambda}{\lambda} \lesssim \|\gamma_\lambda\|_C^2 \|P v\|_2^2 \quad (v \in \mathbf{D}(P)),$$

where

$$\|\gamma_\lambda\|_C^2 = \sup_{R \in \square} \frac{1}{|R|} \iiint_{T(R)} |\gamma_\lambda(x, t)|^2 \frac{dx dt d\lambda}{\lambda}.$$

Here $T(R) := (0, \ell(R)] \times R$ is the Carleson box over R . Hence, it remains to prove the Carleson measure estimate $\|\gamma_\lambda\|_C < \infty$, and the claim in (7.4), to finish the proof of the quadratic estimate in (7.1).

7.2. The T(b) argument. We first construct test functions which belong to the range of P .

Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be a smooth non-negative function with support in $[-1, 1]$ which is 1 on $[-1/2, 1/2]$. Consider $\nu(t) = \eta(t) - \eta(t - z)$ for $z \in \mathbb{R}$, $|z| > 2$. Since $\widehat{\nu}(0) = 0$, we can define $h := (-H_t D_t^{1/2})^{-1} \nu$ by Fourier transform. We claim that $h \in L^2(\mathbb{R})$ satisfies $|h(t) - h(0)| \leq C|t|$ and that we can find some z such that $h(0) \neq 0$. Indeed,

$$h(t) = -\frac{1}{2\pi} \int_{\mathbb{R}} e^{it\tau} \frac{1 - e^{iz\tau}}{i \operatorname{sgn}(\tau) |\tau|^{1/2}} \widehat{\eta}(\tau) d\tau,$$

so that the cheap estimate $|1 - e^{it\tau}| \leq |t\tau|$ yields

$$|h(t) - h(0)| \leq \frac{|t|}{\pi} \int_{\mathbb{R}} |\tau|^{1/2} |\widehat{\eta}(\tau)| d\tau,$$

and likewise, using $|1 - e^{iz\tau}| \leq |z\tau|$ and Plancherel's theorem, it follows that

$$\|h\|_2^2 \leq \frac{|z|^2}{2\pi} \int_{\mathbb{R}} |\tau| |\widehat{\eta}(\tau)|^2 d\tau < \infty.$$

To see the last claim, assume to contrary that $h(0) = 0$ holds for any $|z| > 2$. By differentiating the formula for $h(0)$ with respect to z , we find

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{iz\tau} |\tau|^{1/2} \widehat{\eta}(\tau) d\tau = 0 \quad (|z| > 2).$$

This means that the Fourier transform of $\tau \mapsto |\tau|^{1/2} \widehat{\eta}(\tau)$ is compactly supported, hence this function is analytic. In particular, it is analytic at the origin. But $\tau \mapsto \widehat{\eta}(\tau)$ has the same property and

$\widehat{\eta}(0) = \int_{\mathbb{R}} \eta \geq 1$. Hence $\tau \mapsto |\tau|^{1/2}$ should be analytic at the origin, which is absurd. We now fix z with $|z| > 2$ such that all the properties above hold.

Let $\zeta \in \mathbb{C}^{n+2}$ with $|\zeta| = 1$. Let ζ_i denote the components of ζ and $\zeta_{\parallel} = (\zeta_2, \dots, \zeta_{n+1})$. We need also a parameter $0 < \delta \leq 1$ which is small and will be chosen later. We let χ be a smooth function on \mathbb{R}^n , valued in $[0, 1]$, which is 1 on the cube $[-1/2, 1/2]^n$ and with support in $[-1, 1]^n$.

Fix a parabolic dyadic cube $R = Q \times I$, its center being denoted (x_Q, t_I) . Observe that with our notation $\ell(R) = \ell(Q) = \sqrt{\ell(I)}$. We set

$$\begin{aligned} \chi_Q(x) &:= \chi\left(\frac{x - x_Q}{\ell(Q)}\right) \quad (x \in \mathbb{R}^n), \\ h_{I,\delta}(t) &:= \delta^{-1/2} \ell(I)^{1/2} h\left(\delta \cdot \frac{t - t_I}{\ell(I)}\right) \quad (t \in \mathbb{R}), \\ f_1(x) &:= \zeta_{n+2} - \frac{\delta^{1/2} \zeta_{\parallel} \cdot (x - x_Q)}{h(0)\ell(Q)} \quad (x \in \mathbb{R}^n), \\ f_2(x) &:= \frac{\delta^{1/2} \zeta_1 (x_1 - x_{Q,1})}{h(0)\ell(Q)} \quad (x \in \mathbb{R}^n) \end{aligned}$$

and finally we introduce $L_{R,\delta}^{\zeta} : \mathbb{R}^{n+1} \rightarrow \mathbb{C}^{n+2}$ through

$$L_{R,\delta}^{\zeta}(x, t) := \begin{bmatrix} \chi_Q(x) f_1(x) & h_{I,\delta}(t) \\ \chi_Q(x) f_2(x) & h_{I,\delta}(t) \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix} \quad ((x, t) \in \mathbb{R}^{n+1}).$$

Clearly $L_{R,\delta}^{\zeta} \in \mathbb{L}^2$, but we do not need its \mathbb{L}^2 -norm and so we do not compute it. We claim that $L_{R,\delta}^{\zeta} \in \mathcal{D}(P)$ with

$$(7.6) \quad \|PL_{R,\delta}^{\zeta}\|_2^2 \leq C\delta^{-2}|R|$$

and that

$$(7.7) \quad \iint_R |PL_{R,\delta}^{\zeta} - \zeta|^2 dx dt \leq C\delta|R|$$

with an absolute constant C . Indeed, we first note that

$$PL_{R,\delta}^{\zeta}(x, t) = \begin{bmatrix} \partial_{x_1}(\chi_Q(x) f_2(x)) & h_{I,\delta}(t) \\ -\partial_{x_1}(\chi_Q(x) f_1(x)) & h_{I,\delta}(t) \\ \vdots & \\ -\partial_{x_n}(\chi_Q(x) f_1(x)) & h_{I,\delta}(t) \\ \chi_Q(x) f_1(x) & (-H_t D_t^{1/2}) h_{I,\delta}(t) \end{bmatrix}.$$

Furthermore, we observe by a change of variables that

$$\|h_{I,\delta}\|_{\mathbb{L}^2(\mathbb{R})} = \delta^{-1} \ell(I) \|h\|_{\mathbb{L}^2(\mathbb{R})}$$

and $(-H_t D_t^{1/2}) h_{I,\delta}(t) = \nu(\delta \cdot \frac{t-t_I}{\ell(I)})$, so that

$$\|(-H_t D_t^{1/2}) h_{I,\delta}\|_{\mathbb{L}^2(\mathbb{R})} = \delta^{-1/2} \ell(I)^{1/2} \|\nu\|_{\mathbb{L}^2(\mathbb{R})}.$$

Using that the partial derivatives of the functions of x are bounded (uniformly for $\delta \leq 1$) by $C/\ell(Q)$ with support in $2Q$, we get

$$\|PL_{R,\delta}^{\zeta}\|_2^2 \leq C(\ell(Q)^{n-2} \delta^{-2} \ell(I)^2 + \ell(Q)^n \delta^{-1} \ell(I)) \lesssim \delta^{-2}|R|.$$

Because of the support properties of χ_Q and ν we have for $(x, t) \in R$ that

$$PL_{R,\delta}^\zeta(x, t) = \begin{bmatrix} \partial_{x_1} f_2(x) h_{I,\delta}(t) \\ -\partial_{x_1} f_1(x) h_{I,\delta}(t) \\ \vdots \\ -\partial_{x_n} f_1(x) h_{I,\delta}(t) \\ f_1(x) \end{bmatrix} = \begin{bmatrix} \zeta_1 h(0)^{-1} h(\delta \cdot \frac{t-t_I}{\ell(I)}) \\ \zeta_2 h(0)^{-1} h(\delta \cdot \frac{t-t_I}{\ell(I)}) \\ \vdots \\ \zeta_{n+1} h(0)^{-1} h(\delta \cdot \frac{t-t_I}{\ell(I)}) \\ \zeta_{n+2} - \frac{\delta^{1/2} \zeta_{\parallel} \cdot (x-x_Q)}{h(0)\ell(Q)} \end{bmatrix},$$

so that at the center of the cube $PL_{R,\delta}^\zeta(x_Q, t_I) = \zeta$ and by the Lipschitz property of h at 0 and $|\zeta| = 1$, we get

$$|PL_{R,\delta}^\zeta(x, t) - \zeta| \leq |h(0)|^{-1} (C\delta + \delta^{1/2}) \quad ((x, t) \in R).$$

This completes the proof of (7.6) and (7.7).

We now define the test functions $b_{R,\varepsilon,\delta}^\zeta$ for $\varepsilon, \delta \in (0, 1)$ by

$$b_{R,\varepsilon,\delta}^\zeta := Pv_{R,\varepsilon,\delta}^\zeta \quad v_{R,\varepsilon,\delta}^\zeta := (1 + i\varepsilon\ell MP)^{-1} L_{R,\delta}^\zeta,$$

where $\ell = \ell(R)$.

Lemma 7.5. *There exists $C > 0$ such that for each $\zeta \in \mathbb{C}^{n+2}$ with $|\zeta| = 1$, each parabolic dyadic cube $R \subset \mathbb{R}^{n+1}$ and each $\varepsilon, \delta \in (0, 1)$,*

$$(7.8) \quad \iint_{\mathbb{R}^{n+1}} |v_{R,\varepsilon,\delta}^\zeta - L_{R,\delta}^\zeta|^2 dx dt \leq C(\varepsilon\ell)^2 \delta^{-2} |R|,$$

$$(7.9) \quad \iint_{\mathbb{R}^{n+1}} |b_{R,\varepsilon,\delta}^\zeta - PL_{R,\delta}^\zeta|^2 dx dt \leq C\delta^{-2} |R|,$$

$$(7.10) \quad \left| \iint_R (b_{R,\varepsilon,\delta}^\zeta - \zeta) dx dt \right| \leq C(\varepsilon^{1/3} \delta^{-1} + \delta^{1/2}),$$

$$(7.11) \quad \iint_R |b_{R,\varepsilon,\delta}^\zeta - \zeta|^2 dx dt \leq C\delta^{-2} |R|,$$

$$(7.12) \quad \iiint_{T(R)} |\gamma_\lambda(x, t) S_\lambda b_{R,\varepsilon,\delta}^\zeta(x, t)|^2 \frac{dx dt d\lambda}{\lambda} \leq C\varepsilon^{-2} \delta^{-2} |R|.$$

Proof. Using $(1 + i\lambda MP)^{-1} - 1 = -i\lambda MP(1 + i\lambda MP)^{-1}$, we have

$$v_{R,\varepsilon,\delta}^\zeta - L_{R,\delta}^\zeta = -i\varepsilon\ell(1 + i\varepsilon\ell MP)^{-1} (MPL_{R,\delta}^\zeta).$$

The inequality (7.6) for $PL_{R,\delta}^\zeta$ and the uniform boundedness of $s \mapsto (1 + isMP)^{-1} M$ imply (7.8). Applying P , we see that

$$b_{R,\varepsilon}^\zeta - PL_{R,\delta}^\zeta = -i\varepsilon\ell P(1 + i\varepsilon\ell MP)^{-1} (MPL_{R,\delta}^\zeta).$$

Again the inequality (7.6) and the boundedness of $s \mapsto sP(1 + isMP)^{-1} M$ imply (7.9). Next, (7.11) follows directly from (7.7) and (7.9), keeping in mind that $\delta \in (0, 1)$.

As for (7.12), we first note, using (7.5), (7.6) and the fact that $b_{R,\varepsilon,\delta}^\zeta = Pv_{R,\varepsilon,\delta}^\zeta$, that it suffices to establish the estimate

$$(7.13) \quad \iiint_{T(R)} |\Theta_\lambda b_{R,\varepsilon,\delta}^\zeta(x, t)|^2 \frac{dx dt d\lambda}{\lambda} \leq C\varepsilon^{-2} \delta^{-1} |R|.$$

Now,

$$\begin{aligned} \Theta_\lambda b_{R,\varepsilon,\delta}^\zeta &= \lambda MP(1 + \lambda^2 MPMP)^{-1} MP(1 + i\varepsilon\ell MP)^{-1} L_{R,\delta}^\zeta \\ &= \lambda MP(1 + \lambda^2 MPMP)^{-1} (1 + i\varepsilon\ell MP)^{-1} (MPL_{R,\delta}^\zeta) \\ &= (\lambda/\varepsilon\ell)(1 + \lambda^2 MPMP)^{-1} \varepsilon\ell MP(1 + i\varepsilon\ell MP)^{-1} (MPL_{R,\delta}^\zeta). \end{aligned}$$

Since $(1 + \lambda^2 MPMP)^{-1}$ and $\varepsilon \ell MP(1 + i\varepsilon \ell MP)^{-1}$ are bounded uniformly with respect to λ and $\varepsilon \ell$, we have

$$\|\Theta_\lambda b_{R,\varepsilon,\delta}^\zeta\|_2 \leq C(\lambda/\varepsilon \ell) \|PL_{R,\delta}^\zeta\|_2.$$

Integrating over $\lambda \in (0, \ell]$, we obtain (7.13).

It remains to establish (7.10). Let $\varphi: \mathbb{R}^{n+1} \rightarrow [0, 1]$ be a smooth function which is 1 on $(1-s)Q \times (1-s^2)I$, supported on R , with $\|\nabla_x \varphi\|_\infty \leq C(s\ell)^{-1}$ and $\|\partial_t \varphi\|_\infty \leq C(s\ell)^{-2}$, where $s \in (0, 1)$ still has to be chosen. We can write

$$\iint_R (b_{R,\varepsilon,\delta}^\zeta - \zeta) dx dt = \iint_R (b_{R,\varepsilon,\delta}^\zeta - PL_{R,\delta}^\zeta) dx dt + \iint_R (PL_{R,\delta}^\zeta - \zeta) dx dt.$$

Due to (7.7), the second integral is bounded by $\delta^{1/2}|R|$. Next, we split the first integral as

$$\iint_R \varphi P(v_{R,\varepsilon,\delta}^\zeta - L_{R,\delta}^\zeta) dx dt + \iint_R (1 - \varphi)(b_{R,\varepsilon,\delta}^\zeta - PL_{R,\delta}^\zeta) =: \text{I} + \text{II}.$$

Using (7.9), the properties of φ and the Cauchy-Schwarz inequality, we obtain

$$|\text{II}| \leq Cs^{1/2}\delta^{-1}|R|.$$

The Euclidean norm of $\text{I} \in \mathbb{C}^{n+2}$ can be computed by

$$|\text{I}|^2 = \sum_{j=1}^{n+2} \left| \iint_{\mathbb{R}^{n+1}} (v_{R,\varepsilon,\delta}^\zeta - L_{R,\delta}^\zeta) \overline{P^* \vec{\varphi}_j} dx dt \right|^2$$

with $\vec{\varphi}_j = (0, \dots, 0, \varphi, 0, \dots, 0) \in \mathbb{L}^2$ and the non-zero entry sitting at j -th position. Now, $P^* \vec{\varphi}_j$ can contain two types of terms: the first one are first order derivatives of φ in the x variable, and the second one are $D_t^{1/2} \varphi$ or $H_t D_t^{1/2} \varphi$ acting on the t variable. Both are functions in \mathbb{L}^2 with bounds on the same order of $(s\ell)^{-1}|R|^{1/2}$. For the first one this follows by construction. For the second one, we obtain the order $(s^{-2}\ell^{-2})^{1/2}|R|^{1/2}$ from the L^2 -bounds on φ and $\partial_t \varphi$ and the inequality $\|D_t^{1/2} \varphi\|_2 \leq \|\varphi\|_2^{1/2} \|\partial_t \varphi\|_2^{1/2}$, which follows directly from Plancherel's theorem. Thus, we can apply (7.8) to the effect that

$$|\text{I}| \leq C(\varepsilon/s)\delta^{-1}|R|.$$

Hence, choosing $s = \varepsilon^{2/3}$, we have shown (7.10). \square

In conclusion, setting $b_{R,\varepsilon}^\zeta := b_{R,\varepsilon,\varepsilon^{1/6}}^\zeta$, we obtain

$$\left| \iint_R (b_{R,\varepsilon}^\zeta - \zeta) dx dt \right| \leq C\varepsilon^{1/12} \quad \text{and} \quad \iint_R |b_{R,\varepsilon}^\zeta - \zeta|^2 dx dt \leq C\varepsilon^{-1/3}.$$

At this point, we can run the stopping time argument presented in [9, Sec. 3.4] with this family of test functions instead of $b_{Q,\varepsilon}^w$ there, using Lemma 7.5 exactly as Lemma 3.10 there, to obtain that $\gamma_\lambda(x, t)$ is a parabolic Carleson function. As the argument is really line by line the same with only few minor cosmetic changes, we omit details and refer the reader to [9]. This finishes the proof of Theorem 2.3 modulo the proof of (7.4) that we present below.

7.3. Proof of (7.4). To start the proof we first note, for $\lambda > 0$ fixed and $(x, t) \in \mathbb{R}^{n+1}$, that

$$(\Theta_\lambda P_\lambda - \gamma_\lambda S_\lambda P_\lambda) f(x, t) = \Theta_\lambda \left(g - \iint_R g \right) (x, t),$$

where $g = P_\lambda f$ and R is the only parabolic dyadic cube in \square_λ containing (x, t) . Let us write $R = Q \times I$ as usual. Using the notation of Proposition 2.11 with $C_k(R) = C_k(Q \times I)$ when $k \geq 1$

and $C_0(R) = 2Q \times NI$, we obtain

$$\begin{aligned} \|(\Theta_\lambda P_\lambda - \gamma_\lambda S_\lambda P_\lambda) f\|_2^2 &= \sum_{R \in \square_\lambda} |R| \left| \mathcal{F}_R \left(\Theta_\lambda (g - \mathcal{F}_R g) \right) \right|^2 \\ &\leq \sum_{R \in \square_\lambda} |R| \left(\sum_{k \geq 0} \left(\mathcal{F}_R \left(\Theta_\lambda (1_{C_k(R)} (g - \mathcal{F}_R g)) \right) \right)^2 \right)^{1/2}. \end{aligned}$$

Since $\Theta_\lambda = \frac{1}{2i}(R_{-\lambda} - R_\lambda)M$ and as the bounded multiplication operator M commutes with $1_{C_k(R)}$, we can apply Proposition 2.11 to continue the estimate with

$$\begin{aligned} &\lesssim \sum_{R \in \square_\lambda} |R| \left(\sum_{k \geq 0} N^{-k\varepsilon} \left(\mathcal{F}_{C_k(R)} |g - \mathcal{F}_R g|^2 \right)^{1/2} \right)^2 \\ &\lesssim \sum_{R \in \square_\lambda} \sum_{k \geq 1} |R| N^{-2k\varepsilon} \mathcal{F}_{2^k Q \times N^k I} |g - \mathcal{F}_R g|^2, \end{aligned}$$

where in the second step we have used $C_k(R)$ is contained in $2^{k+1}Q \times N^{k+1}I$ and has comparable measure, and shifted the index k . Now, we write

$$(7.14) \quad g - \mathcal{F}_R g = \left(g - \mathcal{F}_Q g \right) + \left(\mathcal{F}_Q (g - \mathcal{F}_I g) \right).$$

Then, using Poincaré's inequality [36, Lem. 7.12/16] in the x variable,

$$\int_{2^k Q} |g - \mathcal{F}_Q g|^2 \lesssim 2^{2k} \ell(Q)^2 \int_{2^k Q} |\nabla_x g|^2.$$

Hence, integrating over $N^k I$,

$$\begin{aligned} &\sum_{R \in \square_\lambda} \sum_{k \geq 1} |R| N^{-2k\varepsilon} \mathcal{F}_{2^k Q \times N^k I} |g - \mathcal{F}_Q g|^2 \\ &\lesssim \sum_{R \in \square_\lambda} \sum_{k \geq 1} |R| N^{-2k\varepsilon} 2^{2k} \ell(Q)^2 \mathcal{F}_{2^k Q \times N^k I} |\nabla_x g|^2. \end{aligned}$$

Since $\lambda \sim \ell(Q)$ and $\sum_{R \in \square_\lambda} 1_{2^k Q \times N^k I} = 2^{kn} N^k$, we can continue by

$$\lesssim \lambda^2 \sum_{k \geq 1} N^{-2k\varepsilon} 2^{2k} \iint_{\mathbb{R}^{n+1}} |\nabla_x g|^2 \lesssim \lambda^2 \|\nabla_x g\|_2^2,$$

where we have assumed $N^{2\varepsilon} > 4$ by taking N large enough as we may.

For the second term in (7.14), we use the following lemma as a substitute for Poincaré's inequality. Recall that D_t^α is the fractional derivative of order α on \mathbb{R} defined via the Fourier symbol $|\tau|^\alpha$. We write $H^\alpha(\mathbb{R})$ for space of all $h \in L^2(\mathbb{R})$ with $D_t^\alpha h \in L^2(\mathbb{R})$.

Lemma 7.6. *Assume $h \in H^\alpha(\mathbb{R})$ with $\alpha \in (0, 1/2)$ and let $p, q \in [1, \infty)$ be such that $(1-\alpha)p < q \leq p$. Then for any interval J and any $N \geq 2$,*

$$\left(\int_J |h - \mathcal{F}_J h|^p ds \right)^{1/p} \lesssim \ell(J)^\alpha \left(\sum_{l \geq 1} N^{(\alpha-1)l} \int_{N^l J} |D_t^\alpha h|^q ds \right)^{1/q}.$$

The analogous inequality with $H_t D_t^\alpha h$ instead of $D_t^\alpha h$ on the right-hand side also holds and both inequalities remain valid for $\alpha = 1/2$ and $h \in \dot{H}^{1/2}(\mathbb{R})$.

Proof. We treat the estimates with $D_t^\alpha h$ first. If $\alpha < 1/2$, then we can represent h as a classical Riesz potential

$$h(s) = \frac{1}{2\Gamma(\alpha) \cos(\alpha\pi/2)} \int_{\mathbb{R}} \frac{D_t^\alpha h(\sigma)}{|s - \sigma|^{1-\alpha}} d\sigma \quad (\text{a.e. } s \in \mathbb{R}),$$

see for example [47, Sec. 12.1]. If $\alpha = 1/2$, then we use the modified representation obtained in Corollary 3.5. Hence, in any case we have

$$h(s) - h(s') = C_\alpha \int_{\mathbb{R}} \left(\frac{1}{|s - \sigma|^{1-\alpha}} - \frac{1}{|s' - \sigma|^{1-\alpha}} \right) D_t^\alpha h(\sigma) d\sigma \quad (\text{a.e. } s, s' \in J)$$

for a constant C_α depending only on α . Averaging first in s' and then in s yields

$$\left(\int_J |h(s) - \int_J h|^p ds \right)^{1/p} \leq C_\alpha \left(\int_J \left(\int_{\mathbb{R}} \Phi(s, \sigma) |D_t^\alpha h(\sigma)| d\sigma \right)^p ds \right)^{1/p},$$

where

$$\Phi(s, \sigma) := \int_J \left| \frac{1}{|s - \sigma|^{1-\alpha}} - \frac{1}{|s' - \sigma|^{1-\alpha}} \right| ds'$$

is seen to satisfy

$$|\Phi(s, \sigma)| \lesssim \ell(J)^\alpha \frac{1}{\ell(J)} \min \left\{ \left| \frac{s - \sigma}{\ell(J)} \right|^{\alpha-1}, \left| \frac{s - \sigma}{\ell(J)} \right|^{\alpha-2} \right\} \quad (s \in J, \sigma \in \mathbb{R}),$$

using either the triangle inequality (if $\sigma \in 2J$) or the mean-value theorem (if $\sigma \in {}^c(2J)$). In particular, $\int_{\mathbb{R}} \Phi(s, \sigma) d\sigma \lesssim \ell(J)^\alpha$. So, writing $\Phi = \Phi^{1-1/q} \Phi^{1/q}$, we can apply Hölder's inequality to the inner integral to find

$$\left(\int_J \left(\int_{\mathbb{R}} \Phi(s, \sigma) |D_t^\alpha h(\sigma)| d\sigma \right)^p ds \right)^{1/p} \lesssim \ell(J)^{\alpha-\alpha/q} \left(\int_J \left(\int_{\mathbb{R}} \Phi(s, \sigma) |D_t^\alpha h(\sigma)|^q d\sigma \right)^{p/q} ds \right)^{1/p}$$

and since $p \geq q$ by assumption, we obtain from Minkowski's inequality

$$(7.15) \quad \left(\int_J |h(s) - \int_J h|^p ds \right)^{1/p} \lesssim \ell(J)^{\alpha-\alpha/q} \left(\int_{\mathbb{R}} \left(\int_J \Phi(s, \sigma)^{p/q} ds \right)^{q/p} |D_t^\alpha h(\sigma)|^q d\sigma \right)^{1/q}.$$

Putting $C_0(J) := NJ$ and $C_l(J) := N^{l+1}J \setminus N^l J$, $l \geq 1$, the pointwise bounds for Φ imply

$$\left(\int_J \Phi(s, \sigma)^{p/q} ds \right)^{q/p} \lesssim \ell(J)^{\alpha-1} N^{(\alpha-2)l} \quad (\sigma \in C_l(J)),$$

where it is the term for $l = 0$ that requires the assumption $(\alpha - 1)p/q > -1$. This being said, the claim follows upon splitting $\mathbb{R} = \bigcup_{l=0}^{\infty} C_l(J)$ on the right-hand side of (7.15).

The argument is exactly the same if we want to let $H_t D_t^{1/2} h$ appear on the right-hand side: Indeed, for $\alpha < 1/2$ we can again rely on a classical representation [47, Sec. 12.1]:

$$h(s) = \frac{1}{2\Gamma(\alpha) \sin(\alpha\pi/2)} \int_{\mathbb{R}} \frac{\text{sgn}(s - \sigma) H_t D_t^\alpha h(\sigma)}{|s - \sigma|^{1-\alpha}} d\sigma \quad (\text{a.e. } s \in \mathbb{R}).$$

Likewise, for $\alpha = 1/2$ we have $h = c - I_t^{1/2} H_t (H_t D_t^{1/2} h)$ for some constant c (Corollary 3.5) and the required integral representation for $I_t^{1/2} H_t$ has been obtained in Proposition 3.4. \square

Returning to the second term in (7.14), a telescoping sum and Lemma 7.6 applied with $p = q = 2$ and an arbitrary $\alpha \in (0, 1/2)$ yield the bound

$$\begin{aligned} \iint_{2^k Q \times N^k I} \left| \int_Q (g - \int_I g) \right|^2 &\leq \int_Q \int_{N^k I} \left| g - \int_I g \right|^2 \\ &\lesssim \int_Q (k+1)^2 \sum_{j=0}^k \int_{N^j I} \left| g - \int_{N^j I} g \right|^2 \\ &\lesssim \int_Q (k+1)^2 \sum_{j=0}^k \ell(N^j I)^{2\alpha} \sum_{l \geq 1} N^{(\alpha-1)l} \int_{N^l N^j I} |D_t^\alpha g|^2. \end{aligned}$$

So, using $\sum_{R \in \square_\lambda} 1_{Q \times N^{l+j}I} = N^{l+j}$, we obtain

$$\begin{aligned} & \sum_{R \in \square_\lambda} \sum_{k \geq 1} |R| N^{-2k\varepsilon} \iint_{2^k Q \times N^k I} \left| \int_Q (g - \int_I g) \right|^2 \\ & \lesssim \sum_{k \geq 1} N^{-2k\varepsilon} (k+1)^2 \sum_{j=0}^k N^{2j\alpha} \ell(I)^{2\alpha} \sum_{l \geq 1} N^{(\alpha-1)l} \iint_{\mathbb{R}^{n+1}} |D_t^\alpha g|^2. \end{aligned}$$

Choosing $\alpha < \varepsilon$ allows us to sum and we finally obtain a bound by $\ell(I)^{2\alpha} \|D_t^\alpha g\|_2^2$. Since $\lambda^2 \sim \ell(I)$, this completes the proof of (7.4) and thus the proof of Theorem 2.3.

