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# Fujita Kato solution for compressible Navier-Stokes equation with axisymmetric initial data and zero Mach number limit

Boris Haspot \*

## Abstract

In this paper we investigate the question of the existence of global strong solution for the compressible Navier Stokes equations for small initial data such that the rotational part of the velocity  $\mathbb{P}u_0$  belongs to  $\dot{H}^{\frac{N}{2}-1}$ . We show then an equivalence of the so called Fujita Kato theorem to the case of the compressible Navier-Stokes equation when we consider axisymmetric initial data in dimension  $N = 2, 3$ . The main difficulty is related to the fact that in this case the velocity is not Lipschitz, in consequence we have to study carefully the coupling between the rotational and irrotational part of the velocity. In a second part, following the arguments developed in [13] we address the question of convergence to the incompressible model (for *ill-prepared* initial data) when the Mach number goes to zero.

## 1 Introduction

The motion of a slightly compressible barotropic fluid is described by the following system:

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mu \Delta u^\varepsilon - (\lambda + \mu) \nabla \operatorname{div} u^\varepsilon + \frac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \\ (\rho^\varepsilon, u^\varepsilon) - t = 0 = (\rho_0^\varepsilon, u_0^\varepsilon) \end{cases} \quad (1.1)$$

Here  $\rho^\varepsilon = \rho^\varepsilon(t, x) \in \mathbb{R}$  and  $u^\varepsilon = u^\varepsilon(t, x) \in \mathbb{R}^N$  stand for the dimensionless density and velocity field, and the pressure  $P$  is a suitably smooth function of  $\rho^\varepsilon$ . Unless otherwise specified, it will always be assumed that  $x$  belongs to the whole space  $\mathbb{R}^N$  ( $N \geq 2$ ). We denote by  $\lambda$  and  $\mu$  the two Lamé coefficients of the fluid, which are constant and satisfy  $\mu > 0$  and  $\nu = \lambda + 2\mu > 0$ . Such a condition ensures ellipticity for the operator  $\mu \Delta + (\lambda + \mu) \nabla \operatorname{div}$  and is satisfied in the physical cases (where  $\lambda + \frac{2\mu}{N} \geq 0$ ). The initial conditions  $(\rho_0^\varepsilon, u_0^\varepsilon)$  are given.

The parameter  $\varepsilon$  corresponds to the so called Mach number which is given by  $\varepsilon = LT^{-1}\xi^{-1}$  where  $L$  and  $T$  are the typical values of the length and time (before rescaling) and  $\xi$  stands for the sound speed. We refer to [20, 25] or in the introduction of [30] for more explanations on the derivation of the above model.

In this paper our purpose is double, indeed in a first time we are interested in proving the global existence of strong solution for small initial data for suitable initial data such that

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the rotational part of the velocity verifies the same assumption as the so called Fujita Kato theorem and in fact the more general theorem due to Cannone-Meyer-Planchon (see [4]). In a second time we will study the limit of such solution when the Mach number  $\varepsilon$  goes to zero, and we will recover as a limit a solution of the incompressible Navier Stokes equations.

When we study equations derived from the fluids mechanics, a first interesting question is to find the largest functional space  $X$  in local or global well-posedness may be proved and after to compare this space  $X$  with the energy space. It is then natural to understand in which sense a space is *critical* for the existence of strong solution. That it is why it is important to study the scaling of the equations and the notion of self similar solution. In view of the celebrated works by H. Fujita and T. Kato for the incompressible model (*NSI*) (see [18]), one can guess that the critical space for the velocity is  $\dot{H}^{\frac{1}{2}}$ . By critical, we mean that we can solve the system (*NSI*) in functional spaces with norm invariant by the changes of scales which leave (*NSI*) invariant. Indeed we can check that for a solution  $v$  we can associate via the following transformation:

$$(v_0(x), f(t, x)) \rightarrow (lv_0(lx), l^3 f(l^2 t, lx)), \quad v(t, x) \rightarrow lv(l^2 t, lx)$$

a family of solution to (*NSI*) and that the norm in  $\dot{H}^{\frac{N}{2}-1}$  is invariant by the transformation  $v_0(x) \rightarrow lv_0(lx)$ . In the case of barotropic fluids, we can observe that the transformations:

$$\begin{aligned} (\rho_0(x), u_0(x), f(t, x)) &\rightarrow (\rho_0(lx), lu_0(lx), l^3 f(l^2 t, lx)), \\ (\rho(t, x), u(t, x)) &\rightarrow (\rho(l^2 t, lx), lu(l^2 t, lx)). \end{aligned} \tag{1.2}$$

have that property, provided that the pressure term has been changed accordingly. Roughly speaking we expect that such spaces are optimal in term of regularity on the initial data in order to prove the well-posedness of the system (1.1).

It seems then natural to consider initial data  $(b_0, u_0)$  in  $\dot{H}^{\frac{N}{2}} \times (\dot{H}^{\frac{N}{2}-1})^N$ . However  $\dot{H}^{\frac{N}{2}}$  is not a subalgebra of  $L^\infty$ , in particular it does not allow to control the nonlinear term involving the density as the pressure since we can not use classical composition theorem. Furthermore if we wish to propagate any regularity on the density along the time, we have to observe that the density is governed by a transport equation. It requires then that the velocity should be Lipschitz if we want to apply classical theorem on the transport equation. These two restrictions led to consider slightly smaller space, the homogeneous Besov space  $B_{2,1}^{\frac{N}{2}} \times (B_{2,1}^{\frac{N}{2}-1})^N$  (see the definition in section 2) which is also critical according (1.2). Indeed we can verify that the only critical Besov space  $B_{2,r}^{\frac{N}{2}}$  which is embedded in  $L^\infty$  is  $B_{2,1}^{\frac{N}{2}}$ .

Danchin in [8] shows for the first time a result of existence of global strong solution with small initial data in critical space for the scaling of the system. More precisely the initial data are chosen as follows  $(q_0, u_0) \in (B_{2,1}^{\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}-1}) \times B_{2,1}^{\frac{N}{2}-1}$ . The main difficulty consists in obtaining suitable estimates on the linearized system with convection terms. The crucial point is the proof of damping effect on the density which enables to control the pressure term. This last result has been generalized to the case of Besov space constructed on  $L^p$  space by Charve and Danchin in [5], Danchin in [12], Chen et al in [7] and

the author in [21] by using two different methods. We refer also to [24] for the existence of global strong solution for the shallow water system with a smallness condition which is supercritical. In [6] we weaken the previous results in the case of the shallow water system by assuming that the initial velocity verifies only:

$$\mathbb{P}u_0 \in B_{2,2}^{\frac{N}{2}-1}, \mathbb{Q}u_0 \in B_{2,1}^{\frac{N}{2}-1} \text{ and } u_0 \in B_{\infty,1}^{-1}.$$

Here we can observe that the choice of  $u_0$  in  $B_{\infty,1}^{-1}$  is absolutely crucial since it allows to control the Lipschitz norm of the velocity  $u$ . This last result has been extended by Danchin and He in [13] to the case of the system (1.1) with constant viscosity coefficients and with the rotational part of the velocity  $\mathbb{P}u_0$  which belongs only to  $B_{2,r}^{\frac{N}{2}-1}$  with  $r$  suitably chosen and such that  $r > 2$ .

We can mention that all these previous results require a Lipschitz control on the velocity and in particular that  $u_0$  belongs at least to  $B_{\infty,1}^{-1}$ . We would like to avoid this restriction and to consider only the so called Fujita Kato initial data which corresponds to  $\mathbb{P}u_0 \in \dot{H}^{\frac{N}{2}-1}$ . To do this, we are going to consider initial data with geometrical assumption in order to weaken the coupling in the mass equation between the density and the rotational part of the velocity  $\mathbb{P}u$ . More precisely we are going to consider axisymmetric initial data in dimension  $N = 2, 3$ . Let us give now more details on the form of the solution that we are going to consider. If we set  $\rho^\epsilon = 1 + \varepsilon b^\epsilon$ , we are to study:

$$\begin{cases} \partial_t b^\epsilon + \frac{\operatorname{div} u^\epsilon}{\varepsilon} = -\operatorname{div}(b^\epsilon u^\epsilon), \\ \partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon - \frac{\mu \Delta u^\epsilon + (\lambda + \mu) \nabla \operatorname{div} u^\epsilon}{1 + \varepsilon b^\epsilon} + \frac{P'(1 + \varepsilon b^\epsilon)}{1 + \varepsilon b^\epsilon} \frac{\nabla b^\epsilon}{\varepsilon} = 0, \\ (b^\epsilon, u^\epsilon)_{t=0} = (b_0^\epsilon, u_0^\epsilon). \end{cases} \quad (1.3)$$

In dimension  $N = 2$  we are interested in looking at solutions of the form:

$$\rho^\epsilon(t, x) = \rho^\epsilon(t, |x|), \quad u^\epsilon(t, x) = \frac{x}{|x|} u_1^\epsilon(t, |x|) + \frac{x^\perp}{|x|} u_2^\epsilon(t, |x|) = u^{1,\varepsilon}(t, x) + u^{2,\varepsilon}(t, x). \quad (1.4)$$

We observe that the system (1.3) can be rewritten as follows:

$$\begin{cases} \partial_t b^\epsilon + \frac{\operatorname{div} u^{1,\varepsilon}}{\varepsilon} = -\operatorname{div}(b^\epsilon u^{1,\varepsilon}), \\ \partial_t u^{1,\varepsilon} + u^{1,\varepsilon} \cdot \nabla u^{1,\varepsilon} - \frac{\mu \Delta u^{1,\varepsilon} + (\lambda + \mu) \nabla \operatorname{div} u^{1,\varepsilon}}{1 + \varepsilon b^\epsilon} + \frac{P'(1 + \varepsilon b^\epsilon)}{1 + \varepsilon b^\epsilon} \frac{\nabla b^\epsilon}{\varepsilon} = u^{2,\varepsilon} \cdot \nabla u^{2,\varepsilon}, \\ \partial_t u^{2,\varepsilon} + u^{1,\varepsilon} \cdot \nabla u^{2,\varepsilon} + u^{2,\varepsilon} \cdot \nabla u^{1,\varepsilon} - \frac{\mu \Delta u^{2,\varepsilon}}{1 + \varepsilon b^\epsilon} = 0 \\ (b^\epsilon, u^\epsilon)_{t=0} = (b_0^\epsilon, u_0^\epsilon). \end{cases} \quad (1.5)$$

We will also consider the case of axisymmetric solution when  $N = 3$  which corresponds to rewrite  $(\rho, u)$  as follows:

$$u^\epsilon(t, x) = u^{r,\varepsilon}(t, r, z)e_r + u^{\theta,\varepsilon}(t, r, z)e_\theta + u^{z,\varepsilon}(t, r, z)e_z = u_{r,\varepsilon} + u_{\theta,\varepsilon} + u_{z,\varepsilon} \text{ and } \rho(t, x) = \rho^\epsilon(t, r, z), \quad (1.6)$$

with:

$$r = |x'|, \quad x' = (x_1, x_2, 0), \quad e_r = \frac{x'}{|x'|}; \quad e_\theta = \frac{x'^\perp}{|x'|} \quad \text{and} \quad e_z = (0, 0, 1).$$

