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► **To cite this version:**

Benoit Combès, Anne Philippe. Bayesian analysis of multiplicative Gaussian error for multiple ages estimation in optically stimulated luminescence dating. 2016. hal-01346868v1

HAL Id: hal-01346868

<https://hal.science/hal-01346868v1>

Preprint submitted on 19 Jul 2016 (v1), last revised 30 Nov 2016 (v2)

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Bayesian analysis of multiplicative Gaussian error for multiple ages estimation in optically stimulated luminescence dating

Benoit Combès^a, Anne Philippe^{b,*}

^a*IRAMAT-CRP2A, Maison de l'archéologie, Université de Bordeaux Montaigne, France.*

^b*Laboratoire de Mathématiques Jean Leray et ANJA INRIA - Rennes, Université de Nantes, France.*

Abstract

Many dating techniques include significant error terms which are not independent between samples to date. This is typically the case in Optically Stimulated Luminescence (OSL) dating where the conversion from characteristic equivalent doses to the corresponding ages using the annual dosimetry data includes error terms that are common to all produced datings. Dealing with these errors is essential to estimate ages from a set of datings whose chronological ordering is known. In this work, we propose and we study a Bayesian model to address this problem. For this purpose, we first consider a multivariate model with multiplicative Gaussian errors in a Bayesian framework. This model relates the set of characteristic equivalent doses and the set of corresponding ages while taking into account for the systematic and non-systematic errors due to the dosimetry. This model offers the opportunity to deal properly with stratigraphic constraints within OSL datings and with unrelated, systematic-error free, dating (*e.g.* radiocarbon). Then, we use this model to extend an existing Bayesian model for the assessment of characteristic equivalent doses from Single Aliquot and Regenerative (SAR) dose measurements. The resulting overall Bayesian model leads to a joint estimation of all the variables (which include all the dose-response functions and characteristic equivalent doses) of a sequence of, possibly heterogeneous, datings. We also consider a more generic solution consisting in directly using the age model from a set of characteristic equivalent doses estimations with their associated standard errors. We finally give an example of application on a set of 5 OSL datings stratigraphically constrained and observe a good adequacy between the two approaches.

Keywords: optically stimulated luminescence, chronometric dating, Bayesian analysis

1. Introduction

Ordering (or stratigraphic) constraints between samples to date constitute an essential element to build accurate and coherent chronologies. Such a prior information is typically modeled in a Bayesian setting [5] and consists of a basic element of several dedicated softwares *e.g.* OxCal [4], BCal [6] or Chronomodel [11, 19]. These models are designed

*Corresponding author. Anne.Philippe@univ-nantes.fr

to deal with datings associated to additive and independent errors as it is the case for *e.g.* radiocarbon or archaeomagnetic datings. However, paleodosimetric datings do not fit such a frame and to the best of your knowledge, there is still a lack of generic methods to incorporate them in the analyses. Paleodosimetric dating methods [17] rely on the assessment of a characteristic equivalent dose D expressing the total absorbed radiation dose since last resetting event (*e.g.* last exposure to sunlight or heat) of the studied samples. Then, this characteristic equivalent dose can be converted into an age A using the annual dose-rate \dot{d} associated to the environment surrounding the studied sample. Once the characteristic equivalent dose \hat{D} and its standard error σ_D evaluated, the commonly used procedure to perform this conversion consists in [16]:

1. summarizing the distribution of the annual dose-rate \dot{d} with a mean μ_j and standard error σ_j using combination rules,
2. approximating the point estimate and the standard error for the associated age defined by the ratio $A = \frac{D}{\dot{d}}$ using the following quadrature formula:

$$\hat{A} = \frac{\hat{D}}{\mu_j} \text{ and } \sigma_A = \hat{A} \sqrt{\left(\frac{\sigma_D}{\hat{D}}\right)^2 + \left(\frac{\sigma_j}{\mu_j}\right)^2}. \quad (1)$$

This solution has two main limitations:

- A. it provides only a coarse characterization of the statistics of A ,
- B. it does not permit to include additional information as ordering constraints between ages of a given stratigraphic sequence to improve the estimations. Moreover in practice, the dose-rate error $\sigma_{\dot{d}_i}$ for each dating i includes errors that are common to all the datings making existing models to deal with stratigraphic constraints inadequate.

To overcome these two limitations several solutions have been investigated. Rhodes *et al.* [14] proposed to tackle limitation B by inferring the age according to Eq. 1 and then by simply ignoring the systematic part of the error in the analysis of the stratigraphic constraints (using the software Oxcal) [3]. Finally, they used an external criterion to check the coherency of the inference. While pragmatic, this solution does not provide a proper model to provide consistent dating uncertainties. Huntriss [10] proposed to tackle limitation A by simulating a sample from the distribution associated to the age $A = \frac{D}{\dot{d}}$ from two samples simulated according to the distributions associated to D and to the dose-rate \dot{d} . As we will see it, when the distribution for the annual dosimetry is Gaussian (which is nearly the case), our approach extends this one while dealing also with limitation B. Finally, Millard [12] designed a solution to tackle the two limitations. For that purpose, he proposed to model the dosimetry data through a likelihood structure: one latent variable is assigned to each quantity needed for the computation of the dose-rate \dot{d}_i , these latents (that are typically numerous) are related to observations using Gaussian distributions and all are related to the ages to form the model. Priors are set for each latent and the posterior ages