8. REVERSE HÖLDER ESTIMATES AND NON-TANGENTIAL MAXIMAL ESTIMATES

The first part of this section contains the proof of the reverse Hölder estimate for the parabolic conormal differential of reinforced weak solutions alluded to in Theorem 2.14. In the second part we shall derive some consequences of this estimate: the L^2 -bounds for the modified non-tangential maximal function \tilde{N}_* and a.e. convergence of Whitney averages for reinforced weak solutions as stated in Theorems 2.12 and 2.13.

8.1. Proof of Theorem 2.14. For convenience we will fix some notation which will be used throughout the proof but not always restated in all intermediate results.

We let $\Lambda \times Q \times I$ be a parabolic cylinder of sidelength $r > 0$ defined by $\Lambda = (\lambda - r, \lambda + r)$, $Q = B(x, r)$ and $I = (t - r^2, t + r^2]$ and we assume $8r < \lambda$. Hence, the parabolic enlargement $8\Lambda \times 8Q \times 64I$ is contained in \mathbb{R}_+^{n+2} . We fix a smooth cut-off $\eta : \mathbb{R}_+^{n+2} \rightarrow [0, 1]$ with support in $2\Lambda \times 2Q \times 4I$ that is 1 on an enlargement $\frac{3}{2}\Lambda \times \frac{3}{2}Q \times \frac{9}{4}I$. For a reason which will become clear later on, we choose η to have the product form

$$\eta(\mu, y, s) = \eta_\Lambda(\mu) \eta_Q(y) \eta_I(s),$$

where η_I is symmetric about the midpoint of I . To keep control over implicit constants we may also assume that η is obtained from the one fixed cut-off η_0 for the parabolic cylinder with sidelength $r = 1$ centered at the origin $(\lambda, x, t) = (0, 0, 0)$ by the change of variables

$$\eta(\mu, y, s) = \eta_0\left(\frac{\mu - \lambda}{r}, \frac{y - x}{r}, \frac{s - t}{r^2}\right).$$

We denote the translates of an interval J by $J_k = k\ell(J) + J$, $k \in \mathbb{Z}$. For the sake of clarity we give a name to the translation sums

$$\sum(v) := \sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \iiint_{4\Lambda \times 4Q \times I_k} |v|,$$

where v is a measurable function on \mathbb{R}_+^{n+2} . For most of the proof we work with $4\Lambda \times 4Q$ on the right-hand side and assume only $4r < \lambda$. It is only in the final step where we shall enlarge to $8\Lambda \times 8Q$ as in the statement of the theorem. All bounds below will depend only on n and the ellipticity constants of A , but for simplicity we keep on using the symbol \lesssim .

To begin with, let us recall the classical local estimates. Proofs can be found, for example, in Sections 3.2 and 4.1 of [15]. (The proof for Caccioppoli's inequality there assumes $f = 0$ but the argument is the same when $f \neq 0$).

Lemma 8.1 (Caccioppoli). *Let $\Omega \subset \mathbb{R}_+^{n+2}$ be open and $c > 1$. Let $\Lambda \times Q \times I \subset \Omega$ be a parabolic cylinder of sidelength $r > 0$ such that $c\Lambda \times cQ \times c^2I \subset \Omega$. Given $f \in L_{\text{loc}}^2(\Omega)$, every weak solution u to the inhomogeneous problem $\partial_t u - \text{div}_{\lambda, x} A \nabla_{\lambda, x} u = \text{div}_{\lambda, x} f$ in Ω satisfies*

$$\iiint_{\Lambda \times Q \times I} |\nabla_{\lambda, x} u|^2 \lesssim \frac{1}{r^2} \iiint_{c\Lambda \times cQ \times c^2I} |u|^2 + \iiint_{cQ \times c\Lambda \times c^2I} |f|^2,$$

with an implicit constant depending only on c , n and the ellipticity constants of A .

Lemma 8.2 (Reverse Hölder). *Let $\Omega \subset \mathbb{R}_+^{n+2}$ be open and $c > 1$. Suppose that u is a weak solution to $\partial_t u - \operatorname{div}_{\lambda,x} A \nabla_{\lambda,x} u = 0$ in Ω . Then for every parabolic cylinder $\Lambda \times Q \times I \subset \Omega$ such that $c\Lambda \times cQ \times c^2I \subset \Omega$,*

$$\left(\iint_{\Lambda \times Q \times I} |u|^2 \right)^{1/2} \lesssim \iint_{c\Lambda \times cQ \times c^2I} |u|,$$

with an implicit constant depending only on c , n and the ellipticity constants of A .

The following variant of Lemma 7.6 will be our instrument for localising in time.

Lemma 8.3. *Let $p, q \in [1, \infty)$ satisfy $p/2 < q \leq p$. Then for each interval $J \subset \mathbb{R}$ and every $h \in \dot{H}^{1/2}(\mathbb{R})$,*

$$\left(\int_J |h - \int_J h|^p ds \right)^{1/p} \lesssim \sqrt{\ell(J)} \left(\sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \int_{J_k} |D_t^{1/2} h|^q ds \right)^{1/q}.$$

The analogous inequality with $H_t D_t^{1/2} h$ instead of $D_t^{1/2} h$ on the right-hand side also holds.

Proof. The proof follows that of Lemma 7.6 verbatim, the only difference being that in (7.15) we use the uniform estimate

$$\left(\int_J \Phi(s, \sigma)^{p/q} ds \right)^{q/p} \lesssim \frac{1}{\sqrt{\ell(J)}} \frac{1}{1 + |k|^{3/2}} \quad (\sigma \in J_k)$$

and split $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} J_k$ on the right-hand side. \square

With these estimates at hand, we can already prove a reverse Hölder inequality of required type for the spatial gradient of reinforced weak solutions.

Lemma 8.4. *Let u be a reinforced weak solution to (1.1) and let $c = \iint_{2\Lambda \times 2Q \times I} u$. Then*

$$\left(\iint_{2\Lambda \times 2Q \times 4I} |\nabla_{\lambda,x} u|^2 \right)^{1/2} \lesssim \frac{1}{r} \iint_{4\Lambda \times 4Q \times 16I} |u - c| \lesssim \sum (|\nabla_{\lambda,x} u| + |H_t D_t^{1/2} u|),$$

where $H_t D_t^{1/2} u$ could be replaced with $D_t^{1/2} u$ on the right-hand side.

Proof. The first estimate is a direct consequence of Lemmas 8.1 and 8.2 applied to the weak solution $u - c$. For the second estimate we separate averages in space and time by writing

$$u - c = u - \iint_{2\Lambda \times 2Q} u + \iint_{2\Lambda \times 2Q} u - c.$$

Poincaré's inequality in the spatial variables [36, Lem. 7.12/16] allows for an estimate

$$\begin{aligned} \frac{1}{r} \iint_{4\Lambda \times 4Q \times 16I} |u - \iint_{2\Lambda \times 2Q} u| &\lesssim \iint_{4\Lambda \times 4Q \times 16I} |\nabla_{\lambda,x} u| \\ &\lesssim \sum_{|k| \leq 8} \frac{1}{1 + |k|^{3/2}} \iint_{4\Lambda \times 4Q \times I_k} |\nabla_{\lambda,x} u|. \end{aligned}$$

As for the averages in time, we abbreviate

$$f(s) := \iint_{2\Lambda \times 2Q} u(\mu, y, s) d\mu dy.$$

Since the reinforced weak solution u belongs to the class $\dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}_+^{n+1}))$, we obtain $f \in \dot{H}^{1/2}(\mathbb{R})$ and due to Fubini's theorem,

$$(8.1) \quad H_t D_t^{1/2} f(s) = \iint_{2\Lambda \times 2Q} H_t D_t^{1/2} u(\mu, y, s) d\mu dy,$$

see also Lemma 3.10. By a telescoping sum

$$\frac{1}{r} \iint_{4\Lambda \times 4Q \times 16I} \left| \iint_{2\Lambda \times 2Q} u - c \right| = \frac{1}{\sqrt{\ell(I)}} \int_{16I} |f - \int_I f| \lesssim \frac{1}{\sqrt{\ell(16I)}} \int_{16I} |f - \int_{16I} f|,$$

so that Lemma 8.3 applies to the effect that

$$\begin{aligned} \frac{1}{r} \iiint_{4\Lambda \times 4Q \times 16I} \left| \iint_{2\Lambda \times 2Q} u - c \right| &\lesssim \sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \int_{16I + 16k\ell(I)} |H_t D_t^{1/2} f| \\ &\lesssim \sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \int_{I_k} |H_t D_t^{1/2} f|, \end{aligned}$$

where in the last step we have split $16I + 16k\ell(I)$ into translates of I . We conclude by plugging in the identity (8.1). Note that we could also decide to let $D_t^{1/2} f$ appear in (8.1) and our final estimate. \square

Due to the non-locality of the half-order derivative in time, reverse Hölder estimates for $H_t D_t^{1/2} u$ and $D_t^{1/2} u$ are much harder to get. A part of the the following argument was inspired by [39].

Since $H_t D_t^{1/2}$ annihilates constants, we can write the average to be estimated as

$$\iiint_{\Lambda \times Q \times I} |H_t D_t^{1/2} u|^2 = \iiint_{\Lambda \times Q \times I} |H_t D_t^{1/2} (u - c)|^2, \quad \text{where} \quad c := \iiint_{2\Lambda \times 2Q \times I} u$$

and the same can be done with $D_t^{1/2}$ in place of $H_t D_t^{1/2}$. We treat a local version first.

Lemma 8.5. *Let u be a reinforced weak solution to (1.1) and let $c = \iint_{2\Lambda \times 2Q \times I} u$. Then*

$$\left(\iiint_{\Lambda \times Q \times I} |H_t D_t^{1/2} (\eta(u - c))|^2 \right)^{1/2} \lesssim \sum (|\nabla_{\lambda, x} u| + |H_t D_t^{1/2} u|),$$

where $H_t D_t^{1/2}$ can be replaced with $D_t^{1/2}$ on either side of the inequality.

Proof. We begin with a parabolic rescaling of \mathbb{R}_+^{n+1} , setting $\tilde{u}(\mu, y, s) = u(r\mu, ry, r^2s)$ and similarly $\tilde{\eta}$, \tilde{A} , as well as $\tilde{\Lambda} = (r^{-1}\lambda - 1, r^{-1}\lambda + 1)$ and similarly \tilde{Q} , \tilde{I} . Then,

$$\begin{aligned} \frac{1}{r^{n+1}} \iiint_{\Lambda \times Q \times I} |H_t D_t^{1/2} (\eta(u - c))|^2 d\mu dy ds &= \iiint_{\tilde{\Lambda} \times \tilde{Q} \times \tilde{I}} |H_t D_t^{1/2} (\tilde{\eta}(\tilde{u} - c))|^2 d\mu dy ds \\ &\leq \int_{\mathbb{R}} \|H_t D_t^{1/2} (\tilde{\eta}(\tilde{u} - c))\|_{L^2(\mathbb{R}^{n+1})}^2 ds. \end{aligned}$$

Let \mathcal{F}_t denote the Fourier transform in t -direction only. For the moment take for granted that all of the subsequent integrals are finite. We shall check this in the further course of the proof anyway. The Hilbert space valued version of Plancherel's theorem [57, Sec. II.5] yields

$$\begin{aligned} \int_{\mathbb{R}} \|H_t D_t^{1/2} (\tilde{\eta}(\tilde{u} - c))\|_{L^2(\mathbb{R}^{n+1})}^2 ds &= 2\pi \int_{\mathbb{R}} \|i \operatorname{sgn}(\sigma) |\sigma|^{1/2} \mathcal{F}_t(\tilde{\eta}(\tilde{u} - c))\|_{L^2(\mathbb{R}^{n+1})}^2 d\sigma \\ &= 2\pi \int_{\mathbb{R}} |(\mathcal{F}_t(\tilde{\eta}(\tilde{u} - c)), |\sigma| \mathcal{F}_t(\tilde{\eta}(\tilde{u} - c)))|^2 d\sigma \end{aligned}$$

with the duality brackets denoting the $W^{1,2}(\mathbb{R}^{n+1})$ - $W^{-1,2}(\mathbb{R}^{n+1})$ duality. Since these are (isomorphic to) Hilbert spaces as well, we can apply Cauchy-Schwarz' inequality and Plancherel's theorem backwards in order to obtain altogether

$$(8.2) \quad \begin{aligned} &\frac{1}{r^{n+1}} \iiint_{\Lambda \times Q \times I} |H_t D_t^{1/2} (\eta(u - c))|^2 d\mu dy ds \\ &\leq \left(\int_{\mathbb{R}} \|\tilde{\eta}(\tilde{u} - c)\|_{W^{1,2}(\mathbb{R}^{n+1})}^2 ds \right)^{1/2} \left(\int_{\mathbb{R}} \|\partial_t (\tilde{\eta}(\tilde{u} - c))\|_{W^{-1,2}(\mathbb{R}^{n+1})}^2 ds \right)^{1/2}. \end{aligned}$$

Above we instantly used $|i \operatorname{sgn}(\sigma)| \leq 1$ and so the same analysis would apply if we started out with $D_t^{1/2} (\eta(u - c))$. As η is supported in $2\Lambda \times 2Q \times 4I$, the first integral on the right is bounded by

$$\begin{aligned} \int_{\mathbb{R}} \|\tilde{\eta}(\tilde{u} - c)\|_{W^{1,2}(\mathbb{R}^{n+1})}^2 ds &\lesssim \iiint_{2\tilde{\Lambda} \times 2\tilde{Q} \times 4\tilde{I}} |\tilde{u} - c|^2 + |\nabla_{\lambda, x} \tilde{u}|^2 d\mu dy ds \\ &= \frac{1}{r^{n+3}} \iiint_{2\Lambda \times 2Q \times 4I} |u - c|^2 + r^2 |\nabla_{\lambda, x} u|^2 d\mu dy ds. \end{aligned}$$

Next, let $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ and $\chi \in C_0^\infty(\mathbb{R})$, where ϕ acts in the λ - and x variables and χ acts in t -direction. Using that $\tilde{u} - c$ is a weak solution to a parabolic problem (1.1) with coefficients \tilde{A} ,

$$\begin{aligned} & \int_{\mathbb{R}} \left(\iint_{\mathbb{R}^{n+1}} \tilde{\eta}(\tilde{u} - c) \cdot \bar{\phi} \, d\mu \, dy \right) \overline{\partial_t \chi} \, ds \\ &= \iiint_{\tilde{2\Lambda} \times \tilde{2Q} \times \tilde{4I}} (\tilde{u} - c) \cdot \overline{\partial_t(\tilde{\eta}\phi\chi)} - (\tilde{u} - c) \cdot \overline{\phi\chi\partial_t\tilde{\eta}} \, d\mu \, dy \, ds \\ &= \iiint_{\tilde{2\Lambda} \times \tilde{2Q} \times \tilde{4I}} \tilde{A}\nabla_{\lambda,x}(\tilde{u} - c) \cdot \overline{\nabla_{\lambda,x}(\tilde{\eta}\phi\chi)} - (\tilde{u} - c) \cdot \overline{\phi\chi\partial_t\tilde{\eta}} \, d\mu \, dy \, ds \\ &= \int_{\tilde{4I}} \iint_{\tilde{2\Lambda} \times \tilde{2Q}} \left(\tilde{A}\nabla_{\lambda,x}\tilde{u} \cdot \overline{\nabla_{\lambda,x}(\tilde{\eta}\phi)} - (\tilde{u} - c) \cdot \overline{\phi\partial_t\tilde{\eta}} \, d\mu \, dy \right) \chi \, ds. \end{aligned}$$

This implies the bound for the second integral

$$\begin{aligned} \int_{\mathbb{R}} \|\partial_t(\tilde{\eta}(\tilde{u} - c))\|_{W^{-1,2}(\mathbb{R}^{n+1})}^2 \, ds &\lesssim \iiint_{\tilde{2\Lambda} \times \tilde{2Q} \times \tilde{4I}} |\tilde{u} - c|^2 + |\nabla_{\lambda,x}\tilde{u}|^2 \, d\mu \, dy \, ds \\ &= \frac{1}{r^{n+3}} \iiint_{2\Lambda \times 2Q \times 4I} |u - c|^2 + r^2 |\nabla_{\lambda,x}u|^2 \, d\mu \, dy \, ds. \end{aligned}$$

Returning to (8.2) and clearing the powers of r , we find

$$\iiint_{\Lambda \times Q \times I} |H_t D_t^{1/2}(\eta(u - c))|^2 \, d\mu \, dy \, ds \lesssim \iiint_{2\Lambda \times 2Q \times 4I} \frac{1}{r^2} |u - c|^2 + |\nabla_{\lambda,x}u|^2 \, d\mu \, dy \, ds.$$

Lemma 8.2 applied to the weak solution $u - c$ provides control by

$$\frac{1}{r^2} \left(\iiint_{4\Lambda \times 4Q \times 16I} |u - c| \, d\mu \, dy \, ds \right)^2 + \iiint_{2\Lambda \times 2Q \times 4I} |\nabla_{\lambda,x}u|^2 \, d\mu \, dy \, ds$$

and we conclude by Lemma 8.4. \square

We turn to the corresponding non-local terms, which can be written as

$$\iiint_{\Lambda \times Q \times I} |H_t D_t^{1/2}(1 - \eta)(u - c)|^2 = \iiint_{\Lambda \times Q \times I} |H_t D_t^{1/2}(1 - \eta_I)(u - c)|^2$$

and similarly with $D_t^{1/2}$ in place of $H_t D_t^{1/2}$, since η_Λ and η_Q are independent of t and satisfy $\eta_\Lambda \eta_Q = 1$ on $\Lambda \times Q$. Similar to the proof of Lemma 8.4 we separate

$$(8.3) \quad (1 - \eta_I)(u - c) = (1 - \eta_I) \left(u - \iint_{2\Lambda \times 2Q} u \right) + (1 - \eta_I) \left(\iint_{2\Lambda \times 2Q} u - \iiint_{2\Lambda \times 2Q \times I} u \right).$$

Both addends on the right-hand side are in $\dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}_+^{n+1}))$. To see this, we first note that u is contained therein as a reinforced weak solution. As in the proof of Lemma 8.5 we also have

$$\phi u \in L^2(\mathbb{R}; W^{1,2}(\mathbb{R}^{n+1})) \cap H^1(\mathbb{R}; W^{-1,2}(\mathbb{R}^{n+1})) \quad (\phi \in C_0^\infty(\mathbb{R}_+^{n+2}))$$

and thus $\phi u \in \dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}_+^{n+1}))$. In particular, we obtain $\eta_I u \in \dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}_+^{n+1}))$ and for the other terms we note that $\dot{H}^{1/2}$ -regularity in time inherits to averages in space in virtue of Fubini's theorem, see also Lemma 3.10.

Next, we shall see that the estimate of the first term on the right-hand side of (8.3) can once more be reduced to Lemma 8.4.

Lemma 8.6. *Let u be a reinforced weak solution to (1.1). Then*

$$\left(\iiint_{\Lambda \times Q \times I} \left| H_t D_t^{1/2} \left((1 - \eta_I) \left(u - \iint_{2\Lambda \times 2Q} u \right) \right) \right|^2 \right)^{1/2} \lesssim \sum (|\nabla_{\lambda,x}u| + |H_t D_t^{1/2}u|),$$

where $H_t D_t^{1/2}$ can be replaced with $D_t^{1/2}$ on either side of the inequality.

Proof. We introduce $v = (1 - \eta_I)(u - \mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{2\Lambda \times 2Q} u)$. By construction $v(\mu, y, \cdot)$ is supported in ${}^c(\frac{9}{4}I)$ and $v(\mu, y, \cdot) \in \dot{H}^{1/2}(\mathbb{R})$ for almost every μ, y . This justifies to use the representation

$$H_t D_t^{1/2} v(\mu, y, s) = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\operatorname{sgn}(s - \sigma)}{|s - \sigma|^{3/2}} v(\mu, y, \sigma) d\sigma$$

from Corollary 3.7 almost everywhere on $\Lambda \times Q \times I$ and we can decompose $\mathbb{R} = \bigcup_{j \in \mathbb{Z}} I_j$ on the right-hand side to obtain the pointwise estimate

$$|H_t D_t^{1/2} v(\mu, y, s)| \lesssim \frac{1}{r} \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{1 + |j|^{3/2}} \mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{I_j} |v(\mu, y, \sigma)| d\sigma.$$

Due to Minkowski's inequality we can bound the average over $\Lambda \times Q \times I$ by

$$\begin{aligned} \left(\mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{\Lambda \times Q \times I} |H_t D_t^{1/2} v|^2 \right)^{1/2} &\lesssim \frac{1}{r} \left(\mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{\Lambda \times Q} \left(\sum_{j \in \mathbb{Z}} \frac{1}{1 + |j|^{3/2}} \mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{I_j} |v|^2 \right)^{1/2} \right)^{1/2} \\ &\leq \frac{1}{r} \sum_{j \in \mathbb{Z}} \frac{1}{1 + |j|^{3/2}} \left(\mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{\Lambda \times Q \times I_j} |v|^2 \right)^{1/2}. \end{aligned}$$

The very same argument applies to $D_t^{1/2} v$ and yields the same bound. Now, by definition of v and Poincaré's inequality in the spatial variables

$$\mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{\Lambda \times Q \times I_j} |v|^2 \lesssim \mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{2\Lambda \times 2Q \times 4I_j} \left| u - \mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{2\Lambda \times 2Q} u \right|^2 \lesssim r^2 \mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{2\Lambda \times 2Q \times 4I_j} |\nabla_{\lambda, x} u|^2,$$

so that altogether,

$$\left(\mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{\Lambda \times Q \times I} |H_t D_t^{1/2} v|^2 \right)^{1/2} \lesssim \sum_{j \in \mathbb{Z}} \frac{1}{1 + |j|^{3/2}} \left(\mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{2\Lambda \times 2Q \times 4I_j} |\nabla_{\lambda, x} u|^2 \right)^{1/2}.$$

The upshot of all this is that each integral above falls under the scope of Lemma 8.4, giving after an index-shift the bound

$$\sum_{j \in \mathbb{Z}} \frac{1}{1 + |j|^{3/2}} \left(\sum_{k \in \mathbb{Z}} \frac{1}{1 + |k - j|^{3/2}} \mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{4\Lambda \times 4Q \times I_k} |\nabla_{\lambda, x} u| + |H_t D_t^{1/2} u| \right).$$

It remains to remark that the occurring discrete convolution satisfies

$$(8.4) \quad \sum_{j \in \mathbb{Z}} \left(\frac{1}{1 + |j|^{3/2}} \cdot \frac{1}{1 + |k - j|^{3/2}} \right) \lesssim \frac{1}{1 + |k|^{3/2}} \quad (k \in \mathbb{Z})$$

as can be checked by distinguishing the cases $j \leq |k|/2$ and $j > |k|/2$. \square

At this stage of the proof we have reduced the matter to two final reverse Hölder estimates, which are the ones for

$$H_t D_t^{1/2} \left((1 - \eta_I) \left(\mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{2\Lambda \times 2Q} u - \mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{2\Lambda \times 2Q \times I} u \right) \right)$$

and its counterpart with $D_t^{1/2}$ in place of $H_t D_t^{1/2}$. Note that these functions only depend on the t variable. As for the *odd* half-order derivative $H_t D_t^{1/2}$, the estimate will be a consequence of the following lemma. It is at this point where the symmetry assumption on the cut-off η_I comes into play and in fact the argument would not go through for the *even* half-order derivative $D_t^{1/2}$.

Lemma 8.7. *Let J be a bounded interval and $\eta : \mathbb{R} \rightarrow [0, 1]$ be a smooth cut-off function with support in $4J$ that is identically 1 on $\frac{9}{4}J$. Suppose furthermore that η is symmetric about the midpoint of J . If $f \in \dot{H}^{1/2}(\mathbb{R})$ is such that $(1 - \eta)f \in \dot{H}^{1/2}(\mathbb{R})$, then almost everywhere on J ,*

$$\left| H_t D_t^{1/2} \left((1 - \eta) \left(f - \mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_J f \right) \right) \right| \lesssim \sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \mathop{\mathrm{f}}\!\!\!\mathop{\mathrm{f}}\limits_{J_k} |D_t^{1/2} f| + |H_t D_t^{1/2} f| ds.$$

Proof. After a shift we may assume for simplicity that J is centered at the origin. Let $w := H_t D_t^{1/2}((1 - \eta)(f - \underline{f}_J f))$. Thanks to the support properties of η , Corollary 3.7 yields

$$(8.5) \quad w(\tau) = \frac{-1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\operatorname{sgn}(\tau - s)}{|\tau - s|^{3/2}} (1 - \eta(s)) \left(f(s) - \underline{f}_J f \right) ds \quad (\text{a.e. } \tau \in J)$$

and this defines w as a continuous function on J . For the rest of the proof we will consider w only on J and stick with this particular representative.