As previously the system (1.3) can be rewritten as follows:

$$\left\{ \begin{array}{l} \partial_t b^\varepsilon + \frac{\operatorname{div}(u_{r,\varepsilon} + u_{z,\varepsilon})}{\varepsilon} = -\operatorname{div}(b^\varepsilon(u_{r,\varepsilon} + u_{z,\varepsilon})) \\ \partial_t u_{r,\varepsilon} + (u_{r,\varepsilon} + u_{z,\varepsilon}) \cdot \nabla u_{r,\varepsilon} - \frac{\mu \Delta u_{r,\varepsilon} + (\lambda + \mu)[\partial_r \operatorname{div} u_{r,\varepsilon} + \partial_r \operatorname{div} u_{z,\varepsilon}] e_r}{1 + \varepsilon b^\varepsilon} \\ \quad + \frac{P'(1 + \varepsilon b^\varepsilon)}{1 + \varepsilon b^\varepsilon} \frac{\partial_r b^\varepsilon e_r}{\varepsilon} + u_{\theta,\varepsilon} \cdot \nabla u_{\theta,\varepsilon} = 0 \\ \partial_t u_{\theta,\varepsilon} + (u_{r,\varepsilon} + u_{z,\varepsilon}) \cdot \nabla u_{\theta,\varepsilon} - \frac{\mu \Delta u_{\theta,\varepsilon}}{1 + \varepsilon b^\varepsilon} + u_{\theta,\varepsilon} \cdot \nabla u_{r,\varepsilon} = 0, \\ \partial_t u_{z,\varepsilon} + (u_{r,\varepsilon} + u_{z,\varepsilon}) \cdot \nabla u_{z,\varepsilon} - \frac{\mu \Delta u_{z,\varepsilon} + (\lambda + \mu)[\partial_z \operatorname{div} u_{r,\varepsilon} + \partial_z \operatorname{div} u_{z,\varepsilon}] e_z}{1 + \varepsilon b^\varepsilon} \\ \quad + \frac{P'(1 + \varepsilon b^\varepsilon)}{1 + \varepsilon b^\varepsilon} \frac{\partial_z b^\varepsilon e_r}{\varepsilon} = 0. \end{array} \right. \quad (1.7)$$

Setting  $u_{k,\varepsilon} = u_{r,\varepsilon} + u_{z,\varepsilon}$  we can rewrite the previous system as follows:

$$\left\{ \begin{array}{l} \partial_t b^\varepsilon + \frac{\operatorname{div} u_{k,\varepsilon}}{\varepsilon} = -\operatorname{div}(b^\varepsilon u_{k,\varepsilon}) \\ \partial_t u_{k,\varepsilon} + u_{k,\varepsilon} \cdot \nabla u_{k,\varepsilon} - \frac{\mu \Delta u_{k,\varepsilon} + (\lambda + \mu) \nabla \operatorname{div} u_{k,\varepsilon}}{1 + \varepsilon b^\varepsilon} \\ \quad + \frac{P'(1 + \varepsilon b^\varepsilon)}{1 + \varepsilon b^\varepsilon} \frac{\partial_r b^\varepsilon e_r}{\varepsilon} + u_{\theta,\varepsilon} \cdot \nabla u_{\theta,\varepsilon} = 0. \\ \partial_t u_{\theta,\varepsilon} + u_{k,\varepsilon} \cdot \nabla u_{\theta,\varepsilon} - \frac{\mu \Delta u_{\theta,\varepsilon}}{1 + \varepsilon b^\varepsilon} + u_{\theta,\varepsilon} \cdot \nabla u_{r,\varepsilon} = 0. \end{array} \right. \quad (1.8)$$

We can observe that in the system (1.3) and (1.7) the coupling between the density  $b^\varepsilon$  and the velocity  $u^\varepsilon$  is weakened since only the irrotational part of the velocity governs the behavior of the density in the mass equation (except in the 3D case where  $\mathbb{P}u_{z,\varepsilon}$  plays also a role). This remark will be crucial in order to choose  $u_0^2$  or  $u_{\theta,0}$  in  $\dot{H}^{\frac{N}{2}-1}$ .

It will be however mandatory to choose  $u_0^1$ ,  $u_{r,0}$  and  $u_{z,0}$  in  $B_{2,1}^{\frac{N}{2}-1}$  in order to ensure a Lipschitz control on the irrotational part which allows us to estimate the density via the mass equation. Another difficulty will consist in dealing with the coupling between the rotational and the irrotational part of the velocity which appears via the terms  $u^{2,\varepsilon} \cdot \nabla u^{2,\varepsilon}$  and  $u_{\theta,\varepsilon} \cdot \nabla u_{\theta,\varepsilon}$ . We will show some regularizing effects induced on these bilinear convection terms which enables us to propagate the regularity  $B_{2,1}^{\frac{N}{2}-1}$  on the irrotational part of the velocity.

Let us focus now on the zero Mach number limit when  $\varepsilon$  goes to 0. A large amount of literature has been devoted to this main feature. Roughly speaking, two different heuristics have been developed. The case of *well-prepared* data which corresponds to the assumption that  $\rho_0^\varepsilon = 1 + O(\varepsilon^2)$  and  $\operatorname{div} u_0 = O(\varepsilon)$  has been investigated in [27, 28] and [25].

In this paper, we will consider the case of *ill-prepared* data, where it is only assumed that  $\rho_0^\varepsilon = 1 + \varepsilon b_0^\varepsilon$  with  $(b_0^\varepsilon, u_0^\varepsilon, f^\varepsilon)$  uniformly bounded (in a convenient functional space),  $\mathbb{P}u_0^\varepsilon$  tending to some  $v_0$  and  $\mathbb{P}f^\varepsilon$  tending to some  $g$  when  $\varepsilon$  goes to 0. One expects that the limit  $v$  of  $u^\varepsilon$  solves the so called incompressible Navier-Stokes equations:

$$(NSI) \quad \begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi = 0, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0. \end{cases}$$

Let us recall now some results of global strong solution for the system (NSI) which are due to Fujita and Kato in [18] and Cannone, Meyer and Planchon in [4] (see also [13]).

**Theorem 1.1** *Let  $N = 2, 3$  and  $v_0 \in B_{p,r}^{\frac{N}{p}-1}$  with  $\operatorname{div} v_0 = 0$ ,  $p < +\infty$  and  $r \in [1, +\infty]$ . There exists  $\varepsilon_0$  such that if:*

$$\|v_0\|_{B_{p,r}^{\frac{N}{p}-1}} \leq \varepsilon_0 \mu,$$

then (NSI) has a unique global solution  $u$  in the space:

$$\tilde{L}^\infty(\mathbb{R}^+, B_{p,r}^{\frac{N}{p}-1}) \cap \tilde{L}^1(\mathbb{R}^+, B_{p,r}^{\frac{N}{p}+1}).$$

Furthermore we have:

$$\|v\|_{\tilde{L}^\infty(\mathbb{R}^+, B_{p,r}^{\frac{N}{p}-1})} + \mu \|v\|_{\tilde{L}^1(\mathbb{R}^+, B_{p,r}^{\frac{N}{p}+1})} \leq M \|v_0\|_{B_{p,r}^{\frac{N}{p}-1}},$$

for  $M$  a constant depending only on  $N$  and  $p$ .

We can observe that under the condition (1.4) in dimension  $N = 2$  and (1.6) in dimension  $N = 3$  the previous system can be respectively rewritten as follows:

$$\begin{cases} \partial_t u^2 - \mu \Delta u^2 = 0 \\ u^2 \cdot \nabla u^2 = -\nabla \Pi \\ \operatorname{div} u^2 = 0, \end{cases} \quad (1.9)$$

and:

$$\begin{cases} \partial_t u_r + (u_r + u_z) \cdot \nabla u_r - \mu \Delta u_r + u_\theta \cdot \nabla u_\theta + \partial_r \Pi e_r = 0 \\ \partial_t u_\theta + (u_r + u_z) \cdot \nabla u_\theta - \mu \Delta u_\theta + u_\theta \cdot \nabla u_r = 0, \\ \partial_t u_z + (u_r + u_z) \cdot \nabla u_z - \mu \Delta u_z + \partial_z \Pi e_z = 0, \\ \operatorname{div}(u_r + u_z) = 0. \end{cases} \quad (1.10)$$

The main difficulty in order to justify rigorously the convergence is that one has to face the propagation of acoustic waves with the speed  $\varepsilon^{-1}$ , a phenomenon which does not occur in the case of “well-prepared” data.

Nevertheless, several remarkable results have been obtained recently. First of all, for initial data with minimal regularity assumptions, P-L Lions stated in [31] the existence of global law considered is of type  $P(\rho) = a\rho^\gamma$  with certain restrictions on  $\gamma$  depending on the space dimension  $N$ . Since then, convergence results to the incompressible model have

been proved by B. Desjardins, E. Grenier, P.-L. Lions and N. Masmoudi. The case of periodic boundary conditions has been investigated in [32], the case of bounded domains with Dirichlet conditions in [16] and the case of the whole space in [15]. Some *local* weak convergence results are also available in a more general context (see [33]). Roughly, the main difference between the whole space and the periodic case is that in the former case one can utilize the dispersion of sound waves to get strong convergence results whereas in the latter case, the sound waves will oscillate forever, leading only to weak convergence. In the framework of strong periodic solutions, several recent works have to be mentioned. For smooth initial data and no external force, it has been stated in [20] that slightly compressible two-dimensional periodic incompressible Navier-Stokes equations and is unlikely to extend in higher dimension.

More recently Danchin was the first to justify rigorously the zero Mach limit in the framework of critical Besov space for the scaling of the equations (see [9, 10]). More precisely he deals with initial data such that  $(b_0^\varepsilon, u_0^\varepsilon)$  is uniformly bounded in  $\widetilde{B}_{2,1,\varepsilon,\infty}^{\frac{N}{2}-1, \frac{N}{2}} \times B^{\frac{N}{2}-1}$  (see the definition 2.4 for the hybrid Besov spaces). Danchin and He (see [13]) have recently generalized this result to the case of Besov spaces constructed on  $L^p$  Lebesgue spaces with a Lipschitz control on the velocity. To finish we would like to mention a very interesting and original result due to Danchin and Mucha [14] who propose a new approach to pass from the compressible Navier-Stokes equation to the incompressible Navier-Stokes equation by considering the limit of the solution when  $\lambda$  the second coefficient of viscosity goes to  $+\infty$ .

In this paper we would like to extend the previous result [13] for initial data of the type Cannone-Meyer-Planchon see [4] for axisymmetric initial data. From now for the sake of simplicity, we shall also assume that the data  $(b_0^\varepsilon, u_0^\varepsilon)$  do not depend on  $\varepsilon$  and will be merely denoted by  $(b_0, u_0, f)$  (therefore  $v_0 = \mathbb{P}u_0$ ). Our first main results concern the existence of global strong solution for small initial data when the Mach number is fixed  $\varepsilon = 1$ . We refer to the section 2 for the definition of the hybrid Besov space.