are estimated using Markov Chain Monte Carlo methods. A practical drawback of this approach is that each dating laboratory may have its own complex protocol to estimate the dose-rate \dot{d}_i and for each of these protocols one needs to build a dedicated model that must be implemented, validated (particularly with respect to the choice of the associated priors) and made available to the community. In practice, this limits the practical impact of such an approach. Moreover, modeling the dosimetry data using observations can be problematic as some parts of the observations and their associated uncertainties are not strictly speaking associated to an observation but rather on expert information (*e.g.* uncertainty on the water content). Similarly, the systematic component included in the error terms makes that the observations are not independent and the associated Bayesian model is, in most cases, not identifiable. Thus the dosimetry data could be considered as a component of the prior structure of the dating model and not as observations. Notice however that for implementation facilities, Millard proposed then to recast the likelihood structures into priors (and similarly in a companion article [13]).

In this work, we propose a new solution tackling limitations A and B and consisting in a Bayesian model for multiplicative multivariate Gaussian errors. Interestingly, this solution is generic with respect to the laboratory protocol as it only relies on the specification of a few variance and covariance values that can be derived for many dose-rate assessment protocols.

The paper is organized as follows. In Section 2.1, we present the age model to estimate an age from an equivalent dose, a mean dose-rate and its associated standard error. This simple model will be then considered in a multidimensional setting where a part of the dosimetry error is possibly common to several datings (Section 2.2) and where some datings can be constrained to be in a given order (Section 2.3). From that basis, we then consider two situations with optically stimulated luminescence (OSL) datings [1, 20]. In the first situation, we consider that we have the raw OSL measurements at our disposal. We thus propose a model consisting of the addition of a supplemental layer to a Bayesian model to infer a characteristic equivalent dose D from Single Aliquot and Regenerative (SAR) dose measurements we developed in another paper [2] to the age model derived before (Section 3.1). In the second one, we consider that we only have the output characteristic equivalent doses and their standard errors at our disposal. We thus directly consider their conversions to ages (Section 3.2). Then, we give an application of these two models to infer a chronology from a set of luminescence measurements of 5 OSL samples stratigraphically constrained (Section 4). Finally, we summarize our contributions and recommendations in Section 5.

2. Age to equivalent dose relationship: multiplicative Gaussian error

2.1. Individual age model

Once the characteristic equivalent dose D assessed, the age of the studied sediment since last resetting (corresponding to exposure to light) can be obtained by dividing D by the mean annual dose-rate \dot{d} . However in practice, \dot{d} is only known with an error aggregating many sources of errors due to measurement errors, calibration errors and a low level of

awareness of some characteristics of the deposit environment. As a result, even given an accurate assessment of the characteristic equivalent dose D , a significant uncertainty may be associated to the resulting age. Formally, the relationship between the characteristic equivalent dose and the corresponding characteristic age A is handled by the following multiplicative gaussian error model:

$$D = A(\dot{d} + \epsilon), \quad (2)$$

with $\epsilon \sim N(0, \sigma_{\dot{d}}^2)$, which results in the following conditional distribution of D given A :

$$D \sim N(A \cdot \dot{d}, A^2 \cdot \sigma_{\dot{d}}^2), \text{ conditionally to } A. \quad (3)$$

We are interested in estimating the age A from the observed characteristic equivalent dose, and the dose-rate data $\sigma_{\dot{d}}$ and \dot{d} with $A > 0$. One can show that the Jeffreys prior for A is

$$P_A(a) \propto \frac{1}{a} \mathbb{I}_{\mathbb{R}^+}(a). \quad (4)$$

This is a particular case of the prior obtained in [Appendix A.1](#). The prior is improper but the resulting posterior is proper for $d \neq 0$:

$$P_{A|D}(a|d) \propto \frac{1}{\sqrt{2\pi}a^2\sigma_{\dot{d}}} \exp\left(-\frac{(a\dot{d} - d)^2}{2a^2\sigma_{\dot{d}}^2}\right) \mathbb{I}_{\mathbb{R}^+}(a), \quad (5)$$

Interestingly, let $Y = d/X$ ($d \neq 0$) with $X \sim N(\dot{d}, \sigma_{\dot{d}}^2) \cdot I_{]0; \infty[}$ (where $I_{]0; \infty[}$ indicates that the density is truncated on $]0; \infty[$), then for $y \in]0; \infty[$:

$$\begin{aligned} f_Y(y) &= \left| \frac{d}{dy}(d/y) \right| \cdot f_X(d/y) = d/y^2 \cdot f_X(1/y) \\ &\propto d/(\sqrt{2\pi} \cdot \sigma_{\dot{d}} \cdot y^2) \exp\left(-\frac{(\dot{d} - d/y)^2}{2\sigma_{\dot{d}}^2}\right) = d/(\sqrt{2\pi} \cdot \sigma_{\dot{d}} \cdot y^2) \exp\left(-\frac{(y\dot{d} - d)^2}{2y^2 \cdot \sigma_{\dot{d}}^2}\right), \end{aligned}$$

which coincides with the posterior distribution given by Eq 5. So in this specific case, characterizing A consists in simulating the distribution of d/X . As mentioned in the introduction, this simulation based solution has been investigated in a previous work on OSL datings [\[10\]](#).