Step 1: The bound for $w(\tau) - w(0)$. Since $s \notin \frac{9}{4}J$ whenever the integrand in (8.5) is non-zero, we obtain from the mean-value theorem

$$|w(\tau) - w(0)| \lesssim \int_{\mathbb{R}} \frac{|\tau|}{|s|^{5/2}} |1 - \eta(s)| |f(s) - \underline{f}_J f| ds.$$

Decomposing $\mathbb{R} = \bigcup_{k \geq 0} (J_k \cup J_{-k})$, we find

$$(8.6) \quad |w(\tau) - w(0)| \lesssim \frac{1}{\sqrt{\ell(J)}} \sum_{k=1}^{\infty} \frac{1}{1 + |k|^{5/2}} \left(\underline{f}_{J_k} |f - \underline{f}_J f| + \underline{f}_{J_{-k}} |f - \underline{f}_J f| \right).$$

Now, for every $k \geq 0$ the telescoping argument

$$\underline{f}_{J_k} |f - \underline{f}_J f| \leq \underline{f}_{J_k} |f - \underline{f}_{J_k} f| + 6 \sum_{j=1}^k \underline{f}_{J_j \cup J_{j-1}} |f - \underline{f}_{J_j \cup J_{j-1}} f|$$

brings us in a position where Lemma 8.3 applies in the form

$$\begin{aligned} \frac{1}{\sqrt{\ell(J)}} \underline{f}_{J_k} |f - \underline{f}_J f| &\lesssim \sum_{m \in \mathbb{Z}} \frac{1}{1 + |m|^{3/2}} \left(\underline{f}_{J_{k+m}} |H_t D_t^{1/2} f| + \sum_{j=1}^k \underline{f}_{J_{j+m} \cup J_{j-1+m}} |H_t D_t^{1/2} f| \right) \\ &\leq 2 \sum_{m \in \mathbb{Z}} \left(\sum_{j=0}^k \frac{1}{1 + |m - j|^{3/2}} \right) \underline{f}_{J_m} |H_t D_t^{1/2} f|, \end{aligned}$$

where the second step follows from a simple regrouping of terms. For the corresponding average over J_{-k} on the left-hand side we obtain an analogous bound with an inner sum ranging over $-k \leq j \leq 0$, so that altogether we sum over $|j| \leq k$. These estimates inserted on the right-hand side of (8.6) lead us to

$$|w(\tau) - w(0)| \lesssim \sum_{m \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left(\sum_{k \geq |j|} \frac{1}{1 + |k|^{5/2}} \right) \frac{1}{1 + |m - j|^{3/2}} \underline{f}_{J_m} |H_t D_t^{1/2} f|.$$

By a Riemann sum argument on the sum over k and (8.4),

$$\sum_{j \in \mathbb{Z}} \left(\sum_{k \geq |j|} \frac{1}{1 + |k|^{5/2}} \right) \frac{1}{1 + |m - j|^{3/2}} \lesssim \sum_{j \in \mathbb{Z}} \frac{1}{1 + |j|^{3/2}} \frac{1}{1 + |m - j|^{3/2}} \lesssim \frac{1}{1 + |m|^{3/2}},$$

giving the required bound

$$|w(\tau) - w(0)| \lesssim \sum_{m \in \mathbb{Z}} \frac{1}{1 + |m|^{3/2}} \underline{f}_{J_m} |H_t D_t^{1/2} f|.$$

Step 2: The bound for $w(0)$. The ‘kernel’ in the formula (8.5) for $w(0)$ is odd since η is an even function. Hence, we can omit the average $\underline{f}_J f$ and for a later purpose we write

$$w(0) = \frac{1}{2\sqrt{2\pi}} \lim_{S \rightarrow \infty} \int_{(-S, S)} \frac{\operatorname{sgn}(s)}{|s|^{3/2}} (1 - \eta(s)) f(s) ds.$$

Due to Corollary 3.5 we can represent f as a potential

$$f(s) = c + \frac{1}{\sqrt{2\pi}} \left(\int_{(-1, 1)} \frac{g(\rho)}{|s - \rho|^{1/2}} d\rho + \int_{c(-1, 1)} \frac{g(\rho)}{|s - \rho|^{1/2}} - \frac{g(\rho)}{|\rho|^{1/2}} d\rho \right)$$

with $g = D_t^{1/2} f$ and c a constant. Since these integrals converge absolutely for almost every $s \in \mathbb{R}$, see Section 3, we can plug in this representation into the one for $w(0)$ and apply Fubini’s theorem.

During this procedure the terms c and $g(\rho)|\rho|^{-1/2}$ depending not on s vanish again by oddness. So, after a substitution $s = \rho\sigma$,

$$(8.7) \quad w(0) = \lim_{S \rightarrow \infty} \frac{1}{4\pi} \int_{\mathbb{R}} \left(\int_{|\sigma| \leq \frac{S}{|\rho|}} \frac{\operatorname{sgn}(\sigma)}{|\sigma|^{3/2}} \cdot \frac{1 - \eta_J(\sigma\rho)}{|1 - \sigma|^{1/2}} d\sigma \right) \frac{g(\rho)}{\rho} d\rho.$$

Let us inspect the inner integral for fixed $S \geq \ell(J)$: In the case $|\rho| < \ell(J)$ we can simply use the support properties of η to bound this in absolute value by

$$\int_{\frac{9\ell(J)}{8|\rho|} \leq |\sigma|} \frac{1}{|\sigma|^{3/2}} \cdot \frac{1}{|1 - \sigma|^{1/2}} d\sigma \lesssim \frac{|\rho|}{\ell(J)},$$

where we have used the estimate $|\sigma| \leq 9|1 - \sigma|$ on the domain of integration in the second step. If $\ell(J) < |\rho| < 8\ell(J)$, then the estimate above remains true for the simple reason that both sides are comparable to absolute constants. If finally $|\rho| \geq 8\ell(J)$, then we additionally make use of the symmetry of η to rewrite the integral under consideration as

$$\begin{aligned} & \int_{|\sigma| \leq \frac{S}{|\rho|}} \frac{\operatorname{sgn}(\sigma)}{|\sigma|^{3/2}} \left(\frac{1}{|1 - \sigma|^{1/2}} - 1 \right) d\sigma \\ & - \int_{|\sigma| \leq \frac{4\ell(J)}{|\rho|}} \frac{\operatorname{sgn}(\sigma)}{|\sigma|^{3/2}} \left(\frac{1}{|1 - \sigma|^{1/2}} - 1 \right) \eta(\sigma\rho) d\sigma. \end{aligned}$$

The first term is bounded uniformly in S and ρ since $||1 - \sigma|^{-1/2} - 1| \lesssim |\sigma|$ for $|\sigma|$ small, and in the limit for $S \rightarrow \infty$ it converges to the value $C \in \mathbb{R}$ of the corresponding integral over the real line. Similarly, the second term can be controlled by $\ell(J)^{1/2}/|\rho|^{1/2}$ since we have $|\sigma| \leq 1/2$ in the domain of integration. This justifies to use Lebesgue's theorem when passing to the limit in S in (8.7). As a result,

$$(8.8) \quad |w(0)| \lesssim \int_{8J} |g(\rho)| d\rho + \left| C \int_{c(8J)} \frac{g(\rho)}{\rho} d\rho \right| + \sqrt{\ell(J)} \int_{c(8J)} \frac{|g(\rho)|}{|\rho|^{3/2}} d\rho.$$

The truncated Hilbert transform in the middle term can be handled similar to the classical proof of Cotlar's lemma but for the reader's convenience we repeat the short argument. So, we let g_1 be equal to g on $8J$ and zero otherwise. Then for any $0 < \varepsilon < 2\ell(J)$ and $s \in J$,

$$\int_{c(8J)} \frac{g(\rho)}{\rho} d\rho = \int_{c(8J)} \left(\frac{1}{\rho} - \frac{1}{s - \rho} \right) g(\rho) d\rho + \int_{|s - \rho| > \varepsilon} \frac{g(\rho)}{s - \rho} d\rho - \int_{|s - \rho| > \varepsilon} \frac{g_1(\rho)}{s - \rho} d\rho.$$

The mean-value theorem applies to the first term so that in the limit $\varepsilon \rightarrow 0$,

$$\left| \int_{c(8J)} \frac{g(\rho)}{\rho} d\rho \right| \lesssim \int_{c(8J)} \frac{\ell(J)|g(\rho)|}{|\rho|^2} d\rho + |H_t g(s)| + |H_t g_1(s)| \quad (\text{a.e. } s \in J).$$

In view of Tchebychev's inequality and the weak-type bound for the Hilbert transform of $g_1 \in L^1(\mathbb{R})$ we can find some $s \in J$ such that simultaneously

$$|H_t g(s)| \lesssim \int_{8J} |H_t g| \quad \text{and} \quad |H_t g_1(s)| \lesssim \int_{8J} |g|.$$

Returning to (8.8) and plugging in $g = D_t^{1/2} f$,

$$|w(0)| \lesssim \sqrt{\ell(J)} \int_{c(8J)} |D_t^{1/2} f| \frac{d\rho}{|\rho|^{3/2}} + \int_{8J} |H_t D_t^{1/2} f| + |D_t^{1/2} f| d\rho.$$

Now, the usual split up into translates of J gives

$$|w(0)| \lesssim \sum_{m \in \mathbb{Z}} \frac{1}{1 + |m|^{3/2}} \int_{J_m} |D_t^{1/2} f| + |H_t D_t^{1/2} f|,$$

which, together with the outcome of Step 1, establishes the required bound even uniformly on J . The claim follows by averaging over $\tau \in J$. \square

Applying this lemma to $f = \mathbb{f}\mathbb{f}_{2\Lambda \times 2Q} u$ and $\eta = \eta_I$ on $J = I$ completes the reverse Hölder estimate for $H_t D_t^{1/2} u$ with an enlargement to $4\Lambda \times 4Q$ in space on the right-hand side:

Corollary 8.8. *Let u be a reinforced weak solution to (1.1). Then*

$$\left(\mathbb{f}\mathbb{f}\mathbb{f}_{\Lambda \times Q \times I} \left| H_t D_t^{1/2} \left((1 - \eta_I) \left(\mathbb{f}\mathbb{f}_{2\Lambda \times 2Q} u - \mathbb{f}\mathbb{f}\mathbb{f}_{2\Lambda \times 2Q \times I} u \right) \right) \right|^2 \right)^{1/2} \lesssim \sum (|D_t^{1/2} u| + |H_t D_t^{1/2} u|).$$

The missing piece in the proof of Theorem 2.14 is the corresponding estimate with $D_t^{1/2}$ on the left-hand side. We call the reader's attention to the fact that this will not be achieved by an argument for functions on \mathbb{R} but by resorting to the *full* reverse Hölder estimate for $\mathbb{f}\mathbb{f}\mathbb{f}_{2\Lambda \times 2Q \times 4I} |H_t D_t^{1/2} u|^2$ that we have just completed. Since we are using this estimate already for the enlargement $2\Lambda \times 2Q$ in space, we finally obtain an enlargement $8\Lambda \times 8Q$ on the right-hand side as stated in Theorem 2.14.

Lemma 8.9. *Let u be a reinforced weak solution to (1.1). Then*

$$\begin{aligned} & \left(\mathbb{f}\mathbb{f}\mathbb{f}_{\Lambda \times Q \times I} \left| D_t^{1/2} \left((1 - \eta_I) \left(\mathbb{f}\mathbb{f}_{2\Lambda \times 2Q} u - \mathbb{f}\mathbb{f}\mathbb{f}_{2\Lambda \times 2Q \times I} u \right) \right) \right|^2 \right)^{1/2} \\ & \lesssim \sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \mathbb{f}\mathbb{f}\mathbb{f}_{8\Lambda \times 8Q \times I_k} |\nabla_{\lambda, x} u| + |D_t^{1/2} u| + |H_t D_t^{1/2} u| \, d\lambda \, dx \, dt. \end{aligned}$$

Proof. Let $f := \mathbb{f}\mathbb{f}_{2\Lambda \times 2Q} u$ and $h := f - \mathbb{f}_I f$. Rewriting the left-hand side of the estimate in question by means of these two functions reveals that it suffices to prove

$$(8.9) \quad \begin{aligned} & \left(\mathbb{f}_I |D_t^{1/2} (\eta_I h)|^2 \right)^{1/2} + \left(\mathbb{f}_I |D_t^{1/2} f|^2 \right)^{1/2} \\ & \lesssim \sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \mathbb{f}\mathbb{f}\mathbb{f}_{8\Lambda \times 8Q \times I_k} |\nabla_{\lambda, x} u| + |D_t^{1/2} u| + |H_t D_t^{1/2} u| \, d\lambda \, dx \, dt, \end{aligned}$$

where we also used $D_t^{1/2} h = D_t^{1/2} f$ as the half-order derivative annihilates constants. We treat both terms separately.

Step 1: The first term in (8.9). Since $H_t^2 = -1$ as bounded operators on $L^2(\mathbb{R})$,

$$\left(\mathbb{f}_I |D_t^{1/2} (\eta_I h)|^2 \right)^{1/2} \lesssim \left(\frac{1}{\ell(I)} \int_{\mathbb{R}} |H_t D_t^{1/2} (\eta_I h)|^2 \right)^{1/2}.$$

Let η_{4I} have the same properties as η_I but for the interval $4I$ instead of I . In particular, η_{4I} is supported in $16I$ and identically 1 on $9I$, which contains the support of η_I . Using the relation $1 - \eta_{4I} = (1 - \eta_I)(1 - \eta_{4I})$ in the second line of the following computation, we can use commutators to expand

$$\begin{aligned} H_t D_t^{1/2} (\eta_I h) &= H_t D_t^{1/2} h - H_t D_t^{1/2} ((1 - \eta_I)(1 - \eta_{4I} + \eta_{4I})h) \\ &= H_t D_t^{1/2} h - H_t D_t^{1/2} ((1 - \eta_{4I})h) - [H_t D_t^{1/2}, 1 - \eta_I](\eta_{4I} h) - (1 - \eta_I) H_t D_t^{1/2} (\eta_{4I} h) \\ &= \eta_I H_t D_t^{1/2} f - \eta_I H_t D_t^{1/2} ((1 - \eta_{4I})h) + [H_t D_t^{1/2}, \eta_I](\eta_{4I} h). \end{aligned}$$

Pick some $p \in (1, 2)$. From the kernel bounds (6.4) and Young's inequality we obtain that the commutator $[H_t D_t^{1/2}, \eta_I]$ is bounded $L^p(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ with norm controlled by $\ell(I)^{-1/p}$. Thus, the various support assumptions and the definition of h yield

$$\begin{aligned} & \left(\mathbb{f}_I |D_t^{1/2} (\eta_I h)|^2 \right)^{1/2} \lesssim \left(\mathbb{f}_{4I} |H_t D_t^{1/2} f|^2 \right)^{1/2} + \left(\mathbb{f}_{4I} \left| H_t D_t^{1/2} \left((1 - \eta_{4I})(f - \mathbb{f}_I f) \right) \right|^2 \right)^{1/2} \\ & \quad + \frac{1}{\sqrt{\ell(I)}} \left(\mathbb{f}_{16I} |f - \mathbb{f}_I f|^p \right)^{1/p}. \end{aligned}$$

Plugging in the definition of f , the first term is bounded by the L^2 average of $H_t D_t^{1/2} u$ on $2\Lambda \times 2Q \times 4I$, which can be controlled as desired since we have already completed the reverse Hölder estimate stated in Theorem 2.14 for $H_t D_t^{1/2} u$ on the left-hand side with an enlargement factor of 4 in the spatial directions. The second term can be controlled by Lemma 8.7 applied with $\eta = \eta_{4I}$ on $J = 4I$: Indeed, we take the average of f over $\frac{1}{4}J$ instead of J here but the only change in the argument would be to use translates of $\frac{1}{4}J$ in (8.6), too. Finally, for the third term we use a telescopic sum and then Lemma 8.3 with $q = 1$ to obtain

$$\frac{1}{\sqrt{\ell(I)}} \left(\int_{16I} |f - \int_I f|^p \right)^{1/p} \lesssim \frac{1}{\sqrt{\ell(I)}} \left(\int_{16I} |f - \int_{16I} f|^p \right)^{1/p} \lesssim \sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \int_{(16I)_k} |D_t^{1/2} f|,$$

and so all we have to do is to decompose the translates of $16I$ into those of I . We note that this estimate required $p < 2$. This completes the treatment of the first term in (8.9).

Step 2: The second term in (8.9). Using again $H_t^2 = -1$ and the L^2 boundedness of the Hilbert transform, we find

$$\left(\int_I |D_t^{1/2} f|^2 \right)^{1/2} \lesssim \left(\int_{4I} |H_t D_t^{1/2} f|^2 \right)^{1/2} + \left(\int_I |H_t(1_{c(4I)} H_t D_t^{1/2} f)|^2 \right)^{1/2}.$$

From Step 1 we know how to handle the first integral on the right. As for the second one, we put $g := H_t D_t^{1/2} f$ and obtain from the singular integral representation of the Hilbert transform

$$H_t(1_{c(4I)} H_t D_t^{1/2} f)(t) = \int_{c(4I)} \frac{g(\rho)}{\rho - t} d\rho \quad (\text{a.e. } t \in I).$$

Now, we can follow the Cotlar-type argument succeeding (8.8) to deduce the uniform bound

$$|H_t(1_{c(4I)} H_t D_t^{1/2} f)(t)| \lesssim \sum_{m \in \mathbb{Z}} \frac{1}{1 + |m|^{3/2}} \int_{I_m} |D_t^{1/2} f| + |H_t D_t^{1/2} f| \quad (\text{a.e. } t \in I).$$

We conclude by averaging over $t \in I$ and plugging in the definition of f . \square

Remark 8.10. As the reader may have noticed, we could have also written Lemma 8.9 as a statement on real functions, more in the spirit of Lemma 8.7. In fact, the proof above has revealed the following estimate, using the notation and assumptions of Lemma 8.7:

$$\left(\int_J |D_t^{1/2} \left((1 - \eta) \left(f - \int_J f \right) \right)|^2 \right)^{1/2} \lesssim \left(\int_{4J} |H_t D_t^{1/2} f|^2 \right)^{1/2} + \sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \int_{J_k} (|D_t^{1/2} f| + |H_t D_t^{1/2} f|).$$

8.2. Proof of Theorem 2.12. Again $\Lambda \times Q \times I$ denotes a parabolic cylinder in \mathbb{R}_+^{n+2} but to provide for the non-tangential character of the estimates in question we shall now assume its sidelength to be comparable to its distance to the boundary, say $\Lambda = (7\lambda, 9\lambda)$, $Q = B(x, \lambda)$, and $I = (t - \lambda^2, t + \lambda^2)$, for convenience.

Recall that for any measurable function F on the upper half-space \mathbb{R}_+^{n+2} the non-tangential maximal function is given by

$$\tilde{N}_* F(x, t) = \sup_{\lambda > 0} \left(\iiint_{\Lambda \times Q \times I} |F(\mu, y, s)|^2 d\mu dy ds \right)^{1/2},$$

where the supremum is taken over the family of parabolic cubes described above.

It is well-known that the non-tangential maximal function can be controlled by the corresponding square function. For convenience and later reference we include the short argument proving

Lemma 8.11. *For any measurable function F on the upper half-space \mathbb{R}_+^{n+2} the estimate*

$$\sup_{\lambda > 0} \int_{\frac{7}{8}\lambda}^{\frac{9}{8}\lambda} \|F_\mu\|_2^2 d\mu \lesssim \|\tilde{N}_* F\|_2^2 \lesssim \int_0^\infty \|F_\mu\|_2^2 \frac{d\mu}{\mu}$$

holds true where $F_\mu = F(\mu, \cdot)$. If the rightmost integral is finite, then for a.e. $(x, t) \in \mathbb{R}^{n+1}$,

$$\lim_{\lambda \rightarrow 0} \iiint_{\Lambda \times Q \times I} |F|^2 d\mu dy ds = 0.$$

Proof. Spelling out the norm $\|\tilde{N}_* F\|_2^2$ gives up to a multiplicative constant depending only on n ,

$$\iint_{\mathbb{R}^{n+1}} \sup_{\lambda > 0} \left(\iiint_{\mathbb{R}_+^{n+2}} \frac{1_{(8\mu/9, 8\mu/7)}(\lambda) 1_{B(y, \lambda)}(x) 1_{(s-\lambda^2, s+\lambda^2)}(t)}{\lambda^{n+3}} |F(\mu, y, s)|^2 d\mu dy ds \right) dx dt,$$

where 1_E denotes the characteristic function of a set E . Hence, the lower estimate required in the lemma simply follows on pulling the supremum over $\lambda > 0$ outside the double integral and applying Tonelli's theorem. Likewise, for the upper estimate we use

$$\frac{1}{\lambda^{n+3}} \cdot 1_{(8/9\mu, 8/7\mu)}(\lambda) 1_{B(y, \lambda)}(x) 1_{(s-\lambda^2, s+\lambda^2)}(t) \leq \frac{2^{n+3}}{\mu^{n+3}} \cdot 1_{B(y, 2\mu)}(x) 1_{(s-4\mu^2, s+4\mu^2)}(t),$$

pull the supremum over $\lambda > 0$ into the triple integral, and apply Tonelli's theorem.

Now, we apply the first part of the lemma to $1_{(0, 9\varepsilon/8)}(\mu) F(\mu, y, s)$ in place of F , where $\varepsilon > 0$. It follows

$$\iint_{\mathbb{R}^{n+1}} \sup_{0 < \lambda < \varepsilon} \left(\iiint_{\Lambda \times Q \times I} |F(\mu, y, s)|^2 d\mu dy ds \right) dx dt \lesssim \int_0^\varepsilon \|F_\mu\|_2^2 \frac{d\mu}{\mu},$$

proving the a.e. convergence of averages claimed in the second part of the lemma. \square

Let us recall that the backward parabolic conormal differential of reinforced weak solutions is defined by

$$\tilde{D}_A u(\lambda, x, t) = \begin{bmatrix} \partial_{\nu_A} u(\lambda, x, t) \\ \nabla_x u(\lambda, x, t) \\ D_t^{1/2} u(\lambda, x, t) \end{bmatrix}.$$

Using this notation, we can formulate a short hand variant of Theorem 2.14 that is better suited to maximal functions.

Lemma 8.12. *For any reinforced weak solution u to (1.1) and any parabolic cube $\Lambda \times Q \times I$ with $8\Lambda \subset (0, \infty)$,*

$$\left(\iiint_{\Lambda \times Q \times I} |D_A u|^2 d\mu dy ds \right)^{1/2} \lesssim \sum_{m \geq 0} 2^{-m} \iiint_{8\Lambda \times 8Q \times 4^m I} |D_A u| + |\tilde{D}_A u| d\mu dy ds.$$

Proof. Observe that the right hand sides of the reverse Hölder inequalities in Theorem 2.14 involve integrals of $|D_A u| + |\tilde{D}_A u|$. To convert the translates on intervals into the formulation above, use that for any measurable function f on the real line

$$\sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \int_{I_k} |f| \leq \int_I |f| + \sum_{m \geq 0} 4^{-3m/2} \left(\sum_{4^m \leq |k| < 4^{m+1}} \int_{I_k} |f| \right) \leq \sum_{m \geq 0} 2^{-m} \int_{4^m I} |f|. \quad \square$$

Note that if we introduce the operator matrices

$$H_t^\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & H_t \end{bmatrix} \quad \text{and} \quad \tilde{P} = \begin{bmatrix} 0 & \operatorname{div}_x & -H_t D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -D_t^{1/2} & 0 & 0 \end{bmatrix},$$

then $D_A u = H_t^\theta \tilde{D}_A u$, H_t^θ commutes with M and intertwines P and \tilde{P} with $P H_t^\theta = H_t^\theta \tilde{P}$, so that the operator $\tilde{P} M$ has the same operator theoretic properties as $P M$. Also we have the equation $\partial_\lambda \tilde{D}_A u + \tilde{P} M \tilde{D}_A u = 0$ and the link with the parabolic equation guarantees the off-diagonal estimates for $\tilde{P} M$ and $M \tilde{P}$ as well. Alternatively, this can be seen by a verbatim repetition of the proof of Proposition 2.11. Thus, in the following we may argue with $\tilde{D}_A u$ as we do with $D_A u$ and we shall only provide the details involving $D_A u$.

After these preparations we are ready to prove Theorem 2.12.

Step 1: The non-tangential estimate. Let $h \in \overline{\mathbb{R}(PM)}$ and let $F = Sh$ be its semigroup extension $F(\mu, y, s) = e^{-\mu[PM]}h(y, s)$.

The lower bound for \tilde{N}_*f follows from Lemma 8.11 and strong continuity of the semigroup:

$$\|\tilde{N}_*F\|_2^2 \gtrsim \lim_{\lambda \rightarrow 0} \int_{\frac{7}{8}\lambda}^{\frac{9}{8}\lambda} \|e^{-\mu[PM]}h\|_2^2 d\mu = \|h\|_2^2.$$

We turn to the interesting upper estimate. In view of (2.12) we have spectral decompositions $h = h^+ + h^-$, where $h^\pm = \chi^\pm(PM)h \in \mathbb{H}^\pm(PM)$, and thus $F = F^+ + F^-$, where $F^\pm = Sh^\pm$. Since $F^+ = C_0^+h^+$ solves $\partial_\lambda F^+ + PMF^+ = 0$ in the weak sense in \mathbb{R}_+ , Theorem 2.2 yields a representation $F^+ = D_A u$ for some reinforced weak solution u to (1.1). Similarly, for $\lambda < 0$, the reflection $G_\lambda^- = (F^-)_{-\lambda} = (C_0^-h^-)_\lambda$ solves $\partial_\lambda G^- + PMG^- = 0$ on \mathbb{R}_- so that the similar analysis can be performed. So from now on, we may as well assume $h = h^+$ to simplify matters and that $F = C_0^+h = D_A u$ for some reinforced weak solution u to (1.1).

We fix $(x, t) \in \mathbb{R}^{n+1}$ and take $\Lambda \times Q \times I$ as above. Setting $\tilde{F} = \tilde{D}_A u$, Lemma 8.12 asserts

$$\left(\iint_{\Lambda \times Q \times I} |F|^2 d\mu dy ds \right)^{1/2} \lesssim \sum_{m \geq 0} 2^{-m} \iiint_{8\Lambda \times 8Q \times 4^m I} |F| + |\tilde{F}| d\mu dy ds.$$

To control the right hand side, as mentioned above, it is enough to obtain bounds for the terms involving F using the L^2 estimate on h ; those for \tilde{F} would lead to the same estimate with P replaced by \tilde{P} and h replaced by $H_t^\theta h$, which is in L^2 as well.