**Theorem 1.2** *Let  $\varepsilon = 1$ . Let  $N = 2$ . Let  $P$  be a suitably smooth function of the density. Assume that  $\rho_0(x) = \rho_0^1(|x|)$  and  $u_0(x) = \frac{x}{|x|}(u_1)_0(|x|) + (u_2)_0(|x|)$ . Let  $b_0 \in \widetilde{B}_{2,p,1}^{0, \frac{2}{p}}$ ,  $u_0^2 \in B_{p,r}^{\frac{2}{p}-1}$  and  $u_0^1 \in \widetilde{B}_{2,p,1}^{0, \frac{2}{p}-1}$  with  $2 \leq p < 4$  with  $1 \leq r \leq \frac{p}{p-2}$  then there exists  $\varepsilon_0$  such that if:*

$$\|b_0\|_{\widetilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}} + \|u_0^1\|_{\widetilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}} + \|u_0^2\|_{\widetilde{B}_{2,p,2}^{\frac{N}{2}-1, \frac{N}{p}-1}} \leq \varepsilon_0 \quad (1.11)$$

there exists a global solution  $(b, u^1, u^2)$  of the system (1.5) with:

$$\begin{aligned} b &\in \widetilde{L}^\infty(\widetilde{B}_{2,p,1}^{0, \frac{2}{p}}) \cap \widetilde{L}^1(\widetilde{B}_{2,p,1}^{2, \frac{2}{p}}), \\ u^1 &\in \widetilde{L}^\infty(\widetilde{B}_{2,p,1}^{0, \frac{2}{p}-1}) \cap \widetilde{L}^1(\widetilde{B}_{2,p,1}^{2, \frac{2}{p}+1}), \\ u^2 &\in \widetilde{L}^\infty(B_{p,2}^{\frac{2}{p}-1}) \cap \widetilde{L}^1(B_{p,2}^{\frac{2}{p}+1}). \end{aligned} \quad (1.12)$$

If  $p = 2$  the solution is unique.

**Remark 1** *To the best of our knowledge, it is the first result of global strong solution for the compressible Navier-Stokes equation with a velocity  $u$  which is not necessary Lipschitz.*

In addition it allows to choose initial data only a the level of the energy concerning the rotational part.

Let us mention that in [23] we prove the existence of global strong solution with large initial data for the system (1.1) in the case of viscosity coefficients of the type shallow water ( $\mu(\rho) = \mu\rho$  and  $\lambda(\rho) = 0$ ).

**Theorem 1.3** *Let  $\varepsilon = 1$  and  $N = 3$ . Let  $P$  be a suitably smooth function of the density. Assume that  $\rho_0(x) = \rho_0^1(|x|)$  and  $u_0 = (u_r)_0 + (u_\theta)_0 + (u_z)_0$ . Let  $b_0 \in \widetilde{B}_{2,p,1}^{-\frac{1}{2}, \frac{3}{p}}$ ,  $\mathbb{Q}(u_r)_0, \mathbb{Q}(u_z)_0 \in \widetilde{B}_{2,p,1}^{\frac{3}{2}, \frac{3}{p}-1}$ ,  $\mathbb{P}(u_r)_0, \mathbb{P}(u_z)_0 \in B_{p,r}^{\frac{3}{p}-1}$ ,  $u_k \in B_{\infty,1}^{-1}$  and  $(u_\theta)_0 \in B_{p,2}^{\frac{3}{p}-1}$  with  $2 \leq p \leq 4$  and  $1 \leq r \leq \frac{p}{p-2}$  then there exists  $\varepsilon_0$  such that if:*

$$\|b_0\|_{\widetilde{B}_{2,p,1}^{-\frac{1}{2}, \frac{3}{p}}} + \|(\mathbb{Q}(u_r)_0, \mathbb{Q}(u_z)_0)\|_{\widetilde{B}_{2,p,1}^{-\frac{1}{2}, \frac{3}{p}-1}} + \|(\mathbb{P}(u_r)_0, \mathbb{P}(u_z)_0)\|_{B_{p,r}^{\frac{3}{p}-1}} + \|(u_\theta)_0\|_{B_{p,2}^{\frac{3}{p}-1}} + \|u_k\|_{B_{\infty,1}^{-1}} \leq \varepsilon_0 \quad (1.13)$$

there exists a global solution  $(b, u_r, u_\theta, u_z)$  of the system (1.7) with:

$$\begin{aligned} b &\in \widetilde{L}^\infty(\widetilde{B}_{2,p,1}^{-\frac{1}{2}, \frac{3}{p}}) \cap \widetilde{L}^1(\widetilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}}), \\ u_\theta &\in \widetilde{L}^\infty(B_{p,2}^{\frac{3}{p}-1}) \cap \widetilde{L}^1(B_{p,2}^{\frac{3}{p}+1}), \\ \mathbb{Q}u_r, \mathbb{Q}u_z &\in \widetilde{L}^\infty(\widetilde{B}_{2,p,1}^{-\frac{1}{2}, \frac{3}{p}-1}) \cap \widetilde{L}^1(\widetilde{B}_{2,p,1}^{\frac{5}{2}, \frac{3}{p}+1}) \\ \mathbb{P}u_r, \mathbb{P}u_z &\in \widetilde{L}^\infty(\widetilde{B}_{2,p,r}^{-\frac{1}{2}, \frac{3}{p}-1}) \cap \widetilde{L}^1(\widetilde{B}_{2,p,r}^{\frac{5}{2}, \frac{3}{p}+1}) \\ u_k &\in \widetilde{L}^\infty(B_{\infty,1}^{-1}) \cap \widetilde{L}^\infty(B_{\infty,1}^1). \end{aligned} \quad (1.14)$$

If  $p \leq 3$  the solution is unique.

**Remark 2** *This result is a equivalent to the so-called Fujita-Kato theorem and Cannone-Meyer-Planchon theorem (see [18, 4]) for the compressible Navier-Stokes system in the case of axisymmetric initial data.*

**Definition 1.1** *We refer to (2.21) for the definition of low and high frequencies for Besov space. We denote by  $E_{\varepsilon,\nu}^p$  the space of functions  $(b, u)$  such that :*

- $(b^{BF,\varepsilon\nu}, \mathbb{Q}u^{BF,\varepsilon\nu}) \in (\widetilde{C}(B_{2,1}^{\frac{N}{2}-1}) \cap \widetilde{L}^1(B_{2,1}^{\frac{N}{2}+1})) \times (\widetilde{C}(B_{2,1}^{\frac{N}{2}-1}) \cap \widetilde{L}^1(B_{2,1}^{\frac{N}{2}+1})),$
- $(b^{HF,\varepsilon\nu}, \mathbb{Q}u^{HF,\varepsilon\nu}) \in (\widetilde{C}(B_{p,1}^{\frac{N}{p}}) \cap \widetilde{L}^1(B_{p,1}^{\frac{N}{p}})) \times (\widetilde{C}(B_{p,1}^{\frac{N}{p}-1}) \cap \widetilde{L}^1(B_{p,1}^{\frac{N}{p}+1})),$
- $\mathbb{P}u \in \widetilde{C}(B_{p,2}^{\frac{N}{p}-1}) \cap \widetilde{L}^1(B_{p,2}^{\frac{N}{p}+1}).$
- $u_k \in \widetilde{C}(B_{\infty,1}^{-1}) \cap \widetilde{L}^1(B_{\infty,1}^1)$  (only if  $N = 3$ ).

We endow that space with the following norm:

$$\begin{aligned} \|(b, u)\|_{E_{\varepsilon,\nu}^p} &= \|b^{BF,\varepsilon\nu}\|_{\widetilde{C}(B_{2,1}^{\frac{N}{2}-1}) \cap \widetilde{L}^1(B_{2,1}^{\frac{N}{2}+1})} + \|\mathbb{Q}u^{BF,\varepsilon\nu}\|_{\widetilde{C}(B_{2,1}^{\frac{N}{2}-1}) \cap \widetilde{L}^1(B_{2,1}^{\frac{N}{2}+1})} \\ &\varepsilon \|b^{HF,\varepsilon\nu}\|_{\widetilde{C}(B_{p,1}^{\frac{N}{p}-1})} + \varepsilon^{-1} \|b^{HF,\varepsilon\nu}\|_{\widetilde{L}^1(B_{p,1}^{\frac{N}{p}+1})} + \|\mathbb{Q}u^{HF,\varepsilon\nu}\|_{\widetilde{C}(B_{p,1}^{\frac{N}{p}-1}) \cap \widetilde{L}^1(B_{p,1}^{\frac{N}{p}+1})} \\ &+ \|\mathbb{P}u\|_{\widetilde{C}(B_{p,2}^{\frac{N}{p}-1}) \cap \widetilde{L}^1(B_{p,2}^{\frac{N}{p}+1})} + \|u_k\|_{\widetilde{C}(B_{\infty,1}^{-1}) \cap \widetilde{L}^1(B_{\infty,1}^1)} \delta_{N,3}, \end{aligned}$$



with  $\delta_{j,3} = 0$  if  $j \neq 3$  and  $\delta_{j,3} = 1$  if  $j = 3$ .

We can now consider the global convergence of the solutions of (1.1) to (NSI) in the critical functional setting described above.

**Theorem 1.4** *Assume now that the initial data  $(b_0^\varepsilon, u_0^\varepsilon)$  verify the regularity conditions of theorem 1.2 and 1.3. Let  $\tilde{\varepsilon} = \varepsilon\nu$ . There exists a constant  $\eta$  independent of  $\varepsilon$  and  $\nu$  such that if <sup>1</sup>:*

- When  $N = 2$ :

$$C_0^{\varepsilon\nu} = \|(b_0^\varepsilon, (u_0^1)^\varepsilon)\|_{B_{2,1}^0}^{BF,\tilde{\varepsilon}} + \tilde{\varepsilon}\|b_0^\varepsilon\|_{B_{p,1}^{\frac{2}{p}}}^{HF,\tilde{\varepsilon}} + \|(u_0^1)^\varepsilon\|_{B_{p,1}^{\frac{2}{p}-1}}^{HF,\tilde{\varepsilon}} + \|(u_0^2)^\varepsilon\|_{B_{p,2}^{\frac{2}{p}-1}} \leq \eta, \quad (1.15)$$

- When  $N = 3$

$$\begin{aligned} C_0^{\varepsilon\nu} &= \|(b_0^\varepsilon, (\mathbb{Q}u_r, \mathbb{Q}u_z)_0^\varepsilon)\|_{B_{2,1}^{\frac{1}{2}}}^{BF,\tilde{\varepsilon}} + \tilde{\varepsilon}\|b_0^\varepsilon\|_{B_{p,1}^{\frac{3}{p}}}^{HF,\tilde{\varepsilon}} + \|(\mathbb{Q}u_r, \mathbb{Q}u_z)_0^\varepsilon\|_{B_{p,1}^{\frac{3}{p}-1}}^{HF,\tilde{\varepsilon}} \\ &+ \|(\mathbb{P}u_r, \mathbb{P}u_z)_0^\varepsilon\|_{B_{p,2}^{\frac{3}{p}-1}} + \|u_k^\varepsilon\|_{B_{\infty,1}^{-1}} + \|(u_\theta)_0^\varepsilon\|_{B_{p,2}^{\frac{3}{p}-1}} \leq \eta, \end{aligned} \quad (1.16)$$

then the following results hold:

1. *Existence: For all  $0 < \varepsilon \leq \varepsilon_0$ , system (1.1) has a unique solution  $(b^\varepsilon, u^\varepsilon)$  in  $E_{\varepsilon,\nu}^p$  such that for some constant  $M > 0$ :*

$$\|(b^\varepsilon, u^\varepsilon)\|_{E_{\varepsilon,\nu}^p} \leq MC_0^{\varepsilon\nu}.$$

Furthermore  $u_\varepsilon^1, u_r^\varepsilon, u_z^\varepsilon$  converge weakly to 0 and  $u_\varepsilon^2, u_\theta^\varepsilon$  converges in the sense of distribution to  $v$  solution of (NSI).

