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# Non-local length estimators and concave functions

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## Abstract

In a previous work [5], the authors introduced the *Non-Local Estimators* (NLE), a wide class of polygonal estimators including the sparse estimators and a part of the DSS ones. NLE are studied here under concavity assumption and it is shown that concavity almost doubles the multigrid converge rate w.r.t. the general case. A parabola reaching the obtained error bound is exhibited. Moreover, the notion of *biconcavity* relative to a NLE is proposed to describe the case where the digital polygone is also concave. An example is given to show that concavity does not imply biconcavity. Then, an improved error bound is computed under the biconcavity assumption.

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## 1. Introduction

This article is the second of a pair devoted to the study of the multigrid convergence of length estimators. For short, the considered length estimators are based on a polygonal approximation of the digitized function with edges whose discrete sizes tend toward infinity, in mean, as the grid step tends toward zero. Indeed, it is known that length estimators using fixed size edges, even with suitable weights, do not converge in the general case and it is likely that this result could be extend to estimators using edges of bounded sizes, weighted or not. In the first article [5], we introduced the notion of *non local estimator* (NLE), a polygonal estimator using edges whose mean discrete size tend toward infinity and, among the NLE, we considered in particular the *M-sparse estimators* (MSE) whose *true* edge lengths (taking into account the grid step) tend toward zero in mean. We proved that a MSE, or a NLE *close* to a MSE, has the multi-grid convergence property. In the present article, we focus on the improvement brought by the concavity assumption on the multigrid convergence speed for the NLE. Indeed, we know from a previous work [4], that convexity doubles the convergence rate of the *sparse estimators* the most regular MSE. This is not exactly the case in the more general setting of the NLE but nevertheless we show that the convergence is significantly sped up by the concavity for a wide class of continuous functions that satisfy a Lipschitz condition on the left and the right derivative. Moreover, we introduce the notion of *biconcavity* which expresses that both the continuous curve and the polygonal line used

for the length estimation are concave. This notion was implicitly used in [3, theorem 13] to prove the multi-grid convergence of the *maximal digital straight segment estimator* (MDSS). Under the biconcavity assumption, we establish a result that fit our observations on the convergence speed of the MDSS for the natural logarithm function.

The paper is organized as follows. In Section 2, some necessary notations and conventions are recalled, as are the NLE convergence properties in the general case. Two theorems on the multigrid convergence rate of NLE and MSE for concave continuous functions are given in Section 3. An experiment exemplifies the results. Section 4 is devoted to the biconcavity. A sufficient condition for this property is presented and we state our third theorem on the convergence rate. Section 5 concludes the article. The reader will also find in Appendix A an example of a concave function for which our best upper bound for the convergence rate is reached, indicating that this bound cannot be improved in the general case. Moreover an example of a concave function whose digitization family has convex pairs of arbitrary long consecutive chords for an infinity of grid steps is exhibited. Eventually, Appendix B gathers the technical lemmas used in Sections 3 and 4.

## 2. Background and previous results

In this section, we introduce our notations and we recall the notion of Non-Local Estimators (NLE) introduced in [5].

### 2.1. Discretization models

This paper is focused on the digitization of function graphs. So, let us consider a continuous function  $g : [a, b] \rightarrow \mathbb{R}$  ( $a < b$ ), its graph  $\mathcal{C}(g) = \{(x, g(x)) \mid x \in [a, b]\}$  and a positive real number  $r$ , the *resolution*. We assume to have an orthogonal grid in the Euclidean space  $\mathbb{R}^2$  whose set of grid points is  $h\mathbb{Z}^2$  where  $h = 1/r$  is the *grid spacing*. We use the following notations:  $\lfloor x \rfloor$  is the greatest integer less than, or equal to  $x$  and  $\lceil x \rceil$  is the smallest integer greater than  $x$ . For  $i \leq j$  two integers,  $\llbracket i, j \rrbracket$  stands for  $[i, j] \cap \mathbb{Z}$ . The *h-digitization* of the function  $g$  is the discrete function  $\mathcal{D}(g, h) : \llbracket \lceil a/h \rceil, \lfloor b/h \rfloor \rrbracket \rightarrow \mathbb{Z}$  defined by  $\mathcal{D}(g, h)(k) = \lfloor g(kh)/h \rfloor$ . Provided the slope of  $g$  is limited by 1 in modulus, the graph of  $\mathcal{D}(g, h)$  is an 8-connected digital curve. Nevertheless, in this article, we make no assumption on the slope of the function  $g$ .

### 2.2. Non-local length estimators

For any continuous function  $f : [a, b] \rightarrow \mathbb{R}$ ,  $L(f)$  denotes the length of the graph  $\mathcal{C}(f)$  according to Jordan's definition of length:

$$L(f) = \sup_{a=x_0 < x_1 < \dots < x_n=b} \sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2},$$

where the supremum is taken over all possible partitions of  $[a, b]$  and  $n$  is unbounded. The reader can find in [5] a description of the classical length estimators.

Let  $\alpha \in \overline{\mathbb{R}} = [-\infty, +\infty]$  be any non-zero real number. When  $\sigma$  is a partition of some interval  $I \subset \mathbb{R}$ , the  $\alpha$ -th power mean of the  $\sigma$  subinterval length sequence is written  $M_\alpha(\sigma)$ .

- A *pattern function* is a function that maps a discrete curve  $\Gamma$  and a grid spacing  $h$  to a partition of the domain of  $\Gamma$ . Let  $\mathcal{A}$  and  $\mathcal{B}$  be two pattern functions. We say that  $\mathcal{A}$  is *finer* than  $\mathcal{B}$ , we write  $\mathcal{A} \prec \mathcal{B}$ , if for any discrete curve  $\Gamma$  and any grid step  $h$ , the partition  $\mathcal{A}(G, h)$  is finer than the partition  $\mathcal{B}(G, h)$ .
- An  $\alpha$ -*pattern function*  $\mathcal{A}$  on a set of rectifiable functions  $C$  is a pattern function such that, for any function  $g \in C$ ,  $\lim_{h \rightarrow 0} M_\alpha(\mathcal{A}(\mathcal{D}(g, h), h)) = +\infty$ .
- An  $(\alpha, \beta)$ -*pattern function* ( $\beta \in \overline{\mathbb{R}}$ )  $\mathcal{A}$  on  $C$  is an  $\alpha$ -pattern function such that, for any function  $g \in C$ ,  $\lim_{h \rightarrow 0} M_\beta(\mathcal{A}(\mathcal{D}(g, h), h)) \times h = 0$ .
- An  $\alpha$ -pattern function, resp.  $(\alpha, \beta)$ -pattern function, is an  $\alpha$ -pattern function, resp.  $(\alpha, \beta)$ -pattern function, on the set of all rectifiable functions.

The *non-local length estimator* associated to an  $\alpha$ -pattern function  $\mathcal{A}$  maps a pair  $(G, h)$ , consisting of a discrete curve and a grid step, to the length  $L^{\text{NL}}(\mathcal{A}, G, h)$  of an  $h$ -homothetic copy of the polyline whose vertices are the points of  $G$  with abscissas in  $\mathcal{A}(G, h)$ . Given a rectifiable function  $g$ , by abuse of notation, we write  $L^{\text{NL}}(\mathcal{A}, g, h)$  instead of  $L^{\text{NL}}(\mathcal{A}, \mathcal{D}(g, h), h)$  and also  $\mathcal{A}(g, h)$  instead of  $\mathcal{A}(\mathcal{D}(g, h), h)$ . Let  $H : (0, +\infty) \rightarrow \mathbb{N}^*$ . A *sparse estimator with step  $H$*  is a non-local length estimator whose pattern function  $\mathcal{A}$  only depends on the grid step  $h$  and the partition  $\mathcal{A}(G, h)$  has a constant step  $H(h)$  but its last step which is not greater than  $H(h)$ .

The main result without concavity hypothesis is that NLE are convergent for Lipschitz functions. Theorem 1 proved in [5] gives a bound on the error at the grid spacing  $h$  for Lipschitz functions whose derivatives are  $k$ -Lipschitz on any interval included in their domains ( $k > 0$ ).

*Notations.* We present some notations used throughout the remainder of the article. The first ones concern euclidean objects. Thereby, they do not depend upon the grid spacing. The others are related to the grid spacing  $h$  and should be indexed by  $h$ . Nevertheless, as we never have to work with two different grid spacings, the  $h$  index is omitted to lighten the notations.

$I = [a, b]$  is an interval of  $\mathbb{R}$  with a non-empty interior and  $g : I \rightarrow \mathbb{R}$  is a Lipschitz function whose derivative is denoted  $g'$  (from Rademacher's theorem,  $g$  is differentiable almost everywhere). The function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $\varphi(x) = \sqrt{1 + x^2}$ . Thus, one has  $L(g) = \int_{[a, b]} \varphi \circ g'$ .