We introduce $R_\mu := (1 + i\mu PM)^{-1}$ and split $F_\mu = (F_\mu - R_\mu h) + R_\mu h$. After an application of the Cauchy-Schwarz inequality, the same argument as in the proof of Lemma 8.11 yields

$$\begin{aligned} & \iint_{\mathbb{R}^{n+1}} \sup_{\lambda > 0} \left(\sum_{m \geq 0} 2^{-m} \iiint_{8\Lambda \times 8Q \times 4^m I} |F_\mu - R_\mu h| d\mu dy ds \right)^2 dx dt \\ & \lesssim \sum_{m \geq 0} 2^{-m} \iint_{\mathbb{R}^{n+1}} \sup_{\lambda > 0} \left(\iiint_{8\Lambda \times 8Q \times 4^m I} |F_\mu - R_\mu h|^2 d\mu dy ds \right)^2 dx dt \\ & \lesssim \sum_{m \geq 0} 2^{-m} \int_0^\infty \|F_\mu - R_\mu h\|_2^2 \frac{d\mu}{\mu} \\ & = \int_0^\infty \|F_\mu - R_\mu h\|_2^2 \frac{d\mu}{\mu}. \end{aligned}$$

Now, we can use the functional calculus for PM to write $F_\mu - R_\mu h = \psi(\mu PM)h$ with $\psi(z) = e^{-[z]} - (1 + iz)^{-1}$. As ψ is holomorphic on some double sector containing the spectrum of PM and has polynomial decay at 0 and ∞ , the quadratic estimates for PM obtain in Theorem 2.3 yield

$$\int_0^\infty \|F_\mu - R_\mu h\|_2^2 \frac{d\mu}{\mu} = \int_0^\infty \|\psi(\mu PM)h\|_2^2 \frac{d\mu}{\mu} \sim \|h\|_2^2.$$

The remaining term is

$$(8.10) \quad \sum_{m \geq 0} 2^{-m} \iiint_{8\Lambda \times 8Q \times 4^m I} |R_\mu h| d\mu dy ds.$$

Let $q \in (1, 2)$ be such that we have Proposition 2.11 on L^q off-diagonal decay for R_μ at our disposal. With $N_0 \in \mathbb{N}$ as provided by this proposition we define for $J \subset \mathbb{R}$ and $B \subset \mathbb{R}^n$ the annuli

$$C_0(B \times J) = 2B \times N_0 J, \quad C_k(B \times J) = (2^{k+1}B \times N_0^{k+1}J) \setminus (2^k B \times N_0^k J)$$

and we obtain an estimate

$$\begin{aligned} \iint_{8Q \times 4^m I} |R_\mu h| \, dy \, ds &\leq \left(\iint_{8Q \times 4^m I} |R_\mu h|^q \, dy \, ds \right)^{1/q} \\ &\lesssim \sum_{k \geq 0} N_0^{-\varepsilon k} \left(\iint_{C_k(8Q \times 4^m I)} |h|^q \, dy \, ds \right)^{1/q} \\ &\lesssim \sum_{k \geq 0} N_0^{-\varepsilon k} \left(\int_{2^{k+4}Q} \left(\int_{N_0^{k+1}4^m I} |h|^q \, dy \right) \, ds \right)^{1/q} \end{aligned}$$

for some $\varepsilon > 0$ and implicit constants independent of $m \geq 0$ and $\mu \in 8\Lambda$. By means of the q -adapted maximal operators acting separately on each variable by

$$\mathcal{M}_t^q f(x, t) = \sup_{J \ni t} \left(\int_J |f(x, s)|^q \, ds \right)^{1/q} \quad \text{and} \quad \mathcal{M}_x^q f(x, t) = \sup_{B \ni x} \left(\int_B |f(y, t)|^q \, dy \right)^{1/q},$$

where $J \subset \mathbb{R}$ and $B \subset \mathbb{R}^n$ denote intervals and balls, respectively, the ongoing estimate can be completed as

$$\iint_{8Q \times 4^m I} |R_\mu h| \, dy \, ds \lesssim \mathcal{M}_x^q(\mathcal{M}_t^q h)(x, t) \quad (m \geq 0, \mu \in 8\Lambda).$$

Since $q < 2$, the q -adapted maximal operators are bounded on L^2 . Whence, the term defined in (8.10) has non-tangential maximal function bounds

$$\iint_{\mathbb{R}^{n+1}} \sup_{\lambda > 0} \left(\sum_{m \geq 0} 2^{-m} \iint_{8\Lambda \times 8Q \times 4^m I} |R_\mu h| \, d\mu \, dy \, ds \right)^2 \, dx \, dt \lesssim \|\mathcal{M}_x^q \mathcal{M}_t^q h\|_2^2 \lesssim \|h\|_2^2.$$

Altogether, we obtain

$$(8.11) \quad \|\tilde{N}_*(F)\|_2^2 \lesssim \iint_{\mathbb{R}^{n+1}} \sup_{\lambda > 0} \left(\sum_{m \geq 0} 2^{-m} \iint_{8\Lambda \times 8Q \times 4^m I} |F(\mu, y, s)| \, d\mu \, dy \, ds \right)^2 \, dx \, dt \lesssim \|h\|_2^2$$

as required.

Step 2: Almost everywhere convergence of averages. In order to complete the proof of the theorem we have yet to show that for any $h \in \overline{\mathbf{R}(PM)}$ there is convergence

$$(8.12) \quad \lim_{\lambda \rightarrow 0} \iint_{\Lambda \times Q \times I} |Sh(\mu, y, s) - h(x, t)|^2 \, d\mu \, dy \, ds = 0 \quad (\text{a.e. } (x, t) \in \mathbb{R}^{n+1}).$$

As we have the L^2 estimate on the maximal function, it suffices to prove this for h in some dense subset of $\overline{\mathbf{R}(PM)}$.

Lemma 8.13. *There exists $\delta_0 > 0$ depending only on dimension and the ellipticity constants of A such that if $|\frac{1}{p} - \frac{1}{2}| < \delta_0$, then $\{h \in \mathbf{R}(PM) \cap \mathbf{D}(PM) : PMh \in L^p(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})\}$ is dense in $\overline{\mathbf{R}(PM)}$.*

Proof. We take δ_0 as in Lemma 2.10, thereby guaranteeing that all resolvents $R_\mu = (1 + i\mu PM)^{-1}$, $\mu \in \mathbb{R}$, map $\mathbb{L}^p \cap \mathbb{L}^2$ into itself where $\mathbb{L}^p = L^p(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$. For $m \in \mathbb{N}$ define

$$T_m \in \mathcal{L}(\mathbb{L}^2), \quad T_m h = imR_{im}PMR_{i/m}h.$$

Since PM is densely defined and bisectorial, the T_m are uniformly bounded with respect to m and on $\overline{\mathbf{R}(PM)}$ they converge strongly to the identity, see [35, Prop. 3.2.2] for an explicit statement. Now, let $h \in \overline{\mathbf{R}(PM)}$. We can choose functions $h_m \in \mathbb{L}^p \cap \mathbb{L}^2$ such that $h_m \rightarrow h$ in \mathbb{L}^2 . Then also $T_m h_m \rightarrow h$ in \mathbb{L}^2 . However, by definition we have $T_m h_m \in \mathbf{R}(PM) \cap \mathbf{D}(PM)$ and expanding

$$PMT_m h_m = imh_m + R_{i/m}h_m + m^2R_{im}h_m - imR_{im}R_{i/m}h_m,$$

we read off $PMT_m h_m \in \mathbb{L}^p \cap \mathbb{L}^2$ as desired. \square

We complete the proof of the theorem by demonstrating (8.12) if $h \in \mathbf{R}(PM) \cap \mathbf{D}(PM)$ satisfies $PMh \in L^p(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$ for some $p > 2$. As in Step 1 we let $R_\mu = (1 + i\mu PM)^{-1}$ and $\psi(z) = e^{-[z]} - (1 + iz)^{-1}$. We start out by estimating

$$\begin{aligned} \iiint_{\Lambda \times Q \times I} |Sh(\mu, y, s) - h(x, t)|^2 d\mu dy ds &\lesssim \iiint_{\Lambda \times Q \times I} |\psi(\mu PM)h(y, s)|^2 d\mu dy ds \\ &+ \iiint_{\Lambda \times Q \times I} |R_\mu h(y, s) - h(y, s)|^2 d\mu dy ds \\ &+ \iint_{Q \times I} |h(y, s) - h(x, t)|^2 dy ds. \end{aligned}$$

Thanks to Lemma 8.11 and the quadratic estimates for PM , the first term vanishes for a.e. $(x, t) \in \mathbb{R}^{n+1}$ in the limit $\lambda \rightarrow 0$. The third term vanishes for every (parabolic) Lebesgue point of $h \in L^2(\mathbb{R}^{n+1})$, that is, for a.e. $(x, t) \in \mathbb{R}^{n+1}$. Since $h \in \mathbf{D}(PM)$, the middle term can be rewritten as

$$\begin{aligned} \iiint_{\Lambda \times Q \times I} |\mu R_\mu PMh(y, s)|^2 d\mu dy ds &\lesssim \lambda^2 \iiint_{\Lambda \times Q \times I} |R_\mu PMh(y, s)|^2 d\mu dy ds \\ &\lesssim \lambda^2 (\mathcal{M}_x^2 \mathcal{M}_t^2 (PMh)(x, t))^2, \end{aligned}$$

where the second step follows from the L^2 off-diagonal estimates for R_μ similar to the estimate of (8.10). Hence, these terms vanish in the limit $\lambda \rightarrow 0$ if only the maximal function $\mathcal{M}_x^2 \mathcal{M}_t^2 (PMh)(x, t)$ is finite. This happens to be almost everywhere thanks to our additional assumption on h : In fact, \mathcal{M}_t^2 and \mathcal{M}_x^2 are bounded on $L^p(\mathbb{R}^{n+1})$ since $p > 2$ and thus $PMh \in L^p(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$ implies $\mathcal{M}_x^2 \mathcal{M}_t^2 (PMh) \in L^p(\mathbb{R}^{n+1})$.

Remark 8.14. Although it is not needed for our applications, we observe that the almost everywhere convergence (8.12) holds also for $h \in \mathbf{N}(PM)$. In this case, as $(Sh)_\lambda = h$ for all $\lambda > 0$, we can control the averages by the parabolic Hardy-Littlewood maximal operator $\mathcal{M}_{(x,t)}^2 h(x, t)$, use its weak-type (2,2) and Lebesgue differentiation.

8.3. Proof of Theorem 2.13. The estimates for the semigroup extension with respect to MP are easily obtained from those for PM .

Since $M : \overline{\mathbf{R}(PM)} \rightarrow \overline{\mathbf{R}(MP)}$ is an isomorphism, it is a matter of fact for the functional calculus of these type of operators that there is an *intertwining relation*

$$(8.13) \quad e^{-\mu[MP]}h = Me^{-\mu[PM]}M^{-1}h \quad (h \in \overline{\mathbf{R}(MP)}).$$

In fact, we can directly verify $(\lambda - MP)^{-1}M = M(\lambda - PM)^{-1}$ for λ in a suitable double sector in the complex plane and extend this relation to the Cauchy integral defining the semigroups above.

Since M is a bounded multiplication operator, it acts boundedly on $L^2(\Lambda \times Q \times I; \mathbb{C}^{n+2})$ for every Whitney region $\Lambda \times Q \times I$. Let now $h \in \overline{\mathbf{R}(MP)} = M\overline{\mathbf{R}(P)}$. Since $M^{-1}h \in \overline{\mathbf{R}(PM)}$ falls under the scope of Theorem 2.12 and as $M^{-1} : \overline{\mathbf{R}(MP)} \rightarrow L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$ is bounded (though not necessarily a multiplication operator), the upper \tilde{N}_* -bound and the almost everywhere convergence of Whitney averages for the MP -semigroup follow from (8.13) and the corresponding result for the PM -semigroup. Finally the lower bound $\|\tilde{N}_*(e^{-[MP]}h)\|_2 \gtrsim \|h\|_2$ follows from Lemma 8.11 and the strong continuity of the semigroup for MP just as in the case of PM .

9. DUALITY RESULTS FOR λ -INDEPENDENT OPERATORS

In this section we prove Theorem 2.21 and the abstract Green's formula announced in Proposition 2.22. The main step is the following. We use the notation introduced in Section 2.12. The relevant Sobolev spaces have been introduced in Section 2.7. Recall that we gather $\|\cdot\|$ - and θ -components of vectors in a single component denoted r .

Lemma 9.1 (Simultaneous duality). *Let $-1 \leq s \leq 0$. There is a pairing simultaneously realising the pairs of spaces $(\dot{\mathbf{H}}_P^s, \dot{\mathbf{H}}_{P^*}^{-s-1})$, $(\dot{\mathbf{H}}_{PM}^{s,\pm}, \dot{\mathbf{H}}_{P^*M}^{-s-1,\mp})$, $((\dot{\mathbf{H}}_P^s)_\perp, (\dot{\mathbf{H}}_{P^*}^{-s-1})_r)$ and $((\dot{\mathbf{H}}_P^s)_r, (\dot{\mathbf{H}}_{P^*}^{-s-1})_\perp)$ as dual*

spaces. Moreover, in this pairing, the pairs $(\dot{H}_{PM}^{s,\pm}, \dot{H}_{P^*\tilde{M}}^{-s-1,\pm})$, $((\dot{H}_P^s)_\perp, (\dot{H}_{P^*}^{-s-1})_\perp)$ and $((\dot{H}_P^s)_r, (\dot{H}_{P^*}^{-s-1})_r)$ are pairs of orthogonal spaces.

Remark 9.2. In order to make the statement of Lemma 9.1 meaningful, we identify spaces of type X_\perp and X_r as subspaces of X via $X_\perp \cong X_\perp \oplus \{0\}$ and $X_r \cong \{0\} \oplus X_r$.

Proof. With N as defined in (2.28) and P^* the adjoint of P in the canonical $L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$ inner product, N and P^* anti-commute: $P^*N = -NP^*$. So, with $\tilde{M} = NM^*N = NM^*N^{-1}$, we obtain that the adjoint of PM in the complex form

$$(f, Ng) = \iint_{\mathbb{R}^{n+1}} f \cdot \overline{Ng} \, dx \, dt$$

in $L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$ is $-\tilde{M}P^*$. Thus, the dual of the Sobolev space \dot{H}_{PM}^s in the extension of this duality is $\dot{H}_{\tilde{M}P^*}^{-s}$. Next, the formal relation $P^*(\tilde{M}P^*) = (P^*\tilde{M})P^*$ allows one to show that P^* extends to an isomorphism between $\dot{H}_{\tilde{M}P^*}^{-s}$ and $\dot{H}_{P^*\tilde{M}}^{-s-1}$. Therefore $(\dot{H}_{PM}^s, \dot{H}_{P^*\tilde{M}}^{-s-1})$ is a pair of dual spaces for the duality form

$$\beta(f, g) = (f, NP^{*-1}g).$$

As this very argument also applies to $M = 1$, we see that $(\dot{H}_P^s, \dot{H}_{P^*}^{-s-1})$ are dual spaces for the same duality form. As P and P^* swap the \perp - and r -components we have the announced duality for the \perp - and r -spaces as well as the orthogonality. And from the anti-commutation of N and P^* , we derive the announced duality and orthogonality for the spectral spaces. \square

The proof of Theorem 2.21 now follows by applying abstract results. We shall make the same identifications as in Remark 9.2.

Proof of Theorem 2.21. We have a pair of projectors $(\chi^+(PM), \chi^-(PM))$ associated with the splitting $\dot{H}_P^s = \dot{H}_{PM}^{s,+} \oplus \dot{H}_{PM}^{s,-}$ and due to the identifications made above, the component maps (N_\perp, N_r) become the projectors associated with $\dot{H}_P^s = (\dot{H}_P^s)_\perp \oplus (\dot{H}_P^s)_r$. In the duality β their dual pairs are $(\chi^-(P^*\tilde{M}), \chi^+(P^*\tilde{M}))$ and (N_r, N_\perp) corresponding to the dual splittings $\dot{H}_{P^*}^{-s-1} = \dot{H}_{P^*\tilde{M}}^{-s-1,-} \oplus \dot{H}_{P^*\tilde{M}}^{-s-1,+}$ and $\dot{H}_{P^*}^{-s-1} = (\dot{H}_{P^*}^{-s-1})_r \oplus (\dot{H}_{P^*}^{-s-1})_\perp$, respectively.

Let us consider for example the Neumann problem $(N)_{\mathcal{E}_s}^{\mathcal{L}}$. By Theorem 2.15 this is well-posed if and only if $N_\perp : \dot{H}_{PM}^{s,+} \rightarrow (\dot{H}_P^s)_\perp$ is an isomorphism. In the duality above, the adjoint of $N_\perp : \dot{H}_{PM}^{s,+} \rightarrow (\dot{H}_P^s)_\perp$ is $\chi^-(P^*\tilde{M}) : (\dot{H}_{P^*}^{-s-1})_r \rightarrow \dot{H}_{P^*\tilde{M}}^{-s-1,-}$. Applying an abstract result on pairs of projections [18, Lem. 13.2], we have that invertibility of the latter is exactly equivalent to invertibility of $N_\perp : \dot{H}_{P^*\tilde{M}}^{-s-1,+} \rightarrow (\dot{H}_{P^*}^{-s-1})_\perp$. By the analogue of Theorem 2.15 for the backward equation, this in turn is equivalent to well-posedness of $(N)_{\mathcal{E}_{-1-s}^*}$.

The regularity problem can be handled in the same manner. Finally, preservation of compatible well-posedness can be obtained unravelling this argument. \square

A proof of the abstract Green's formula can be given by an approximation procedure as in [18, Thm. 1.7]. There, it was mentioned that one can also use the simultaneous duality above and this is the approach that we shall present here.

Proof of Proposition 2.22. Set $h = D_A u|_{\lambda=0} \in \dot{H}_{PM}^{s,+}$ and $g = \tilde{D}_A^* w|_{\lambda=0} \in \dot{H}_{P^*\tilde{M}}^{-s-1,+}$. We have $(h, NP^{*-1}g) = 0$ by the orthogonality relation and we claim that

$$(h, NP^{*-1}g) = \langle \partial_{\nu_A} u|_{\lambda=0}, w|_{\lambda=0} \rangle - \langle u|_{\lambda=0}, \partial_{\nu_A^*} w|_{\lambda=0} \rangle.$$

Indeed, \perp and r -component of h and $NP^{*-1}g$ can be computed in appropriate dual spaces described in the argument below and we write $(h, NP^{*-1}g) = \langle h_\perp, (NP^{*-1}g)_\perp \rangle + \langle h_r, (NP^{*-1}g)_r \rangle$. First, $h_\perp = \partial_{\nu_A} u|_{\lambda=0} \in (\dot{H}_P^s)_\perp = \dot{H}_{\partial_t - \Delta_x}^{s/2}$ and using Lemma 2.26 and the isomorphism property of π_{P^*} ,

$$(NP^{*-1}g)_\perp = w|_{\lambda=0} \in (\pi_{P^*} \dot{H}_{\tilde{M}P^*}^{-s})_\perp = (\dot{H}_{P^*}^{-s})_\perp = \dot{H}_{-\partial_t - \Delta_x}^{-s/2}.$$

To compute the other part, we may symmetrise the argument by changing s to $-s - 1$ and observing that $(h, NP^{*-1}g) = -(NP^{-1}h, g)$ and that NP^{-1} swaps r - and \perp -components. Hence $\langle h_r, (NP^{*-1}g)_r \rangle = -\langle (NP^{-1}h)_\perp, g_\perp \rangle$ formally and it remains to identify what this means. We have $(NP^{-1}h)_\perp = u|_{\lambda=0} \in (\dot{H}_P^{s+1})_\perp = \dot{H}_{\partial_t - \Delta_x}^{s/2+1/2}$ and $g_\perp = \partial_{\nu_{A^*}} w|_{\lambda=0} \in (\dot{H}_{P^*}^{-s-1})_\perp = \dot{H}_{-\partial_t - \Delta_x}^{-s/2-1/2}$. The result follows. \square

10. THE OPERATOR $\text{sgn}(PM)$

We provide the rather algebraic proofs of the results on $\text{sgn}(PM)$ stated in Section 2.13. The appearing Sobolev spaces have been introduced in Section 2.7. We keep on representing vectors by \perp - and r -components and operators on spaces $X = \dot{H}_P^s$ by the associated (2×2) -matrices and let $i_\perp : X_\perp \rightarrow X_\perp \oplus \{0\}$ and $i_r : X_r \rightarrow \{0\} \oplus X_r$ be the canonical embeddings. Then the maps N_\perp, N_r are just the coordinate maps with this representation. From (2.31) we obtain

$$(10.1) \quad \begin{aligned} 1 \pm s_{\perp\perp}(PM) &= 2N_\perp \chi^\pm(PM) i_\perp, & s_{\perp r}(PM) &= 2N_\perp \chi^+(PM) i_r, \\ s_{r\perp}(PM) &= 2N_r \chi^+(PM) i_\perp, & 1 \pm s_{rr}(PM) &= 2N_r \chi^\pm(PM) i_r. \end{aligned}$$

With this notation set up, we are ready to give the

- Proof of Theorem 2.23.* (i) Based on Theorem 2.15 and its counterpart for the lower half-space, $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ is well-posed on *both* half-spaces if, and only if, the two operators $N_r : \dot{H}_{PM}^{s,\pm} \rightarrow (\dot{H}_P^s)_r$ are invertible. By an abstract principle for pairs of complementary projections [18, Lem. 13.6], this in turn is equivalent to invertibility of the composite map $N_r \chi^+(PM) i_\perp : (\dot{H}_P^s)_\perp \rightarrow (\dot{H}_P^s)_r$, which in view of (10.1) exactly means that $s_{r\perp}(PM) : (\dot{H}_P^s)_\perp \rightarrow (\dot{H}_P^s)_r$ is invertible. In order to avoid confusion, we note that in [18] the results are equivalently formulated using the projectors $P_\perp = i_\perp \circ N_\perp$, $P_r = i_r \circ N_r$ of X onto $X_\perp \oplus \{0\}$ and $\{0\} \oplus X_r$, respectively.
- (ii) Theorem 2.15 yields again equivalence of the Neumann problem $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ being well-posed on both half-spaces with invertibility of the two operators $N_\perp : \dot{H}_{PM}^{s,\pm} \rightarrow (\dot{H}_P^s)_\perp$. By the same reasoning as before, this is equivalent to $s_{\perp r} : (\dot{H}_P^s)_r \rightarrow (\dot{H}_P^s)_\perp$ being invertible.
- (iii) Let us abbreviate $s_{\perp\perp}(PM)$ by $s_{\perp\perp}$ and so on. Spelling out the identity $\text{sgn}(PM)^2 = 1$ reveals

$$\begin{bmatrix} s_{\perp\perp}^2 + s_{\perp r} s_{r\perp} & s_{\perp\perp} s_{\perp r} + s_{\perp r} s_{rr} \\ s_{r\perp} s_{\perp\perp} + s_{rr} s_{r\perp} & s_{r\perp} s_{\perp r} + s_{rr}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In particular, we have $s_{\perp r} s_{r\perp} = (1 + s_{\perp\perp})(1 - s_{\perp\perp})$ and $s_{r\perp} s_{\perp r} = (1 + s_{rr})(1 - s_{rr})$. Now, suppose that $s_{\perp r}$ and $s_{r\perp}$ are invertible. Then we have invertible products of commuting operators $(1 + s_{\perp\perp})(1 - s_{\perp\perp})$ and $(1 + s_{rr})(1 - s_{rr})$, which implies that the four factors are invertible. Conversely, if the four operators $1 \pm s_{\perp\perp}$, $1 \pm s_{rr}$ are invertible, then both products $s_{\perp r} s_{r\perp}$, $s_{r\perp} s_{\perp r}$ are invertible, which implies that $s_{\perp r}$, $s_{r\perp}$ themselves are invertible.

- (iv) This is just the conjunction of the previous three equivalences. \square

Finally, we give the proof of Proposition 2.25. The reader may recall the definition of the Neumann-to-Dirichlet map $\Gamma_{ND}^{\mathcal{L},+}$ on the upper half-space from Section 2.13.

Proof of Proposition 2.25. The four statements are all proved in exactly the same way. We restrict ourselves to the Neumann problem $(N)_{\mathcal{E}_s}^{\mathcal{L}}$ on the upper half-space. Invoking Proposition 2.16, we have to check that $N_\perp : \dot{H}_{PM}^{s,+} \rightarrow (\dot{H}_P^s)_\perp$ is an isomorphism and that its inverse agrees with the one obtained on $(\dot{H}_P^{-1/2})_\perp$. Due to (2.32), (2.33) and our assumption that $1 - s_{rr}(PM) : (\dot{H}_P^s)_r \rightarrow (\dot{H}_P^s)_r$ is invertible and that the inverse is compatible on $(\dot{H}_P^{-1/2})_r$, we obtain that

$$\left[\Gamma_{ND}^{\mathcal{L},+} \right] : (\dot{H}_P^s)_\perp \cap (\dot{H}_P^{-1/2})_\perp \rightarrow \dot{H}_{PM}^{s,+} \cap \dot{H}_{PM}^{-1/2,+}$$

is well-defined and a two-sided inverse for $N_\perp : \dot{H}_{PM}^{s,+} \cap \dot{H}_{PM}^{-1/2,+} \rightarrow (\dot{H}_P^s)_\perp \cap (\dot{H}_P^{-1/2})_\perp$. Again by assumption it extends to a bounded operator $(\dot{H}_P^s)_\perp \rightarrow \dot{H}_{PM}^{s,+}$. This yields the sought-after inverse for $N_\perp : \dot{H}_{PM}^{s,+} \rightarrow (\dot{H}_P^s)_\perp$ and compatibility on $(\dot{H}_P^{-1/2})_\perp$ follows by construction. \square

11. INVERTING PARABOLIC OPERATORS BY LAYER POTENTIALS

Here, we provide the remaining proofs for results on layer potentials discussed in Section 2.14, where the relevant notation has been introduced.