2. *Convergence:*

- If  $N = 3$  then choosing  $r = 2$  we have:

$$\begin{aligned} \mu\|\mathbb{P}u^\varepsilon - v\|_{\tilde{L}^1(B_{p,2}^{\frac{4}{p}+\frac{1}{2}})} + \|\mathbb{P}u^\varepsilon - v\|_{\tilde{L}^\infty(B_{p,2}^{\frac{4}{p}-\frac{3}{2}})} &\leq \\ M(\|\mathbb{P}u_0^\varepsilon - v_0\|_{B_{p,2}^{\frac{4}{p}-\frac{3}{2}}} + C_0^{\varepsilon\nu}\tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}}) & \end{aligned}$$

$$\nu^{\frac{1}{2}}(\|b^\varepsilon\|_{\tilde{L}^2(B_{p,1}^{\frac{4}{p}-\frac{1}{2}})} + \|\mathbb{Q}u^\varepsilon\|_{\tilde{L}^2(B_{p,1}^{\frac{4}{p}-\frac{1}{2}})}) \leq MC_0^{\varepsilon\nu}\tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}}.$$

- If  $N = 2$  then we have:

$$\begin{aligned} \|u_\varepsilon^2 - v\|_{\tilde{L}^1(B_{p,2}^{\frac{c+2}{p}-\frac{c}{2}+1})} + \|u_\varepsilon^2 - v\|_{\tilde{L}^\infty(B_{p,2}^{\frac{c+2}{p}-\frac{c}{2}-1})} \\ \leq M(\|(u_0^2)^\varepsilon - v_0\|_{B_{p,2}^{\frac{c+2}{p}-\frac{c}{2}-1}} + C_0^{\varepsilon\nu}\tilde{\varepsilon}^{c(\frac{1}{2}-\frac{1}{p})}). \end{aligned}$$

$$\nu^{\frac{1}{2}}(\|b^\varepsilon\|_{\tilde{L}^2(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}})} + \|u_\varepsilon^1\|_{\tilde{L}^2(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}})}) \leq MC_0^{\varepsilon\nu}\tilde{\varepsilon}^{c(\frac{1}{2}-\frac{1}{p})}.$$

with  $c$  verifying  $0 \leq c < \frac{1}{2}$  and  $c < \frac{8-2p}{p-2}$ .

---

<sup>1</sup>We refer to the next section and in particular (2.21), (2.22) for the notations.

**Remark 3** It would be easy to generalize the previous result when  $N = 3$  to the case  $(\mathbb{P}v_r)_0, (\mathbb{P}v_z)_0 \in B_{p,r_1}^{\frac{N}{p}-1}$  with  $1 \leq r_1 \leq \frac{p}{p-2}$ . For simplicity we just take  $r_1 = 2$ .

It would be also possible to deal with different Lebesgue index on  $u_r, u_z$  and  $u_\theta$ .

Our paper is organized as follows. In the first section 2, we introduce the homogeneous hybrid Besov spaces (see also [2]) and recall the basis of Littlewood-Paley theory and the paraproduct. In section 3, we prove the theorem 1.2 and 1.3. In the last section, following the arguments of [13] we study the zero Mach limit of the solution when  $\varepsilon$  goes to 0 using Strichartz estimates in order to deal with the irrotational part which is governed roughly speaking by a wave equation.

## 2 Homogeneous and hybrid Besov spaces

### 2.1 Homogeneous Besov space and paraproduct

As usual, the Fourier transform of  $u$  with respect to the space variable will be denoted by  $\mathcal{F}(u)$  or  $\widehat{u}$ . To build the Littlewood-Paley decomposition, we need to fix a smooth radial function  $\chi$  supported in (for example) the ball  $B(0, \frac{4}{3})$ , equal to 1 in a neighborhood of  $B(0, \frac{2}{3})$  and such that  $r \mapsto \chi(r.e_r)$  is nonincreasing over  $\mathbb{R}_+$ . So that if we define  $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$ , then  $\varphi$  is compactly supported in the annulus  $\{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  and we have that,

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1. \quad (2.17)$$

Then we can define the *dyadic blocks*  $(\Delta_l)_{l \in \mathbb{Z}}$  by  $\Delta_l := \varphi(2^{-l}D)$  (that is  $\widehat{\Delta_l u} = \varphi(2^{-l}\xi)\widehat{u}(\xi)$ ) so that, formally, we have

$$u = \sum_l \dot{\Delta}_l u \quad (2.18)$$

As (2.17) is satisfied for  $\xi \neq 0$ , the previous formal equality holds true for tempered distributions *modulo polynomials*. A way to avoid working modulo polynomials is to consider the set  $\mathcal{S}'_h$  of tempered distributions  $u$  such that

$$\lim_{l \rightarrow -\infty} \|\dot{S}_l u\|_{L^\infty} = 0,$$

where  $\dot{S}_l$  stands for the low frequency cut-off defined by  $\dot{S}_l := \chi(2^{-l}D)$ . If  $u \in \mathcal{S}'_h$ , (2.18) is true and we can write that  $\dot{S}_l u = \sum_{q \leq l-1} \dot{\Delta}_q u$ . We can now define the homogeneous

Besov spaces used in this article:

**Definition 2.2** For  $s \in \mathbb{R}$  and  $1 \leq p, r \leq \infty$ , we set

$$\|u\|_{B_{p,r}^s} := \left( \sum_l 2^{rls} \|\Delta_l u\|_{L^p}^r \right)^{\frac{1}{r}} \quad \text{if } r < \infty \quad \text{and} \quad \|u\|_{B_{p,\infty}^s} := \sup_l 2^{ls} \|\Delta_l u\|_{L^p}.$$

We then define the space  $B_{p,r}^s$  as the subset of distributions  $u \in \mathcal{S}'_h$  such that  $\|u\|_{B_{p,r}^s}$  is finite.

Once more, we refer to [2] (chapter 2) for properties of the inhomogeneous and homogeneous Besov spaces.

In this paper, we shall mainly work with functions or distributions depending on both the time variable  $t$  and the space variable  $x$ . We shall denote by  $\mathcal{C}(I; X)$  the set of continuous functions on  $I$  with values in  $X$ . For  $p \in [1, \infty]$ , the notation  $L^p(I; X)$  stands for the set of measurable functions on  $I$  with values in  $X$  such that  $t \mapsto \|f(t)\|_X$  belongs to  $L^p(I)$ .

The Littlewood-Paley decomposition enables us to work with spectrally localized (hence smooth) functions rather than with rough objects. We naturally obtain bounds for each dyadic block in spaces of type  $L_T^\rho L^p$ . Going from those type of bounds to estimates in  $L_T^\rho \dot{B}_{p,r}^s$  requires to perform a summation in  $\ell^r(\mathbb{Z})$ . When doing so however, we *do not* bound the  $L_T^\rho \dot{B}_{p,r}^s$  norm for the time integration has been performed *before* the  $\ell^r$  summation. This leads to the following notation:

**Definition 2.3** For  $T > 0$ ,  $s \in \mathbb{R}$  and  $1 \leq r, \sigma \leq \infty$ , we set

$$\|u\|_{\tilde{L}_T^\sigma \dot{B}_{p,r}^s} := \|2^{js} \|\dot{\Delta}_q u\|_{L_T^\sigma L^p}\|_{\ell^r(\mathbb{Z})}.$$

One can then define the space  $\tilde{L}_T^\sigma \dot{B}_{p,r}^s$  as the set of tempered distributions  $u$  over  $(0, T) \times \mathbb{R}^d$  such that  $\lim_{q \rightarrow -\infty} \dot{S}_q u = 0$  in  $L^\sigma([0, T]; L^\infty(\mathbb{R}^d))$  and  $\|u\|_{\tilde{L}_T^\sigma \dot{B}_{p,r}^s} < \infty$ . The letter  $T$  is omitted for functions defined over  $\mathbb{R}^+$ . All the properties of continuity for the product and composition which are true in Besov spaces remain true in the above spaces. The time exponent just behaves according to Hölder's inequality.

Let us now recall a few nonlinear estimates in Besov spaces. Formally, any product of two distributions  $u$  and  $v$  may be decomposed into

$$uv = T_u v + T_v u + R(u, v), \text{ where} \quad (2.19)$$

$$T_u v := \sum_l \dot{S}_{l-1} u \dot{\Delta}_l v, \quad T_v u := \sum_l \dot{S}_{l-1} v \dot{\Delta}_l u \quad \text{and} \quad R(u, v) := \sum_l \sum_{|l'-l| \leq 1} \dot{\Delta}_l u \dot{\Delta}_{l'} v.$$

The above operator  $T$  is called ‘‘paraproduct’’ whereas  $R$  is called ‘‘remainder’’. The decomposition (2.19) has been introduced by Bony in [3].

In this article we will frequently use the following estimates (we refer to [2] section 2.6, [10], for general statements, more properties of continuity for the paraproduct and remainder operators, sometimes adapted to  $\tilde{L}_T^\sigma \dot{B}_{p,r}^s$  spaces): under the same assumptions there exists a constant  $C > 0$  such that if  $1/p_1 + 1/p_2 = 1/p$ , and  $1/r_1 + 1/r_2 = 1/r$ :

$$\begin{aligned} \|\dot{T}_u v\|_{B_{2,1}^s} &\leq C \|u\|_{L^\infty} \|v\|_{\dot{B}_{2,1}^s}, \\ \|\dot{T}_u v\|_{B_{p,r}^{s+t}} &\leq C \|u\|_{\dot{B}_{p_1,r_1}^t} \|v\|_{\dot{B}_{p_2,r_2}^s} \quad (t < 0), \\ \|\dot{R}(u, v)\|_{B_{p,r}^{s_1+s_2-\frac{N}{2}}} &\leq C \|u\|_{\dot{B}_{p_1,r_1}^{s_1}} \|v\|_{\dot{B}_{p_2,r_2}^{s_2}} \quad (s_1 + s_2 > 0). \end{aligned} \quad (2.20)$$

Let us now turn to the composition estimates. We refer for example to [2] (Theorem 2.59, corollary 2.63):

**Proposition 2.1** 1. Let  $s > 0$ ,  $u \in \dot{B}_{2,1}^s \cap L^\infty$  and  $F \in W_{loc}^{[s]+2,\infty}(\mathbb{R}^d)$  such that  $F(0) = 0$ . Then  $F(u) \in \dot{B}_{2,1}^s$  and there exists a function of one variable  $C_0$  only depending on  $s$ ,  $d$  and  $F$  such that

$$\|F(u)\|_{B_{2,1}^s} \leq C_0(\|u\|_{L^\infty})\|u\|_{B_{2,1}^s}.$$

2. If  $u$  and  $v \in \dot{B}_{2,1}^{\frac{N}{2}}$  and if  $v - u \in \dot{B}_{2,1}^s$  for  $s \in ]-\frac{N}{2}, \frac{N}{2}]$  and  $G \in W_{loc}^{[s]+3,\infty}(\mathbb{R}^d)$ , then  $G(v) - G(u)$  belongs to  $\dot{B}_{2,1}^s$  and there exists a function of two variables  $C$  only depending on  $s$ ,  $d$  and  $G$  such that

$$\|G(v) - G(u)\|_{B_{2,1}^s} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty}) \left( |G'(0)| + \|u\|_{B_{2,1}^{\frac{N}{2}}} + \|v\|_{B_{2,1}^{\frac{N}{2}}} \right) \|v - u\|_{\dot{B}_{2,1}^s}.$$

Let us end this section by recalling the following estimates for the heat equation:

**Proposition 2.2** Let  $s \in \mathbb{R}$ ,  $(p, r) \in [1, +\infty]^2$  and  $1 \leq \rho_2 \leq \rho_1 \leq +\infty$ . Assume that  $u_0 \in \dot{B}_{p,r}^s$  and  $f \in \tilde{L}_T^{\rho_2}(\dot{B}_{p,r}^{s-2+2/\rho_2})$ . Let  $u$  be a solution of:

$$\begin{cases} \partial_t u - \mathcal{A}u = f \\ u|_{t=0} = u_0, \end{cases}$$

with  $\mathcal{A} = \mu\Delta + (mu + \lambda)\nabla\text{div}$ . Then there exists  $C > 0$  depending only on  $N, \mu, \rho_1$  and  $\rho_2$  such that:

$$\|u\|_{\tilde{L}_T^{\rho_1}(B_{p,r}^{s+2/\rho_1})} \leq C(\|u_0\|_{B_{p,r}^s} + \|f\|_{\tilde{L}_T^{\rho_2}(B_{p,r}^{s-2+2/\rho_2})}).$$

If in addition  $r$  is finite then  $u$  belongs to  $C([0, T], \dot{B}_{p,r}^s)$ .