Given some grid spacing  $h > 0$ ,  $A$ , resp.  $B$ , is the smallest, resp. largest, integer such that  $Ah \in I$ , resp.  $Bh \in I$ . The functions  $g_l, g_c, g_r$  are resp. the restrictions of the function  $g$  to the intervals  $[a, Ah]$ ,  $[Ah, Bh]$ ,  $[Bh, b]$ . For any pattern function  $\mathcal{A}$ , we write  $M_\alpha^{\mathcal{A}}$ , instead of  $M_\alpha(\mathcal{A}(g, h))$  when there is no ambiguity. The number of subintervals in the partition  $\mathcal{A}(g, h)$  is denoted  $N^{\mathcal{A}}$ , or just  $N$  when possible and the integers defining the partition  $\mathcal{A}(g, h)$  are  $A = a_0 < a_1 < \dots < a_N = B$  ( $A = b_0 < b_1 < \dots < b_N = B$  for the partition  $\mathcal{B}(g, h)$ ). In particular, for a sparse estimator with step  $H$  and a real  $\alpha$ , the mean  $M_\alpha(\mathcal{A}(G, h))$  lies between  $H(h)$  and  $H(h)(1 - 1/N)^{1/\alpha}$ . Finally, two piecewise affine functions,  $g_c^{\mathcal{A}}$  and  $\lfloor g_c^{\mathcal{A}} \rfloor$ , are defined. They interpolate the continuous function  $g_c$  and its discretization (actually, the  $h$ -homothetic copy of the discrete curve  $\mathcal{D}(g, h)$ ) according to the pattern function  $\mathcal{A}$ . The graph of  $g_c^{\mathcal{A}}$ , resp.  $\lfloor g_c^{\mathcal{A}} \rfloor$ , is the polyline linking the points  $(a_i h, g(a_i h))_{i=0}^N$  which are on  $\mathcal{C}(g)$ , resp. the grid points  $(a_i h, \lfloor \frac{g(a_i h)}{h} \rfloor h)_{i=0}^N$ .

We recall the following theorem proved in [5].

**Theorem 1.** *Let  $g: [a, b] \rightarrow \mathbb{R}$  be a  $k_1$ -Lipschitz function and  $\mathcal{A}$  be a 1-pattern function. If there exist a  $(1, \beta)$ -pattern function  $\mathcal{B}$ ,  $\beta \in [1, +\infty]$ , and a real  $\omega$  such that, for any grid spacing  $h$ ,  $\|\lfloor g_c^{\mathcal{A}} \rfloor - \lfloor g_c^{\mathcal{B}} \rfloor\|_\infty \leq \omega h$ , then*

- if  $\beta = +\infty$ , the non-local estimation  $L^{\text{NL}}(g, \mathcal{A}, h)$  converges toward the length of the curve  $\mathcal{C}(g)$  as  $h$  tends to 0;
- if  $g'$  is  $k_2$ -Lipschitz on each interval included in its domain, we have

$$L(g) - L^{\text{NL}}(\mathcal{A}, g, h) \leq Sh + ThM_1^{\mathcal{B}}(1 + (C^{\mathcal{B}})^2) + U\mathcal{H}^{\mathcal{B}} + V\left(\frac{1}{M_1^{\mathcal{A}}} + \frac{1}{M_1^{\mathcal{B}}}\right), \quad (1)$$

where  $S = 2\varphi(k_1)$ ,  $T = k_2(b - a)/2$ ,  $U = \varphi(k_1) - 1$ ,  $V = (1 + 2\omega)\varphi'(k_1 + 1/M_1^{\mathcal{A}})(b - a)$  and  $\mathcal{H}^{\mathcal{B}}$  is the measure of the union of the  $\mathcal{B}(g, h)$  subintervals on which  $g$  is not differentiable.

Furthermore, if  $\mathcal{B}(g, h) \subseteq \mathcal{A}(g, h)$ , the term  $1/M_1^{\mathcal{A}} + 1/M_1^{\mathcal{B}}$  in the right hand side of Equation 1 can be replaced by  $1/M_1^{\mathcal{B}}$ .

Apart from the first one, the upper bounds that appear in the right hand side of Equation 1 can be improved in the case of concave functions.

### 3. Concave functions length estimation

In this section, we assume that the function  $g$  is concave on  $[a, b]$ . This implies in particular that  $g$  admits left and right derivatives, noted  $d_\ell g$  and  $d_r g$ , at any point of  $(a, b)$  and is Lipschitz continuous on any closed subinterval of  $(a, b)$ . We assume moreover that  $g$  is Lipschitz on  $[a, b]$ . Then,  $g$  is differentiable in  $a$  and  $b$ . Under this new hypothesis, we can improve the bound on the convergence speed of the estimated length toward the true length of the curve  $\mathcal{C}(g)$ .

### 3.1. General case

Let  $\mathcal{A}$  be a pattern function. The functions  $g_l, g_r, g_c^A$  and  $[g_c^A]$  are those defined in Paragraph *Notations* of Section 2.2. Firstly, we recall a bound on the errors due to the loss of the true left and right extremities of the curve  $\mathcal{C}(g)$ . Its proof can be found in[5].

**Proposition 2 (Curve extremity error).** *For any  $k$ -Lipschitz function  $g$ , we have*

$$L(g_l) + L(g_r) \leq 2\varphi(k)h.$$

Propositions 3 and 4 are improvements of Propositions 3 and 4 of [5] for concave curves. The first one gives an upper bound on the *discretization* error.

**Proposition 3 (Error between curve and curve chords).** *If there exists a real  $k > 0$  such that  $d_r g(x) - d_l g(y) \leq k(y - x)$  for any  $x, y$  such that  $Ah \leq x < y \leq Bh$  then*

$$L(g_c) - L(g_c^A) \leq \sum_{i=1}^N \frac{k^2}{4} (a_i - a_{i-1})^3 h^3 \leq \frac{k^2 (b-a) M_3^3}{4M_1} h^2. \quad (2)$$

PROOF. Note that the proof appeal to a technical result, Lemma 12, which is stated, and proved, in Appendix B.

Let us consider the partition  $\sigma = h \cdot \mathcal{A}(g, h)$  of the interval  $[Ah, Bh]$  and the piecewise affine function  $g_c^{A+}: [Ah, Bh] \rightarrow \mathbb{R}$  defined by

$$g_c^{A+}(x) = \min(g(x_{i-1}) + d_r g(x_{i-1})(x - x_{i-1}), g(x_i) - d_l g(x_i)(x_i - x)) ,$$

where  $[x_{i-1}, x_i]$  is the subinterval of the partition  $\sigma$  that contains  $x$ . Note that  $g_c^{A+}(x_i), 0 \leq i \leq N$ , is uniquely defined and is equal to  $g(x_i)$ .

Since  $g$  is concave, we have on the one hand  $d_r g(x_{i-1}) \leq g' \leq d_l g(x_i)$  on any subinterval  $[x_{i-1}, x_i]$  of  $\sigma$  and, on the other hand,  $g_c^A \leq g_c \leq g_c^{A+}$  on  $[Ah, Bh]$ . Therefore, we can apply Lemma 11 and Lemma 12 on each subinterval of the partition  $\sigma$ . Together with the hypothesis on the derivatives of  $g$ , this leads to the following inequalities.

$$\begin{aligned} L(g_c) - L(g_c^A) &\leq L(g_c^{A+}) - L(g_c^A) \\ &\leq \sum_{i=1}^N (x_i - x_{i-1}) \frac{(d_r g(x_{i-1}) - d_l g(x_i))^2}{4} \\ &\leq \sum_{i=1}^N \frac{k^2}{4} (x_i - x_{i-1})^3 \\ &\leq \frac{k^2 h^3 N}{4} M_3^3 \\ &\leq \frac{k^2 h^2 (b-a)}{4} \frac{M_3^3}{M_1} . \end{aligned}$$

Hence, the result holds.  $\square$

Inequality (2) has to be compared to the following one obtained in [5, Proposition 3] for a function  $g$  differentiable with a derivative  $k$  Lipschitz continuous:

$$L(g_c) - L(g_c^A) \leq \frac{k(b-a)}{2} h M_2 .$$

When the partition  $\mathcal{A}(g, h)$  is roughly even,  $M_3^3/M_1 \approx M_2^2$  and the upper bound is squared under the concavity assumption. In the worst case, we also note that

$$\frac{M_2^3}{M_1} = \frac{\sum (a_{i+1} - a_i)^3}{\sum (a_{i+1} - a_i)} \leq \frac{\sum (a_{i+1} - a_i) M_{+\infty}^2}{\sum (a_{i+1} - a_i)} \leq M_{+\infty}^2 .$$

**Example 1.** *The result given by Proposition 3 is illustrated on Fig. 1 with the natural logarithm on the interval  $[1, 2]$ , the sparse estimators with steps  $H(h) = h^{-\gamma}$  where  $\gamma \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$  and the MDSS estimator. The grid steps used for the plot are  $h = (2/3)^n$ ,  $n \in [1, 40]$ . Then, for any  $\gamma$ ,  $M_\alpha \approx h^{-\gamma}$  (precisely,  $h^{-\gamma}(1-h)^\alpha \leq M_\alpha \leq h^{-\gamma}$ ) and Eq. (2) gives the following expression for the discretization error*

$$L(g_c) - L(g_c^A) = \frac{1}{4} h^{2(1-\gamma)} .$$

*In Figure 1, we see that the above formula gives the right convergence rate though the constant is too high. This was expected mainly because Eq. (2) involves an upper bound for the second derivative while this derivative is not constant. Regarding the MDSS estimator, we just know from [1] that  $\Omega(h^{-1/3}) \leq M_1 \leq \mathcal{O}(h^{-1/3} \log(h^{-1}))$ . So, we plotted two lines  $\propto h^{4/3}$  and  $\propto h^{4/3} \log^2(h^{-1})$ .*

The following proposition gives an upper bound on the *quantization error*. The pattern functions have been introduced in [5] to report on the behavior of two families of length estimators: sparse estimators [4] that use domain partitions  $\mathcal{A}(G, h)$  that only depends upon the parameter  $h$  and MDSS (Maximum Digital Straight Segments) that use domain partitions that only depend upon the discrete function  $G$  (local estimators domain partitions depend neither upon  $h$  nor upon  $G$  and fail to converge). Since MDSS domain partitions does not depend upon the grid step, one cannot assert anything about the 'true length' of the subsegments of a MDSS so the underlying pattern function of a MDSS is not an  $(\alpha, \beta)$ -pattern function. Nevertheless, since by definition a MDSS is close to the curve, the resulting digital curve segmentation is not far from the segmentation produced by some  $(\alpha, \beta)$ -pattern function. This is the reason why in the next proposition and in Theorem 6, we appeal to two pattern functions that are close to each other.