Proof of Theorem 2.29. Let $f \in \dot{H}_{\partial_t - \Delta_{\lambda,x}}^{-1/2}$ and $u \in \dot{H}_{\partial_t - \Delta_{\lambda,x}}^{1/2}$. Note that $D_A u \in L^2(\mathbb{R}; \overline{\mathbb{R}(P)})$. Following the strategy of proof of Theorem 2.2, we see that $u = \mathcal{L}^{-1}f$ if and only if $\partial_\lambda F + PMF = \begin{bmatrix} f \\ 0 \end{bmatrix}$ in the weak sense, where $F = D_A u$. Recall that for vectors $F \in \mathbb{C}^{n+2}$ we gather here F_\parallel and F_θ in one component denoted F_r . Replacing $\begin{bmatrix} f \\ 0 \end{bmatrix}$ by the differential equation and integrating by parts, it follows that we have, in $L^2(\mathbb{R}; \overline{\mathbb{R}(P)})$ and for almost every $\lambda > 0$,

$$\begin{aligned} \int_{\varepsilon < |\lambda - \mu| < R} D_A \mathcal{S}_{\lambda - \mu} f_\mu \, d\mu &= + e^{-\varepsilon PM} \chi^+(PM) F_{\lambda - \varepsilon} + e^{\varepsilon PM} \chi^-(PM) F_{\lambda + \varepsilon} \\ &\quad - e^{-RPM} \chi^+(PM) F_{\lambda - R} - e^{RPM} \chi^-(PM) F_{\lambda + R}. \end{aligned}$$

Applying $-P^{-1}$ to both sides, we have an equality in $L^2(\mathbb{R}; \dot{H}_P^1)$. Finally, taking the \perp -component, we obtain in $L^2(\mathbb{R}; \dot{H}_{\partial_t - \Delta_x}^{1/2}) = E'$,

$$\begin{aligned} \int_{\varepsilon < |\lambda - \mu| < R} \mathcal{S}_{\lambda - \mu} f_\mu \, d\mu &= -(P^{-1} e^{-\varepsilon PM} \chi^+(PM) F_{\lambda - \varepsilon})_\perp - (P^{-1} e^{\varepsilon PM} \chi^-(PM) F_{\lambda + \varepsilon})_\perp \\ &\quad + (P^{-1} e^{-RPM} \chi^+(PM) F_{\lambda - R})_\perp + (P^{-1} e^{RPM} \chi^-(PM) F_{\lambda + R})_\perp. \end{aligned}$$

This defines uniformly bounded operators from $\dot{H}_{\partial_t - \Delta_{\lambda,x}}^{-1/2}$ to E' . Next, we take strong limits in E' . The terms corresponding to $R \rightarrow \infty$ tend to 0 using the bounded holomorphic functional calculus of PM and that $F_{\lambda \pm R}$ tend to 0 in $L^2(\mathbb{R}; \dot{H}_P^1)$. The terms corresponding to $\varepsilon \rightarrow 0$, converge to

$$-(P^{-1} \chi^+(PM) F)_\perp - (P^{-1} \chi^-(PM) F)_\perp = -(P^{-1} F)_\perp = u.$$

This proves the limit in the statement. \square

Proof of Theorem 2.30. To \mathcal{L} there correspond P and $M = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}$ with $A = \begin{bmatrix} A_{\parallel\parallel} & 0 \\ 0 & 1 \end{bmatrix}$ as in Section 5. Let $f \in \dot{H}_{\partial_t - \Delta_x}^{-1/2}$. As $\begin{bmatrix} f \\ 0 \end{bmatrix} \in \dot{H}_P^{-1}$, we find, using the functional calculus of PM on \dot{H}_P^{-1} ,

$$\int_\varepsilon^R \mathcal{S}_\lambda f \, d\lambda = \left((PMP)^{-1} (e^{-RPM} - e^{-\varepsilon PM}) \chi^+(PM) \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp.$$

Similarly, we obtain

$$\int_{-R}^{-\varepsilon} \mathcal{S}_\lambda f \, d\lambda = - \left((PMP)^{-1} (e^{\varepsilon PM} - e^{RPM}) \chi^-(PM) \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp.$$

Here, we have used that by construction PMP is an isomorphism from \dot{H}_P^1 onto \dot{H}_P^{-1} . Hence, taking strong limits for the semigroup on each spectral space of \dot{H}_P^{-1} , and adding terms we obtain

$$\lim_{\varepsilon \rightarrow 0, R \rightarrow \infty} \int_{\varepsilon \leq |\lambda| \leq R} \mathcal{S}_\lambda f \, d\lambda = - \left((PMP)^{-1} (\chi^+(PM) + \chi^-(PM)) \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp = - \left((PMP)^{-1} \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_\perp.$$

From the form of \mathcal{L} we see that PMP is block diagonal and the scalar block is precisely $\partial_t - \operatorname{div}_x A(x, t) \nabla_x = L$. This gives us the conclusion. \square

12. WELL-POSEDNESS RESULTS FOR λ -INDEPENDENT COEFFICIENTS

In this section we shall prove our well-posedness results formulated in Theorem 2.32 and obtain the resolution of the Kato problem for parabolic operators. We also supply proofs of the related Propositions 2.20 and 2.34. We shall need again the Sobolev spaces introduced in Section 2.7.

12.1. The proof of Theorem 2.6. For $A_{\parallel\parallel}(x, t)$ any bounded and uniformly elliptic $(n \times n)$ -matrix we let

$$M := \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_{\parallel\parallel} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad PM = \begin{bmatrix} 0 & \operatorname{div}_x A_{\parallel\parallel} & -D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{1/2} & 0 & 0 \end{bmatrix}$$

As PM has a bounded holomorphic functional calculus on $\overline{\mathbb{R}(P)}$ by Theorem 2.3, the operator $\operatorname{sgn}(PM)$ is a bounded involution on this space. Therefore PM and $[PM] = \operatorname{sgn}(PM)PM$ share the same domain with $\|PMh\|_2 \sim \|[PM]h\|_2$. But $[PM] = \sqrt{(PM)^2}$ by definition and

$$(PM)^2 = \begin{bmatrix} L & 0 & 0 \\ 0 & -\nabla_x \operatorname{div}_x A_{\parallel\parallel} & \nabla_x D_t^{1/2} \\ 0 & -H_t D_t^{1/2} \operatorname{div}_x A_{\parallel\parallel} & \partial_t \end{bmatrix}.$$

Specialising to the first component, we see that the domain of \sqrt{L} is $H^{1/2}(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}; W^{1,2}(\mathbb{R}^n))$ together with the homogeneous estimate

$$\|\sqrt{L}\|_2 \sim \|\nabla_x f\|_2 + \|H_t D_t^{1/2} f\|_2 = \|\nabla_x f\|_2 + \|D_t^{1/2} f\|_2$$

This solves the parabolic Kato problem.

12.2. The block case (i). More generally, we assume that the coefficients $A(x, t)$ are in block form

$$A := \begin{bmatrix} A_{\perp\perp} & 0 \\ 0 & A_{\parallel\parallel} \end{bmatrix}.$$

Since results for this case will be re-used in the context of more general coefficients, we shall write $A(x, t) = A_b(x, t)$ and $M = M_b$ to avoid confusion. As usual we concatenate \perp - and \parallel -components of vectors to a single component denoted r . Similarly, we represent operators acting on them as (2×2) -matrices. For example, $M = M_b$ can be written in block form

$$M_b = \begin{bmatrix} A_{\perp\perp}^{-1} & 0 \\ 0 & M_{rr} \end{bmatrix}, \quad \text{where} \quad M_{rr} := \begin{bmatrix} A_{\parallel\parallel} & 0 \\ 0 & 1 \end{bmatrix}.$$

For $\operatorname{sgn}(PM_b)$ we have already introduced such a representation as a (2×2) -matrix in (2.30):

$$\operatorname{sgn}(PM_b) = \begin{bmatrix} s_{\perp\perp}(PM_b) & s_{\perp r}(PM_b) \\ s_{r\perp}(PM_b) & s_{rr}(PM_b) \end{bmatrix},$$

which by the holomorphic functional calculus is bounded on $\overline{\mathbb{R}(P)}$ and satisfies $\operatorname{sgn}(PM_b)^2 = 1$. On the other hand, M_b commutes with the matrix N introduced in (2.28) since both are in block form. Hence, $PM_b N = -N PM_b$, which carries over to the functional calculus by $\operatorname{sgn}(PM_b) N = -N \operatorname{sgn}(PM_b)$. These two properties of $\operatorname{sgn}(PM_b)$ can only hold if its diagonal blocks vanish and its off-diagonal blocks are inverses of each other, that is,

$$(12.1) \quad \operatorname{sgn}(PM_b) = \begin{bmatrix} 0 & s_{\perp r}(PM_b) \\ s_{\perp r}(PM_b)^{-1} & 0 \end{bmatrix}.$$

From Section 2.7 we know that the bounded functional calculus for PM_b extends to the Sobolev spaces $\dot{H}_{PM_b}^s = \dot{H}_P^s$, $-1 \leq s \leq 0$, and by density of $\overline{\mathbb{R}(P)} \cap \dot{H}_P^s$ in \dot{H}_P^s , the formula (12.1) extends, too. In particular, $1 \pm s_{\perp\perp}(PM_b)$ and $1 \pm s_{rr}(PM_b)$ all act as the identity on the respective components of \dot{H}_P^s . Thus, Proposition 2.25 yields compatible well-posedness of $(R)_{\mathcal{E}_s}^{\mathcal{L}}$ and $(N)_{\mathcal{E}_s}^{\mathcal{L}}$.

12.3. The triangular cases (ii) and (iv). We adopt notation from the previously discussed block case. It suffices to prove well-posedness at regularity $s = 0$: Indeed, in view of compatible well-posedness for block matrices, we can perturb along $\mathbb{A} := \{\tau 1 + (1 - \tau)A : 0 \leq \tau \leq 1\}$ to the heat equation and obtain *compatible* well-posedness at regularity $s = 0$ from Proposition 2.20. Then the intermediate cases follow from interpolation with (compatible) well-posedness in the energy class by means of Theorem 2.18.

Appealing to Theorem 2.23, we shall check invertibility of $s_{\perp r}(PM)$ and $s_{r\perp}(PM)$ for A lower- and upper-triangular, respectively. The absence of one of the entries of A will allow us to transfer invertibility from regularity $s = -1/2$ to $s = 0$. In the context of elliptic systems this strategy has been put into place in [16]. The argument presented below constitutes a simplification even in the elliptic case. In the following we utilize the block coefficients $A_b := \begin{bmatrix} A_{\perp\perp} & 0 \\ 0 & A_{\parallel\parallel} \end{bmatrix}$ and the corresponding multiplication operator M_b . In the case of triangular matrices A we will see that the Sobolev spaces adapted to PM_b carry substantial information also on PM . Note that

$$PM_b = \begin{bmatrix} 0 & \operatorname{div}_x A_{\parallel\parallel} & -D_t^{1/2} \\ -\nabla_x A_{\perp\perp}^{-1} & 0 & 0 \\ -H_t D_t^{1/2} A_{\perp\perp}^{-1} & 0 & 0 \end{bmatrix}$$

acts independently on \perp - and r -components and hence so does $(PM_b)^2$. So, by the very definition of homogeneous Sobolev spaces we can decompose

$$\dot{H}_{PM_b}^s = (\dot{H}_{PM_b}^s)_{\perp} \oplus (\dot{H}_{PM_b}^s)_r$$

for any $s \in \mathbb{R}$, see Section 2.7, and besides the usual isomorphism $PM_b : \dot{H}_{PM_b}^s \rightarrow \dot{H}_{PM_b}^{s-1}$ we have the restricted isomorphisms

$$(12.2) \quad PM_b : (\dot{H}_{PM_b}^s)_{\perp} \oplus \{0\} \xrightarrow{\cong} \{0\} \oplus (\dot{H}_{PM_b}^{s-1})_r$$

and

$$(12.3) \quad PM_b : \{0\} \oplus (\dot{H}_{PM_b}^s)_r \xrightarrow{\cong} (\dot{H}_{PM_b}^{s-1})_{\perp} \oplus \{0\}.$$

The fact that we use these operators associated with the coefficients A_b different from A reflects that well-posedness, in contrast to *a priori* representations, cannot be comprised only by the functional calculus arising from A . Below, we shall frequently and without further mentioning use Lemma 2.7 to the effect that $\dot{H}_{PM_b}^s = \dot{H}_P^s = \dot{H}_{PM}^s$ holds for $-1 \leq s \leq 0$.

Proof of (ii). As we have seen, it suffices to prove that $s_{\perp r}(PM) : (\dot{H}_P^0)_r \rightarrow (\dot{H}_P^0)_{\perp}$ is invertible if A is lower-triangular. Taking into account (12.2), this is actually equivalent to

$$T_r(PM) : f \mapsto N_r PM_b \begin{bmatrix} s_{\perp r}(PM)f \\ 0 \end{bmatrix}$$

providing an isomorphism $(\dot{H}_{PM_b}^0)_r \rightarrow (\dot{H}_{PM_b}^{-1})_r$ and it is this property that we shall check. From Corollary 2.24 and (12.2) we can infer

$$(12.4) \quad T_r(PM) : (\dot{H}_{PM_b}^{-1/2})_r \xrightarrow{\cong} (\dot{H}_{PM_b}^{-3/2})_r.$$

Since $\operatorname{sgn}(PM) \in \mathcal{L}(\dot{H}_P^{-1/2})$ depends Lipschitz continuously on M , see the discussion before Theorem 2.19, also $T_r(PM)$ depends Lipschitz continuously on M as a bounded operator between the Sobolev spaces above. Now, we use the lower-triangular structure of M for the one and only time to see that for $f \in (\dot{H}_{PM_b}^{1/2})_r \cap (\dot{H}_{PM_b}^{-1/2})_r$ we can also write

$$(12.5) \quad \begin{aligned} T_r(PM)f &= N_r PM_b \begin{bmatrix} s_{\perp r}(PM)f \\ 0 \end{bmatrix} = N_r PM \operatorname{sgn}(PM) \begin{bmatrix} 0 \\ f \end{bmatrix} \\ &= N_r \operatorname{sgn}(PM) PM \begin{bmatrix} 0 \\ f \end{bmatrix} = s_{r\perp}(PM) N_{\perp} PM_b \begin{bmatrix} 0 \\ f \end{bmatrix}. \end{aligned}$$

Hence, (12.3) and Corollary 2.24 yield that $T_r(PM)$ extends by density to an isomorphism

$$(12.6) \quad T_r(PM) : (\dot{H}_{PM_b}^{1/2})_r \xrightarrow{\cong} (\dot{H}_{PM_b}^{-1/2})_r$$

and by the same argument as before we have Lipschitz continuous dependence on M .

Of course we want to interpolate between (12.4) and (12.6) but interpolation of invertibility requires that the two obtained inverses agree on their common domain of definition. For the moment, we can at least state for all $f \in (\dot{H}_{PM_b}^{1/2})_r \cap (\dot{H}_{PM_b}^{-1/2})_r$ the comparability

$$\|T_r(PM)f\|_{(\dot{H}_{PM_b}^{-1/2})_r} + \|T_r(PM)f\|_{(\dot{H}_{PM_b}^{-3/2})_r} \sim \|f\|_{(\dot{H}_{PM_b}^{1/2})_r} + \|f\|_{(\dot{H}_{PM_b}^{-1/2})_r}.$$

Hence,

$$(12.7) \quad T_r(PM) : (\dot{H}_{PM_b}^{1/2})_r \cap (\dot{H}_{PM_b}^{-1/2})_r \rightarrow (\dot{H}_{PM_b}^{-1/2})_r \cap (\dot{H}_{PM_b}^{-3/2})_r$$

is bounded from above and below. Here, the intersection spaces carry their natural sum norms. Since the bounded maps in (12.4) and (12.6) depend Lipschitz continuously on M , the same holds true for $T_r(PM)$ when considered a bounded map between these two intersection spaces. We claim that in the block case $M = M_b$ it is also onto. Indeed, since the diagonal entries of $\text{sgn}(PM_b)$ vanish by (12.1), we can improve the calculation in (12.5) to the effect that

$$\begin{bmatrix} 0 \\ T_r(PM_b)f \end{bmatrix} = PM_b \text{sgn}(PM_b) \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

Now, for each $g \in (\dot{H}_{PM_b}^{-1/2})_r \cap (\dot{H}_{PM_b}^{-3/2})_r$ we can define $f \in (\dot{H}_{PM_b}^{1/2})_r \cap (\dot{H}_{PM_b}^{-1/2})_r$ via

$$\begin{bmatrix} 0 \\ f \end{bmatrix} := (PM_b)^{-1} \text{sgn}(PM_b) \begin{bmatrix} 0 \\ g \end{bmatrix}$$

and obtain $T_r(PM_b)f = g$. By the method of continuity, perturbing M to M_b along $M(\tau) = \tau M_b + (1-\tau)M$, we obtain that the map in (12.7) is an isomorphism. In particular, this implies that the inverses obtained in (12.4) and (12.6) are compatible.

Eventually, real interpolation midway between (12.4) and (12.6) for $T_r(PM)$ and its unambiguously defined inverse yields the required isomorphism

$$T_r(PM) : (\dot{H}_{PM_b}^0)_r \xrightarrow{\cong} (\dot{H}_{PM_b}^{-1})_r.$$

The interpolation argument uses [17, Thm. 5.3] for the real interpolation of the full $\dot{H}_{PM_b}^s$ -spaces and [23, Thm. 6.4.2] for the transfer to their complemented subspaces $(\dot{H}_{PM_b}^s)_r$. \square

The proof for upper-triangular matrices follows along the same lines and we shall only give the main steps.

Proof of (iv). First, the question of invertibility of $s_{r\perp}(PM) : (\dot{H}_P^0)_\perp \rightarrow (\dot{H}_P^0)_r$ is equivalent to proving that

$$T_\perp(PM) : f \mapsto N_\perp PM_b \begin{bmatrix} 0 \\ s_{r\perp}(PM)f \end{bmatrix}$$

acts as an isomorphism $(\dot{H}_{PM_b}^0)_\perp \rightarrow (\dot{H}_{PM_b}^{-1})_\perp$. Due to Corollary 2.24 and (12.3) it certainly is invertible from $(\dot{H}_{PM_b}^{-1/2})_\perp$ into $(\dot{H}_{PM_b}^{-3/2})_\perp$. The fact that M is upper-triangular lets us discover the identity

$$\begin{aligned} T_\perp(PM)f &= N_\perp PM_b \begin{bmatrix} 0 \\ s_{r\perp}(PM)f \end{bmatrix} = N_\perp PM \text{sgn}(PM) \begin{bmatrix} f \\ 0 \end{bmatrix} \\ &= N_\perp \text{sgn}(PM) PM \begin{bmatrix} f \\ 0 \end{bmatrix} = s_{r\perp}(PM) N_r PM_b \begin{bmatrix} f \\ 0 \end{bmatrix} \end{aligned}$$

for $f \in (\dot{H}_{PM_b}^{1/2})_\perp \cap (\dot{H}_{PM_b}^{-1/2})_\perp$. Therefore $T_\perp(PM)$ extends to an isomorphism from $(\dot{H}_{PM_b}^{1/2})_\perp$ to $(\dot{H}_{PM_b}^{-1/2})_\perp$ and we conclude by interpolation as in the proof of (ii). \square

12.4. The triangular cases (iii) and (v). This is a consequence of Theorem 2.21: (iii) is equivalent to (ii) and (v) is equivalent to (iv).

12.5. The Hermitian case (vi). From the proof presented below it will become clear that we do neither use real coefficients nor equations but self-adjointness. It will also become clear at what instance we have to assume that coefficients do not depend on the time variable. The argument follows that of [54] using some integral identities to obtain Rellich estimates.

Let $h \in \mathbf{H}^+(PM)$. Then $F = F(\lambda, \cdot) = e^{-\lambda PM} h$ satisfies $\partial_\lambda F = -PMF$ in the strong sense with strong limits $\lim_{\lambda \rightarrow \infty} \partial_\lambda^k F = 0$, $k \geq 0$, and $\lim_{\lambda \rightarrow 0} F = h$ in $\mathbb{L}^2 = L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$. Let N be the reflection matrix introduced in (2.28). A computation shows that

$$P^*N + NP = \begin{bmatrix} 0 & 0 & (1+H_t)D_t^{1/2} \\ 0 & 0 & 0 \\ (1-H_t)D_t^{1/2} & 0 & 0 \end{bmatrix}.$$

Next, by definition of M in (2.7), the Hermitian condition $A^* = A$ is equivalent to $M^*N = NM$. Using the Hermitian inner product (\cdot, \cdot) on \mathbb{L}^2 , we obtain

$$\begin{aligned} \partial_\lambda(NF, MF) &= (NPMF, MF) + (NF, MPMF) = (MF, P^*NMF) + (NMF, PMF) \\ &= (MF, (P^*N + NP)MF) = 2 \operatorname{Re}((1+H_t)F_\theta, D_t^{1/2}(MF)_\perp). \end{aligned}$$

By integration in λ and the limits above,

$$(Nh, Mh) = -2 \int_0^\infty \operatorname{Re}((1+H_t)F_\theta, D_t^{1/2}(MF)_\perp) d\lambda.$$

Using $D_t^{1/2}(MF)_\perp = -H_t \partial_\lambda F_\theta$, and integration by parts with vanishing boundary terms,

$$\begin{aligned} \int_0^\infty \|D_t^{1/4}(MF)_\perp\|_2^2 d\lambda &= -2 \operatorname{Re} \int_0^\infty \lambda (D_t^{1/4} \partial_\lambda (MF)_\perp, D_t^{1/4} (MF)_\perp) d\lambda \\ &= 2 \operatorname{Re} \int_0^\infty \lambda (\partial_\lambda (MF)_\perp, H_t \partial_\lambda F_\theta) d\lambda \\ &\leq 2 \left(\int_0^\infty \|\lambda H_t \partial_\lambda F_\theta\|_2^2 \frac{d\lambda}{\lambda} \right)^{1/2} \left(\int_0^\infty \|(\lambda M \partial_\lambda F)_\perp\|_2^2 \frac{d\lambda}{\lambda} \right)^{1/2} \\ &\leq C \|h\|_2^2, \end{aligned}$$

where the last step follows from the boundedness of H_t and M on \mathbb{L}^2 and the square function estimates for PM , see Theorem 2.3.

Thus, by the Cauchy-Schwarz inequality and the boundedness of H_t ,

$$(12.8) \quad |(Nh, Mh)| \lesssim \|h\|_2 \left(\int_0^\infty \|D_t^{1/4} F_\theta\|_2^2 d\lambda \right)^{1/2}.$$

The remaining integral on the right is the difficult term. In order to justify some calculations, we use the following lemma which we shall prove in Section 12.8 below.

Lemma 12.1. *Assume that the coefficients of \mathcal{L} are Lipschitz continuous with respect to (x, t) and that u is a reinforced weak solution of (1.1) with $D_A u \in \mathcal{H}_{\text{loc}}$. Then $\nabla_{\lambda, x}^2 u$, $\partial_t u$ and $\nabla_x \partial_t u$ are in $L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}))$ and for any $r > 0$,*

$$(12.9) \quad \int_r^{2r} \|\nabla_{\lambda, x}^2 u\|_2^2 + \|\partial_t u\|_2^2 + \|\nabla_x \partial_t u\|_2^2 d\lambda \leq C \int_{r/2}^{4r} \|\nabla_{\lambda, x} u\|_2^2 d\lambda$$

with C depending on r , the ellipticity constants of A and $\|\nabla A\|_\infty$. In particular, a similar estimate holds for $\nabla_x H_t D_t^{1/2} u$.

Assuming the hypothesis of Lemma 12.1 and using the correspondence of F with a reinforced weak solution u , we observe that $F_\theta = H_t D_t^{1/2} u$ and $D_t^{1/2} F_\theta = \partial_t u$. It will be simpler to come back to

the usage of u in the following calculation. Note that $\operatorname{div}_{\lambda,x} A \nabla_{\lambda,x} u = \partial_t u \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}))$ and that

$$\operatorname{div}_{\lambda,x} A \nabla_{\lambda,x} u = \partial_\lambda (A \nabla_{\lambda,x} u)_\perp + \operatorname{div}_x (A \nabla_{\lambda,x} u)_\parallel.$$

Since F is the semigroup extension of h , we know that $(A \nabla_{\lambda,x} u)_\perp = F_\perp$ is smooth in λ as a function valued in \mathbb{L}^2 . Therefore we have $\operatorname{div}_x (A \nabla_{\lambda,x} u)_\parallel \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}))$ and thanks to the lemma also $\nabla_x F_\theta = \nabla_x H_t D_t^{1/2} u$ is contained in this space. This allows us to obtain for almost every $\lambda > 0$ the following identities:

$$\begin{aligned} \|D_t^{1/4} F_\theta\|_2^2 &= (D_t^{1/2} F_\theta, F_\theta) \\ &= (\partial_t u, H_t D_t^{1/2} u) \\ &= (\partial_\lambda (A \nabla_{\lambda,x} u)_\perp + (\operatorname{div}_x (A \nabla_{\lambda,x} u)_\parallel, H_t D_t^{1/2} u) \\ &= (\partial_\lambda (A \nabla_{\lambda,x} u)_\perp, H_t D_t^{1/2} u) - ((A \nabla_{\lambda,x} u)_\parallel, \nabla_x H_t D_t^{1/2} u) \\ &= \frac{d}{d\lambda} ((A \nabla_{\lambda,x} u)_\perp, H_t D_t^{1/2} u) \\ &\quad - ((A \nabla_{\lambda,x} u)_\perp, \partial_\lambda H_t D_t^{1/2} u) - ((A \nabla_{\lambda,x} u)_\parallel, \nabla_x H_t D_t^{1/2} u) \\ &= -\frac{d}{d\lambda} (F_\perp, F_\theta) - (A \nabla_{\lambda,x} u, H_t D_t^{1/2} \nabla_{\lambda,x} u) \\ &= -\frac{d}{d\lambda} (F_\perp, F_\theta) - \frac{1}{2} (\nabla_{\lambda,x} u, [A, H_t D_t^{1/2}] \nabla_{\lambda,x} u), \end{aligned}$$

where we used the self-adjointness of A and the skew-adjointness of $H_t D_t^{1/2}$ in the last line and $[A, H_t D_t^{1/2}]$ denotes the commutator between the multiplication by A and $H_t D_t^{1/2}$. Integrating the equality above and recalling the limits for F , we obtain

$$(12.10) \quad \int_0^\infty \|D_t^{1/4} F_\theta\|_2^2 d\lambda = -(h_\perp, h_\theta) - \int_0^\infty (\nabla_{\lambda,x} u, [A, H_t D_t^{1/2}] \nabla_{\lambda,x} u) d\lambda,$$

in the sense that the left hand integral exists provided the right hand integral exists.

By Murray's theorem [52], the commutator between a bounded function a on the real line and $D_t^{1/2}$ is bounded on $L^2(\mathbb{R})$ if and only if $D_t^{1/2} a \in \text{BMO}(\mathbb{R})$ with norm on the order of $\|D_t^{1/2} a\|_{\text{BMO}}$. This is the same upon replacing $D_t^{1/2}$ with $H_t D_t^{1/2}$. Actually, it can nowadays be seen as an easy corollary of the T(1) theorem. Thus, on assuming $\|D_t^{1/2} A\|_{L^\infty(\text{BMO})} < \infty$ and recalling $\int_0^\infty \|F\|_2^2 d\lambda \sim \|h\|_{\dot{H}_P^{-1/2}}^2$ from Theorem 2.8 if additionally $h \in \dot{H}_P^{-1/2}$, we would obtain

$$(12.11) \quad \int_0^\infty \|D_t^{1/4} F_\theta\|_2^2 d\lambda \lesssim |(h_\perp, h_\theta)| + \|D_t^{1/2} A\|_{L^\infty(\text{BMO})} \|h\|_{\dot{H}_P^{-1/2}}^2.$$

Here, we wrote $L^\infty(\text{BMO}) := L^\infty(\mathbb{R}^{n+1}; \text{BMO}(\mathbb{R}))$. However, there is no chance in our situation that the integral involving the commutator converges just assuming $h \in \mathbf{H}^+(PM)$. It is at this point that we have to assume that A does not depend on the t variable. In this case $[A, H_t D_t^{1/2}] = 0$ and we obtain from (12.8) and (12.10) the estimate

$$|(Nh, Mh)| \lesssim \|h\|_2 \|h_\perp\|_2^{1/2} \|h_\theta\|_2^{1/2}.$$

By definition of N in (2.28) we have

$$|(h, Mh) - 2(h_\perp, (Mh)_\perp)| = |(Nh, Mh)| = |(h, Mh) - 2(h_\parallel, (Mh)_\parallel) - 2(h_\theta, h_\theta)|.$$

Taking into account the accretivity condition $\|h\|_2^2 \lesssim \operatorname{Re}(h, Mh)$, we deduce the Rellich estimate

$$\|h_\perp\|_2^2 \sim \|h\|_2^2 \sim \|h_\parallel\|_2^2 + \|h_\theta\|_2^2 = \|h_r\|_2^2.$$

This proves that $N_\perp : \mathbf{H}^+(PM) \rightarrow (\mathbf{H}_P)_\perp$ and $N_r : \mathbf{H}^+(PM) \rightarrow (\mathbf{H}_P)_r$ have lower bounds. Up to now we have made the qualitative assumption that A is Lipschitz continuous in (x, t) . However, since the Lipschitz norm of A does not enter the estimate above, we may remove this extra assumption

by an approximation argument that we shall present in the proof of Proposition 2.34 below. For the time being we admit that this is possible.