Note that in practice, the domain of validity for A is limited to a particular period of the history (often called the study period) $[a_{min}, a_{max}]$ with $0 < a_{min} < a_{max}$. This information can modeled by modifying the prior as follows:

$$P_A(a) \propto \frac{1}{a} \mathbb{I}_{[a_{min}, a_{max}]}(a). \quad (6)$$

Notice that bounding the support of the prior is motivated by the application, but is also needed to ensure the existence of the posterior moments. In the following synthetic experiments, we fix values for a_{min} and a_{max} such that they largely contain the ages of

interest: the true ages are chosen in the interval $[100; 200]$ while $[a_{min}; a_{max}]$ is set to $[1; 400]$.

Figure 1 shows how the posterior behaves as a function of σ_d . Particularly, when σ_d increases the distribution becomes more asymmetric so that its mode is shifted toward the left while its mean is shifted toward the right. This emphasizes the potential error committed using Eq. 1.

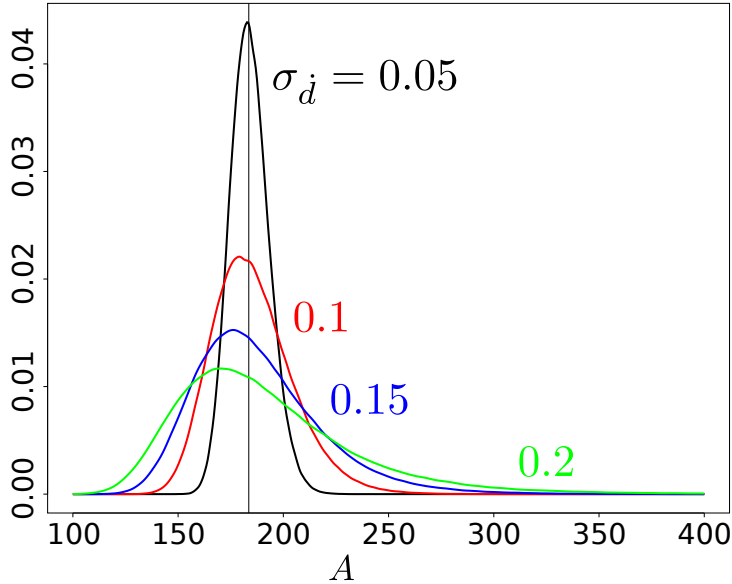


Figure 1: Posterior distribution of A for $\sigma_d = 0.05, 0.1, 0.15, 0.2$ with $\dot{d} = 1$. The vertical line represents the true value of D/\dot{d} .

2.2. Multivariate model : sequence of ages

Let then consider the model for N s ages indexed as $A_i, i \in [1, N]$. For this purpose, we need to explicit the content of σ_d . In practice, the dose-rate is estimated as combinations of physical quantities that are known with a given uncertainty. A part of this uncertainty is specific to the measured sample (*e.g.* counting errors due to the spectrometer) and another -the systematic error- is related to the measurement device and is common to all the datings of the studied sequence (*e.g.* error on the calibration of the measurements device). Without loss of generality, we will consider in the following, a simple case composed of these two sources of error (a more sophisticated instantiation leading to structurally the same model will be considered in Section 4):

$$D_i = A_i \cdot \dot{d}_i + A_i \cdot \epsilon_{d,i} + A_i \cdot \alpha_i \cdot \epsilon_{d,c} \quad (7)$$

where $\epsilon_{d,i} \sim N(0, \sigma_{d,i}^2)$ and $\epsilon_{d,c} \sim N(0, \sigma_{d,c}^2)$ and $\alpha_i > 0$ is known and gives, for each dating i , the degree of contamination to the term of systematic error. Assuming that the

noise terms $\epsilon_{d,c}$ and $\epsilon_{d,i}$ (for all i) are independent, the joint conditional distribution of the $(D_i)_i$ can be written as:

$$(D_i)_i | (A_i)_i \sim \mathcal{N}((A_i \cdot \dot{d}_i)_i, \Sigma), \quad (8)$$

where the covariance matrix Σ depends from $(A_i)_i$ and writes:

$$\Sigma_{i,i} = A_i^2(\sigma_{d,i}^2 + \alpha_i^2 \sigma_{d,c}^2), \quad (9)$$

and for $i \neq j$,

$$\Sigma_{i,j} = A_i A_j \alpha_i \alpha_j \sigma_{d,c}^2. \quad (10)$$

This covariance matrix is positive definite as long as at least N of the variances $(\sigma_{d,i}^2)_{i=1,N}$ and $\sigma_{d,c}^2$ are not null (assuming that the α_i are strictly positive). An example of design for matrix Σ from a practical situation is discussed in Section 4.1 and described in Appendix B.

Again, the Jeffreys prior for (A_i) writes:

$$P_{(A_i)_i}((a_i)_i) \propto \prod_{i=1}^N 1/a_i \text{ for } 0 < a_{min} < a_i < a_{max} \text{ and } 0 \text{ else.} \quad (11)$$

The proof is relegated in Appendix A.1. This choice of prior leads to the following posterior distribution:

$$P_{(A_i)_i | (D_i)_i}((a_i)_i | (d_i)_i) \propto P_{(D_i)_i | (A_i)_i}((d_i)_i | (a_i)_i) P_{(A_i)_i}((a_i)_i), \quad (12)$$

that will be characterized using a Metropolis-within-Gibbs sampler [15].