## 2.2 Hybrid Besov spaces

As explained, in the compressible Navier-Stokes system, the density fluctuation has two distinct behaviours in some low and high frequencies, separated by a frequency threshold. This leads to the definition of the hybrid Besov spaces. Let us begin with the spaces that are introduced by R. Danchin in [8, 10] or [2]:

**Definition 2.4** For  $\alpha > 0$ ,  $r \in [0, \infty]$  and  $s \in \mathbb{R}$  we denote

$$\|u\|_{\tilde{B}_{2,p,1,\alpha,r}^{s_1,s_2}} \stackrel{def}{=} \sum_{l \leq -\frac{\ln \alpha}{\ln 2}} 2^{ls_1} \|\dot{\Delta}_l u\|_{L^p} + \alpha^{1-\frac{2}{r}} \sum_{l \geq -\frac{\ln \alpha}{\ln 2}} 2^{ls_2} \|\dot{\Delta}_l u\|_{L^p}^r.$$

We will note in the sequel:

$$z^{BF,\alpha} = \sum_{l \leq -\frac{\ln \alpha}{\ln 2}} \Delta_l z \quad \text{and} \quad z^{BF,\alpha} = \sum_{l > -\frac{\ln \alpha}{\ln 2}} \Delta_l z. \quad (2.21)$$

and:

$$\|z\|_{B_{p,r}^{BF,\alpha}} = \|z^{BF,\alpha}\|_{B_{p,r}^s} \quad \text{and} \quad \|z\|_{B_{p,r}^{HF,\alpha}} = \|z^{BF,\alpha}\|_{B_{p,r}^s}. \quad (2.22)$$

**Remark 1** As stated in [2] we have the equivalence

$$\frac{1}{2} \left( \|u\|_{\dot{B}_{2,1}^{s-1}} + \alpha \|u\|_{\dot{B}_{2,1}^s} \right) \leq \|u\|_{\tilde{B}_\alpha^{s,\infty}} \leq \|u\|_{\dot{B}_{2,1}^{s-1}} + \alpha \|u\|_{\dot{B}_{2,1}^s}.$$

Let us start now by recalling some estimates in Besov space for the following system:

$$\begin{cases} \partial_t h + u \cdot \nabla h + \operatorname{div} u = F', \\ \partial_t u - \mu \Delta u - \mu \nabla \operatorname{div} u + a \nabla h = G', \\ (h, u)_{/t=0} = (h_0, u_0), \end{cases} \quad (2.23)$$

with  $(F', G')$  external forces. More precisely in [21, 10, 7] it has been proved the following proposition by some different methods. In particular in [21], we introduce the notion of *effective velocity* which enables us to diagonalize the system (2.23).

**Proposition 2.3** *Let  $2 \leq p \leq 4$ . Let  $(h^2, u^2)$  the solution of (2.23). There exists a constant  $C$  depending only on  $\mu, N, s$  and  $s'$  such that the following estimate holds:*

$$\begin{aligned} & \| (h, u)(t) \|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}} \times B_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} + \| (h, u) \|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}} \times \tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}+1})} \\ & \leq C e^{V(T)} \left( \| (h_0, u_0) \|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}} \times B_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}} + \| (F', G') \|_{\tilde{L}^1(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}} \times B_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} \right). \end{aligned}$$

with:

$$V(T) = \int_0^t \| \nabla u(s) \|_{\tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}}} ds.$$

### 2.3 Dispersive estimate

Following B. Desjardins and E. Grenier in [15], we shall use some dispersive inequalities for the wave equation: the so-called Strichartz estimates (see [34, 19, 26] and the references therein). Here is the statement that we need. The reader may refer to [10] for a proof.

**Proposition 2.4** *Let  $(b, d)$  be a solution of the following system:*

$$(W^\varepsilon) \quad \begin{cases} \partial_t b + \varepsilon^{-1} \Lambda d = F, \\ \partial_t d - \varepsilon^{-1} \Lambda b = G. \end{cases}$$

*Then for any  $s \in \mathbb{R}$  and positive  $T$  (possibly infinite), the following estimate holds:*

$$\| (b, d) \|_{L_T^r(B_{p,1}^{s+N(\frac{1}{p}-\frac{1}{2})+\frac{1}{r}})} \lesssim \varepsilon^{\frac{1}{r}} \left( \| (b_0, d_0) \|_{B_{2,1}^s} + \| (F, G) \|_{L_T^1(B_{2,1}^s)} \right)$$

with  $p \geq 2$ ,  $\frac{2}{r} \leq \min(1, (N-1)(\frac{1}{2} - \frac{1}{p}))$  and  $(r, p, N) \neq (2, \infty, 3)$ .

Now, we can expect proposition 2.4 combined with estimates for the heat equation to provide us with uniform estimates for  $(b^\varepsilon, u^\varepsilon)$ , thus uniform bounds for the right-hand side of  $(W^\varepsilon)$ . According to proposition 2.4, this should give us some convergence result. No further arguments are needed to get global convergence in the small in critical spaces, or local convergence in the large for more regular data. In the next two sections, we shall develop these ideas and give complete statements of our convergence results.

### 3 Proof of the theorem 1.2 and 1.3

Let us start with the proof of the theorem 1.2, the proof of theorem 1.3 will follow the same lines. We fix now the cut off frequency in (2.4) which is equal now to  $l_0$ .

#### 3.1 Proof of the existence

Our goal now consists in proving the existence of global strong solution for the system (1.5) with  $\varepsilon = 1$  under the smallness condition (1.11). Let us briefly explain the main arguments of the proof.

1. We smooth out the data and get a sequence of local solutions  $(b_n, u_n^1, u_n^2)_{n \in \mathbb{N}}$  on  $[0, T_n]$  to (1.5) by using the result of [24].
2. We prove uniform estimates on  $(b_n, u_n^1, u_n^2)$  on  $[0, T_n]$  in  $E_{1,\nu}^p$  by using the proposition 2.3 and by studying the behavior of  $u_n^1$  in low and high frequencies. We deduce then that  $T_n = +\infty$ .
3. We use compactness arguments to prove that the sequence  $(b_n, u_n^1, u_n^2)$  converges, up to extraction, to a solution of (1.5).

#### Construction of approximate solutions

We smooth out the data as follows:

$$(b_0)_n = S_n b_0 \quad (u_0^1)_n = S_n u_0^1 \quad \text{and} \quad (u_0^2)_n = S_n u_0^2 \quad .$$

We observe that  $(u_0^1)_n$  and  $(u_0^2)_n$  are always invariant by rotation since the Fourier transform conserves the property of radial functions and since we work with  $\varphi$  a radial function in section 2.

$$S_n u_0^1(x) = \frac{x}{|x|} S_n \left( \frac{u_0^1(y) \cdot y}{|y|} \right)(x) \quad \text{and} \quad S_n u_0^2(x) = \frac{x^\perp}{|x|} S_n \left( \frac{u_0^2(y) \cdot y}{|y|} \right)(x).$$

Note that we have:

$$\forall l \in \mathbb{Z}, \quad \|\Delta_l (b_n)_0\|_{L^p} \leq \|\Delta_l b_0\|_{L^p} \quad \text{and} \quad \|(b_n)_0\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}} \leq \|b_0\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}},$$

and similar properties for  $(u_0^1)_n, (u_0^2)_n$ , a fact which will be used repeatedly during the next steps. Now, according [24], one can solve (1.5) with the smooth data  $((b_0)_n, (u_0^1)_n, (u_0^2)_n)$ . We get a solution  $(b_n, u_n^1, u_n^2)$  on a non trivial time interval  $[0, T_n]$  which verifies by uniqueness the condition (1.4) such that:

$$\begin{aligned} b_n &\in \tilde{C}([0, T_n], \tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}) \quad u_n^1 \in \tilde{C}([0, T_n], \tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}) \cap \tilde{L}^1([0, T_n], \tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}+1}), \\ u_n^2 &\in \tilde{C}([0, T_n], B_{p,2}^{\frac{N}{p}-1}) \cap \tilde{L}^1([0, T_n], B_{p,2}^{\frac{N}{p}+1}). \end{aligned} \quad (3.24)$$

By uniqueness we can observe that the solution  $(b_n, u_n^1, u_n^2)$  remains rotationally invariant, and we have:

$$\begin{cases} \partial_t b_n + \operatorname{div} u_n^1 + u_n^1 \cdot \nabla b_n = -b_n \operatorname{div} u_n^1, \\ \partial_t u_n^1 + u_n^1 \cdot \nabla u_n^1 - \frac{\mu \Delta u_n^1 + (\lambda + \mu) \nabla \operatorname{div} u_n^1}{1 + b_n} + \frac{P'(1 + b_n)}{1 + b_n} \nabla b_n = -u_n^2 \cdot \nabla u_n^2, \\ \partial_t u_n^2 + u_n^1 \cdot \nabla u_n^2 + u_n^2 \cdot \nabla u_n^1 - \frac{\mu \Delta u_n^2 + (\lambda + \mu) \nabla \operatorname{div} u_n^2}{1 + b_n} = 0, \\ (b_n, u_n^1, u_n^2)_{t=0} = (S_n b_0, S_n u_0^1, S_n u_0^2). \end{cases} \quad (3.25)$$

### Uniform bounds

In this part, we aim at getting uniform estimates on  $(b_n, u_n^1, u_n^2)$  in the following space  $F_T$  with the norm  $\|\cdot\|_{F_T}$ :

$$F_T = (\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}) \cap \tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}})) \times (\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}) \cap \tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}+1})) \\ \times (\tilde{L}_T^\infty(B_{p,2}^{\frac{N}{p}-1}) \cap \tilde{L}_T^1(B_{p,2}^{\frac{N}{p}+1})).$$

$$\|(b_n, u_n^1, u_n^2)\|_{F_T} = \|b_n\|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}) \cap \tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}})} + \|u_n^1\|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}) \cap \tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}+1})} \\ + \|u_n^2\|_{\tilde{L}_T^\infty(B_{p,2}^{\frac{N}{p}-1}) \cap \tilde{L}_T^1(B_{p,2}^{\frac{N}{p}+1})}.$$