Thus, we can apply Proposition 2.20 with $\mathbb{A} = \{\tau 1 + (1 - \tau)A : 0 \leq \tau \leq 1\}$ to obtain compatible well-posedness of $(R)_{\mathcal{E}_0}^{\mathcal{L}}$ and $(N)_{\mathcal{E}_0}^{\mathcal{L}}$ from the analogous result for the heat equation in Section 12.2. By reversing time, the same result holds for the backward equation with coefficients $A(x)$ and duality (Theorem 2.21) along with $A = A^*$ yields compatible well-posedness of $(R)_{\mathcal{E}_{-1}}^{\mathcal{L}}$ and $(N)_{\mathcal{E}_{-1}}^{\mathcal{L}}$ for the forward equation. Finally, by Theorem 2.18 and (compatible) well-posedness in the energy classes for $s = -1/2$, we obtain the intermediate cases.

12.6. The case of constant coefficients (vii). In this case, the maps N_{\perp} and N_r are Fourier multiplier operators in the context of the Mihlin multiplier theorem, and their invertibility is done explicitly in [54], when $s = 0$. The other cases follow by reversing time, duality and interpolation just as in the Hermitian case above.

12.7. Proof of Proposition 2.34. We recall from (2.23) that for u an energy solution to (1.1) with bounded and measurable coefficients and h the boundary data of $D_A u$, there is comparability

$$(12.12) \quad \|h_r\|_{\dot{H}_{\partial_t - \Delta_x}^{-1/4}} \sim \|h\|_{\dot{H}_P^{-1/2}} \sim \|h_{\perp}\|_{\dot{H}_{\partial_t - \Delta_x}^{-1/4}}.$$

Here, $h_r = [\nabla_x u, H_t D_t^{1/2} u]|_{\lambda=0}$ belongs to $(\dot{H}_{\partial_t - \Delta_x}^{-1/4})^{n+1}$ and we forget the power of $n + 1$ by abuse of notation. Moreover, implicit constants depend only on ellipticity of A and dimension since these estimates were obtained from the Lax-Milgram lemma. Given $h \in \mathbf{H}^+(PM) \cap \dot{H}_{PM}^{-1/2,+}$, we can follow the proof of (iv) and obtain from (12.8) and (12.11) that

$$|(Nh, Mh)| \lesssim \|h\|_2 \left(|(h_{\perp}, h_{\theta})|^{1/2} + \|h\|_{\dot{H}_P^{-1/2}} \right).$$

Taking into account (2.23), the same reasoning as before reveals

$$(12.13) \quad \|h_r\|_2^2 + \|h_r\|_{\dot{H}_{\partial_t - \Delta_x}^{-1/4}}^2 \sim \|h\|_2^2 + \|h\|_{\dot{H}_{\partial_t - \Delta_x}^{-1/4}}^2 \sim \|h_{\perp}\|_2^2 + \|h_{\perp}\|_{\dot{H}_{\partial_t - \Delta_x}^{-1/4}}^2,$$

where implicit constants depend only on ellipticity of A , dimension and $\|D_t^{1/2} A\|_{L^{\infty}(\text{BMO})}$. So far, this estimate holds under the *a priori* assumption that A is Lipschitz continuous in (x, t) . Let us now remove this additional regularity and assume $D_t^{1/2} A \in L^{\infty}(\text{BMO})$ only.

By convolution with smooth positive kernels we can find a sequence of Lipschitz matrices A_j with uniform ellipticity bounds that converge towards A almost everywhere. Since $\|D_t^{1/2} A_j\|_{L^{\infty}(\text{BMO})} \leq \|D_t^{1/2} A\|_{L^{\infty}(\text{BMO})}$ holds due to the translation invariance of the $L^{\infty}(\text{BMO})$ -norm, the bounds (12.13) are uniform in j for $h_j \in \mathbf{H}^+(PM_j) \cap \dot{H}_{PM_j}^{-1/2,+}$, where M_j is the matrix corresponding to A_j .

We want to obtain the same equivalence for all $h \in \mathbf{H}^+(PM) \cap \dot{H}_{PM}^{-1/2,+}$. To this end we shall use in a crucial way well-posedness of the BVPs in the energy class, see Section 2.3. Consider for example the Neumann problem. Let $f \in L^2(\mathbb{R}^{n+1}) \cap \dot{H}_{\partial_t - \Delta_x}^{-1/4}$ and let u and u_j be the unique energy solutions to (1.1) with coefficients A and A_j and conormal derivative equal to f , respectively. Theorem 2.8 creates two vectors $h, h_j \in \dot{H}_P^{-1/2,+}$ with $h_{\perp} = (h_j)_{\perp} = f$. Due to (12.12) the $(h_j)_r$ are uniformly bounded in $\dot{H}_{\partial_t - \Delta_x}^{-1/4}$. The hidden coercivity estimate (2.22) implies that the u_j are uniformly bounded in the energy space $\dot{E}(\mathbb{R}_+^{n+2})$, which means that the $D_{A_j} u_j$ are uniformly bounded in $L^2(\mathbb{R}_+^{n+2}; \mathbb{C}^{n+2})$. Without loss of generality we may assume weak convergence. Passing to the limit in the variational formulation

$$\iiint_{\mathbb{R}_+^{n+2}} A_j \nabla_{\lambda, x} u_j \cdot \overline{\nabla_{\lambda, x} v} + H_t D_t^{1/2} u_j \cdot \overline{D_t^{1/2} v} \, d\lambda \, dx \, dt = -\langle f, v|_{\lambda=0} \rangle \quad (v \in \dot{E}),$$

we discover that the weak limit solves the equation with coefficients A and by well-posedness in the energy class we deduce $D_{A_j} u_j \rightarrow D_A u$ weakly in $L^2(\mathbb{R}_+^{n+2}; \mathbb{C}^{n+2})$. Now, Lemma 3.14 yields

$u_j|_{\lambda=0} \rightarrow u|_{\lambda=0}$ weakly in $\dot{H}_{\partial_t-\Delta_x}^{1/4}$ and hence $(h_j)_r \rightarrow h_r$ weakly in $\dot{H}_{\partial_t-\Delta_x}^{-1/4}$. Thus, we can pass to the limit inferior in (12.13) for h_j and get the one-sided inequality

$$\|h_r\|_2^2 + \|h_r\|_{\dot{H}_{\partial_t-\Delta_x}^{-1/4}}^2 \lesssim \|h_\perp\|_2^2 + \|h_\perp\|_{\dot{H}_{\partial_t-\Delta_x}^{-1/4}}^2.$$

For the reverse inequality we argue similarly using the energy solutions with fixed regularity data.

As usual, (12.13) and the method of continuity (see the proof of Proposition 2.20 below for the specific application) allows us to conclude that the regularity problem with data $u|_{\lambda=0} \in \dot{H}_{\partial_t-\Delta_x}^{1/2} \cap \dot{H}_{\partial_t-\Delta_x}^{1/4}$ and the Neumann problem with data $\partial_{\nu_A} u|_{\lambda=0} \in \dot{H}_{\partial_t-\Delta_x}^0 \cap \dot{H}_{\partial_t-\Delta_x}^{-1/4}$ are well-posed in the class of solutions with $D_A u \in \mathcal{E}_0 \cap \mathcal{E}_{-1/2}$. Indeed, by Theorems 2.5 and 2.8 this amounts to showing that $N_\bullet : \mathbf{H}^+(PM) \cap \dot{H}_{PM}^{-1/2,+} \rightarrow (\overline{\mathbf{R}(P)} \cap \dot{H}_P^{-1/2})_\bullet$ is an isomorphism, where \bullet designates either r or \perp . Now, (12.13) provides lower bounds for this map and if A is in block form, then Proposition 2.16 and Theorem 2.32(i) yield invertibility. This yields (i) and the first claim in (iii).

We perform a duality argument to conclude the remaining assertions. First, the same analysis applies to the regularity problem for the backward in time equation with coefficients $A^*(x, t)$ since the hypothesis $D_t^{1/2} A \in L^\infty(\text{BMO})$ is preserved in changing t to $-t$ and in taking adjoints. By the argument of Theorem 2.21, each $f \in L^2(\mathbb{R}^{n+1}) + \dot{H}_{\partial_t-\Delta_x}^{1/4}$ corresponds to a unique $h \in \dot{H}_{PM}^{-1,+} + \dot{H}_{PM}^{-1/2,+}$ (which is the dual of $\dot{H}_{P^*M}^{0,+} \cap \dot{H}_{P^*M}^{-1/2,+}$ in that duality) with $h_r = [\nabla_x f, H_t D_t^{1/2} f]$. Then $F = e^{-\lambda PM} h$ is the parabolic conormal differential of a reinforced weak solution u with $u|_{\lambda=0} = f$. By construction u is the sum of two solutions $u_1 + u_2$ with $D_A u_1 \in \mathcal{E}_{-1}$ and $D_A u_2 \in \mathcal{E}_{-1/2}$ and owing to Theorem 2.8 it is unique under all weak solutions with this property. This proves (ii). The corresponding statement on the Neumann problem is obtained in the same way.

12.8. Proof of Lemma 12.1. This is a consequence of Caccioppoli-type arguments for weak solutions. Assume first that u is a weak solution of $\partial_t u - \text{div}_{\lambda,x} A(x, t) \nabla_{\lambda,x} u = 0$. We let $\Lambda \times Q \times I$ be a parabolic cylinder in \mathbb{R}_+^{n+2} of size r (not necessary of Whitney type) and $\tilde{\Lambda} \times \tilde{Q} \times \tilde{I}$ denote an enlargement contained in \mathbb{R}_+^{n+2} .

Since A is Lipschitz continuous with respect to $X = (\lambda, x)$, Caccioppoli's inequality (Lemma 8.1) and the classical method of difference quotients imply that for $i = 0, \dots, n$ the function $v = \partial_{X_i} u$ is a weak solution to $\partial_t v - \text{div}_{\lambda,x} A \nabla_{\lambda,x} v = \text{div}_{\lambda,x} (\partial_{X_i} A) \nabla_{\lambda,x} u$. This being said, we can sum up Caccioppoli's inequality for all these solutions and obtain

$$\iiint_{\Lambda \times Q \times I} |\nabla_{\lambda,x}^2 u|^2 \lesssim \frac{1}{r^2} \iiint_{\tilde{\Lambda} \times \tilde{Q} \times \tilde{I}} |\nabla_{\lambda,x} u|^2 + \|\nabla_{\lambda,x} A\|_\infty^2 \iiint_{\tilde{\Lambda} \times \tilde{Q} \times \tilde{I}} |\nabla_{\lambda,x} u|^2.$$

Since now u has second order derivatives in X within $L_{\text{loc}}^2(\mathbb{R}_+^{n+2})$, we can interpret the equation $\partial_t u = \text{div}_{\lambda,x} A \nabla_{\lambda,x} u$ in $L_{\text{loc}}^2(\mathbb{R}_+^{n+2})$. It follows that $\iiint_{\Lambda \times Q \times I} |\partial_t u|^2$ can be bounded in the same way as $\iiint_{\Lambda \times Q \times I} |\nabla_{\lambda,x}^2 u|^2$ above. Eventually, we use that A is also Lipschitz continuous with respect to t to conclude likewise that $v = \partial_t u$ is a weak solution to $\partial_t v - \text{div}_{\lambda,x} A \nabla_{\lambda,x} v = \text{div}_{\lambda,x} (\partial_t A) \nabla_{\lambda,x} u$. Caccioppoli's inequality for this solution v reads

$$\iiint_{\Lambda \times Q \times I} |\nabla_{\lambda,x} \partial_t u|^2 \lesssim \frac{1}{r^2} \iiint_{\tilde{\Lambda} \times \tilde{Q} \times \tilde{I}} |\partial_t u|^2 + \|\partial_t A\|_\infty^2 \iiint_{\tilde{\Lambda} \times \tilde{Q} \times \tilde{I}} |\nabla_{\lambda,x} u|^2.$$

Adjusting the enlargements, we have obtained

$$\iiint_{\Lambda \times Q \times I} |\partial_t u|^2 + |\nabla_{\lambda,x}^2 u|^2 + |\nabla_{\lambda,x} \partial_t u|^2 \lesssim \iiint_{\tilde{\Lambda} \times \tilde{Q} \times \tilde{I}} |\nabla_{\lambda,x} u|^2,$$

with an implicit constant depending on dimension, ellipticity, r and $\|\nabla_{\lambda,x} A\|_\infty + \|\partial_t A\|_\infty$.

If in addition u is a reinforced weak solution with $D_A u \in \mathcal{H}_{\text{loc}}$, then we can sum these estimates in x and t . More precisely, fix $\Lambda = (r, 2r)$, $\tilde{\Lambda} = (r/2, 4r)$ and choose translates $Q_k \times I_k$ of $Q \times I$ forming a partition of \mathbb{R}^{n+1} . Then the cubes $\tilde{Q}_k \times \tilde{I}_k$ have finite overlap and summing up, we obtain (12.9). By interpolation in time between, using for example Plancherel's theorem, the estimates for $\nabla_{\lambda,x} u$ and $\partial_t \nabla_{\lambda,x} u$, we obtain the one for $D_t^{1/2} \nabla_{\lambda,x} u$ and also for $H_t D_t^{1/2} \nabla_{\lambda,x} u$ as the Hilbert transform is isometric.

12.9. Proof of Proposition 2.20. Given $A \in \mathbb{A}$, there is a continuous path $\gamma : [0, 1] \rightarrow \mathbb{A}$ such that $\gamma(0) = A_0$ and $\gamma(1) = A$. We put $\gamma(\tau) := A_\tau$, let M_τ be the corresponding matrix defined in (2.7) and $\chi_\tau^+ := \chi^+(PM_\tau)$ the associated spectral projection. With this notation, the assumption is that all maps $N_\bullet : \chi_\tau^+ \dot{H}_P^s \rightarrow (\dot{H}_P^s)_\bullet$ have lower bounds and that invertibility holds for $\tau = 0$. In the following, we always consider subspaces of \dot{H}_P^s as Hilbert spaces with the inherited norm. Due to Theorem 2.15 and Proposition 2.16, the claim amounts to proving invertibility for all τ and that the inverses are compatible with those at regularity $s = -1/2$ provided this is true for $\tau = 0$. Let us recall from the discussion before Theorem 2.19 that $\chi_\tau^+ : \dot{H}_P^s \rightarrow \chi_\tau^+ \dot{H}_P^s$ is bounded and depends continuously on $\tau \in [0, 1]$.

First, we claim there exists $\delta > 0$ such that $\chi_\tau^+ : \chi_\sigma^+ \dot{H}_P^s \rightarrow \chi_\tau^+ \dot{H}_P^s$ is invertible whenever $|\sigma - \tau| \leq \delta$. Indeed, choosing δ sufficiently small according to uniform continuity of $\tau \mapsto \chi_\tau^+$, we can define bounded operators $\chi_\tau^+ \dot{H}_P^s \rightarrow \chi_\sigma^+ \dot{H}_P^s$ by $(1 - \chi_\sigma^+(\chi_\sigma^+ - \chi_\tau^+))^{-1} \chi_\sigma^+$ and $\chi_\sigma^+(1 - \chi_\tau^+(\chi_\tau^+ - \chi_\sigma^+))^{-1}$, respectively, via a convergent Neumann series. Using the relations

$$(1 - \chi_\sigma^+(\chi_\sigma^+ - \chi_\tau^+))\chi_\sigma^+ = \chi_\sigma^+ \chi_\tau^+ \chi_\sigma^+, \quad \chi_\tau^+ \chi_\sigma^+ = \chi_\tau^+(1 - \chi_\tau^+(\chi_\tau^+ - \chi_\sigma^+)),$$

these turn out to be left- and right inverses for $\chi_\tau^+ : \chi_\sigma^+ \dot{H}_P^s \rightarrow \chi_\tau^+ \dot{H}_P^s$.

Now, we start with $\sigma = 0$ and obtain for $|\tau| \leq \delta$ that $N_\bullet : \chi_\tau^+ \dot{H}_P^s \rightarrow (\dot{H}_P^s)_\bullet$ is invertible if and only if the composite map $N_\bullet \chi_\tau^+ : \chi_0^+ \dot{H}_P^s \rightarrow (\dot{H}_P^s)_\bullet$ between *fixed* spaces is invertible. The latter has lower bounds for any $|\tau| \leq \delta$ since this holds for $N_\bullet : \chi_\tau^+ \dot{H}_P^s \rightarrow (\dot{H}_P^s)_\bullet$; and it is invertible for $\tau = 0$ again by assumption and since χ_0^+ is a projection. Thus, the method of continuity yields invertibility for all $|\tau| \leq \delta$. Iterating this process finitely many times, we obtain invertibility for every $\tau \in [0, 1]$.

Finally, we turn to compatible well-posedness. From Theorem 2.15 and well-posedness of the BVPs in the energy class, we can infer for every τ that $N_\bullet : \chi_\tau^+ \mathbb{H}_P^{-1/2} \rightarrow (\mathbb{H}_P^{-1/2})_\bullet$ is invertible. Together with the assumption, this entails lower bounds for $N_\bullet : \chi_\tau^+ (\mathbb{H}_P^s \cap \mathbb{H}_P^{-1/2}) \rightarrow (\mathbb{H}_P^s \cap \mathbb{H}_P^{-1/2})_\bullet$, the intersections being Hilbert spaces for the sum of the two norms. Now, the crucial observation is that invertibility between these intersection spaces is equivalent to compatibility of the two inverses on $(\dot{H}_P^s)_\bullet$ and $(\dot{H}_P^{-1/2})_\bullet$ since their intersection is dense in both of them. This being said, we can simply repeat the argument above, replacing systematically \dot{H}_P^s by $\dot{H}_P^s \cap \dot{H}_P^{-1/2}$, to deduce that the inverses of N_\bullet are compatible for all τ provided they are for $\tau = 0$.

13. UNIQUENESS FOR DIRICHLET PROBLEMS

Here, we give the proofs of the uniqueness results on Dirichlet problems with non-tangential maximal function control stated in Theorems 2.37 and 2.40. For simplicity, we often fix Whitney parameters in the definition of \tilde{N}_* in such a way that $\Lambda = (\lambda, 2\lambda)$, $Q = B(x, \lambda)$, $I = (t - \lambda^2, t + \lambda^2)$ is a Whitney cylinder of sidelength $\lambda > 0$, where the center $(x, t) \in \mathbb{R}^{n+1}$ will be clear from the context. We shall freely use that different choices of Whitney parameters yield non-tangential maximal functions with comparable L^2 -norms, see Section 2.10. We also introduce a variant of the non-tangential maximal function

$$\tilde{N}_{*,1}(u)(x, t) := \sup_{\lambda > 0} \iiint_{\Lambda \times Q \times I} |u(\mu, y, s)| d\mu dy ds \quad ((x, t) \in \mathbb{R}^{n+1}),$$

where the subscript is reminiscent of the fact that we are using L^1 -averages instead of L^2 -averages. Of course, $\tilde{N}_{*,1}(u) \leq \tilde{N}_*(u)$ due to Hölder's inequality and if $\mathcal{L}u = 0$ in the weak sense, then $\tilde{N}_*(u)$ and $\tilde{N}_{*,1}(u)$ are comparable in L^2 -norm thanks to the reverse Hölder estimate in Lemma 8.2.

In both theorems we have to deduce that a certain weak solution to $\mathcal{L}u = 0$ vanishes on \mathbb{R}_+^{n+2} . In order to bring into play the assumptions on the dual equations, we shall use test functions of the form $\phi = \mathcal{L}^*H$. Similar to Section 2.3 we use the homogeneous energy space $\dot{E}(\mathbb{R}^{n+2})$ (considered modulo constants) to define

$$\langle \mathcal{L}^*H, v \rangle := \iiint_{\mathbb{R}^{n+2}} A^*(x, t) \nabla_{\lambda, x} \phi \cdot \overline{\nabla_{\lambda, x} v} - H_t D_t^{1/2} \phi \cdot \overline{D_t^{1/2} v} d\lambda dx dt \quad (v \in \dot{E}(\mathbb{R}^{n+2})),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing on $\dot{E}(\mathbb{R}^{n+2})$. Hidden coercivity reveals that $\mathcal{L}^* : \dot{E}(\mathbb{R}^{n+2}) \rightarrow \dot{E}(\mathbb{R}^{n+2})^*$ is an isomorphism and we say that H is the energy solution to $\mathcal{L}^*H = \phi$ on \mathbb{R}^{n+2} . A similar discussion applies to energy solutions of the inhomogeneous equation for \mathcal{L} . We remark that we work on the whole space \mathbb{R}^{n+2} to avoid the discussion of boundary values.

13.1. Proof of Theorem 2.37. For the sake of clean arrangement we shall postpone proofs of technical lemmas used throughout the main argument until Section 13.2.

Let us assume that u is a weak solution to $\mathcal{L}u = 0$ in \mathbb{R}_+^{n+2} that satisfies $\tilde{N}_*(u) \in L^2(\mathbb{R}^{n+1})$ and for almost every $(x, t) \in \mathbb{R}^{n+1}$,

$$(13.1) \quad \lim_{\lambda \rightarrow 0} \iint_{W((x,t),\lambda)} |u| = 0,$$

where for the argument we prefer to use $W((x, t), \lambda) := (\lambda/2, 4\lambda) \times B(x, \lambda) \times (t - 4\lambda^2, t + 4\lambda^2)$ as the Whitney regions defining \tilde{N}_* . We need to show $u = 0$ almost everywhere.

Let $\phi \in C_0^\infty(\mathbb{R}_+^{n+2})$. For a reason that will become clear in the course of the proof, we test u not against ϕ but against $\partial_\lambda^2 \phi$ and we shall show

$$(13.2) \quad (u, \partial_\lambda^2 \phi) = 0,$$

where here and throughout (\cdot, \cdot) denotes the inner product on $L^2(\mathbb{R}_+^{n+2})$. This suffices to conclude: Indeed, $\partial_\lambda^2 u = 0$ implies $u(\lambda, x, t) = f(x, t) + \lambda g(x, t)$ for some $f, g \in L_{\text{loc}}^2(\mathbb{R}^{n+1})$ and from (13.1) and Lebesgue's differentiation theorem we deduce $f = 0$ almost everywhere. As now $u(\lambda, x, t) = \lambda g(x, t)$, we obtain for any $\lambda > 0$ by means of Tonelli's theorem

$$(13.3) \quad \iint_{\mathbb{R}^{n+1}} |g(x, t)|^2 dx dt = \frac{1}{\lambda^2} \iint_{\mathbb{R}^{n+1}} \left(\iint_{W((x,t),\lambda)} |u|^2 \right) dx dt \leq \iint_{\mathbb{R}^{n+1}} \frac{1}{\lambda^2} \tilde{N}_*(u)(x, t)^2 dx dt.$$

Since $\tilde{N}_*(u) \in L^2(\mathbb{R}^{n+1})$ by assumption, we discover $g = 0$ almost everywhere on letting $\lambda \rightarrow \infty$.

We turn to the proof of (13.2). Let $\varepsilon \in (0, 1)$ and $R \in (1, \infty)$ be two degrees of freedom that will be chosen sufficiently small and large, respectively. Let $\eta = \eta(\lambda)$ be smooth with $\eta(\lambda) = 0$ for $\lambda \leq 1$ and $\eta(\lambda) = 1$ for $\lambda \geq 2$ and put $\theta(\lambda) = \eta(\lambda/\varepsilon) - \eta(\lambda/R)$. For ε and R sufficiently small and large, respectively, we have $(u, \partial_\lambda^2 \phi) = (\theta u, \partial_\lambda^2 \phi)$ since ϕ has compact support in \mathbb{R}_+^{n+2} . Since $\partial_\lambda^2 \phi \in \dot{E}(\mathbb{R}^{n+2})^*$, we can introduce the dual equation by writing $\partial_\lambda^2 \phi = \mathcal{L}^*v$ for some $v \in \dot{E}(\mathbb{R}^{n+2})$. We have

Lemma 13.1. *The functions θu and θv are test functions for the reinforced weak formulation of parabolic equations, that is, they are in the closure of $C_0^\infty(\mathbb{R}_+^{n+2})$ for the norm $\|\nabla_{\lambda,x} \cdot\|_2 + \|D_t^{1/2} \cdot\|_2$.*

Since θ does not depend on t , we obtain in particular $u \in \dot{H}^{1/2}(\mathbb{R}; L_{\text{loc}}^2(\mathbb{R}_+^{n+1}))$, that is, u is a reinforced weak solution, see also Corollary 3.6. In this way we can justify the calculation

$$\begin{aligned} (u, \partial_\lambda^2 \phi) &= (\theta u, \mathcal{L}^*v) \\ &= (\nabla_{\lambda,x}(\theta u), A^* \nabla_{\lambda,x}v) + (H_t D_t^{1/2}(\theta u), D_t^{1/2}v) \\ &= (u \nabla_{\lambda,x} \theta, A^* \nabla_{\lambda,x}v) - (\nabla_{\lambda,x}u, A^* \nabla_{\lambda,x} \theta v) + (\nabla_{\lambda,x}u, A^* \nabla_{\lambda,x}(\theta v)) + (H_t D_t^{1/2}u, D_t^{1/2}(\theta v)) \\ &= (u \nabla_{\lambda,x} \theta, A^* \nabla_{\lambda,x}v) - (\nabla_{\lambda,x}u, A^* \nabla_{\lambda,x} \theta v), \end{aligned}$$

where in the third step we have used that $\theta = \theta(\lambda)$ is real valued and the final step follows from the reinforced formulation of the equation $\mathcal{L}u = 0$. At this stage of the proof we want to bring the existence hypothesis into play. So, we need further properties of v stated in the following

Lemma 13.2. *Let $\phi \in C_0^\infty(\mathbb{R}^{n+2})$. The energy solution to $\mathcal{L}^*v = \partial_\lambda^2 \phi$ satisfies $v \in C^\infty(\mathbb{R}; L^2 \cap \dot{H}_{\partial_t - \Delta_x}^{1/2})$. Moreover, $\tilde{N}_{*,1}(\frac{v-v_0}{\lambda}) \in L^2(\mathbb{R}^{n+1})$, where $v_0 := v|_{\lambda=0}$.*

Since in particular $v_0 \in \dot{H}_{\partial_t - \Delta_x}^{1/2}$, the assumption provides a reinforced weak solution to $\mathcal{L}^*w = 0$ that satisfies $\tilde{N}_*(\tilde{D}_{A^*}w) \in L^2(\mathbb{R}^{n+1})$ and $w|_{\lambda=0} = v_0$ in the sense of $\dot{H}_{\partial_t - \Delta_x}^{1/2}$. The backward conormal differential \tilde{D}_{A^*} has been introduced in Section 2.12. Due to $v_0 \in L^2(\mathbb{R}^{n+1})$ we can realise w by an

additive constant such that $w|_{\lambda=0} = v_0$ holds also in $L^2(\mathbb{R}^{n+1})$. Upon replacing v with w , the same line of reasoning as above then reveals

$$0 = (\theta u, \mathcal{L}^* w) = (u \nabla_{\lambda,x} \theta, A^* \nabla_{\lambda,x} w) - (\nabla_{\lambda,x} u, A^* \nabla_{\lambda,x} \theta w),$$

invoking the following lemma to justify calculations.