Figure 2 shows how the inference on the same age A_1 is modified when the number N of studied ages increases. Obviously, the studied model is different from the one in which the inferences on the ages are independent from each others, however, observing that the marginal posteriors are stable when N increases is satisfactory.

2.3. Sequence of ages with ordering constraints

In practice, the previously developed model is of few interest as none of the datings A_i interacts directly with the others and consequently the resulting inference is not so different from what we obtain from a set independent inferences. In this section, we consider the informative case where one knows that the ages are stratigraphically constrained *i.e.* they (or some of them) are in a known chronological order. This case can be implemented by adding a support constraint for the vector $(A_i)_i$ into its prior [7]. For that purpose, we consider \mathcal{B} , the set of N -uplets satisfying the imposed stratigraphy (that can be incompletely stated *i.e.* some of the ages can be not fully ordered) and state:

$$P_{(A_i)_i}((a_i)_i) \propto \prod_{i=1}^N \left(\frac{1}{a_i} \right) \mathbb{I}_{\mathcal{B}}((a_i)_i), \quad (13)$$

where $\mathbb{I}_{\mathcal{B}}((a_i)_i) = 1$ if $(a_i)_i \in \mathcal{B}$ and 0 else.

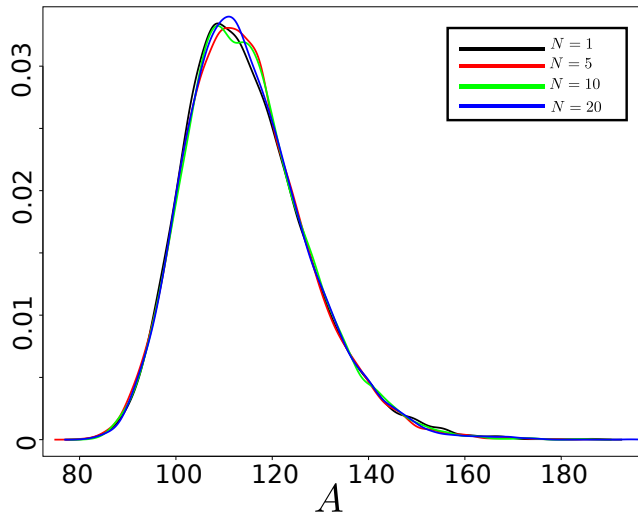


Figure 2: **Posterior distribution for the age A_1 when the number of ages in the sequence increases** ($N = 1, 5, 10, 20$). We fix for all i $\sigma_{d,i} = \alpha_i \sigma_{d,c} = 0.1$ and the \hat{d}_i are randomly draw from a $U(0.5, 1.5)$.

Moreover, it is possible to allow some of the datings of the sequence not to be affected by the systematic errors (or by different systematic terms). In practice, this situation typically happens when some of the datings are performed by different laboratories making the errors on the assessment of the dose-rates \hat{d}_i independent from each others or when some other unrelated dating methods (*e.g.* radiocarbon dating) are used. Here for a sake of simplicity, we will consider the first case. In practice, it implies to set to zero the covariance terms of Σ relating two ages/dosimetries being estimated by two different laboratories.

Let first consider the case where there is no systematic-error free dating. In the experiments displayed in Figures 3 and 4, we used $\sigma_{d,i} = 0.1$, $\alpha_i \sigma_{d,c} = 0.1$ and $(\hat{d}_i)_i$ randomly draw from a $U(0.5, 1.5)$. Figure 3 shows how the error of estimation for a given age is modified when increasing the number of stratigraphically constrained ages. Particularly, one observes that the mean and the variance error decrease slowly with the number of observations. This is related to the systematic error term that cannot be inferred without additional information. Figure 4 shows typical posteriors on a given age with an increasing number of stratigraphically constrained ages.

Let then consider the effect of adding some systematic-error free datings. In the experiments displayed in Figures 5 and 6, we consider three settings: in the first one, we consider 5 datings without any stratigraphic constraints, all affected by the systematic error term. In the second one, we consider the same 5 datings in stratigraphic constraint (again, all affected by the systematic error term). In the third one, that two of them are not affected by the systematic-error term (while keeping the same overall variance level). Figure 5 shows how the error of estimation for a given age is modified when considering these different settings. Particularly, one observes that whereas the number of datings is the same in all cases and whereas the overall error level is similar, the estimation error is,

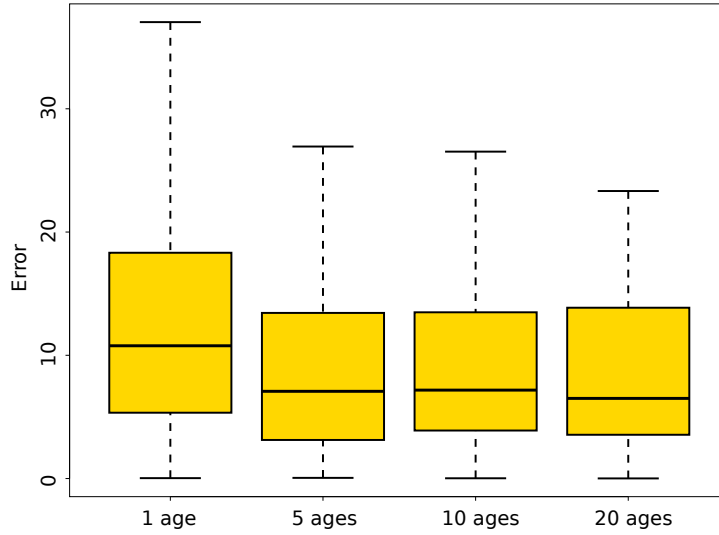


Figure 3: **Box plot of the estimation error for a given age when the number of ages in the sequence increases ($N = 1, 5, 10, 20$) over a set of 100 realizations.** From left to right, statistics on the 100 experiments for 1, 5, 10 and 20 stratigraphically constrained ages.