**Proposition 4 (Error between curve chords and grid chords).** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two pattern functions such that  $\mathcal{B} \prec \mathcal{A}$  and  $g_c^B - \lfloor g_c^A \rfloor \leq \omega h$  for some  $\omega > 0$ . Then*

$$|L(g_c^B) - L(\lfloor g_c^A \rfloor)| \leq U \sum_{i=1}^{N^B} \frac{h}{b_i - b_{i-1}} + V h \leq U \frac{b-a}{M_{-1}^B M_1^B} + V h , \quad (3)$$

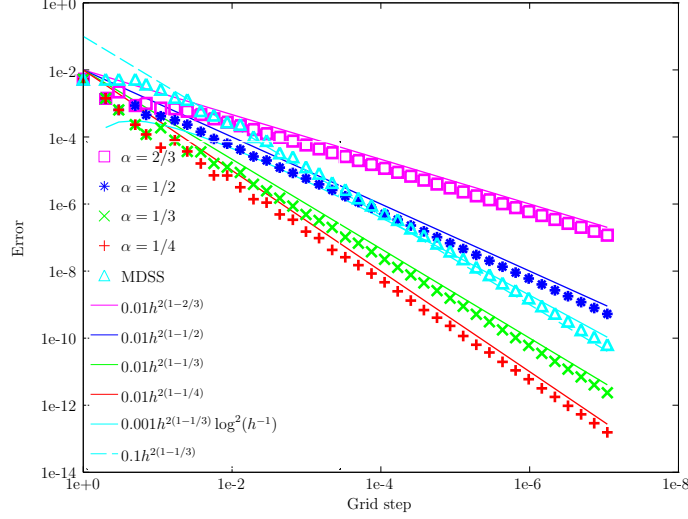


Figure 1:  $|L(g_c) - L(g_c^A)|$  (see text).

where  $U = \omega^2$  and  $V = \max(g'(a), g'(a) - 2g'(b))$ .

PROOF. From the hypotheses, we have

$$\lfloor g_c^A \rfloor \leq g_c^B \leq \lfloor g_c^A \rfloor + \omega h .$$

Let  $s_1$  and  $s_2$  be the slopes of the first and last segments of  $g_c^B$ . Since  $g$  is concave,  $g'(a) \geq s_1 \geq s_2 \geq g'(b)$ . From Lemma 14, applied with  $f_1 = \lfloor g_c^A \rfloor$ ,  $f_2 = g_c^B$ ,  $\sigma = h\mathcal{B}(g, h)$ ,  $p = N^B$  and  $e = \omega h$ , we derive

$$\begin{aligned} |L(g_c^B) - L(\lfloor g_c^A \rfloor)| &\leq U \frac{N^B}{hM_{-1}^B} h^2 + V_0 h \quad \text{where } V_0 = \max(s_1, s_1 - 2s_2) \\ &\leq U \frac{b-a}{M_{-1}^B M_1^B} + V h . \end{aligned}$$

□

**Example 2.** The result given by Proposition 4 is illustrated on Fig. 2 with the same function and patterns as in Example 1, taking each time  $\mathcal{A} = \mathcal{B}$  (and  $\omega = 1$ ). With the sparse estimators, we have, for any  $\gamma$  and  $\alpha$ ,  $M_\alpha = \Theta(h^{-\gamma})$ . For the MDSS estimator, we assume that, for any  $\alpha$ ,  $M_\alpha$  is in  $\Theta(h^{-1/3})$  or in  $\Theta(h^{-1/3} \log(h^{-1}))$ . Then, Eq. 12 gives the following upper bounds for the error  $L(g_c^A) - L(\lfloor g_c^A \rfloor)$ :

- $\mathcal{O}(h^{\min(1, 2\gamma)})$  for the sparse estimators;
- $\mathcal{O}(h^{2/3})$ , or  $\mathcal{O}(h^{2/3}/\log^2(h^{-1}))$ , for the MDSS estimator.



The lines in Fig. 2 correspond to these upper bounds. Though the behavior of the quantization error is less regular than the behavior of the discretization error, the observed convergence rates for the quantization errors fit again our upper bounds. Also, note that the observed constants, hidden in the big  $O$ , are smaller than the ones calculated from Eq. 12 (from a factor of about 10).

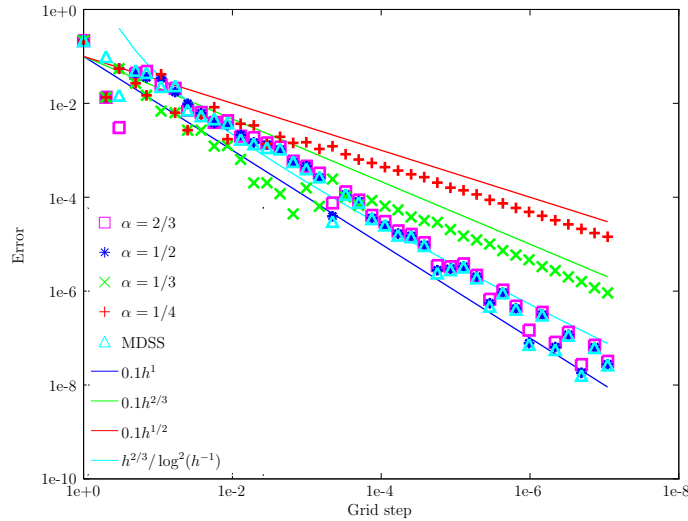


Figure 2:  $|L(g_c^A) - L(\lfloor g_c^A \rfloor)|$  (see text).

From Propositions 2, 3 and 4, we derive the following theorems on the convergence speed when the function  $g$  is concave. Compared to Theorem 1, concavity almost squares the convergence speed. In particular, the step optimal size for fixed size algorithms remains unchanged ( $H_\gamma(h) = \Theta(h^{-\frac{1}{2}})$ ) but the speed is improved up to  $h$ .

**Theorem 5.** *Let  $\mathcal{A}$  be a  $(-1, 1)$ -pattern function. Let  $g: [a, b] \rightarrow \mathbb{R}$  be a Lipschitz and concave function such that  $\{(d_r g(x) - d_\ell g(y))/(y-x) \mid x < y \in [a, b]\}$  is bounded. Then  $L^{\text{NL}}(\mathcal{A}, g, h)$  converges toward  $L(g)$  as  $h$  tends to zero and*

$$L(g) - L^{\text{NL}}(\mathcal{A}, g, h) = \mathcal{O}\left(\frac{h^2 (M_3)^3}{M_1}\right) + \mathcal{O}\left(\frac{1}{M_{-1} M_1}\right). \quad (4)$$

PROOF. The function  $g$  satisfies the hypothesis of Propositions 2, 3 and 4. So

we have

$$\begin{aligned} |L(g) - L(g_c)| &= \mathcal{O}(h) , \\ |L(g_c) - L(g_c^{\mathcal{B}})| &= \mathcal{O}\left(\frac{h^2(M_3)^3}{M_1}\right) , \\ |L(g_c^{\mathcal{B}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)| &= \mathcal{O}\left(\frac{1}{M_{-1}M_1}\right) + \iota(h) . \end{aligned}$$

Since  $\alpha \mapsto M_\alpha$  is non decreasing, we derive

$$h^2 \frac{(M_3)^3}{M_1} \times \frac{1}{M_{-1}M_1} \geq h^2 ,$$

Thus, we can see that either

$$h^2 \frac{(M_3)^3}{M_1} \geq h \text{ or } \frac{1}{M_{-1}M_1} \geq h .$$

Hence, Eq. (4) holds.

Since  $\mathcal{A}$  is an  $(-1, 1)$ -pattern function, on the one hand  $M_{-1}$  tends toward  $+\infty$ . On the other hand, from the classical inequality  $\|x\|_r \leq n^{1/r-1/p} \|x\|_p$  for  $x \in \mathbb{R}^n$  and  $0 < r < p$ , we derive that,

$$\frac{h^2(M_2)^3}{M_1} \leq h^2 N^{3/2} M_1^2 \leq (b-a)^{3/2} (hM_1)^{1/2} .$$

Then, since  $\lim_{h \rightarrow +\infty} hM_1 = 0$ , we conclude straightforwardly that  $L^{\text{NL}}(\mathcal{A}, g, h)$  converges toward  $L(g)$ .  $\square$

**Theorem 6.** *Let  $\mathcal{A}$  be a 1-pattern function. Let  $g: [a, b] \rightarrow \mathbb{R}$  be a Lipschitz and concave function such that  $\{(d_r g(x) - d_\ell g(y))/(y-x) \mid x < y \in [a, b]\}$  is bounded. If, as  $h$  tends toward zero, the Hausdorff distance between  $\mathcal{D}(g, h)$  and  $\lfloor g_c^{\mathcal{A}} \rfloor$  is bounded<sup>1</sup>, then  $L^{\text{NL}}(\mathcal{A}, g, h)$  converges toward  $L(g)$  and*

$$L(g) - L^{\text{NL}}(\mathcal{A}, g, h) = \mathcal{O}(h) + \mathcal{O}\left(\frac{1}{M_1^{\mathcal{A}}}\right) . \quad (5)$$

PROOF. Let  $h > 0$  and  $(a_i)_{i=0}^N = \mathcal{A}(g, h)$ . We subdivide each subinterval of the partition  $\mathcal{A}(g, h)$  in fixed size segments whose sizes are  $\ell$  and a last segment whose size is not greater than  $\ell$  (we do a sparse estimation of each subinterval). Then, the pattern function  $\mathcal{B}$  is defined by  $\mathcal{B}(g, h) = (b_i)_{i=0}^{N^{\mathcal{B}}}$  where  $b_0 = a_0 = a$  and, for any  $i \in \llbracket 1, N^{\mathcal{B}} \rrbracket$ ,  $b_i = \min(b_{i-1} + \ell, a_j)$  with  $j = \min\{k \mid a_k > b_{i-1}\}$ .