Lemma 13.3. *The function θw is in the closure of $C_0^\infty(\mathbb{R}_+^{n+2})$ for the norm $\|\nabla_{\lambda,x} \cdot\|_2 + \|D_t^{1/2} \cdot\|_2$.*

Putting things together, we conclude

$$(u, \partial_\lambda^2 \phi) = (u \nabla_{\lambda,x} \theta, A^* \nabla_{\lambda,x} (v - w)) - (\nabla_{\lambda,x} u, A^* \nabla_{\lambda,x} \theta (v - w)).$$

Here, both u and $v - w$ vanish at $\lambda = 0$ and in fact this convergence at the boundary was the reason to introduce w . On recalling our choice of θ , we obtain the straightforward bound

$$(13.4) \quad |(u, \partial_\lambda^2 \phi)| \lesssim I_\varepsilon + I_R + \text{II}_\varepsilon + \text{II}_R,$$

where

$$I_\sigma := \int_\sigma^{2\sigma} \iint_{\mathbb{R}^{n+1}} |u| |\nabla_{\lambda,x} (v - w)|, \quad \text{II}_\sigma := \int_\sigma^{2\sigma} \iint_{\mathbb{R}^{n+1}} |\nabla_{\lambda,x} u| |v - w|.$$

We prepare the limiting arguments $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$. Define parabolic cylinders $\frac{1}{2}W((x, t), \lambda) := (\lambda, 2\lambda) \times B(x, \lambda) \times (t - \lambda^2, t + \lambda^2)$, of which $W((x, t), \lambda)$ are enlargements. By an averaging trick in (x, t) and Tonelli's theorem we conclude

$$(13.5) \quad \begin{aligned} I_\sigma &= \int_\sigma^{2\sigma} \iint_{\mathbb{R}^{n+1}} \left(\iint_{B(x, \sigma) \times (t - \sigma^2, t + \sigma^2)} |u| |\nabla_{\lambda,x} (v - w)| \, dy \, ds \right) dx \, dt \, d\mu \\ &\leq \iint_{\mathbb{R}^{n+1}} \left(\iiint_{\frac{1}{2}W((x, t), \sigma)} |u|^2 \, d\mu \, dy \, ds \right)^{1/2} \left(\iiint_{\frac{1}{2}W((x, t), \sigma)} |\nabla_{\lambda,x} (v - w)|^2 \, d\mu \, dy \, ds \right)^{1/2} dx \, dt \\ &\lesssim \iint_{\mathbb{R}^{n+1}} \left(\iiint_{W((x, t), \sigma)} |u| \, d\mu \, dy \, ds \right) \left(\iiint_{W((x, t), \sigma)} \left| \frac{v - w}{\mu} \right| \, d\mu \, dy \, ds \right) dx \, dt \\ &\leq \left\| \tilde{N}_{*,1}(u) \right\|_{L^2(\mathbb{R}^{n+1})} \left\| \tilde{N}_{*,1} \left(\frac{v - w}{\lambda} \right) \right\|_{L^2(\mathbb{R}^{n+1})}, \end{aligned}$$

where in the second to last step we have used the reverse Hölder inequality (Lemma 8.2) on u and Caccioppoli's inequality (Lemma 8.1) followed by the reverse Hölder inequality on $v - w$. The latter can be justified for σ sufficiently small or large, in which case $\mathcal{L}^*(v - w) = \partial_\lambda^2 \phi = 0$ holds on a neighbourhood of $W((x, t), \sigma)$. By interchanging the roles of u and $v - w$ we also get

$$(13.6) \quad \begin{aligned} \text{II}_\sigma &\lesssim \iint_{\mathbb{R}^{n+1}} \left(\iiint_{W((x, t), \sigma)} \left| \frac{u}{\mu} \right| \, d\mu \, dy \, ds \right) \left(\iiint_{W((x, t), \sigma)} |v - w| \, d\mu \, dy \, ds \right) dx \, dt \\ &\lesssim \left\| \tilde{N}_{*,1}(u) \right\|_{L^2(\mathbb{R}^{n+1})} \left\| \tilde{N}_{*,1} \left(\frac{v - w}{\lambda} \right) \right\|_{L^2(\mathbb{R}^{n+1})}. \end{aligned}$$

In both estimates $\tilde{N}_{*,1}(u) \in L^2(\mathbb{R}^{n+1})$ holds by assumption and writing $v - w = (v - v_0) - (w - v_0)$, we obtain $\tilde{N}_{*,1}(\frac{v-w}{\lambda}) \in L^2(\mathbb{R}^{n+1})$ from Lemma 13.2 and the subsequent result. This is a parabolic version of [46, Thm. 3.1] but in contrast to its elliptic counterpart, it is limited to solutions to the homogeneous equation as will become clear from the proof given in Section 13.2.

Lemma 13.4. *Let w be a reinforced weak solution to $\mathcal{L}^* w = 0$ on \mathbb{R}_+^{n+2} satisfying $\tilde{N}_*(\tilde{D}_{A^*} w) \in L^2(\mathbb{R}^{n+1})$. Then there exists a trace $w_0 \in L_{\text{loc}}^2(\mathbb{R}^{n+1})$ such that for almost every $(x, t) \in \mathbb{R}^{n+1}$,*

$$\lim_{\lambda \rightarrow 0} \iiint_{\Lambda \times Q \times I} |w - w_0(x, t)| \, d\mu \, dy \, ds = 0.$$

Furthermore, $\tilde{N}_{*,1}(\frac{w-w_0}{\lambda}) \in L^2(\mathbb{R}^{n+1})$ and $\nabla_x w_0 = (\tilde{D}_{A^*} w)|_{\lambda=0}$. An analogous statement holds for reinforced weak solutions to $\mathcal{L}u = 0$ upon replacing \tilde{D}_{A^*} by D_A .

The upshot is that we can compute limits in (13.5) and (13.6) by means of Lebesgue's theorem. From (13.1) we conclude $I_\varepsilon \rightarrow 0$ and $II_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. In order to obtain $I_R \rightarrow 0$ and $II_R \rightarrow 0$ as $R \rightarrow \infty$, we use $\tilde{N}_{*,1}(u) \in L^2(\mathbb{R}^{n+1})$ and the generic lemma on non-tangential maximal functions:

Lemma 13.5. *If $u \in L^1_{\text{loc}}(\mathbb{R}_+^{n+2})$ satisfies $\tilde{N}_{*,1}(u) \in L^2(\mathbb{R}^{n+1})$, then for every $(x, t) \in \mathbb{R}^{n+1}$,*

$$\lim_{\lambda \rightarrow \infty} \iint_{\Lambda \times Q \times I} |u| = 0.$$

Going back to (13.4), we see that (13.2) holds and the proof of Theorem 2.37 is complete.

13.2. Non-tangential estimates for energy solutions. Here, we supply the auxiliary results used in the proof of Theorem 2.37. As a general observation, but also to set a strategy of proof that will be re-used several times, we begin with an extension of Lemma 3.14 to the effect that the trace of the homogeneous energy space can also be understood in the sense of non-tangential convergence. This is a parabolic analogue of [4, Thm. 6.42].

Lemma 13.6. *For each $v \in \dot{E}(\mathbb{R}_+^{n+2})$ the trace $v_0 := v|_{\lambda=0} \in \dot{H}_{\partial_t - \Delta_x}^{1/4}$ can be realised in $L^2_{\text{loc}}(\mathbb{R}^{n+1})$ such that $\tilde{N}_{*,1}(\frac{v-v_0}{\sqrt{\lambda}}) \in L^2(\mathbb{R}^{n+1})$ and for almost every $(x, t) \in \mathbb{R}^{n+1}$,*

$$\lim_{\lambda \rightarrow 0} \iint_{\Lambda \times Q \times I} |v - v_0(x, t)| \, d\mu \, dy \, ds = 0.$$

Proof. Abbreviating $F := |\nabla_{\lambda,x} v| + |H_t D_t^{1/2} v|$, we can infer from Poincaré's inequality as in the proof of Lemma 8.4,

$$(13.7) \quad \iint_{\Lambda \times Q \times I} \left| v - \iint_{\Lambda \times Q \times I} v \right| \lesssim \lambda \sum_{k \in \mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \iint_{\Lambda \times Q \times I_k} |F| \leq \lambda \sum_{m \geq 0} 2^{-m} \iint_{\Lambda \times Q \times 4^m I} |F|,$$

where the second step follows by a simple rearrangement of terms as in the proof of Lemma 8.12. Let \mathcal{M}_x and \mathcal{M}_t denote again the maximal operators in the x and t variable, respectively. By Hölder's inequality,

$$(13.8) \quad \begin{aligned} \iint_{\Lambda \times Q \times I} \left| v - \iint_{\Lambda \times Q \times I} v \right| &\lesssim \lambda \sum_{m \geq 0} 2^{-m} \left(\int_{\Lambda} \left(\iint_{Q \times 4^m I} |F| \right)^2 \right)^{1/2} \\ &\lesssim \sqrt{\lambda} \left(\int_0^\infty (\mathcal{M}_x \mathcal{M}_t F_\mu)(x, t)^2 \, d\mu \right)^{1/2} =: \sqrt{\lambda} V(x, t). \end{aligned}$$

Since $F \in L^2(\mathbb{R}_+^{n+2})$ by assumption on v , we obtain $V \in L^2(\mathbb{R}^{n+1})$ from the L^2 boundedness of the maximal operators. In particular, we may concentrate on fixed $(x, t) \in \mathbb{R}^{n+1}$ for which $V(x, t)$ is finite.

Let $h(\lambda) := \iint_{\Lambda \times Q \times I} v$, noting that for us $\Lambda \times Q \times I$ depends only on λ as we have fixed (x, t) . From (13.8) we derive that for every $\delta \in (0, 1)$ there is a constant C_δ such that for every $\lambda > 0$ we have an estimate $|h(\lambda) - h(\delta\lambda)| \leq C_\delta \sqrt{\lambda}$. This implies that $\lim_{\lambda \rightarrow 0} h(\lambda) =: w_0(x, t)$ exists. Therefore we can use a telescopic sum of the estimates (13.8) for Whitney regions of sidelength $2^{-k}\lambda$, $k \in \mathbb{N}$, to obtain the stronger estimate

$$(13.9) \quad \iint_{\Lambda \times Q \times I} |v - w_0(x, t)| \, d\mu \, dy \, ds \lesssim \sqrt{\lambda} V(x, t).$$

Moreover, given $(x, t), (y, s) \in \mathbb{R}^{n+1}$, we can use (13.8) on a Whitney region of length $\lambda \sim |x - y| + |t - s|^{1/2}$ to get

$$|w_0(x, t) - w_0(y, s)| \lesssim (|x - y| + |t - s|^{1/2})^{1/2} (V(x, t) + V(y, s)).$$

In particular, this implies $w_0 \in L^2_{\text{loc}}(\mathbb{R}^{n+1}) \cap \mathcal{S}'(\mathbb{R}^{n+1})$ and together with (13.9) we finally obtain

$$\iint_{\Lambda \times Q \times I} |v - w_0| \, d\mu \, dy \, ds \lesssim \sqrt{\lambda} (V(x, t) + \mathcal{M}_x \mathcal{M}_t(V)(x, t)).$$

Since $V \in L^2(\mathbb{R}^{n+1})$, this implies $\tilde{N}_{*,1}(\frac{v-w_0}{\sqrt{\lambda}}) \in L^2(\mathbb{R}^{n+1})$.

In order to conclude, we have yet to check that $w_0 = v_0$ holds on \mathbb{R}^{n+1} in the sense of distributions modulo constants. To this end let $\phi \in C_0^\infty(\mathbb{R}^{n+1})$. Using Fubini's theorem, we perform an averaging trick in (x, t) and write

$$\iint_{\mathbb{R}^{n+1}} \left(\int_\lambda^{2\lambda} v(\mu, y, s) \, d\mu - w_0(y, s) \right) \phi(y, s) \, dy \, ds = \sqrt{\lambda} \iint_{\mathbb{R}^{n+1}} \left(\iiint_{W((x,t),\lambda)} \frac{v-w_0}{\sqrt{\lambda}} \phi \right) dx \, dt,$$

where $W((x, t), \lambda) := (\lambda, 2\lambda) \times B(x, \lambda) \times (t - \lambda^2, t + \lambda^2)$. The integral on the right is bounded uniformly in λ since $\tilde{N}_{*,1}(\frac{v-w_0}{\sqrt{\lambda}}) \in L^2(\mathbb{R}^{n+1})$ and ϕ has compact support. Thus, $\int_\lambda^{2\lambda} v \, d\mu \rightarrow w_0$ in the sense of distributions as $\lambda \rightarrow 0$. On the other hand, due to Lemma 3.14 we have $v(\lambda, \cdot) \rightarrow v_0$ as $\lambda \rightarrow 0$ in $\dot{H}_{\partial_t - \Delta_x}^{1/4}$ and hence in the sense of distributions modulo constants. \square

We adapt this strategy to give the proofs of Lemmas 13.4 and 13.2.

Proof of Lemma 13.4. Since notation is set up for the forward equation throughout the paper, we prefer to argue for a reinforced weak solution to $\mathcal{L}u = 0$ that satisfies $\tilde{N}_*(D_A u) \in L^2(\mathbb{R}^{n+1})$. This reduction is justified by the usual transposition of results between forward and backward equations, see also Section 2.12. The proof follows that of Lemma 13.6 verbatim, once we have shown that there exists $U \in L^2(\mathbb{R}^{n+1})$ such that

$$\iiint_{\Lambda \times Q \times I} \left| u - \iiint_{\Lambda \times Q \times I} u \right| \lesssim \lambda U(x, t).$$

Here, $\Lambda \times Q \times I$ is a fixed Whitney region of sidelength λ centered at (x, t) . Let $F = D_A u$. From (13.7) we know that we can take

$$U(x, t) := \sup_{\Lambda' \times Q' \times I'} \sum_{m=0}^{\infty} 2^{-m} \iiint_{\Lambda' \times Q' \times 4^m I'} |F|,$$

with the supremum over all Whitney regions $\Lambda' \times Q' \times I'$ centered at (x, t) . However, this is *not* the non-tangential maximal function that we control by assumption and it is for this reason that we have to assume that u is a reinforced weak solution to the homogeneous equation. Indeed, F is a solution to the first order system (2.10), see Theorem 2.2, and as $\tilde{N}_*(F) \in L^2(\mathbb{R}^{n+1})$ implies the Dini condition $\sup_{\lambda > 0} \int_\lambda^{2\lambda} \|F\|_{L^2(\mathbb{R}^{n+1})} \, d\mu < \infty$, Theorem 2.5 provides a representation $F(\lambda, \cdot) = e^{-\lambda[PM]} h$ for some $h \in H^+(PM)$. Now, $\|U\|_2 \lesssim \|h\|_2 < \infty$ follows from (8.11). \square

Proof of Lemma 13.2. For any $\psi \in C_0^\infty(\mathbb{R}^{n+2})$ we have $\partial_\lambda \psi \in \dot{E}(\mathbb{R}^{n+2})^*$. Thus, can define $\mathcal{L}^{*-1}(\partial_\lambda \psi)$ by hidden coercivity and there is an *a priori* estimate

$$\|\mathcal{L}^{*-1}(\partial_\lambda \psi)\|_{\dot{E}} \lesssim \|\partial_\lambda \psi\|_{\dot{E}^*} \leq \|\psi\|_2.$$

In particular, we have $\partial_\lambda \mathcal{L}^{*-1}(\partial_\lambda \psi) \in L^2(\mathbb{R}^{n+2})$. As \mathcal{L}^* has λ -independent coefficients, we can use the method of difference quotients in $\dot{E}(\mathbb{R}^{n+2})$ and Caccioppoli's estimate (Lemma 8.1) to deduce

$$\partial_\lambda \mathcal{L}^{*-1}(\partial_\lambda \psi) = \mathcal{L}^{*-1}(\partial_\lambda^2 \psi) \in \dot{E}(\mathbb{R}^{n+2}).$$

Applying these observations with $\psi = \phi$, $\partial_\lambda \phi$ reveals $v, \partial_\lambda v \in \dot{E}(\mathbb{R}^{n+2}) \cap L^2(\mathbb{R}^{n+2})$, that is, the four functions $v, \nabla_x v, H_t D_t^{1/2} v, \partial_\lambda v$ as well as their λ -derivatives are in $L^2(\mathbb{R}^{n+2})$. Integration in λ shows that they can be defined continuously with values in $L^2(\mathbb{R}^{n+1})$. Thus, $v \in C(\mathbb{R}; L^2 \cap \dot{H}_{\partial_t - \Delta_x}^{1/2})$ and $\partial_\lambda v \in C(\mathbb{R}; L^2)$. Now, we can iterate this argument taking $\psi = \partial_\lambda^k \phi$, $k \geq 2$, to obtain $C^\infty(\mathbb{R}; L^2 \cap \dot{H}_{\partial_t - \Delta_x}^{1/2})$.

Let us turn to the non-tangential estimate. Due to Lemma 13.6 we only need to check it for Whitney regions $\Lambda \times Q \times I$ centered at $(x, t) \in \mathbb{R}^{n+1}$ with sidelength $\lambda \leq 1$. The same lemma asserts that v attains its trace v_0 in the sense of almost everywhere convergence of Whitney averages.

Therefore we can apply (13.7) iteratively on the Whitney regions of sidelength $2^{-k}\lambda$, $k \in \mathbb{N}$, centered at (x, t) to find by a telescopic sum

$$(13.10) \quad \iiint_{\Lambda \times Q \times I} |v - v_0| \lesssim \lambda \sup_{k \in \mathbb{N}} \left(\sum_{m \geq 0} 2^{-m} \iint_{2^{-k}Q \times 4^{m-k}I} \int_{2^{-k}\lambda}^{2^{-k+1}\lambda} |\partial_\lambda v| + |\nabla_x v| + |H_t D_t^{1/2} v| \right).$$

In order to treat the integrals over $|\nabla_x v| + |H_t D_t^{1/2} v|$, we write

$$v(\mu, y, s) = v_0(y, s) + \int_0^\mu \partial_\lambda v(\sigma, y, s) \, d\sigma \quad ((\mu, y, s) \in \mathbb{R}_+^{n+2})$$

to obtain, taking into account $2^{-k+1}\lambda \leq 2$,

$$\begin{aligned} \iint_{2^{-k}Q \times 4^{m-k}I} \int_{2^{-k}\lambda}^{2^{-k+1}\lambda} |\nabla_x v| + |H_t D_t^{1/2} v| &\lesssim \iint_{2^{-k}Q \times 4^{m-k}I} \int_0^2 |\nabla_x \partial_\lambda v| + |H_t D_t^{1/2} \partial_\lambda v| \\ &\quad + \iint_{2^{-k}Q \times 4^{m-k}I} |\nabla_x v_0| + |H_t D_t^{1/2} v_0|. \end{aligned}$$

For the integrals over $\partial_\lambda v$ in (13.10) a similar use of the fundamental theorem of calculus leads us to

$$\iint_{2^{-k}Q \times 4^{m-k}I} \int_{2^{-k}\lambda}^{2^{-k+1}\lambda} |\partial_\lambda v| \lesssim \iint_{2^{-k}Q \times 4^{m-k}I} \int_0^2 |\partial_\lambda^2 v| + \iint_{2^{-k}Q \times 4^{m-k}I} |\partial_\lambda v|_{\lambda=0}.$$

Due to $v \in C^\infty(\mathbb{R}; L^2 \cap \dot{H}_{\partial_t - \Delta_x}^{1/2})$ we have

$$F_0 := |\nabla_x v_0| + |H_t D_t^{1/2} v_0| + |\partial_\lambda v|_{\lambda=0} \in L^2(\mathbb{R}^{n+1})$$

and

$$F := \int_0^2 |\nabla_x \partial_\lambda v| + |H_t D_t^{1/2} \partial_\lambda v| + |\partial_\lambda^2 v| \, d\mu \in L^2(\mathbb{R}^{n+1}).$$

The previous two estimates then show that the right-hand side of (13.10) can be controlled by $\lambda \mathcal{M}_x \mathcal{M}_t(F + F_0)(x, t)$ and L^2 boundedness of the maximal operators yields the claim. \square

Next, we supply a general fact on weak solutions implying in particular that every weak solution to $(D)_2^L$ is reinforced. This was announced already in Section 2.17 and used in Lemma 13.1.

Lemma 13.7. *Let $\phi \in C_0^\infty(\mathbb{R}_+^{n+2})$ and let u be a weak solution to $\mathcal{L}u = \phi$ on \mathbb{R}_+^{n+2} that satisfies $\tilde{N}_{*,1}(u/\omega) \in L^2(\mathbb{R}^{n+1})$ for some locally bounded, measurable $\omega : (0, \infty) \rightarrow (0, \infty)$ depending only on λ . Then for any $\theta \in C_0^\infty(0, \infty)$, depending only on λ , it holds $\theta u \in L^2(\mathbb{R}_+^{n+2}) \cap \dot{E}(\mathbb{R}_+^{n+2})$.*

Proof. The statement we have to prove is purely qualitative and so we shall not care about dependence of implicit constants. First, we pick $a > 0$ and $c_0 < 1$ such that $\text{supp}(\theta) \subset (c_0^{-1}a, c_0a)$. It will turn out convenient to use $W((x, t), \lambda) := ((2c_0)^{-1}\lambda, 2c_0\lambda) \times B(x, 2\lambda) \times (t - 4\lambda^2, t + 4\lambda^2)$ as Whitney cylinder around (x, t) to define $\tilde{N}_{*,1}$. We also introduce $\frac{1}{2}W((x, t), \lambda) := ((c_0)^{-1}\lambda, c_0\lambda) \times B(x, \lambda) \times (t - \lambda^2, t + \lambda^2)$. By an averaging trick in (x, t) , we find

$$(13.11) \quad \begin{aligned} \iiint_{\mathbb{R}_+^{n+2}} |u\theta|^2 \, d\lambda \, dx \, dt &\lesssim \int_{c_0^{-1}a}^{c_0a} \iint_{\mathbb{R}^{n+1}} \left(\iint_{B(x,a) \times (t-a^2, t+a^2)} |u|^2 \, dy \, ds \right) \, dx \, dt \, d\lambda \\ &\lesssim \iint_{\mathbb{R}^{n+1}} \left(\iint_{\frac{1}{2}W((x,t),a)} |u|^2 \, d\lambda \, dy \, ds \right) \, dx \, dt. \end{aligned}$$

Since ϕ is compactly supported, there is a parabolic cube $Q \times I \subseteq \mathbb{R}_+^{n+2}$ such that $\mathcal{L}u = 0$ in a neighbourhood of $W((x, t), a)$ for every $(x, t) \in {}^c(Q \times I)$. This makes the reverse Hölder estimate for weak solutions (Lemma 8.2) applicable and we deduce

$$(13.12) \quad \iint_{\frac{1}{2}W((x,t),a)} |u|^2 \lesssim \left(\iint_{W((x,t),a)} |u| \right)^2 \lesssim \tilde{N}_{*,1}(u/\omega)(x, t)^2 \quad ((x, t) \in {}^c(Q \times I)).$$

Note that we have used the local boundedness of ω in the second step. Thus,

$$\iiint_{\mathbb{R}_+^{n+2}} |u\theta|^2 \lesssim \iint_{Q \times I} \left(\iiint_{\frac{1}{2}W((x,t),a)} |u|^2 \right) dx dt + \iint_{c(Q \times I)} \tilde{N}_{*,1}(u/\omega)^2(x,t) dx dt.$$

Here the second integral is finite by assumption on u and so is the first one since u as a weak solution is locally square-integrable. So far, this proves $\theta u \in L^2(\mathbb{R}_+^{n+2})$. Next, we claim $\nabla_{\lambda,x}(\theta u) \in L^2(\mathbb{R}_+^{n+2}; \mathbb{C}^{n+1})$. Indeed,

$$\nabla_{\lambda,x}(\theta u) = u \nabla_{\lambda,x} \theta + \theta \nabla_{\lambda,x} u,$$

where the first term can be treated just as θu above and for the second one the only change in the argument is to apply Caccioppoli's estimate (Lemma 8.1) to eliminate the gradient of u before using a reverse Hölder estimate as in (13.12). Hence, $\theta u \in L^2(\mathbb{R}; W^{1,2}(\mathbb{R}_+^{n+1}))$.

At this point, we have all the ingredients to follow the interpolation argument given in the proof of Lemma 8.5 to first show $\partial_t(\theta u) \in L^2(\mathbb{R}; W^{-1,2}(\mathbb{R}_+^{n+1}))$ and then deduce the missing piece of information $H_t D_t^{1/2}(\theta u) \in L^2(\mathbb{R}_+^{n+2})$. The fact that here u solves the inhomogeneous equation $\mathcal{L}u = \phi$ does not pose any difficulties in this argument since we have $\theta\phi \in L^2(\mathbb{R}_+^{n+2})$. \square

As a direct application we obtain the

Proofs of Lemma 13.1 and Lemma 13.3. Let us start with θv and let $v_0 := v|_{\lambda=0}$. On noting that for almost every $(\lambda, x, t) \in \mathbb{R}_+^{n+2}$,

$$\left| \frac{v(\lambda, x, t)}{\max\{1, \lambda\}} \right| \leq \left| \frac{v(\lambda, x, t) - v_0(x, t)}{\lambda} \right| + |v_0(x, t)|,$$

we deduce the pointwise bound

$$\tilde{N}_{*,1} \left(\frac{v}{\max\{1, \lambda\}} \right) \leq \tilde{N}_{*,1} \left(\frac{v - v_0}{\lambda} \right) + \mathcal{M}_x \mathcal{M}_t(v_0).$$

By Lemma 13.2 and L^2 boundedness of the maximal operators the right-hand side is in $L^2(\mathbb{R}^{n+1})$. Thus, Lemma 13.7 yields $\theta v \in L^2(\mathbb{R}_+^{n+2}) \cap \dot{E}(\mathbb{R}_+^{n+2})$. In this *inhomogeneous* energy space we can approximate θv by the usual cut-off and mollification procedure. Since θv is supported away from the boundary $\lambda = 0$, these approximants are in $C_0^\infty(\mathbb{R}_+^{n+2})$ as required.