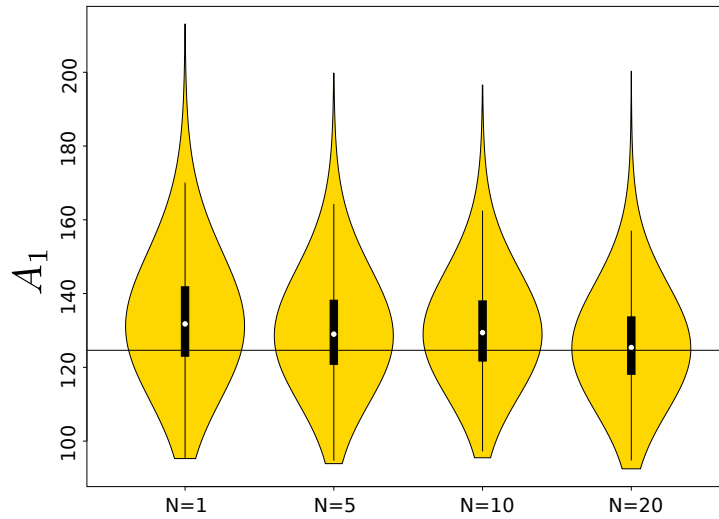


Figure 4: **Violin plot of the posterior samples for a given age when the number of ages in the sequence increases ($N = 1, 5, 10, 20$).** The horizontal line is the true value for the age.

as expected, highly reduced when considering independent datings. Figure 6 shows typical posteriors from this set of experiments. It illustrates how the posterior variance diminishes when introducing systematic-error free datings.

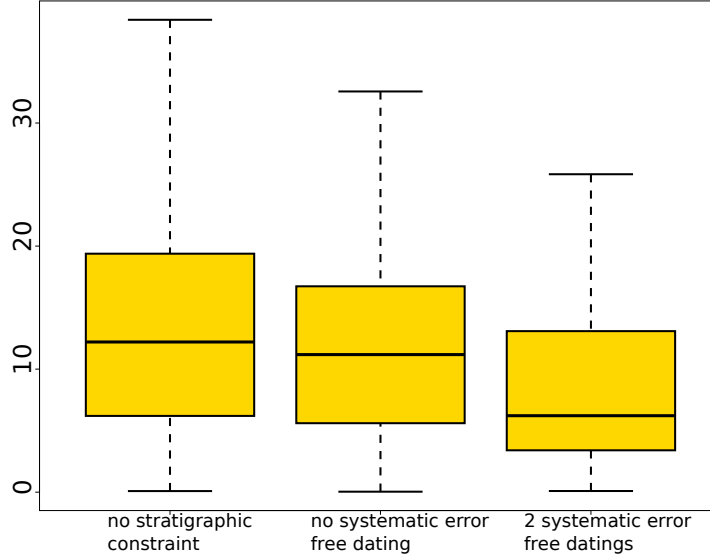


Figure 5: **Box plot of the estimation error on the same age A_3 from a set of 100 realizations.** From left to right, statistics on the 100 experiments for 5 ages no stratigraphically constrained, 5 stratigraphically constrained ages and 5 stratigraphically constrained ages whose two are free of systematic error.

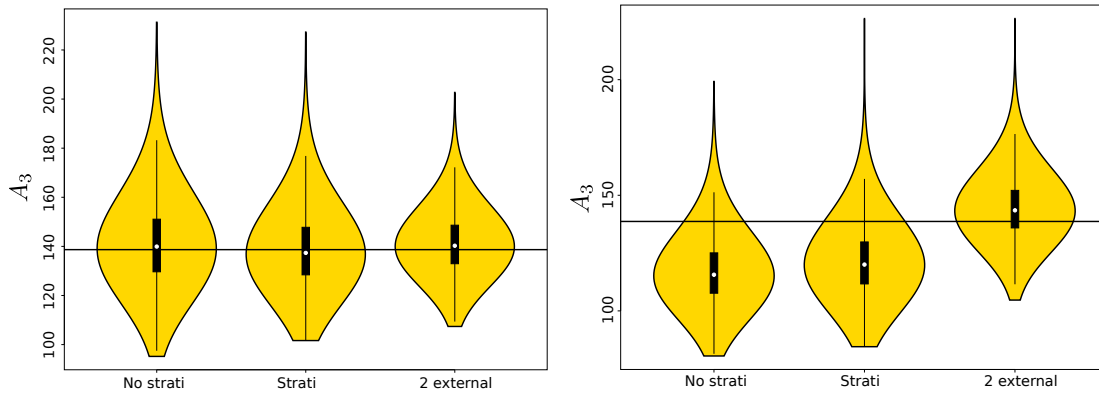


Figure 6: **Violin plots of the posterior samples for A_3 for $N = 5$ ages for two typical cases: one where the systematic component of the error is low (left plot) and one where the error is high (right plot).** The left density is the posterior when no setting stratigraphic constraint, the middle density is the posterior when setting stratigraphic constraints and only correlated ages and the right density, the posterior when setting stratigraphic constraints but with two systematic-error free datings among the five. The horizontal line is the true value for A_3 .