Let  $k = \max\{(d_r g(x) - d_\ell g(y))/(y-x) \mid x < y \in [a, b]\}$ . From Proposition 2, we have

$$|L(g) - L(g_c)| = \mathcal{O}(h) . \quad (6)$$

<sup>1</sup>Actually, instead of  $\lfloor g_c^{\mathcal{A}} \rfloor$ , we should use the function  $x \mapsto \lfloor g_c^{\mathcal{A}} \rfloor(hx)/h$ .

From Proposition 3, we derive

$$|L(g_c) - L(g_c^{\mathcal{B}})| \leq \sum_{i=1}^{N^{\mathcal{B}}} \frac{k^2}{4} (b_i - b_{i-1})^3 h^3 \leq \frac{k^2}{4} N^{\mathcal{B}} (\ell h)^3 ,$$

where

$$\begin{aligned} N^{\mathcal{B}} &= \sum_{i=1}^{N^{\mathcal{A}}} \left\lceil \frac{a_i - a_{i-1}}{\ell} \right\rceil \\ &\leq \sum_{i=1}^{N^{\mathcal{A}}} \frac{a_i - a_{i-1}}{\ell} + N^{\mathcal{A}} \\ &\leq \frac{B - A}{\ell} + \frac{B - A}{M_1^{\mathcal{A}}} . \end{aligned}$$

Thus,

$$N^{\mathcal{B}} \leq (b - a) \left( \frac{1}{\ell h} + \frac{1}{h M_1^{\mathcal{A}}} \right) . \quad (7)$$

Then

$$|L(g_c) - L(g_c^{\mathcal{B}})| \leq \frac{k^2}{4} (b - a) \left( \ell^2 h^2 + \frac{\ell^3 h^2}{M_1^{\mathcal{A}}} \right) . \quad (8)$$

The functions  $\lfloor g_c^{\mathcal{A}} \rfloor$  and  $g_c^{\mathcal{B}}$  are piecewise affine. Thus,

$$\begin{aligned} \|\lfloor g_c^{\mathcal{A}} \rfloor - g_c^{\mathcal{B}}\|_{\infty} &= \max_{i \in \llbracket 0, N^{\mathcal{B}} \rrbracket} (|\lfloor g_c^{\mathcal{A}} \rfloor(hb_i) - g_c^{\mathcal{B}}(hb_i)|) \\ &\leq \max_{i \in \llbracket 0, N^{\mathcal{B}} \rrbracket} (|\lfloor g_c^{\mathcal{A}} \rfloor(hb_i) - h\mathcal{D}(g, h)(b_i)|) + h \\ &\leq O(h) \quad (\text{from the hypotheses}) , \end{aligned}$$

because  $\lfloor g_c^{\mathcal{A}} \rfloor$  is close to the digitization set for the Hausdorff distance while  $g$  is Lipschitz. Then, the hypotheses of Proposition 4 are satisfied. We derive that there exists two constants  $U$  and  $V$ , depending on  $g$  and  $\mathcal{A}$  such that

$$\begin{aligned} |L(g_c^{\mathcal{B}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)| &\leq U \sum_{i=1}^{N^{\mathcal{B}}} \frac{h}{(b_i - b_{i-1})} + Vh \\ &\leq U \left( (N^{\mathcal{B}} - N^{\mathcal{A}}) \times \frac{h}{\ell} + N^{\mathcal{A}} \times h \right) + U h \\ &\leq U h \left( \frac{N^{\mathcal{B}}}{\ell} + N^{\mathcal{A}} \right) + V h . \end{aligned}$$

Hence, Equation (7) implies

$$|L(g_c^{\mathcal{B}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)| \leq U(b - a) \left( \frac{1}{\ell^2} + \frac{1}{\ell M_1^{\mathcal{A}}} + \frac{1}{M_1^{\mathcal{A}}} \right) + Vh . \quad (9)$$

Eventually, we obtain the following upper bound:

$$|L(g) - L(\lfloor g_c^A \rfloor)| \leq 2\varphi(k_1)h + \frac{k^2}{4} \left( \ell^2 h^2 + \frac{\ell^3 h^2}{M_1^A} \right) + U(b-a) \left( \frac{1}{\ell^2} + \frac{1}{\ell M_1^A} + \frac{1}{M_1^A} \right) + Vh$$

Taking  $\ell = h^{-1/2}$ , we obtain the result:

$$|L(g) - L(\lfloor g_c^A \rfloor)| = \mathcal{O}(h) + \mathcal{O}(1/M_1^A) .$$

Note that, if we assume a uniform distribution of the integers  $(a_i - a_{i-1}) \bmod \ell$  in the interval  $\llbracket 0, \ell - 1 \rrbracket$ , the expected value of  $\sum_{i=1}^{N^B} \frac{h}{(b_i - b_{i-1})}$  is in  $\mathcal{O}\left(\left(b-a\right)\left(\frac{1}{\ell^2} + \frac{1}{\ell M_1^A} + \frac{1}{M_1^A}\right)\right)$  for large enough  $N^A$ . Then, together with  $\ell = h^{-1/2}$ , Equation 7 becomes  $|L(g) - L(\lfloor g_c^A \rfloor)| = \mathcal{O}(h) + \mathcal{O}(h^{1/2}/M_1^A)$ .  $\square$

On our example with the logarithm, the observed errors for the MDSS method (see Figure 3) is in  $\mathcal{O}(h)$  which is better than the expected convergence rate  $\mathcal{O}(h) + \mathcal{O}(h^{1/2}/M_1^A)$  (and *a fortiori* better than the worst case convergence rate  $\mathcal{O}(h) + \mathcal{O}(1/M_1^A)$ ) since  $M_1$  for the MDSS pattern function lies between  $\mathcal{O}(h^{-1/3})$  and  $\mathcal{O}(h^{-1/3} \log(h^{-1}))$  [1].

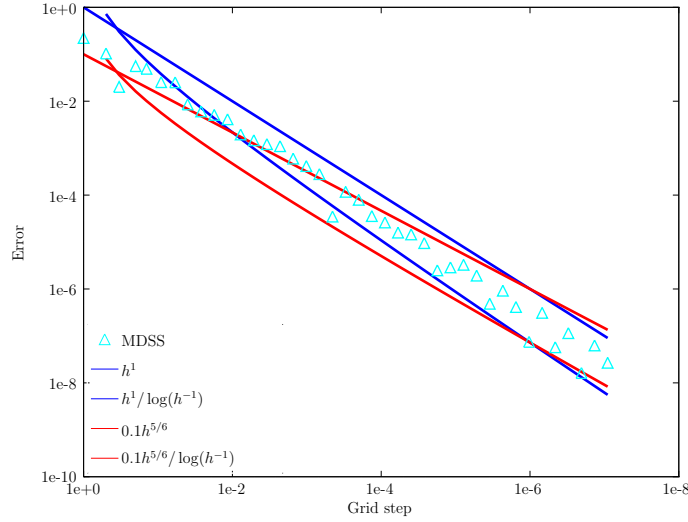


Figure 3:  $|L(g) - L(\lfloor g_c^{\text{MDSS}} \rfloor)|$  (see text).

In the next section, we introduce the notion of *biconcavity* which corresponds to the actual behavior of MDSS and we show that this property speeds up the convergence rate and explain the observed convergence rate of the MDSS.

#### 4. Biconcavity

When the function  $g$  is concave, the piecewise affine function  $g_c^A$  is clearly also concave. Nevertheless, the second piecewise function  $\lfloor g_c^A \rfloor$  is not necessarily concave. When, below some threshold  $h_0$ , the function  $\lfloor g_c^A \rfloor$  is concave for any  $h > 0$ , we say that  $g$  is *biconcave relative to  $\mathcal{A}$* . In Appendix Appendix A.2, we exhibit a concave function that is not biconcave relative to any local estimator. Nevertheless, it follows from the very definition of  $\lfloor g_c^A \rfloor$  that its hypograph is *digitally convex* (the convex hull of the hypograph does not contain more integer points than the hypograph itself) and it was proved in [2] that the MDSS of the boundary of digitally convex body of  $\mathbb{Z}^2$  are monotonic. Hence, continuous concave functions are biconcave relative to the MDSS pattern function. This section gives a sufficient condition to get the biconcavity property and studies the consequences on the convergence speed of such a property.