Referring to Lemma 13.4 instead, the same argument applies to θw and for θu we may simply use the assumption $\tilde{N}_*(u) \in L^2(\mathbb{R}^{n+1})$. \square

We complete the treatment of postponed auxiliary results with the

Proof of Lemma 13.5. As usual, let $\Lambda \times Q \times I$ be a Whitney region centered around (x, t) with side-length λ . By a change of Whitney parameters we may assume $\tilde{N}_{*,1}$ is defined with respect to the larger regions $\Lambda \times 2Q \times 4I$. With this convention, we have for every $(z, r) \in Q \times I$,

$$\iiint_{\Lambda \times Q \times I} |u(y, s)| dy ds \leq 2^{n+2} \tilde{N}_{*,1}(u)(z, r).$$

Hence,

$$\iiint_{\Lambda \times Q \times I} |u(y, s)| dy ds \leq 2^{n+2} \iint_{Q \times I} \tilde{N}_{*,1}(u)(z, r) dz dr \lesssim \lambda^{-1-n/2} \|\tilde{N}_{*,1}(u)\|_{L^2(\mathbb{R}^{n+1})}$$

and the conclusion follows. \square

13.3. Proof of Theorem 2.40. Let $f \in L^2(\mathbb{R}^{n+1}) \subset L^2(\mathbb{R}^{n+1}) + \dot{H}_{\partial_t - \Delta_x}^{1/4}$ and let u be the corresponding solution to $\mathcal{L}u = 0$ provided by part (ii) of Proposition 2.34, written as the sum of two solutions $u_1 + u_2$ to the same equation with $D_A u_1 \in \mathcal{E}_{-1}$ and $D_A u_2 \in \mathcal{E}_{-1/2}$. Thanks to Theorem 2.8 there are traces

$$u_{1,0} := u_1|_{\lambda=0} \in L^2(\mathbb{R}^{n+1}), \quad u_{2,0} := u_2|_{\lambda=0} \in \dot{H}_{\partial_t - \Delta_x}^{1/4},$$

which by construction satisfy $u_{1,0} + u_{2,0} = f$ in the sense of distributions modulo constants. In particular, we may modify u by an additive constant to obtain also $u_{2,0} = f - u_{1,0} \in L^2(\mathbb{R}^{n+1})$.

Next, Theorem 2.9 provides a representation $u_1(\lambda, \cdot) = c - (e^{-\lambda[MP]\tilde{h}})_\perp$ for some unique $c \in \mathbb{C}$ and $\tilde{h} \in \mathbf{H}^+(MP)$. Since $u_{1,0} \in L^2(\mathbb{R}^{n+1})$, we must have $c = 0$, in which case Theorem 2.13 yields $\tilde{N}_*(u_1) \in L^2(\mathbb{R}^{n+1})$ and a.e. convergence of Whitney averages of u_1 towards $u_{1,0}$ as $\lambda \rightarrow 0$. As for u_2 , we recall that $D_A u_2 \in \mathcal{E}_{-1/2}$ precisely means $u_2 \in \dot{\mathbf{E}}(\mathbb{R}_+^{n+2})$. Thus, Lemma 13.6 yields $\tilde{N}_{*,1}(\frac{u_2 - u_{2,0}}{\sqrt{\lambda}}) \in L^2(\mathbb{R}^{n+1})$ and a.e. convergence of Whitney averages of u_2 to $u_{2,0}$ as $\lambda \rightarrow 0$. Adding up, we obtain Whitney average convergence of u towards the boundary data f . Moreover, on splitting for almost every $(\lambda, x, t) \in \mathbb{R}_+^{n+2}$,

$$\left| \frac{u(\lambda, x, t)}{\max\{1, \sqrt{\lambda}\}} \right| \leq |u_1(\lambda, x, t)| + \left| \frac{u_2(\lambda, x, t) - u_{2,0}(x, t)}{\sqrt{\lambda}} \right| + |u_{2,0}(x, t)|,$$

we get a pointwise bound

$$\tilde{N}_{*,1}\left(\frac{u}{\max\{1, \sqrt{\lambda}\}}\right) \leq \tilde{N}_{*,1}(u_1) + \tilde{N}_{*,1}\left(\frac{u_2 - u_{2,0}}{\sqrt{\lambda}}\right) + \mathcal{M}_x \mathcal{M}_t(u_{2,0})$$

and we have seen that all three maximal functions are in $L^2(\mathbb{R}^{n+1})$. The reverse Hölder inequality in Lemma 8.2 finally yields $\tilde{N}_*\left(\frac{u}{\max\{1, \sqrt{\lambda}\}}\right) \in L^2(\mathbb{R}^{n+1})$ as required.

It remains to prove uniqueness. Let u be a weak solution to $\mathcal{L}u = 0$ with $\tilde{N}_*\left(\frac{u}{\max\{1, \sqrt{\lambda}\}}\right) \in L^2(\mathbb{R}^{n+1})$ such that for almost every $(x, t) \in \mathbb{R}^{n+1}$,

$$\lim_{\lambda \rightarrow 0} \iint_{\Lambda \times Q \times I} |u| = 0.$$

In proving $u = 0$, we can follow the proof of Theorem 2.37. Indeed, given $\phi \in C_0^\infty(\mathbb{R}_+^{n+2})$, we can define $v := \mathcal{L}^{*-1}(\partial_\lambda^2 \phi)$ as before. Since the regularity assumption on A is preserved under a reversal of time and taking adjoints, we also have at hand Proposition 2.34 for the adjoint equation. In particular, since Lemmas 3.14 and 13.2 pay for $v|_{\lambda=0} \in \dot{\mathbf{H}}_{\partial_t - \Delta_x}^{1/2} \cap \dot{\mathbf{H}}_{\partial_t - \Delta_x}^{1/4}$, we can find a reinforced weak solution to $\mathcal{L}^* w = 0$ with trace $w|_{\lambda=0} = v|_{\lambda=0}$ and the same estimate $\tilde{D}_{A^*} u \in \mathcal{E}_0$ as before. As an additional feature, $\tilde{D}_{A^*} u \in \mathcal{E}_{-1/2}$, that is, $w \in \dot{\mathbf{E}}(\mathbb{R}_+^{n+2})$.

Now that the same setup is settled, there are only two instances in the proof of Theorem 2.37 at which the precise form of the non-tangential control for u matters, namely (13.3) and the limiting argument relying on (13.5) and (13.6). For the former, having L^2 -control only on $\tilde{N}_*(u/\sqrt{\lambda})$ is sufficient. For the latter, we need a different argument only when $\sigma = R$ is large, in which case we simply distribute the factor $1/\mu$ to obtain

$$\mathbb{I}_\sigma + \mathbb{II}_\sigma \lesssim \iint_{\mathbb{R}^{n+1}} \left(\iint_{W((x,t),\sigma)} \left| \frac{u}{\sqrt{\mu}} \right| d\mu dy ds \right) \left(\iint_{W((x,t),\sigma)} \left| \frac{v-w}{\sqrt{\mu}} \right| d\mu dy ds \right) dx dt.$$

Here, the averages on $u/\sqrt{\lambda}$ are under control by assumption and for those of $(v-w)/\sqrt{\lambda}$ we can use the additional information on w : As $v-w \in \dot{\mathbf{E}}(\mathbb{R}_+^{n+2})$ has trace 0, the required L^2 -control is due to Lemma 13.6. This being said, we can complete the proof as in the case of Theorem 2.37.

14. MISCELLANEOUS GENERALISATIONS AND OPEN PROBLEMS

14.1. Systems. All of our results apply to parabolic systems without any changes but in the ellipticity condition. More precisely, we can deal with systems of m equations given by

$$\partial_t u^\alpha - \sum_{i,j=0}^n \sum_{\beta=1}^m \partial_i (A_{i,j}^{\alpha,\beta}(\lambda, x, t) \partial_j u^\beta) = 0, \quad \alpha = 1, \dots, m,$$

in \mathbb{R}_+^{n+2} , where $\partial_0 = \partial_\lambda$ and $\partial_i = \partial_{x_i}$ if $i = 1, \dots, n$, and where we impose the following assumptions on the matrix $A(\lambda, x, t) = (A_{i,j}^{\alpha,\beta}(\lambda, x, t))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m}$. First,

$$A(\lambda, x, t) = (A_{i,j}^{\alpha,\beta}(\lambda, x, t))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m} \in L^\infty(\mathbb{R}^{n+2}; \mathcal{L}(\mathbb{C}^{m(1+n)}))$$

is bounded and measurable. Second, A satisfies, for some $\kappa > 0$ and uniformly for all λ , the ellipticity condition

$$\operatorname{Re} \iint_{\mathbb{R}^{n+1}} (A(\lambda, x, t) f(x, t) \cdot \overline{f(x, t)}) dx dt \geq \kappa \sum_{i=0}^n \sum_{\alpha=1}^m \iint_{\mathbb{R}^{n+1}} |f_i^\alpha(x, t)|^2 dx dt,$$

for all $f = (f_j^\alpha)_{j=0, \dots, n}^{\alpha=1, \dots, m} \in L^2(\mathbb{R}^{n+1}; \mathbb{C}^{m(1+n)})$ with $\operatorname{curl}_x (f_j^\alpha)_{j=1, \dots, n} = 0$ for all α .

14.2. Adding lower order terms to the elliptic part. There is no real difficulty to adapt our techniques and proofs to the case of (strictly elliptic) equations with lower order terms:

$$\mathcal{L}u = \partial_t u - \operatorname{div}_X (A(X, t) \nabla_X u + bu) + c \nabla_X u + du = 0$$

where b, c are $n + 1$ vectors and d is a scalar, all coefficients are bounded and measurable, and we assume pointwise ellipticity (in the case of one equation). In this case the associated parabolic conormal differential becomes

$$D_A u(\lambda, x, t) = \begin{bmatrix} \partial_{\nu_A} u(\lambda, x, t) + b_\perp u(\lambda, x, t) \\ \nabla_x u(\lambda, x, t) \\ u(\lambda, x, t) \\ H_t D_t^{1/2} u(\lambda, x, t) \end{bmatrix}$$

and the corresponding parabolic Dirac operator is given by

$$P = \begin{bmatrix} 0 & \operatorname{div}_x & 1 & -D_t^{1/2} \\ -\nabla_x & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -H_t D_t^{1/2} & 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} B_{\perp\perp} & B_{\perp\parallel} & \tilde{b}_\perp & 0 \\ B_{\parallel\perp} & B_{\parallel\parallel} & \tilde{b}_\parallel & 0 \\ \tilde{c}_\perp & \tilde{c}_\parallel & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$\begin{bmatrix} B_{\perp\perp} & B_{\perp\parallel} & \tilde{b}_\perp \\ B_{\parallel\perp} & B_{\parallel\parallel} & \tilde{b}_\parallel \\ \tilde{c}_\perp & \tilde{c}_\parallel & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ A_{\perp\perp} & A_{\perp\parallel} & b_\perp \\ A_{\parallel\perp} & A_{\parallel\parallel} & b_\parallel \end{bmatrix} \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} & b_\perp \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}.$$

The upshot is the proof of the square function implied in Theorem 2.3. One can either adapt the proof given for the pure second order case or prove a lower order perturbation result. We do not get into details.

14.3. The setup of cylindrical domains. We may replace $(X, t) \in \mathbb{R}_+^{n+2}$ by $(X, t) \in B \times \mathbb{R}$, where B is the unit ball of \mathbb{R}^{n+1} . In that case, the normal independence of the coefficients means that they do not depend on the radial variable as in [46]. Adapting [20] to the parabolic setting, one can develop the first order approach. Again the proof of the square function in Theorem 2.3 is the main part and should be handled by mixing arguments from here with those of [20].

We remark that if t belongs to the unbounded line, then this makes it impossible to treat the Rellich estimates in the self-adjoint case with t -dependent coefficients via Fredholm methods even though the angular variable lives on the sphere. However, one should be able to treat boundary value problems with initial condition on bounded intervals $[0, T]$ imposing that the commutators between coefficients and half-order time derivatives are compact, uniformly in the angular variable. We leave this idea and further details to interested readers.

14.4. Degenerate parabolic equations and systems. In the setting of the parabolic upper half-space, it may also be interesting to consider the case of degenerate equations and systems. To expand a bit on this in the case of equations, one may consider equations

$$(14.1) \quad \lambda_1(X, t) \partial_t u - \operatorname{div}_X A(X, t) \nabla_X u = 0,$$

with

$$\kappa \lambda_2(X, t) |\xi|^2 \leq \operatorname{Re}(A(X, t) \xi \cdot \bar{\xi}), \quad |A(X, t) \xi \cdot \zeta| \leq C \lambda_2(X, t) |\xi| |\zeta|,$$

where the weights λ_1 and λ_2 are non-negative measurable functions subject to conditions.

In the case when $\lambda_1(X, t) = w(x) = \lambda_2(X, t)$ for some $w \in A_2(\mathbb{R}^n)$, one may try to develop a weighted version of our paper by following [21]. Again, the matter is to prove the weighted analogue to Theorem 2.3 for λ -independent coefficients. We claim that this is possible. It is an interesting open problem to construct test functions in the case when the weight also depends on the t variable and is, for instance, an A_2 weight for the parabolic doubling space \mathbb{R}^{n+1} .

Although not directly related, it is here worth noting that in the case of real parabolic equations, interior regularity of weak solutions to (14.1), when $\lambda_1(X, t) = w(X, t) = \lambda_2(X, t)$, or $\lambda_1(X, t) = 1$ and $\lambda_2(X, t) = w(X, t)$, was studied in [30, 31] under various integrability conditions on the weight. We remark that the conclusions there in terms of Muckenhoupt conditions are not exactly as for the elliptic case, showing that the A_2 condition is not the optimal one to obtain regularity.

14.5. Boundary value problems with other spaces of data. Another possible development is the formalisation of the first order setup to approach boundary value problems with data in other spaces than the L^2 based Sobolev spaces.

Starting with L^p spaces, it amounts to first study *a priori* estimates enjoyed by the Cauchy extension and next to identify trace spaces of solutions having some square function control and prove that their conormal differentials are given by a Cauchy extension.

As a consequence of the results in Section 2.9, in particular, Lemma 2.10 and Proposition 2.11, one can extend the bounded holomorphic functional calculus of PM to L^p for p near 2 and this implies (after some work) the alluded square function estimates. However, if one wants to develop results covering larger ranges of p , one should look at Hardy space theory and uniqueness results as studied in [18, 19] for the elliptic situation. This theory has been extended in [4] to cover fractional Besov-Hardy-Sobolev spaces. While the strategy is in place, the details could reveal more complicated again due to the presence of half-order derivatives in time.

14.6. Higher order problems and more. One can also study other types of problems either for the purpose of boundary value problems with λ independent coefficients or to prove the Kato square root estimate.

We have in mind parabolic systems with higher order elliptic part in divergence form. The first order method for the elliptic case was discussed in [9].

One could also think of replacing ∂_t by other operators. If we replace it by ∂_t^2 , we come back to an elliptic problem. But we could also think of ∂_t^3 or fractional derivatives. In principle, the algebraic formalism can be set but the conditions to proceed with the analysis are unclear.

Finally, one could try to treat elliptic parts of some fractional order but in this case the scary thing is the lack of spatial off-diagonal estimates.

REFERENCES

- [1] D.R. ADAMS and L.I. HEDBERG. *Function Spaces and Potential Theory*. Grundlehren der Mathematischen Wissenschaften, vol. 314, Springer, Berlin, 1996.
- [2] M.A. ALFONSECA, P. AUSCHER, A. AXELSSON, S. HOFMANN, and S. KIM. *Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients*. *Adv. Math.* **226** (2011), no. 5, 4533–4606.
- [3] A. AMENTA. *Tent spaces over metric measure spaces under doubling and related assumptions*. In *Operator theory in harmonic and non-commutative analysis*, *Oper. Theory Adv. Appl.*, vol. 240, Birkhäuser/Springer, Cham, 2014, 1–29.
- [4] A. AMENTA and P. AUSCHER. *Elliptic Boundary Value Problems with Fractional Regularity Data: The First Order Approach*. CRM Monographs Series, vol. 37, 2018.
- [5] P. AUSCHER and A. AXELSSON. *Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I*. *Invent. Math.* **184** (2011), no. 1, 47–115.
- [6] P. AUSCHER and A. AXELSSON. *Remarks on maximal regularity*. *Progr. Nonlinear Differential Equations Appl.*, vol. 80, Birkhäuser, Basel, 2011, 45–55.
- [7] P. AUSCHER, A. AXELSSON, and S. HOFMANN. *Functional calculus of Dirac operators and complex perturbations of Neumann and Dirichlet problems*. *J. Funct. Anal.* **255** (2008), no. 2, 374–448.
- [8] P. AUSCHER, A. AXELSSON, and A. MCINTOSH. *Solvability of elliptic systems with square integrable boundary data*. *Ark. Mat.* **48** (2010), no. 2, 253–287.

- [9] P. AUSCHER, A. AXELSSON, and A. MCINTOSH. *On a quadratic estimate related to the Kato conjecture and boundary value problems*. In Harmonic analysis and partial differential equations, Contemp. Math., vol. 505, Amer. Math. Soc., Providence RI, 2010, 105–129.
- [10] P. AUSCHER, S. BORTZ, M. EGERT and O. SAARI. *Non-local self-improving properties: A functional analytic approach*, Tunis. J. Math. **1** (2019), no. 2, 151–183.
- [11] P. AUSCHER and M. EGERT. *On non-autonomous maximal regularity for elliptic operators in divergence form*. Arch. Math. **107** (2016), no. 3, 271–284.
- [12] P. AUSCHER, M. EGERT, and K. NYSTRÖM. *The Dirichlet problem for second order parabolic operators in divergence form*. J. Éc. polytech. Math. **5** (2018), 407–441.
- [13] P. AUSCHER, S. HOFMANN, M. LACEY, A. MCINTOSH, and P. TCHAMITCHIAN. *The solution of the Kato square root problem for second order elliptic operators on \mathbb{R}^n* . Ann. of Math. (2) **156** (2002), no. 2, 633–654.
- [14] P. AUSCHER and J.M. MARTELL. *Weighted norm inequalities, off-diagonal estimates and elliptic operators. II. Off-diagonal estimates on spaces of homogeneous type*. J. Evol. Equ. **7** (2007), no. 2, 265–316.
- [15] P. AUSCHER, S. MONNIAUX, and P. PORTAL. *On existence and uniqueness for non-autonomous parabolic Cauchy problems with rough coefficients*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **19** (2019), no. 2, 387–471.
- [16] P. AUSCHER, A. MCINTOSH, and M. MOURGOLOU. *On L^2 solvability of BVPs for elliptic systems*. J. Fourier Anal. Appl. **19** (2013), no. 3, 478–494.
- [17] P. AUSCHER, A. MCINTOSH, and A. NAHMOD. *Holomorphic functional calculi of operators, quadratic estimates and interpolation*. Indiana Univ. Math. J. **46** (1997), no. 2, 375–403.
- [18] P. AUSCHER and M. MOURGOLOU. *Representation and uniqueness for boundary value elliptic problems via first order systems*. Rev. Mat. Iberoam. **35** (2019), no. 1, 241–315.
- [19] P. AUSCHER and S. STAHLHUT. *Functional Calculus for first order systems of Dirac type and boundary value problems*. Mém. Soc. Math. France (N.S.) **144** (2016).
- [20] P. AUSCHER and A. ROSÉN. *Weighted maximal regularity estimates and solvability of nonsmooth elliptic systems II*. Anal. PDE **5** (2012), no. 5, 983–1061.
- [21] P. AUSCHER, A. ROSÉN, and D. RULE. *Boundary value problems for degenerate elliptic equations and systems*. Ann. Sci. Éc. Norm. Supér (4) **48** (2015), no. 4, 951–1000.
- [22] A. AXELSSON, S. KEITH, and A. MCINTOSH. *Quadratic estimates and functional calculi of perturbed Dirac operators*. Invent. Math. **163** (2006), no. 3, 455–497.
- [23] J. BERGH and J. LÖFSTRÖM. *Interpolation Spaces. An Introduction*. Grundlehren der Mathematischen Wissenschaften, no. 223, Springer, Berlin, 1976.
- [24] G. BOURDAUD. *Localisation et multiplicateurs des espaces de Sobolev homogènes*. Manuscripta Math. **60** (1988), no. 1, 93–130.
- [25] R. BROWN. *The method of layer potentials for the heat equation in Lipschitz cylinders*. Amer. J. Math. **111** (1989), no. 2, 339–379.
- [26] R. BROWN. *The initial-Neumann problem for the heat equation in Lipschitz cylinders*. Trans. Amer. Math. Soc. **320** (1990) (1), 1–52.
- [27] A. CASTRO, K. NYSTRÖM, and O. SANDE. *Boundedness of single layer potentials associated to divergence form parabolic equations with complex coefficients*. Calc. Var. Partial Differ. Equ. **55** (2016), no. 5, Article:124.
- [28] P. CEREJEIRAS, U. KÄHLER, and F. SOMMEN. *Parabolic Dirac operators and the Navier-Stokes equations over time-varying domains*. Math. Methods Appl. Sci. **28** (2005), no. 14, 1715–1724.
- [29] P. CEREJEIRAS, F. SOMMEN, and N. VIEIRA. *Fischer decomposition and special solutions for the parabolic Dirac operator*. Math. Methods Appl. Sci. **30** (2007), no. 9, 1057–1069.
- [30] F. CHIARENZA and R. SERAPIONI. *Degenerate parabolic equations and Harnack inequality*. Ann. Mat. Pura Appl. (4) **137** (1984) 149–162.
- [31] F. CHIARENZA and R. SERAPIONI. *Pointwise estimates for degenerate parabolic equations*. Appl. Anal. **23** (1987) 287–299.
- [32] M. COSTABEL. *Boundary integral operators for the heat equation*. Integral equations and Operator Theory **13** (1990), no. 4, 498–552.
- [33] M. DINDOS and S. HWANG. *The Dirichlet problem for second order parabolic operators satisfying Carleson condition*. Rev. Mat. Iberoam. **34** (2018), no. 2, 767–810.
- [34] M. DINDOS, S. PETERMICHL and J. PIPHER. *BMO solvability and the A_∞ condition for second order parabolic operators*. Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), no. 5, 1155–1180.
- [35] M. EGERT. *On Kato’s conjecture and mixed boundary conditions*. PhD Thesis, Sierke Verlag, Göttingen, 2015.
- [36] D. GILBARG and N.S. TRUDINGER. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics, Springer, Berlin, 2001.
- [37] M. HAASE. *The Functional Calculus for Sectorial Operators*. Operator Theory: Advances and Applications, vol. 169, Birkhäuser, Basel, 2006.
- [38] S. HOFMANN. *Parabolic singular integrals of Calderon-type, rough operators and caloric layer potentials*. Duke Math. J. **90** (1997), no. 2, 209–259.
- [39] S. HOFMANN and J.L. LEWIS. *L^2 solvability and representation by caloric layer potentials in time-varying domains*. Ann. of Math. (2) **144** (1996), no. 2, 349–420.

- [40] S. HOFMANN and J.L. LEWIS. *The Dirichlet problem for parabolic operators with singular drift terms*. Mem. Amer. Math. Soc. **151** (2001) no. 719.
- [41] Y. HUANG. *Weighted tent spaces with Whitney averages: factorization, interpolation and duality*. Math. Z. **282** (2016), no. 2, 913–933.
- [42] T. HYTÖNEN and A. ROSÉN. *On the Carleson duality*. Ark. Mat. **51** (2013), no. 2, 293–313.
- [43] N. KALTON, S. MAYBORODA, and M. MITREA. *Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin spaces and applications to problems in partial differential equations*. In Interpolation theory and applications, Contemp. Math., vol. 455, Amer. Math. Soc., Providence, RI, 2007, 121–177.
- [44] S. KAPLAN. *Abstract boundary value problems for linear parabolic equations*. Ann. Scuola Norm. Sup. Pisa (3) **20** (1966), 395–419.
- [45] T. KATO. Perturbation Theory for Linear Operators. Classics in Mathematics, Springer, Berlin, 1995.
- [46] C.E. KENIG and J. PIPHER. *The Neumann problem for elliptic equations with nonsmooth coefficients*. Invent. Math. **113** (1993), no. 3, 447–509.
- [47] A.A. KILBAS, O.I. MARICHEV, and S.G. SAMKO. Fractional Integrals and Derivatives. Theory and applications. Gordon and Breach Science Publishers, Yverdon, 1993.
- [48] T. KUROKAWA. *Riesz potentials, higher Riesz transforms and Beppo Levi spaces*. Hiroshima Math. J. **18** (1988), no. 3, 541–597.
- [49] J.L. LEWIS and M.A.M. MURRAY. *The method of layer potentials for the heat equation in time-varying domains*. Mem. Amer. Math. Soc. **114** (1995), no. 545.
- [50] P.I. LIZORKIN. *Multipliers of Fourier integrals and estimates of convolutions in spaces with mixed norm. Applications*. Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 218–247.
- [51] A. MCINTOSH. *Operators which have an H^∞ functional calculus*. In Miniconference on operator theory and partial differential equations, Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 14, Austral. Nat. Univ., Canberra, 1986, 210–231.
- [52] M.A.M. MURRAY. *Commutators with fractional differentiation and BMO Sobolev spaces*. Indiana Univ. Math. J. **34** (1985), no. 1, 205–215.
- [53] K. NYSTRÖM. *Square functions estimates and the Kato problem for second order parabolic operators in \mathbb{R}^{n+1}* . Adv. Math. **293** (2016), 1–36.
- [54] K. NYSTRÖM. *L^2 Solvability of boundary value problems for divergence form parabolic equations with complex coefficients*. J. Differential Equations **262** (2016), no. 3, 2808–2939.
- [55] A. ROSÉN. *Layer potentials beyond singular integral operators*. Publ. Mat. **57** (2013), no. 2, 429–454.
- [56] I. ŠNEĬBERG. *Spectral properties of linear operators in interpolation families of Banach spaces*. Mat. Issled. **9** (1974), no. 2, 214–229, 254–255.
- [57] E.M. STEIN. Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series, no. 30, Princeton University Press, Princeton NJ, 1970.
- [58] G. VERCHOTA. *Layer potentials and regularity for the Dirichlet problem for Laplace’s equation in Lipschitz domains*. J. Funct. Anal. **59** (1984), no. 3, 572 – 611.

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