3. Two models for OSL dating

In this section, we consider the prior we developed in the previous section as a component of a full optically stimulated luminescence (OSL) dating model (Section 3.1). Then, we consider a two-steps approach consisting in assessing the posterior ages $(A_i)_i$ from measured characteristic equivalent doses and their standard errors $(\hat{D}_i, \sigma_{D,i})$ that are computed separately using another model (Section 3.2).

3.1. Full model

The model presented in this section is a combination of the previously developed age model with a model for the estimation of characteristic equivalent doses $(D_i)_i$ from normalised luminescence observations (denoted as $(n_i^{k,j})_{i,j,k}, (\sigma_{n_i^{k,j}})_{i,j,k}$) presented elsewhere [2]. In practice, it simply consists in adding the prior structure $P((d_i)_i|(a_i)_i)P((a_i)_i)$ as defined in Eq. 8 and 13 to the marginal model $P((d_i)_i|(n_i^{k,j})_{i,j,k}, (\sigma_{n_i^{k,j}})_{i,j,k})$. The directed acyclic graph representing the overall 4-stages hierarchical model is given in Figure 7. Notations are introduced in the caption (see the corresponding paper [2] for more details, the notations have been a bit recasted for simplicity). For a sake of completeness, we also recall here the corresponding posterior distribution (removing the exogenous parameters $\dot{d}_{lab}, t_i^{k,j}, \dot{d}_i, \sigma_{\dot{d},i}, \sigma_{\dot{d},c}, \alpha_i$ from the notations and the random variables indices from the density function for a sake of simplicity):

$$\begin{aligned} & P((a_i)_i, (d_i)_i, (\sigma_{d_i})_i, (d_i^{1,j})_{i,j}, (\sigma_{f,i}^j)_{i,j}, (\theta_i^j)_{i,j}, (q_i^{k,j})_{i,j,k} | (n_i^{k,j})_{i,j,k}, (\sigma_{n_i^{k,j}})_{i,j,k}) \\ \propto & \prod_{i,k,j} \left(P(n_i^{k,j} | q_i^{k,j}, \sigma_{n_i^{k,j}}) \right) \\ & \prod_{i,k,j} \left(P(q_i^{k,j} | \sigma_{f,i}^j, \theta_i^j, d_i^{1,j}) \right) \prod_{i,j} \left(P(d_i^{1,j} | d_i, \sigma_{d,i}) P(\sigma_{d,i} | d_i) P(\sigma_{f,i}^j) P(\theta_i^j) \right) \\ & P((d_i)_i | (a_i)_i) P((a_i)_i), \end{aligned}$$

with the age model $P((d_i)_i|(a_i)_i)P((a_i)_i)$ as defined in Eq. 8 and 13. The other densities (discussed in our previous paper) write:

$$\begin{aligned} P(n_i^{k,j} | q_i^{k,j}, \sigma_{n_i^{k,j}}) & \propto \exp - \frac{(n_i^{k,j} - q_i^{k,j})^2}{2\sigma_{n_i^{k,j}}^2} \\ P(q_i^{k,j} | \sigma_{f,i}^j, \theta_i^j, d_i^{1,j}) & \propto \frac{1}{\sqrt{2\pi}\sigma_{f,i}^j} \exp - \frac{(f_{\theta_i^j}(d_i^{k,j}) - q_i^{k,j})^2}{2\sigma_{f,i}^{j^2}}, \text{ with } d_i^{k,j} = t_i^{k,j} \cdot \dot{d}_{lab} \text{ for } k \geq 2 \end{aligned}$$

$$P(\sigma_{f,i}^j) \propto e^{-20\sigma_{f,i}^j} \mathbb{I}_{\mathbb{R}^+}(\sigma_{f,i}), \quad P(\theta_i^j = [u_i^j, v_i^j, w_i^j, x_i^j]) \propto \exp\left(\frac{(u_i^j - 6.5)^2}{170}\right) \mathbb{I}_{\mathbb{R}^+}(u^j)$$

$$\exp\left(\frac{(v_i^j - 50)^2}{10^6}\right) \mathbb{I}_{\mathbb{R}^+}(v^j) \exp\left(\frac{(w_i^j - 0.002)^2}{0.0002}\right) \mathbb{I}_{\mathbb{R}^+}(w^j) \exp\left(\frac{(z_i^j - 0.5)^2}{12.5}\right) \mathbb{I}_{[-w^j, \infty]}(z^j),$$

$$P(d_i^{1,j} | d_i, \sigma_{d,i}) \propto \left[\frac{\sigma_{d,i}}{(d_i^{1,j} - d_i)^2 + \sigma_{d,i}^2} \right], \quad P(\sigma_{d,i} | d_i) = \frac{0.16d_i}{\sigma_{d,i}^2 + (0.16d_i)^2} \mathbb{I}_{\mathbb{R}^+}(\sigma_{d,i}).$$

The inference is then performed using a Metropolis-within-Gibbs sampler. Some results will be discussed in Section 4.