**Proposition 7.** *Let  $\mathcal{A}$  be pattern function and let  $g: [a, b] \rightarrow \mathbb{R}$  be a concave function such that, for some constant  $k > 0$ , it is true that  $d_r g(x) - d_l g(y) \geq k(y - x)$  for any  $x, y \in [a, b]$  such that  $x < y$ . If one of the following condition holds, then the piecewise affine function  $\lfloor g_c^A \rfloor$  is concave.*

- (i)  $hM_{-\infty}^2 \geq 2/k$ ,
- (ii)  $hM_{-\infty}^2 \geq 1/k$  and  $\mathcal{A}(g, h)$  is a constant sequence.

PROOF. Let  $\delta_i = a_i - a_{i-1}$  for  $1 \leq i \leq N$ . The piecewise affine function  $\lfloor g_c^A \rfloor$  is concave iff, for any  $i \in \llbracket 1, N-1 \rrbracket$ ,

$$\frac{\lfloor g_c^A \rfloor(ha_{i+1}) - \lfloor g_c^A \rfloor(ha_i)}{h\delta_{i+1}} \leq \frac{\lfloor g_c^A \rfloor(ha_i) - \lfloor g_c^A \rfloor(ha_{i-1})}{h\delta_i}. \quad (10)$$

Since, for any  $k \in \llbracket 0, N \rrbracket$ ,  $\lfloor g_c^A \rfloor(ha_k)$  is a multiple of  $h$ , Equation (10) can be rewritten as

$$\begin{aligned} \delta_i(\lfloor g_c^A \rfloor(ha_{i+1}) - \lfloor g_c^A \rfloor(ha_i)) - \delta_{i+1}(\lfloor g_c^A \rfloor(ha_i) - \lfloor g_c^A \rfloor(ha_{i-1})) \\ < h \gcd(\delta_i, \delta_{i+1}). \end{aligned}$$

Thus, from the very definition of the function  $\lfloor g_c^A \rfloor$ , we derive that Equation (10) is true whenever

$$\delta_i(g(ha_{i+1}) - g(ha_i) + h) - \delta_{i+1}(g(ha_i) - g(ha_{i-1}) + h) \leq h \gcd(\delta_i, \delta_{i+1}). \quad (11)$$

Now, from the hypotheses, we derive that, for any  $x, y \in [a, b]$  such that  $x < y$ ,  $g(y) - g(x) = \int_x^y g'(t) dt \leq \int_x^y d_r g(t) - k(t-x) dt \leq d_r g(x)(y-x) - \frac{1}{2}k(y-x)^2$ . Alike,  $d_l g(y)(y-x) + \frac{1}{2}k(y-x)^2 \leq g(y) - g(x)$ . Thus, Equation (??) is true whenever

$$h\delta_i\delta_{i+1}(d_r g(ha_i) - \frac{1}{2}kh\delta_{i+1} - d_l g(ha_i) - \frac{1}{2}kh\delta_i) \leq h(\gcd(\delta_i, \delta_{i+1}) - \delta_i - \delta_{i+1}).$$

Noting that  $d_r g(ha_i) \leq d_\ell g(ha_i)$ , we get the following sufficient inequality

$$hM_{-\infty}^2 k(\delta_{i+1} + \delta_i) \geq 2(\delta_{i+1} + \delta_i - \gcd(\delta_i, \delta_{i+1})) .$$

That is

$$hM_{-\infty}^2 k \geq 2\left(1 - \frac{\gcd(\delta_i, \delta_{i+1})}{\delta_{i+1} + \delta_i}\right) .$$

Proposition 7 follows straightforwardly.  $\square$

The next proposition is an improvement of Proposition 4 in case of biconcavity. It is a consequence of Lemma 15.

**Proposition 8.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two pattern functions such that  $\mathcal{B} \prec \mathcal{A}$ ,  $\lfloor g_c^{\mathcal{A}} \rfloor$  is concave and  $\|\lfloor g_c^{\mathcal{A}} \rfloor - \lfloor g_c^{\mathcal{B}} \rfloor\|_\infty \leq \omega h$  for some  $\omega > 0$ . Then*

$$|L(g_c^{\mathcal{B}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)| \leq U h , \quad (12)$$

where  $U = \max(\alpha, \alpha - 2\beta)$  with  $\alpha = \varphi'(g'(a) + 1)$  and  $\beta = \varphi'(g'(b) - 1)$ .

PROOF. From the hypotheses, we have

$$(\lfloor g_c^{\mathcal{A}} \rfloor - \omega h) \leq g_c^{\mathcal{B}} \leq (\lfloor g_c^{\mathcal{A}} \rfloor - \omega h) + (2\omega + 1)h .$$

Moreover,  $g_c^{\mathcal{B}}$  is concave (for  $g$  is concave).

Let  $s_1^{\mathcal{A}}$  and  $s_2^{\mathcal{A}}$ , resp.  $s_1^{\mathcal{B}}$  and  $s_2^{\mathcal{B}}$ , be the slopes of the first and last segments of  $\lfloor g_c^{\mathcal{A}} \rfloor$ , resp.  $g_c^{\mathcal{B}}$ . From Lemma 15, applied with  $f_1 = \lfloor g_c^{\mathcal{A}} \rfloor - \omega h$ ,  $f_2 = g_c^{\mathcal{B}}$ ,  $\sigma = h\mathcal{B}(g, h)$ ,  $p = N^{\mathcal{B}}$  and  $e = (2\omega + 1)h$ , we derive

$$|L(g_c^{\mathcal{B}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)| \leq U_0 h ,$$

where  $U_0 = \max(\varphi'(s_1), \varphi'(s_1) - 2\varphi'(s_2))$  with  $s_i, i \in \{1, 2\}$ , lying between  $s_i^{\mathcal{A}}$  and  $s_i^{\mathcal{B}}$ .

Let  $(a_i)_{i=0}^N = \mathcal{A}(g, h)$ ,  $\delta_1 = a_1 - a_0$  and  $\delta_N = a_N - a_{N-1}$ . It can easily be seen that  $s_1^{\mathcal{A}} < s_1^{\mathcal{B}} + 1/\delta_1$  and  $s_2^{\mathcal{A}} > s_2^{\mathcal{B}} - 1/\delta_N$ . Then, since  $g$  is concave,  $s_1^{\mathcal{A}} < g'(a) + 1/\delta_1 \leq g'(a) + 1$  and  $s_2^{\mathcal{A}} > g'(b) - 1/\delta_N \geq g'(b) - 1$ .

Thus,  $s_1 \leq \max(s_1^{\mathcal{A}}, s_1^{\mathcal{B}}) < g'(a) + 1$  and  $s_2 \geq \min(s_2^{\mathcal{A}}, s_2^{\mathcal{B}}) > g'(b) - 1$ . As the function  $\varphi'$  is increasing, we get  $\varphi'(s_1) < \alpha$  and  $\varphi'(s_2) > \beta$ , then  $U_0 < U$  and the result holds.  $\square$

The following theorem is the consequence of Proposition 8 on the convergence speed of the non-local estimators.

**Theorem 9.** *Let  $\mathcal{A}$  be a 1-pattern function Let  $g: [a, b] \rightarrow \mathbb{R}$  be a Lipschitz function, biconcave relative to  $\mathcal{A}$ , such that  $\{(d_r g(x) - d_\ell g(y))/(y-x) | x < y \text{ in } [a, b]\}$  is bounded from above by a positive constant. If, as  $h$  tends toward zero, the Hausdorff distance between  $\mathcal{D}(g, h)$  and  $\lfloor g_c^{\mathcal{A}} \rfloor$  is bounded, then*

$$L(g) - L^{\text{NL}}(g, h) = \mathcal{O}(h) + \mathcal{O}\left(\frac{h^{2/3}}{M_1^{\mathcal{A}}}\right) .$$

PROOF. The proof is similar as the proof of Theorem 6 except that we invoke Proposition 8 instead of Proposition 4. Then, in Equation (9), the term  $(b - a) \left( \frac{1}{\ell^2} + \frac{1}{\ell M_1^A} + \frac{1}{M_1^A} \right)$  vanishes and we get

$$|L(g) - L(\lfloor g_c^A \rfloor)| \leq 2\varphi(k_1)h + \frac{k^2}{4} \left( \ell^2 h^2 + \frac{\ell^3 h^2}{M_1^A} \right) + Vh .$$

Taking  $\ell = h^{-4/9}$ , we obtain the result:

$$|L(g) - L(\lfloor g_c^A \rfloor)| = \mathcal{O}(h) + \mathcal{O}\left(\frac{h^{2/3}}{M_1^A}\right) .$$

□

Observe that, for the MDSS pattern function on the set of  $C^3$  functions with positive curvature, we have ([1])  $\Omega(h^{-1/3}) \leq M_1 \leq \mathcal{O}(h^{-1/3} \log(h^{-1}))$ . Then  $|L(g) - L(\lfloor g_c^{\text{MDSS}} \rfloor)| = \mathcal{O}(h \log(h^{-1}))$ .

## 5. Conclusion

In this paper, thanks to the concavity assumption, we improve previous results on the multigrid convergence rate of the NLE, a class of estimators that rely on a polygonal interpolation of the continuous function digitization. Furthermore, we introduce the notion of *biconcavity*, which, is satisfied by the MDSS estimator and by the sparse estimators with enough large pattern sizes. Biconcavity allow further improvement of the convergence rate, up to  $\mathcal{O}(h)$  in the worst case, which is optimal with a square grid whose step is  $h$ . Actually, the NLE framework with its pattern functions appears to be an efficient tool to study the multigrid convergence of the length estimators.