Let us start by estimating the incompressible part  $u_n^2$  in  $\tilde{L}_T^\infty(B_{p,2}^{\frac{N}{p}-1}) \cap \tilde{L}_T^1(B_{p,2}^{\frac{N}{p}+1})$ . We observe that  $u_n^2$  verify the following equation:

$$\partial_t u_n^2 - \mu \Delta u_n^2 - (\lambda + \mu) \nabla \operatorname{div} u_n^2 = -u_n^1 \cdot \nabla u_n^2 - u_n^2 \cdot \nabla u_n^1 - (\mu \Delta u_n^2 + (\lambda + \mu) \nabla \operatorname{div} u_n^2) \frac{b_n}{1 + b_n}.$$

According to proposition 2.2, we have:

$$\|u_n^2\|_{\tilde{L}_T^\infty(B_{p,2}^{\frac{N}{p}-1}) \cap \tilde{L}_T^1(B_{p,2}^{\frac{N}{p}+1})} \lesssim \|u_0^2\|_{B_{p,2}^{\frac{N}{p}-1}} + \|u_n^1 \cdot \nabla u_n^2\|_{\tilde{L}_T^1(B_{p,2}^{\frac{N}{p}-1})} + \|u_n^2 \cdot \nabla u_n^1\|_{\tilde{L}_T^1(B_{p,2}^{\frac{N}{p}-1})} \\ + \|\Delta u_n^2 \frac{b_n}{1 + b_n}\|_{\tilde{L}_T^1(B_{p,2}^{\frac{N}{p}-1})}. \quad (3.26)$$

From (2.20) and (2.1) we get:

$$\|u_n^1 \cdot \nabla u_n^2\|_{\tilde{L}_T^1(B_{p,2}^{\frac{N}{p}-1})} \lesssim \|(b_n, u_n^1, u_n^2)\|_{F_T}^2 \\ \|u_n^2 \cdot \nabla u_n^1\|_{\tilde{L}_T^1(B_{p,2}^{\frac{N}{p}-1})} \lesssim \|(b_n, u_n^1, u_n^2)\|_{F_T}^2 \\ \|\Delta u_n^2 \frac{b_n}{1 + b_n}\|_{\tilde{L}_T^1(B_{p,2}^{\frac{N}{p}-1})} \lesssim \|b_n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \|u_n^2\|_{\tilde{L}_T^1(B_{p,2}^{\frac{N}{p}+1})}. \quad (3.27)$$

Let us consider now the behavior of  $(b_n, u_n^1)$  which verify: the system (3.25):

$$\begin{cases} \partial_t b_n + \operatorname{div} u_n^1 + u_n^1 \cdot \nabla b_n = F_n^1, \\ \partial_t u_n^1 - \mu \Delta u_n^1 - (\lambda + \mu) \nabla \operatorname{div} u_n^1 + P'(1) \nabla b_n = F_n^2, \\ (b_n, u_n^1)_{t=0} = (S_n b_0, S_n u_0^1). \end{cases} \quad (3.28)$$

with:

$$\begin{aligned}
F_n^1 &= -b_n \operatorname{div} u_n^1, \\
F_n^2 &= -u_n^2 \cdot \nabla u_n^2 - u_n^1 \cdot \nabla u_n^1 - (\mu \Delta u_n^1 + (\lambda + \mu) \nabla \operatorname{div} u_n^1) \frac{b_n}{1 + b_n} \\
&\quad + (P'(1) - \frac{P'(1 + b_n)}{1 + b_n}) \nabla b_n,
\end{aligned} \tag{3.29}$$

Using proposition 2.3 and 2.2, we have the following estimate on  $(b_n, u_n^1, u_n^2)$ :

$$\begin{aligned}
&\|b_n\|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}) \cap \tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}})} + \|u_n^1\|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}) \cap \tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}+1})} \\
&\leq C e^{V_n(T)} (\|b_0\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}} + \|u_0^1\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}} + \|F_n^1\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}})} + \|F_n^2\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})}),
\end{aligned} \tag{3.30}$$

with:

$$V_n(T) = C \int_0^t \|\nabla u_n^1(s)\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}}} ds. \tag{3.31}$$

Therefore, it is only a matter of proving appropriate estimates for  $F_n^1$  and  $F_n^2$  by using proposition 2.20. Let us start with  $F_n^1$ , we have according to the proposition 2.20:

$$\|F_n^1\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}})} \lesssim \|b_n\|_{\tilde{L}^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}})} \|\operatorname{div} u_n^1\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}})}. \tag{3.32}$$

Similarly concerning  $F_n^2$ , we have from proposition 2.20 and 2.1:

$$\begin{aligned}
&\|(\mu \Delta u_n^1 + (\lambda + \mu) \nabla \operatorname{div} u_n^1) \frac{b_n}{1 + b_n}\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} \\
&\quad \leq C (\|b_n\|_{L^\infty}) \|b_n\|_{\tilde{L}^\infty(\tilde{B}_{p,1}^{\frac{N}{2}-1, \frac{N}{p}})} \|u_n^1\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}+1})} \\
&\| (P'(1) - \frac{P'(1 + b_n)}{1 + b_n}) \nabla b_n \|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} \lesssim \|b_n\|_{\tilde{L}^\infty(\tilde{B}_{p,1}^{\frac{N}{2}-1, \frac{N}{p}})} \|b_n\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}})} \\
&\|u_n^1 \cdot \nabla u_n^1\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} \lesssim \|u^1\|_{\tilde{L}_T^2(\tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}})} \|\nabla u_n^1\|_{\tilde{L}_T^2(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})}.
\end{aligned} \tag{3.33}$$

As in [13] we set  $p^* = \frac{2p}{p-2}$  such that  $\frac{1}{p} + \frac{1}{p^*} = \frac{1}{2}$ . Since  $2 \leq p < 4$  we deduce that  $p \leq p^*$ , we can now sea with the convection term  $u_n^2 \cdot \nabla u_n^2$ . We have then:

$$(u_n^2 \cdot \nabla u_n^2)_i = T_{(u_n^2)_j} \partial_j (u_n^2)_i + T_{\partial_j (u_n^2)_i} (u_n^2)_j + \partial_j R((u_n^2)_j, (u_n^2)_i). \tag{3.34}$$

Next we have since  $\frac{1}{p} + \frac{1}{p^*} = \frac{1}{2}$  and  $\frac{1}{2} + \frac{1}{2} = 1$ , using classical paraproduct law (see [2], Chap 2) and interpolation since  $B_{p,2}^{\frac{N}{p}-1} \hookrightarrow B_{p,2}^{\frac{N}{p^*}-1}$ :

$$\|T_{(u_n^2)_j} \partial_j (u_n^2)_i\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} \lesssim \|u_n^2\|_{\tilde{L}^\infty(B_{p,2}^{\frac{N}{p^*}-1})} \|\nabla u_n^2\|_{\tilde{L}^1(B_{p,2}^{\frac{N}{p}})}. \tag{3.35}$$



Similarly we have:

$$\|T_{\partial_j(u_n^2)_i}(u_n^2)_j\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} \|\nabla u_n^2\|_{\tilde{L}^\infty(B_{p,2}^{\frac{N}{p}-2})} \|\nabla u_n^2\|_{\tilde{L}^1(B_{p,2}^{\frac{N}{p}+1})}. \quad (3.36)$$

For the remainder term we have:

$$\|\partial_j R((u_n^2)_j, (u_n^2)_i)\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} \lesssim \|u_n^2\|_{\tilde{L}^2(B_{4,2}^{\frac{d}{4}})}^2. \quad (3.37)$$

We proceed similarly in high frequencies, and using (3.35), (3.36) and (3.37) we get:

$$\|u_n^2 \cdot \nabla u_n^2\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} \lesssim \|u_n^2\|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,2}^{\frac{N}{2}-1, \frac{N}{p}-1})} \|u_n^2\|_{\tilde{L}_T^1(\tilde{B}_{2,p,2}^{\frac{N}{2}+1, \frac{N}{p}+1})}. \quad (3.38)$$

Finally combining (3.32), (3.33) and (3.38) we obtain that:

$$\|(b_n, u_n^1, u_n^2)\|_{F_T} \leq C e^{\|(b_n, u_n^1, u_n^2)\|_{F_T}} (\|(b_0)_n\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}} + \|(u_0^1)_n\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}} + \|(u_0^2)_n\|_{\tilde{B}_{2,p,2}^{\frac{N}{2}-1, \frac{N}{p}-1}} + \|(b_n, u_n^1, u_n^2)\|_{F_T}^2). \quad (3.39)$$

By a basic bootstrap argument we deduce that the sequence  $(b_n, u_n^1, u_n^2)_{n \in \mathbb{N}}$  is uniformly bounded in  $n$  in  $F_{T_n}$ . Since  $(\nabla u_n^1, \nabla u_n^2)$  is uniformly bounded in  $\tilde{L}_{T_n}^1(\tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}}) \times \tilde{L}_{T_n}^1(\tilde{B}_{2,p,2}^{\frac{N}{2}, \frac{N}{p}})$  we deduce by using classical continuation criterion (see [11, 24]) that for any  $n$ ,  $T_n = +\infty$ . It is now classical to prove that the sequence  $(b_n, u_n^1, u_n^2)_{n \in \mathbb{N}}$  converges in the sense of the distributions to a solution  $(b, u^1, u^2)$  of (1.5) (see for more details [2]). The uniqueness of the solution is also classical (see [2]).

The proof of the theorem 1.3 follows exactly the same lines.

## 4 Low Mach limit, Proof of the theorem 1.4

We are now going to prove the theorem 1.4 by following essentially the arguments of the proof of Danchin and He in [13].

### 4.1 Global uniform estimates in $\varepsilon$

Let us make the following change of functions

$$c(t, x) = \varepsilon b^\varepsilon(\varepsilon^2 t, \varepsilon x) \quad \text{and} \quad v(t, x) = \varepsilon u^\varepsilon(\varepsilon^2 t, \varepsilon x). \quad (4.40)$$

Then  $(c, v)$  must verify:

$$\begin{cases} \partial_t c + \operatorname{div} v = -\operatorname{div}(cv), \\ \partial_t v + v \cdot \nabla v - \frac{\mu \Delta v + (\lambda + \mu) \nabla \operatorname{div} v}{1+c} + \frac{P'(1+c)}{1+c} = 0, \\ (c, v)_{t=0} = (c_0, v_0). \end{cases} \quad (4.41)$$

Using the theorem 1.2, 1.3 and by scaling invariance, we know that there exists  $\eta > 0$  such that if:

$$C_0^{\varepsilon\nu} \leq \eta$$

then there exists a global strong solution  $(c, v)$  for the system (4.41). In addition we have:

$$\|(b^\varepsilon, u^\varepsilon)\|_{E_{\varepsilon, \nu}^p} \lesssim \|(c, v)\|_{E_{1, \nu}^p} \lesssim C_0^{\varepsilon \nu}. \quad (4.42)$$

## 4.2 The incompressible limit, weak convergence

Using the uniform estimates (4.42), we can show that the limit of  $(b^\varepsilon, \mathbb{Q}u^\varepsilon)$  when  $\varepsilon$  goes to 0 converges to 0 in the sense of the distribution. Similarly  $\mathbb{P}u^\varepsilon$  converges to  $v$  in the sense of the distribution. The proof follows the same line than [13] (see also [33]).