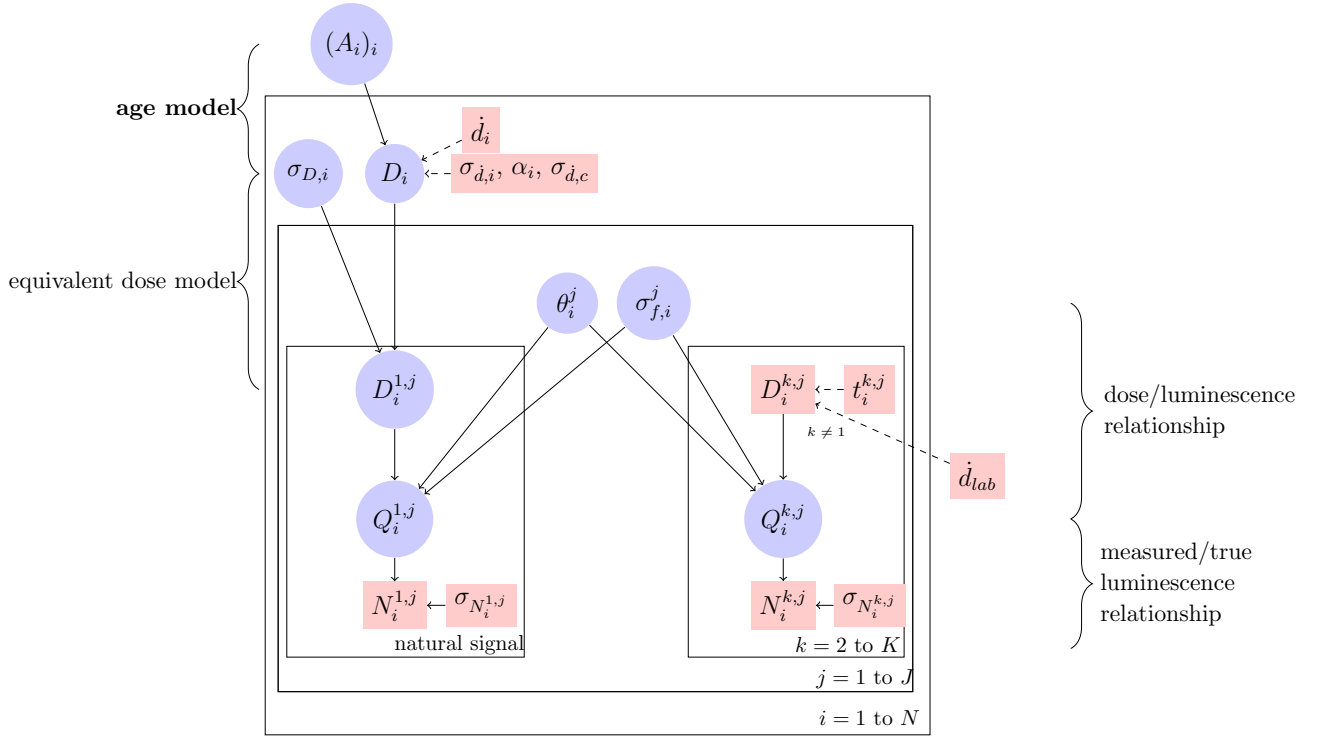


Figure 7: DAG representation of the inference model. **Sequence level:** The vector of ages $(A_i)_i$ is related to the vector of characteristic equivalent doses $(D_i)_i$ by the age model taking into account for the different kinds of errors on the dosimetry. **Sample i level:** For each sample i , the characteristic equivalent dose D_i and dispersion $\sigma_{D,i}$ are common to all aliquots of the sample. **Aliquot j level:** Each aliquot j is associated with a natural dose $D_i^{1,j}$, a set of regenerative dose $D_i^{k,j}$ ($k \geq 2$) and a dose response function $f_{\theta_i^j}$ with its error term $\sigma_{f,i}^j$. **Dose k level:** This function links the natural and regenerative doses $D_i^{k,j}$ and the true normalized signal $Q_i^{k,j}$ while each of this signal is related the corresponding observation $N_i^{k,j}$. Plain arrows represent stochastic relationships, dashed arrows represent deterministic relationships, blue circles represent model parameters and red boxes represent observations and exogenous parameters. In this DAG, we represented a single aliquot j and a single sample i level but there are many of them.

3.2. Two-steps model

In this section, we consider that we want to infer a set of ages from a set of inferred characteristic equivalent doses with their associated standard errors $\hat{D}_i, \sigma_{D,i}$, the dose-rate

values and standard errors and a set of stratigraphic constraints. This situation typically arises when the chronological scenario is built using several pool of datings published by other research groups. In such a case, the high-level results (such as the $\hat{D}_i, \sigma_{D,i}$) are commonly given in the corresponding publications but most of the times the raw luminescence data are not available. Moreover, such an approach allows users of other non-Bayesian approaches to estimate the $\hat{D}_i, \sigma_{D,i}$ (such as Galbraith's approaches [9, 8]) to account properly for ordering constraints on their data.