## Appendix A. Counterexamples

### Appendix A.1. An inferior bound for the convergence speed of a concave function

We present in this section an example of rectification of a parabola by a sparse estimator where the bound found in Theorem 5 is reached.

Let  $H = h^{-\gamma}$  be the step of the sparse estimator, the pattern function of which is noted  $\mathcal{A}$ . Let  $g$  be the function defined on the interval  $I = [\frac{1}{16}, \frac{19}{48}]$  by  $g(x) = (\frac{19}{48})^2 - x^2$ . The function  $g$  clearly satisfies the hypotheses of Theorem 5 and  $M_k^A = \mathcal{O}(h^{-\gamma})$  for any non-zero real number  $k$ . Then, from Theorem 5 we get

$$L(g) - L^{\text{NL}}(\mathcal{A}, g, h) = \mathcal{O}(h^{2(1-\gamma)}) + \mathcal{O}(h^{2\gamma}) .$$

Thereby, the best choice for  $H$  is  $h^{-1/2}$  which gives  $L(g) - L^{\text{NL}}(\mathcal{A}, g, h) = \mathcal{O}(h)$ . Let  $g_c^A$  and  $\lfloor g_c^A \rfloor$  be the piecewise affine functions defined in Section 2.2. Then, we shall prove below that the lengths of their curves satisfy  $L(\lfloor g_c^A \rfloor) + 0.06h \leq$

$L(g_c^A) \leq L(g)$  for any  $h = (12(8p+1))^{-2}$  where  $p \in \mathbb{N}$ . Observe that the bounds of the interval  $I$  are multiple of  $h$ . Hence, there is no error due to the bounds (*i.e.*  $g_c^-g$ ). Moreover, the length of  $I$  is a multiple of  $hH$ . Then, the function  $g$  verifies the condition (ii) of Prop. 7 and is then biconcave relative to  $\mathcal{A}$ . Eventually, for any  $p \in \mathbb{N}$  and  $h = (12(8p+1))^{-2}$ , we get  $L(g) - L^{\text{NL}}(\mathcal{A}, g, h) \geq 0.07h$ .

*Detailed calculus.*

Let  $h = \frac{1}{144(8p+1)^2}$  ( $p \in \mathbb{N}$ ) and  $H = h^{-\frac{1}{2}} = 12(8p+1)$ .

Thereby, here we have

$$\begin{aligned} A &= 9(8p+1)^2 \quad \text{and} \quad Ah = \frac{1}{16} \quad , \\ B &= 57(8p+1)^2 \quad \text{and} \quad Bh = \frac{19}{48} \quad , \\ N &= \left\lceil \frac{\frac{19}{48} - \frac{1}{16}}{hH} \right\rceil = 4(8p+1) \quad , \\ \forall i \in \llbracket 0, N \rrbracket, \quad ha_i &= \frac{1}{16} + ihH = \frac{1}{16} + i\sqrt{h} \quad . \end{aligned}$$

Furthermore, we have

$$g(ha_i) = \lfloor g_c^A \rfloor (ha_i) + (i \bmod 2) \times \frac{h}{2} \quad . \quad (\text{A.1})$$

We also set

$$\begin{aligned} c &= \frac{h}{2} \quad , \\ z_i &= h \frac{(a_i + a_{i+1})}{2} \quad , \\ y_i &= g(ha_{i+1}) - g(ha_i) \\ &= -2\sqrt{h} z_i \quad . \end{aligned}$$

Then, from (A.1), we derive

$$\begin{aligned} L(g_c^A) - L(\lfloor g_c^A \rfloor) &= \sum_{i=0}^{N/2-1} \left( \sqrt{h + y_{2i}^2} + \sqrt{h + y_{2i+1}^2} \right) \\ &\quad - \left( \sqrt{h + (y_{2i} - c)^2} + \sqrt{h + (y_{2i+1} + c)^2} \right) \end{aligned}$$

On the one hand

$$\begin{aligned} \sqrt{h + y_{2i}^2} - \sqrt{h + (y_{2i} - c)^2} &= -\frac{h}{4} \frac{8z_{2i} + \sqrt{h}}{\sqrt{1 + 4z_{2i}^2} + \sqrt{1 + 4(z_{2i} + \frac{1}{4}\sqrt{h})^2}} \\ &\geq -\frac{h}{8} \frac{8z_{2i} + \sqrt{h}}{\sqrt{1 + 4z_{2i}^2}} \end{aligned}$$



On the other hand

$$\begin{aligned}\sqrt{h + y_{2i+1}^2} - \sqrt{h + (y_{2i+1} + c)^2} &= \frac{h}{4} \frac{8z_{2i+1} - \sqrt{h}}{\sqrt{1 + 4z_{2i+1}^2} + \sqrt{1 + 4(z_{2i+1} - \frac{1}{4}\sqrt{h})^2}} \\ &\geq \frac{h}{8} \frac{8z_{2i+1} - \sqrt{h}}{\sqrt{1 + 4z_{2i+1}^2}}\end{aligned}$$

By summing,

$$\begin{aligned}L(g_c^A) - L(\lfloor g_c^A \rfloor) &\geq \\ &h \sum_{i=0}^{16p+1} \left( \frac{z_{2i+1}}{\sqrt{1 + 4z_{2i+1}^2}} - \frac{z_{2i}}{\sqrt{1 + 4z_{2i}^2}} \right) - \frac{h\sqrt{h}}{8} \sum_{i=0}^{32p+3} \frac{1}{\sqrt{1 + 4z_i^2}}.\end{aligned}$$

Since the function  $f_1(x) = \frac{x}{\sqrt{1+4x^2}}$  is monotonically increasing and concave, one has

$$\begin{aligned}\sum_{i=0}^{16p+1} (f_1(z_{2i+1}) - f_1(z_{2i})) &\geq \frac{1}{2} \sum_{i=0}^{32p+3} (f_1(z_{i+1}) - f_1(z_i)) \\ &\geq \frac{1}{2} (f_1(z_{32p+4}) - f_1(z_0)) .\end{aligned}$$

Moreover, the function  $f_2(x) = \frac{1}{\sqrt{1+4x^2}}$  is monotonically decreasing and convex. Thus the Riemann sum  $\sum_{i=0}^{32p+3} \frac{1}{\sqrt{1+4z_i^2}} \times \sqrt{h}$  is bounded by the integral  $\int_{\frac{1}{16}}^{\frac{19}{48}} f_2(x) dx$ . It follows that

$$\begin{aligned}L(g_c^A) - L(\lfloor g_c^A \rfloor) &\geq \frac{h}{2} \left( f_1\left(\frac{19}{48} + \frac{\sqrt{h}}{2}\right) - f_1\left(\frac{1}{16} + \frac{\sqrt{h}}{2}\right) \right. \\ &\quad \left. - \frac{1}{8} \arg \sinh\left(\frac{19}{24}\right) + \frac{1}{8} \arg \sinh\left(\frac{1}{8}\right) \right).\end{aligned}$$

Since  $\sqrt{h} \leq \frac{1}{12}$  for any  $p \in \mathbb{N}$ , we obtain

$$L(g_c^A) - L(\lfloor g_c^A \rfloor) > 0.076h.$$

Eventually, for any  $h = \frac{1}{(12(8p+1))^2}$ , we have shown that

$$L(g) \geq L(g_c^A) \geq L(\lfloor g_c^A \rfloor) + 0.07h.$$

This example shows that for some non-local estimators, the obtained bounds are tight and therefore cannot be improved in the general case.

*Appendix A.2. Biconcavity*

In this section, we exhibit a concave function whose discretizations contain arbitrary long convex pairs of chords. The counterexamples rely on the following theorem proved in [6]. This theorem asserts that, given a function  $x \mapsto ax^2 + bx + c$ , the distribution in  $[0, h]$  of the values of the expression  $(a(kh)^2 + b(kh) + c) \bmod h$ ,  $k \in \mathbb{N}$ , which are the errors resulting from the quantization in  $h\mathbb{Z}$ , tends toward the equidistribution.

**Theorem 10 ([6, Lemma 2 and Prop. 3]).** *Let  $a, b \in \mathbb{R}$ ,  $a < b$ . Let  $g : [a, b] \rightarrow \mathbb{R}$  be a polynomial function of degree 2. Then, for all interval  $I \subseteq [0, 1]$ ,*

$$\lim_{h \rightarrow 0} \frac{\text{card}\{x \in h\mathbb{Z} \cap [a, b] \mid g(x) \bmod h \in hI\}}{\text{card}(h\mathbb{Z} \cap [a, b])} = \mu(I) ,$$

where  $\mu(I)$  is the classical length of  $I$ .

Let us consider the function  $g(x) = 2x - x^2$ ,  $x \in [0, 1]$ , which is concave. We denote by  $\lfloor g \rfloor$  the function  $x \in [0, 1] \mapsto \lfloor g(x) \rfloor \in \mathbb{Z}$ . Let  $H$  be a positive integer.

Thanks to Theorem 10, we prove that, for each grid spacing  $h$  below some threshold, we can choose an integer  $p$  such that the finite difference  $\lfloor g \rfloor((p + H)h) - \lfloor g \rfloor(ph)$  is less than or equal to the grid spacing  $h$  while the finite difference  $\lfloor g \rfloor((p + 2)Hh) - \lfloor g \rfloor(ph)$  is greater than twice the grid spacing  $h$ . Thus, the graph of  $\lfloor g \rfloor$  has a convex pair of consecutive chords.