## 4.3 Convergence of $b^\varepsilon$ and $\mathbb{Q}u^\varepsilon$ to zero

**The case  $N = 3$ :** The starting point is the dispersive inequality given in proposition 2.4 for  $(W^\varepsilon)$ . Indeed denoting  $u_k^\varepsilon = u_r^\varepsilon + u_z^\varepsilon$  we have:

$$\begin{cases} \partial_t b^\varepsilon + \varepsilon^{-1} \operatorname{div} \mathbb{Q}u_k^\varepsilon = F^\varepsilon, \\ \partial_t \mathbb{Q}u_k + \varepsilon^{-1} \nabla b^\varepsilon = G^\varepsilon, \end{cases} \quad (4.43)$$

with  $F^\varepsilon = -\operatorname{div}(b^\varepsilon u^\varepsilon)$  and  $G^\varepsilon = \nu \Delta \mathbb{Q}u_k^\varepsilon - \mathbb{Q}(u_k^\varepsilon \cdot \nabla u_k^\varepsilon) - \mathbb{Q}(\frac{\varepsilon b}{1+\varepsilon b} \mathcal{A}u_k^\varepsilon) + \frac{K(\varepsilon b^\varepsilon) \nabla b^\varepsilon}{\varepsilon} - \mathbb{Q}(u_\theta^\varepsilon \cdot \nabla u_\theta^\varepsilon)$ .

Let us assume that  $\varepsilon = \nu = 1$  for the moment. Proposition 2.4 gives for all  $p_1 \in [2, +\infty)$ :

$$\|(b, \mathbb{Q}u)\|_{\tilde{L}^{\frac{2p_1}{p_1-2}}(B_{p_1,1}^{\frac{2}{p_1}-\frac{1}{2}})}^{BF} \lesssim \|(b_0, \mathbb{Q}u_0)\|_{B_{2,1}^{\frac{1}{2}}}^{BF} + \|(F, G)\|_{L^1(B_{2,1}^{\frac{N}{2}-1})}^{BF}.$$

We have seen in the previous section that:

$$\begin{aligned} \|(F, G)\|_{L^1(B_{2,1}^{\frac{N}{2}-1})}^{BF} &\leq C_0 \\ \|(b, \mathbb{Q}u)\|_{L^1(B_{2,1}^{\frac{N}{2}+1})}^{BF} &\leq C_0. \end{aligned} \quad (4.44)$$

Using interpolation result with:

$$[L^1(B_{2,1}^{\frac{5}{2}}), \tilde{L}^{\frac{2p_1}{p_1-2}}(B_{p_1,1}^{\frac{2}{p_1}-\frac{1}{2}})]_{\frac{p_1}{p_1+2}} = \tilde{L}^2(B_{p,1}^{\frac{4}{p}-\frac{1}{2}}) \quad \text{with } p = \frac{p_1+2}{2}.$$

Similarly we have for  $\alpha \leq 0$  small enough:

$$[L^1(B_{2,1}^{\frac{5}{2}}), \tilde{L}^{\frac{2p_1}{p_1-2}}(B_{p_1,1}^{\frac{2}{p_1}-\frac{1}{2}})]_{\frac{2p_1(1+\alpha)}{(p_1+2)(2+\alpha)}} = \tilde{L}^{2+\alpha}(B_{p,1}^{\frac{4}{p}-\frac{1}{2}-\frac{\alpha}{2+\alpha}}) \quad \text{with } p = \frac{2(2+\alpha)(p_1+2)}{8+6\alpha-\alpha p_1}.$$

We deduce from (4.44) that:

$$\begin{aligned} \nu^{\frac{1}{2}} \|(b, \mathbb{Q}u)\|_{\tilde{L}^2(B_{p,1}^{\frac{4}{p}-\frac{1}{2}})}^{BF} &\lesssim C_0 \quad \text{for all } p \in [2, +\infty) \\ \|(b, \mathbb{Q}u)\|_{\tilde{L}^{2+\alpha}(B_{p,1}^{\frac{4}{p}-\frac{1}{2}-\frac{\alpha}{2+\alpha}})}^{BF} &\lesssim C_0 \quad \text{for all } p \in [2, +\infty) \text{ with } p = \frac{2(2+\alpha)(p_1+2)}{8+6\alpha-\alpha p_1}. \end{aligned}$$

Coming back to the original variables in (4.40) we get:

$$\begin{aligned} \nu^{\frac{1}{2}} \|(b^\varepsilon, \mathbb{Q}u^\varepsilon)\|_{\tilde{L}^2(B_{p,1}^{\frac{4}{p}-\frac{1}{2}})}^{BF, \tilde{\varepsilon}} &\lesssim \tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}} C_0^{\varepsilon\nu} \\ \|(b, \mathbb{Q}u)\|_{\tilde{L}^{2+\alpha}(B_{p,1}^{\frac{4}{p}-\frac{1}{2}-\frac{\alpha}{2+\alpha}})}^{BF} &\lesssim C_0^{\varepsilon\nu} \varepsilon^{\frac{1}{2}-\frac{1}{p}}. \end{aligned}$$

We have by using (1.16) and the definition of the cut-off in (2.21):

$$\begin{aligned} \|(b^\varepsilon, \mathbb{Q}u^\varepsilon)\|_{\tilde{L}^2(B_{p,1}^{\frac{4}{p}-\frac{1}{2}})} &\lesssim \|(b^\varepsilon, \mathbb{Q}u^\varepsilon)\|_{\tilde{L}^2(B_{p,1}^{\frac{4}{p}-\frac{1}{2}})}^{BF, \tilde{\varepsilon}} + \|(b^\varepsilon, \mathbb{Q}u^\varepsilon)\|_{\tilde{L}^2(B_{p,1}^{\frac{4}{p}-\frac{1}{2}})}^{HF, \tilde{\varepsilon}} \\ &\lesssim \nu^{-1/2} \tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}} C_0^{\varepsilon\nu} + \tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}} \|(b^\varepsilon, \mathbb{Q}u^\varepsilon)\|_{\tilde{L}^2(B_{p,1}^{\frac{N}{2}})}^{HF, \tilde{\varepsilon}}, \\ &\lesssim \nu^{-1/2} \tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}} C_0^{\varepsilon\nu}. \end{aligned}$$

Similarly we have:

$$\|(b, \mathbb{Q}u)\|_{\tilde{L}^{2+\alpha}(B_{p,1}^{\frac{4}{p}-\frac{1}{2}-\frac{\alpha}{2+\alpha}})} \lesssim C_0^{\varepsilon\nu} \varepsilon^{\frac{1}{2}-\frac{1}{p}}.$$

**The case  $N = 2$ :**

As previously we have:

$$\begin{cases} \partial_t b^\varepsilon + \varepsilon^{-1} \operatorname{div} u_\varepsilon^1 = F^\varepsilon, \\ \partial_t u_\varepsilon^1 + \varepsilon^{-1} \nabla b^\varepsilon = G^\varepsilon, \end{cases} \quad (4.45)$$

with  $F^\varepsilon = -\operatorname{div}(b^\varepsilon u_\varepsilon^1)$  and  $G^\varepsilon = \nu \Delta u_\varepsilon^1 - (u_\varepsilon^1 \cdot \nabla u_\varepsilon^1) - (\frac{\varepsilon b^\varepsilon}{1+\varepsilon b} \mathcal{A} u_\varepsilon^1) + \frac{K(\varepsilon b^\varepsilon) \nabla b^\varepsilon}{\varepsilon} - u_\varepsilon^2 \cdot \nabla u_\varepsilon^2$ . Taking  $\varepsilon = \nu = 1$  for the moment and using again proposition (2.4) to (1.3) we have:

$$\|(b, u^1)\|_{\tilde{L}^r(B_{p_1,1}^{\frac{2}{r}-1+\frac{1}{r}})}^{BF} \lesssim \|(b_0, u_0^1)\|_{B_{2,1}^{\frac{1}{2}}}^{BF} + \|(F, G)\|_{L^1(B_{2,1}^{\frac{N}{2}-1})}^{BF} \quad \text{for } \frac{2}{r} \leq \frac{1}{2} - \frac{1}{p_1}.$$

We take now  $r$  such that  $\frac{1}{r} = c(\frac{1}{2} - \frac{1}{p_1})$ . We have seen in the previous section that:

$$\begin{aligned} \|(F, G)\|_{L^1(B_{2,1}^{\frac{N}{2}-1})}^{BF} &\leq C_0 \\ \|(b, u^1)\|_{L^1(B_{2,1}^{\frac{N}{2}+1})}^{BF} &\leq C_0. \end{aligned} \quad (4.46)$$

Using interpolation as in the previous case, we deduce that for  $\alpha < 0$ :

$$\begin{aligned} \|(b, u^1)\|_{\tilde{L}^2(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}})}^{BF} &\lesssim C_0 \quad \text{with } p = \frac{4p_1 + (4 - 2p_1)c}{p_1 + 2 + (2 - p_1)c} \\ \|(b, u^1)\|_{\tilde{L}^{2+\alpha}(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}-\frac{\alpha}{2+\alpha}})}^{BF} &\lesssim C_0 \quad \text{with } \frac{1}{p} = \frac{p_1 + 2 + c(2 - p_1)}{4p_1 + c(4 - 2p_1)} + \alpha \frac{2 - p_1}{2(2 + \alpha)(2p_1 - c(p_1 - 2))}. \end{aligned}$$

We recall that since  $c \in [0, \frac{1}{2}]$  and  $p_1 \in [2, +\infty]$  we can deal with  $p \in [2, 4)$ . Coming back to the original variables as in (4.40) it yields:

$$\begin{aligned} \nu^{\frac{1}{2}} \|(b^\varepsilon, u^{1,\varepsilon})\|_{\tilde{L}^2(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}})}^{BF, \tilde{\varepsilon}} &\lesssim \tilde{\varepsilon}^{c(\frac{1}{2}-\frac{1}{p})} C_0^{\varepsilon\nu}, \\ \|(b, u^{1,\varepsilon})\|_{\tilde{L}^{2+\alpha}(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}-\frac{\alpha}{2+\alpha}})}^{BF} &\lesssim \tilde{\varepsilon}^{c(\frac{1}{2}-\frac{1}{p})} C_0^{\varepsilon\nu}. \end{aligned}$$

Proceeding as previously we get for  $\alpha < 0$  small enough:

$$\begin{aligned} \|(b^\varepsilon, u_\varepsilon^1)\|_{\tilde{L}^2(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}})} &\lesssim \nu^{-1/2} \tilde{\varepsilon}^{c(\frac{1}{2}-\frac{1}{p})} C_0^{\varepsilon\nu}, \\ \|(b, u^{1,\varepsilon})\|_{\tilde{L}^{2+\alpha}(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}-\frac{\alpha}{2+\alpha}})} &\lesssim \tilde{\varepsilon}^{c(\frac{1}{2}-\frac{1}{p})} C_0^{\varepsilon\nu}. \end{aligned}$$

#### 4.4 Convergence of the incompressible part

Let us give now the proof of the convergence of  $\mathbb{P}u^\varepsilon$  to  $v$ . Setting  $^2 \omega^\varepsilon = \mathbb{P}u^\varepsilon - v$  we observe that in dimension  $N = 2$  and  $N = 3$ ,  $\omega^\varepsilon$  verifies:

$$\partial_t \omega^\varepsilon - \mu \Delta \omega^\varepsilon = -u^{1,\varepsilon} \cdot \nabla u^{2,\varepsilon} - u^{2,\varepsilon} \cdot \nabla u^{1,\varepsilon} - \frac{\mu \varepsilon b^\varepsilon}{1 + \varepsilon b^\varepsilon} \Delta u^{2,\varepsilon}. \quad (4.47)$$

and:

$$\partial_t \omega^\varepsilon - \mu \Delta \omega^\varepsilon = -\mathbb{P}(\omega^\varepsilon \cdot \nabla v) - \mathbb{P}(\mathbb{P}u^\varepsilon \cdot \nabla \omega^\varepsilon) - \mathbb{P}(Qu^\varepsilon \cdot \nabla \mathbb{P}u^\varepsilon) - \mathbb{P}(u^\varepsilon \cdot \nabla Qu^\varepsilon) - \mathbb{P}\left(\frac{\varepsilon b}{1 + \varepsilon b} \mathcal{A}u^\varepsilon\right). \quad (4.48)$$

**The case  $N = 3$ :**

Let us apply now the proposition 2.2 to the system (4.48), we get:

$$\begin{aligned} Y_\varepsilon &= \|\omega^\varepsilon\|_{\tilde{L}^\infty(B_{p,2}^{\frac{4}{p}-\frac{3}{2}})} + \|\omega^\varepsilon\|_{\tilde{L}^1(B_{p,2}^{\frac{4}{p}+\frac{1}{2}})} \\ &\lesssim \|\omega_0^\varepsilon\|_{B_{p,2}^{\frac{4}{p}-\frac{3}{2}}} + \|G^\varepsilon\|_{\tilde{L}^1(B_{p,2}^{\frac{4}{p}-\frac{3}{2}})}, \end{aligned}$$

with  $G^\varepsilon = -\mathbb{P}(\omega^\varepsilon \cdot \nabla v) - \mathbb{P}(\mathbb{P}u^\varepsilon \cdot \nabla \omega^\varepsilon) - \mathbb{P}(Qu^\varepsilon \cdot \nabla \mathbb{P}u^\varepsilon) - \mathbb{P}(u^\varepsilon \cdot \nabla Qu^\varepsilon) - \mathbb{P}\left(\frac{\varepsilon b}{1 + \varepsilon b} \mathcal{A}u^\varepsilon\right)$ . Next it remains essentially to estimate  $\|G^\varepsilon\|_{\tilde{L}^1(B_{p,2}^{\frac{4}{p}-\frac{3}{2}})}$ . According to (2.20) and the fact

<sup>2</sup>It implies that  $\omega^\varepsilon = u^{2,\varepsilon} - u^2$  when  $N = 2$  and  $\omega^\varepsilon = \mathbb{P}u_{k,\varepsilon} + u_{\theta,\varepsilon} - u_k - u_\theta$  when  $N = 3$ , we refer to (1.9) and (1.10) for the definition of  $u^2$ ,  $u_k$  and  $u_\theta$ .

that  $\frac{4}{p} - \frac{1}{2} \leq \frac{3}{p}$  and  $\frac{4}{p} - \frac{3}{2} + \frac{3}{p} > 0$  we have:

$$\begin{aligned} \|\mathbb{P}u^\varepsilon \cdot \nabla \omega^\varepsilon\|_{L^1(B_{p,2}^{\frac{4}{p}-\frac{3}{2}})} &\lesssim \|\mathbb{P}u^\varepsilon\|_{\tilde{L}^\infty(B_{p,2}^{\frac{3}{p}-1})} \|\nabla \omega^\varepsilon\|_{L^1(B_{p,2}^{\frac{4}{p}-\frac{1}{2}})} + \|\mathbb{P}u^\varepsilon\|_{\tilde{L}^\infty(B_{p,2}^{\frac{3}{p}+1})} \|\nabla \omega^\varepsilon\|_{\tilde{L}^\infty(B_{p,2}^{\frac{4}{p}-\frac{5}{2}})}, \\ \|\omega^\varepsilon \cdot \nabla v\|_{L^1(B_{p,2}^{\frac{4}{p}-\frac{3}{2}})} &\lesssim \|\nabla v\|_{\tilde{L}^\infty(B_{p,2}^{\frac{3}{p}-2})} \|\omega^\varepsilon\|_{\tilde{L}^1(B_{p,2}^{\frac{4}{p}+\frac{1}{2}})} + \|\nabla v\|_{\tilde{L}^1(B_{p,2}^{\frac{3}{p}})} \|\omega^\varepsilon\|_{\tilde{L}^\infty(B_{p,2}^{\frac{4}{p}-\frac{3}{2}})}, \\ \|u^\varepsilon \cdot \nabla \mathbb{Q}u^\varepsilon\|_{L^1(B_{p,2}^{\frac{4}{p}-\frac{3}{2}})} &\lesssim \|\nabla \mathbb{Q}u^\varepsilon\|_{\tilde{L}^{2+\alpha}(B_{p,1}^{\frac{4}{p}-\frac{3}{2}-\frac{\alpha}{2+\alpha}})} \|u^\varepsilon\|_{\tilde{L}^{\frac{2+\alpha}{1+\alpha}}(B_{p,2}^{\frac{3}{p}+2+\alpha})}, \\ \|\mathbb{Q}u^\varepsilon \cdot \nabla \mathbb{P}u^\varepsilon\|_{L^1(B_{p,2}^{\frac{4}{p}-\frac{3}{2}})} &\lesssim \|\mathbb{Q}u^\varepsilon\|_{\tilde{L}^2(B_{p,1}^{\frac{4}{p}-\frac{1}{2}})} \|\nabla \mathbb{P}u^\varepsilon\|_{\tilde{L}^2(B_{p,2}^{\frac{3}{p}-1})}. \end{aligned}$$

Similarly we have:

$$\|\mathbb{P}(\frac{\varepsilon b}{1+\varepsilon b} \mathcal{A}u^\varepsilon)\|_{L^1(B_{p,2}^{\frac{4}{p}-\frac{3}{2}})} \lesssim (1 + \|\varepsilon b^\varepsilon\|_{\tilde{L}^\infty(B_{p,1}^{\frac{3}{p}})}) \|\varepsilon b^\varepsilon\|_{\tilde{L}^\infty(B_{p,1}^{\frac{4}{p}-\frac{1}{2}})} \|u^\varepsilon\|_{L^1(B_{p,2}^{\frac{3}{p}+1})}.$$

Furthermore using the cut-off (2.22), we have:

$$\begin{aligned} \|\varepsilon b^\varepsilon\|_{\tilde{L}^\infty(B_{p,1}^{\frac{4}{p}-\frac{1}{2}})} &\lesssim \nu^{-1} \tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}} \|b^\varepsilon\|_{\tilde{L}^\infty(B_{2,1}^{\frac{1}{2}})}^{BF, \tilde{\varepsilon}} + \tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}} \|\varepsilon b^\varepsilon\|_{\tilde{L}^\infty(B_{p,1}^{\frac{3}{p}})}^{HF, \tilde{\varepsilon}} \\ &\lesssim \nu^{-1} \tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}} C_0^{\varepsilon\nu}. \end{aligned}$$

Combining all the above estimates gives with  $\alpha > 0$  small:

$$Y_\varepsilon \lesssim \|\omega_0^\varepsilon\|_{B_{p,2}^{\frac{4}{p}-\frac{3}{2}}} + \nu^{-1} \tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}} (1 + \nu^{-1} C_0^{\varepsilon\nu}) (C_0^{\varepsilon\nu})^2 + \mu^{-1} (\|v\|_{\tilde{L}^\infty(B_{p,2}^{\frac{3}{p}-1})} + \mu \|v\|_{\tilde{L}^1(B_{p,2}^{\frac{3}{p}+1})}) Y_\varepsilon.$$

Using the theorem 1.1 and the smallness assumption, we deduce that:

$$Y_\varepsilon \lesssim \|\omega_0^\varepsilon\|_{B_{p,2}^{\frac{4}{p}-\frac{3}{2}}} + \tilde{\varepsilon}^{\frac{1}{2}-\frac{1}{p}} C_0^{\varepsilon\nu}.$$

which concludes the proof of the case  $N = 3$ .

**The case  $N = 2$ :**

As previously applying proposition 2.2 to the system (4.47), we obtain:

$$\begin{aligned} Y_\varepsilon &= \|\omega^\varepsilon\|_{\tilde{L}^\infty(B_{p,2}^{\frac{c+2}{p}-\frac{c}{2}-1})} + \|\omega^\varepsilon\|_{\tilde{L}^1(B_{p,2}^{\frac{c+2}{p}-\frac{c}{2}+1})} \\ &\lesssim \|\omega_0^\varepsilon\|_{B_{p,2}^{\frac{c+2}{p}-\frac{c}{2}-1}} + \|G^\varepsilon\|_{\tilde{L}^1(B_{p,2}^{\frac{4}{p}-\frac{c}{2}-1})}, \end{aligned}$$

with  $G^\varepsilon = -u^{1,\varepsilon} \cdot \nabla u^{2,\varepsilon} - u^{2,\varepsilon} \cdot \nabla u^{1,\varepsilon} - \frac{\mu \varepsilon b^\varepsilon}{1+\varepsilon b^\varepsilon} \Delta u^{2,\varepsilon}$ . According to (2.20) and the fact that  $\frac{c+2}{p} - \frac{c}{2} - 1 + \frac{2}{p} > 0$  we have:

$$\begin{aligned} \|u^{2,\varepsilon} \cdot \nabla u^{1,\varepsilon}\|_{L^1(B_{p,2}^{\frac{c+2}{p}-\frac{c}{2}-1})} &\lesssim \|\nabla u^{1,\varepsilon}\|_{\tilde{L}^{2+\alpha}(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}-1-\frac{\alpha}{2+\alpha}})} \|u^{2,\varepsilon}\|_{\tilde{L}^{\frac{2+\alpha}{1+\alpha}}(B_{p,2}^{\frac{2}{p}+2+\alpha})}, \\ \|u^{1,\varepsilon} \cdot \nabla u^{2,\varepsilon}\|_{L^1(B_{p,2}^{\frac{c+2}{p}-\frac{c}{2}-1})} &\lesssim \|u^{1,\varepsilon}\|_{\tilde{L}^2(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}})} \|\nabla u^{2,\varepsilon}\|_{\tilde{L}^2(B_{p,2}^{\frac{2}{p}-1})}. \end{aligned}$$

Similarly we have:

$$\left\| \frac{\varepsilon b^\varepsilon}{1 + \varepsilon b} \Delta u^{2,\varepsilon} \right\|_{\tilde{L}^1(B_{p,2}^{\frac{c+2}{p}-\frac{c}{2}-1})} \lesssim (1 + \|\varepsilon b^\varepsilon\|_{\tilde{L}^\infty(B_{p,1}^{\frac{2}{p}})}) \|\varepsilon b^\varepsilon\|_{\tilde{L}^\infty(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}})} \|u^{2,\varepsilon}\|_{\tilde{L}^1(B_{p,2}^{\frac{2}{p}+1})}.$$

Furthermore using the cut-off (2.22), we have:

$$\begin{aligned} \|\varepsilon b^\varepsilon\|_{\tilde{L}^\infty(B_{p,1}^{\frac{c+2}{p}-\frac{c}{2}})} &\lesssim \nu^{-1} \tilde{\varepsilon}^{c(\frac{1}{2}-\frac{1}{p})} \|b^\varepsilon\|_{\tilde{L}^\infty(B_{2,1}^0)}^{BF,\tilde{\varepsilon}} + \tilde{\varepsilon}^{c(\frac{1}{2}-\frac{1}{p})} \|\varepsilon b^\varepsilon\|_{\tilde{L}^\infty(B_{p,1}^{\frac{3}{p}})}^{HF,\tilde{\varepsilon}} \\ &\lesssim \nu^{-1} \tilde{\varepsilon}^{c(\frac{1}{2}-\frac{1}{p})} C_0^{\varepsilon\nu}. \end{aligned}$$

We conclude by using the same arguments than for the case  $N = 3$ .

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