*Detailed calculus.*

According to Theorem 10 with  $[a, b] = [1 - \frac{17}{24H}, 1 - \frac{16}{24H}]$  and  $I = [\frac{4}{12}, \frac{7}{12})$ , it exists a real  $h_0 > 0$  such that, for any  $h \in (0, h_0)$ , one has

$$\text{card}\{n \in J \mid g(nh) - \lfloor g \rfloor(nh) \in [\frac{4h}{12}, \frac{7h}{12})\} \geq \frac{1}{5} \text{card } J ,$$

where  $J = \llbracket \frac{a}{h}, \frac{b}{h} \rrbracket$ .

Since  $\text{card } J \rightarrow +\infty$  as  $h \rightarrow 0$ , there exists  $h_1 > 0$  such that for any  $h < h_1$ , one can find  $n_0 \in \mathbb{N}$  such that  $\llbracket n_0H, (n_0 + 2)H \rrbracket \subset J$  and  $g(n_0hH) - \lfloor g \rfloor(n_0hH) \in [\frac{4h}{12}, \frac{7h}{12})$ .

Let  $h < h_1$ . Noting that  $\frac{16}{12H} \leq g'(x) \leq \frac{17}{12H}$  on  $[a, b]$ , we claim that

$$\begin{aligned} & \lfloor g \rfloor((n_0 + 1)hH) - \lfloor g \rfloor(n_0hH) \\ & < g((n_0 + 1)hH) - (g(n_0hH) - \frac{7}{12}h) \\ & < \frac{17}{12H} \times hH + \frac{7}{12}h \\ & < 2h . \end{aligned}$$

As the left hand side of the above inequalities is a multiple of  $h$ , we get

$$\lfloor g \rfloor((n_0 + 1)hH) - \lfloor g \rfloor(n_0hH) \leq h .$$

In the same way, we obtain

$$\begin{aligned}
& \lfloor g \rfloor((n_0 + 2)hH) - \lfloor g \rfloor(n_0hH) \\
& > g((n_0 + 2)hH) - h - (g(n_0hH) - \frac{4}{12}) \\
& > \frac{16}{12H} \times hH - \frac{2}{3}h \\
& > 2h .
\end{aligned}$$

Thus,

$$\lfloor g \rfloor((n_0 + 2)hH) - \lfloor g \rfloor(n_0hH) \geq 3h .$$

Finally, we have

$$\lfloor g \rfloor((n_0 + 2)hH) - \lfloor g \rfloor(n_0hH) > 2 (\lfloor g \rfloor((n_0 + 1)hH) - \lfloor g \rfloor(n_0hH)) .$$

That is, the function  $\lfloor g \rfloor$  is strictly convex on  $[n_0hH, (n_0 + 2)hH]$ .

## Appendix B. Technical lemmas

**Lemma 11.** *Let  $f$  be a Lipschitz continuous function defined on an interval  $[a, b]$ . Let  $m, M$  be two real numbers such that  $m \leq f'(t) \leq M$  for any  $t \in [a, b]$  where the derivative of  $f$  is defined. Then, the length  $L(f)$  of the graph of  $f$  is less than, or equal to, the length of the polylines joining the points  $A(a, f(a))$  and  $B(b, f(b))$  with segments of slopes  $m$  or  $M$ .*

PROOF. We assume without loss of generality that  $[a, b] = [0, 1]$ . Let  $s$  be the slope of the line from  $A$  to  $B$ . Since  $f$  is Lipschitz continuous, it is almost everywhere differentiable and the slope  $s$  is equal to the integral of  $f'$  on  $[0, 1]$ . Thus,  $m \leq s \leq M$  and there exists  $k \in [0, 1]$  such that  $s = (1 - k)m + kM$ . Moreover,

$$L(f) = \int_0^1 \varphi \circ f'(t) dt.$$

and it can easily be seen that the length of any polyline joining the points  $A$  and  $B$  with segments of slopes  $m$  or  $M$  is  $L = (1 - k)\varphi(m) + k\varphi(M)$ .

We shall prove that  $L(f) - s \leq L - s$ , that is

$$\int_0^1 \psi \circ f'(t) dt \leq (1 - k)\psi(m) + k\psi(M)$$

where  $\psi(x) = \varphi(x) - x$ . Observe that the function  $\psi$  is positive, decreasing and convex.

Let  $\psi \circ g$  be a simple function such that  $\psi \circ g \leq \psi \circ f'$  (since  $\psi$  is bijective from  $\mathbb{R}$  to  $]0, +\infty[$ , any positive simple function can be written as  $\psi \circ g$ ). From  $\psi \circ g \leq \psi \circ f'$ , we derive that  $g \geq f'$ . Thus,  $g \geq m$ . Furthermore, even if it

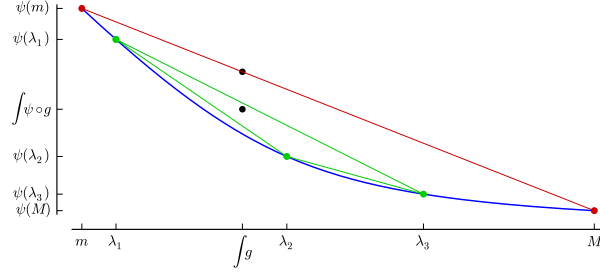


Figure B.4: An illustration of the first inequality in (B.1). We assume  $g = \sum_{i=0}^n \lambda_i 1_{E_i}$  where, for any  $i$ ,  $m \leq \lambda_i \leq M$ , the measurable sets  $E_i$  are pairwise disjoint and  $\sum_{i=0}^n \mu(E_i) = 1$  (here,  $\mu$  is the Lebesgue measure on  $\mathbb{R}$ ). Thus, the point with coordinates  $(\int g, \int \psi \circ g)$  is the barycenter of the weighted points  $((\lambda_i, \psi(\lambda_i)), \mu(E_i))$  while the point with coordinates  $(\int g, (1 - k_1)\psi(m) + k_1\psi(M))$  is the barycenter of the weighted points  $((m, \psi(m)), 1 - k_1), ((M, \psi(M)), k_1)$ .

means replacing  $g$  by  $\inf(g, M)$ , we may assume that  $g \leq M$ . Now, let  $k_1$  be the real in  $[0, 1]$  such that

$$\int_0^1 g(t) dt = (1 - k_1)m + k_1M.$$

As  $g \geq f'$ , we have  $k_1 \geq k$  and, since  $\psi$  is convex and decreasing,

$$\int_0^1 \psi \circ g(t) dt \leq (1 - k_1)\psi(m) + k_1\psi(M) \leq (1 - k)\psi(m) + k\psi(M). \quad (\text{B.1})$$

The first inequality in Equation B.1 is illustrated, and commented, in Figure B.4.

Eventually,

$$\int_0^1 \psi \circ f'(t) dt = \max_g \int_0^1 \psi \circ g(t) dt \leq (1 - k)\psi(m) + k\psi(M).$$

**Lemma 12.** *Let  $ABC$  be a triangle in  $\mathbb{R}^2$  ( $A \neq C$ ) with edges of slopes  $-\infty < \alpha < \beta < \gamma < +\infty$ . We assume that the edge  $AC$  have slope  $\beta$ . Then,*

$$\frac{AB + BC - AC}{AC} \leq \frac{(\gamma - \alpha)^2}{4\varphi(\beta)}$$

Fig. B.5 illustrates the configuration studied in Lemma 12.

PROOF. Let  $k \in (0, 1)$  such that  $\beta = k\gamma + (1 - k)\alpha$ . Let  $m$  be the abscissa of  $\mathbf{AC}$ . It can be seen that the vectors  $\mathbf{AB}$ ,  $\mathbf{BC}$  and  $\mathbf{AC}$  have coordinates

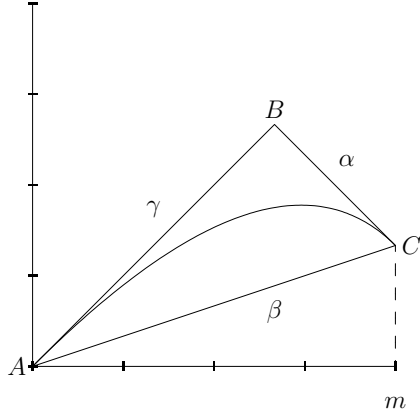


Figure B.5:  $\alpha, \beta, \gamma$  are the slopes of the segments  $BC, CA, AB$ .

$(km, km\gamma), ((1-k)m, (1-k)m\alpha)$  and  $(m, m\beta)$ . Thus,

$$\begin{aligned}
 AB + BC - AC &= m(k\varphi(\gamma) + (1-k)\varphi(\alpha) - \varphi(\beta)) \\
 &= m\left(k(\varphi(\gamma) - \varphi(k\gamma + (1-k)\alpha)) + \right. \\
 &\quad \left. (1-k)(\varphi(\alpha) - \varphi(k\gamma + (1-k)\alpha))\right) \\
 &= mk(1-k)(\gamma - \alpha)(\varphi'(\xi_1) - \varphi'(\xi_2)) \\
 &= mk(1-k)(\gamma - \alpha)(\xi_1 - \xi_2)\varphi''(\xi).
 \end{aligned}$$

where  $\xi_1, \xi_2, \xi$  lie between  $\alpha$  and  $\gamma$ .

Hence,

$$AB + BC - AC \leq \frac{m(\gamma - \alpha)^2}{4}, \quad (\text{B.2})$$

for  $\|\varphi''\|_\infty = 1$ . As  $AC = m\varphi(\beta)$ , the result holds.

**Lemma 13.** *Let  $(u_n)_{n \in \mathbb{N}}$  a monotonically non-increasing sequence of real non negative numbers and  $(c_n)_{n \in \mathbb{N}}$  a sequence of reals in an interval  $I$  such that  $\sum_{i=0}^j c_i \in I$  for any integer  $j$ . Then,  $\sum_{i=0}^j c_i u_i \in u_0 I$  for any integer  $j$ .*

PROOF. If  $u_0 = 0$ , then  $u_n = 0$  for any  $n$  and the result is obvious. From now, we assume  $u_0 > 0$ . Let  $n \in \mathbb{N}$  and  $S = \sum_{i=0}^n c_i u_i$ . We set  $C_j = \sum_{i=0}^j c_i$  for any  $j \leq n$ ,  $p_i = \frac{u_i - u_{i+1}}{u_0}$  for any  $i \leq n-1$  and  $p_n = \frac{u_n}{u_0}$ . The reals  $p_i$  are all non-negative and their sum equals 1. We can easily check that

$$\begin{aligned}
 S &= \sum_{i=0}^{n-1} \left( \sum_{j=0}^i c_j \right) (u_i - u_{i+1}) + \left( \sum_{j=0}^n c_j \right) u_n \\
 &= u_0 \left( \sum_{i=0}^n p_i C_i \right)
 \end{aligned}$$

The last equality above shows that the real  $\frac{1}{u_0} S$  is the barycenter –with non-negative weights– of numbers in the interval  $I$ . Thus, the result holds.

**Lemma 14.** *Let  $f_1$  and  $f_2$  be two piecewise affine functions defined on  $[c, d] \subset \mathbb{R}$ , ( $c < d$ ), with a common partition  $\sigma = (x_i)_{i=0}^p$  having  $p$  steps and such that  $f_1 \leq f_2 \leq f_1 + e$  for some constant  $e > 0$ . If furthermore  $f_2$  is concave, then*

$$|L(f_1) - L(f_2)| \leq \sum_{i=1}^p \frac{1}{x_i - x_{i-1}} e^2 + Ue \leq \frac{p}{M_{-1}(\sigma)} e^2 + Ue .$$

where  $U = \max(\varphi'(s_{2,0}), \varphi'(s_{2,0}) - 2\varphi'(s_{2,p-1}))$  is a constant which depends on the slopes  $s_{2,0}$  and  $s_{2,p-1}$  of the first and the last segments of  $f_2$ .

PROOF. Let  $\sigma = (x_i)_{i=0}^p$  be the common partition for  $f_1$  and  $f_2$ . We write  $m_i$  for  $x_{i+1} - x_i$  and  $s_{1,i}$ , resp.  $s_{2,i}$ , for the slope of  $f_1$ , resp.  $f_2$ , on the interval  $[x_i, x_{i+1}]$ . Then,

$$\begin{aligned} L(f_1) - L(f_2) &= \sum_{i=0}^{p-1} m_i (\varphi(s_{1,i}) - \varphi(s_{2,i})) \\ &= \sum_{i=0}^{p-1} \varphi'(s_{0,i}) m_i (s_{1,i} - s_{2,i}) \quad \text{where } s_{0,i} \in [s_{1,i}, s_{2,i}] \\ &= \sum_{i=0}^{p-1} \varphi'(s_{2,i}) m_i (s_{1,i} - s_{2,i}) + \\ &\quad \sum_{i=0}^{p-1} (\varphi'(s_{0,i}) - \varphi'(s_{2,i})) m_i (s_{1,i} - s_{2,i}) . \end{aligned}$$

Let give an upper bound for  $C = \left| \sum_{i=0}^{p-1} \varphi'(s_{2,i}) m_i (s_{1,i} - s_{2,i}) \right|$ . Since the function  $f_2$  is concave, the sequence  $(s_{2,i})_{i=0}^{p-1}$  is non-increasing as is the sequence  $(\varphi'(s_{2,i}))_{i=0}^{p-1}$  (for the function  $\varphi'$  is increasing). Hence, we can apply Lemma 13 with the settings

$$\begin{aligned} c_i &= m_i (s_{1,i} - s_{2,i}) \\ &= (f_1(x_{i+1}) - f_2(x_{i+1})) - (f_1(x_i) - f_2(x_i)), \\ u_i &= \varphi'(s_{2,i}) - \varphi'(s_{2,p-1}), \\ I &= [-e, e]. \end{aligned}$$

Lemma 13 induces that  $\left| \sum_{i=0}^{p-1} u_i c_i \right| \leq u_0 e$ . Then, we get

$$\begin{aligned} C &\leq \left| \sum_{i=0}^{p-1} u_i c_i \right| + \left| \sum_{i=0}^{p-1} \varphi'(s_{2,p-1}) c_i \right| \\ &\leq u_0 e + |\varphi'(s_{2,p-1})| |(f_1(d) - f_2(d)) - (f_1(c) - f_2(c))| \\ &\leq u_0 e + |\varphi'(s_{2,p-1})| e \\ &\leq Ue \end{aligned}$$

where  $U = \max(\varphi'(s_{2,0}), \varphi'(s_{2,0}) - 2\varphi'(s_{2,p-1}))$ .

We now look at the sum  $D = \sum_{i=0}^{p-1} (\varphi'(s_{0,i}) - \varphi'(s_{2,i})) m_i (s_{1,i} - s_{2,i})$ . The function  $\varphi'$  is 1-Lipschitz ( $\varphi''(x) = (1+x^2)^{-3/2}$ ), so we have

$$|\varphi'(s_{0,i}) - \varphi'(s_{2,i})| \leq |s_{0,i} - s_{2,i}| \leq |s_{1,i} - s_{2,i}| .$$

Then,

$$\begin{aligned} D &\leq \sum_{i=0}^{p-1} m_i (s_{1,i} - s_{2,i})^2 \\ &\leq \sum_{i=0}^{p-1} \frac{c_i^2}{m_i} \\ &\leq \sum_{i=0}^{p-1} \frac{1}{m_i} e^2 . \end{aligned}$$

Eventually, we get

$$|L(f_1) - L(f_2)| \leq Ue + \sum_{i=0}^{p-1} \frac{1}{m_i} e^2 . \quad (\text{B.3})$$

□

**Lemma 15.** *Let  $f_1$  and  $f_2$  be two concave piecewise affine functions defined on  $[c, d] \subset \mathbb{R}$  such that  $f_1 \leq f_2 \leq f_1 + e$  for some  $e > 0$ . Then*

$$|L(f_1) - L(f_2)| \leq Ue . \quad (\text{B.4})$$

where  $U = \max(\varphi'(\alpha), \varphi'(\alpha) - 2\varphi'(\beta))$  with  $\alpha$ , resp.  $\beta$ , lying between the slopes of the first, resp. last, segments of  $\mathcal{C}(f_1)$  and  $\mathcal{C}(f_2)$ .

PROOF. Let  $\sigma = (x_k)_{k=0}^p$  be a common partition for  $f_1$  and  $f_2$ . We write  $m_k$  for  $x_{k+1} - x_k$  and  $s_{1,k}$ , resp.  $s_{2,k}$ , for the slope of  $f_1$ , resp.  $f_2$ , on the interval  $[x_k, x_{k+1}]$ . Since  $f_1$  and  $f_2$  are concave, the sequences  $(s_{1,k})$  and  $(s_{2,k})$  are monotonically non-increasing. Then,

$$L(f_1) - L(f_2) = \sum_{k=0}^{p-1} m_k (\varphi(s_{1,k}) - \varphi(s_{2,k})) = \sum_{k=0}^{p-1} \varphi'(z_k) m_k (s_{1,k} - s_{2,k}) ,$$

where  $z_k \in (s_{1,k}, s_{2,k})$ .

Let  $i < j$  be two integers in  $\llbracket 0, p-1 \rrbracket$ . Since  $s_{1,i} > s_{1,j}$ ,  $s_{2,i} > s_{2,j}$  and, by definition,  $\varphi'(z_i)$  and  $\varphi'(z_j)$  are the slopes of two chords of the convex curve  $\mathcal{C}(\varphi)$  between the points of abscissas  $s_{1,i}$ ,  $s_{1,j}$  for the former and between the points of abscissas  $s_{2,i}$ ,  $s_{2,j}$  for the latter, we derive that  $\varphi'(z_i) > \varphi'(z_j)$ . Thereby, the sequence  $(\varphi'(z_k))$  is monotonically non-increasing.

Now, from Lemma 13, taking

$$\begin{aligned}
c_k &= m_k(s_{1,k} - s_{2,k}) \\
&= (f_1(x_{k+1}) - f_2(x_{k+1})) - (f_1(x_k) - f_2(x_k)), \\
u_k &= \varphi'(z_k) - \varphi'(z_{p-1}) \text{ and} \\
I &= [-e, e] ,
\end{aligned}$$

we derive from (12) that

$$\begin{aligned}
|L(f_1) - L(f_2)| &= \left| \sum_{k=0}^{p-1} (u_k + \varphi'(z_{p-1})) c_k \right| \\
&\leq \left| \sum_{k=0}^{p-1} u_k c_k \right| + |\varphi'(z_{p-1})| \sum_{k=0}^{p-1} c_k \\
&\leq u_0 e + |\varphi'(z_{p-1})| e \\
&\leq U e ,
\end{aligned}$$

where  $U = \varphi'(z_0) - \varphi'(z_{p-1}) + |\varphi'(z_{p-1})| = \max(\varphi'(z_0), \varphi'(z_0) - 2\varphi'(z_{p-1}))$ .  $\square$

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