3.2.1. Individual age model

As for the previous model, the ages are related to the true characteristic equivalent dose through:

$$D_i = A_i \dot{d}_i + A_i \epsilon_{d,i}, \quad (14)$$

which are themselves related to the inferred characteristic equivalent dose \hat{D}_i by:

$$\hat{D}_i = D_i + \epsilon_{D,i}, \quad (15)$$

where $\epsilon_{d,i} \sim N(0, \sigma_{d,i}^2)$ and $\epsilon_{D,i} \sim N(0, \sigma_{D,i}^2)$ are independent which results in:

$$\hat{D}_i | A_i \sim \mathcal{N}(A_i \cdot \dot{d}_i, A_i^2 \cdot \sigma_{d,i}^2 + \sigma_{D,i}^2). \quad (16)$$

For this specific model, we obtain the Jeffreys prior (proof in [Appendix A.2](#)):

$$p_{A_i}(a_i) \propto \sqrt{\frac{a_i^2(2\sigma_{d,i}^4 + \dot{d}_i^2 \sigma_{d,i}^2) + \sigma_{D,i}^2 \dot{d}_i^2}{(a_i^2 \sigma_{d,i}^2 + \sigma_{D,i}^2)^2}} \text{ for } 0 < a_{min} < a_i < a_{max} \text{ and } 0 \text{ else.} \quad (17)$$

When $\sigma_{D,i} = 0$, the graph of this function (up to a multiplicative constant) is equivalent to the one of $a_i \mapsto 1/a_i$. When $\sigma_{D,i}^2 > 2\sigma_{d,i}^2$ (which will be the case for our application), the graph is strictly decreasing and decreases less sharply when $\sigma_{D,i}$ increases.

3.2.2. Multidimensional model : sequence of ages

In this section, we consider a multidimensional extension of the model presented below analogously to what we performed in Section 2. For that, we modified Eq. 14 as

$$D_i = A_i \dot{d}_i + A_i \epsilon_{d,i} + A_i \alpha_i \epsilon_{d,c}, \quad (18)$$

where $\epsilon_{d,i} \sim N(0, \sigma_{d,i}^2)$, $\epsilon_{D,i} \sim N(0, \sigma_{D,i}^2)$ and $\epsilon_{d,c} \sim N(0, \sigma_{d,c}^2)$ are independent which results in:

$$(\tilde{D}_i)_i | (A_i)_i \sim \mathcal{N}((A_i \cdot \dot{d}_i)_i, \Sigma + I_{\sigma_{D,i}^2}), \quad (19)$$

where $I_{\sigma_{D,i}^2}$ is the diagonal matrix such that $I_{i,i} = \sigma_{D,i}^2$. Then to be consistent with Eq. 17, we set the prior:

$$P_{(A_i)}((a_i)) \propto \prod \sqrt{\frac{a_i^2(2(\sigma_{d,i}^2 + \alpha_i^2 \sigma_{d,c}^2))^2 + \dot{d}_i^2(\sigma_{d,i}^2 + \alpha_i^2 \sigma_{d,c}^2) + \sigma_{D,i}^2 \dot{d}_i^2}{(a_i^2(\sigma_{d,i}^2 + \alpha_i^2 \sigma_{d,c}^2) + \sigma_{D,i}^2)^2}} \mathbb{I}_{\mathcal{B}}(a_1, \dots, a_N), \quad (20)$$

we can then write the posterior:

$$P_{(A_i)_i | (\tilde{D}_i)_i}((a_i)_i | (\tilde{d}_i)_i) \propto P_{(\tilde{D}_i)_i | (A_i)_i}((\tilde{d}_i)_i | (a_i)_i) P_{(A_i)_i}((a_i)_i), \quad (21)$$

that can be characterized using a Metropolis-within-Gibbs sampler. Some results will be discussed in Section 4.

	OSL8	OSL7	OSL6	OSL4	OSL1
\dot{d}_i (Gy)	1.40	1.30	1.25	1.33	1.13
$\sigma_{\dot{d}_i}$ (Gy)	0.03	0.03	0.03	0.03	0.03
$\alpha_{\dot{d}_i} \sigma_{\dot{d}_c}$ (Gy)	0.08	0.09	0.09	0.09	0.10

Table 1: **Summary of the quantities defining Σ for the five studied samples.**

4. Application

In this section, we use the two methods presented in Section 3.1 and 3.2 to process a set of real measurements consisting of 5 OSL datings stratigraphically constrained [18]. Before starting, below is an overview of the data.

4.1. Data and parameters

The five studied sediments respectively called OSL1, OSL4, OSL6, OSL7 and OSL8 (we kept the names given in the original publication) have been sampled from a sedimentary pile corresponding to the Middle Stone Age. This leads to the following upper and lower bound for the ages: $a_{min} = 25ka$ and $a_{max} = 280ka$. The stratigraphy imposes the ages to be ordered such that $A_1 \geq A_4 \geq A_6 \geq A_7 \geq A_8$. Details on the measurements protocol can be found in the original paper [18]. In the following, we focus our interest on how the dose-rate is typically characterized for this specific example. The dose-rate \dot{d}_i is considered as the sum of its β , γ and cosmic components (the α component is supposed to be negligible):

$$\dot{d}_i = \dot{d}_{\beta,i} + \dot{d}_{\gamma,i} + \dot{d}_{cos,i} \quad (22)$$

In practice, these three quantities are estimated through the combination of several physical quantities that are themselves not completely known. More specifically, for each sample i , one assesses values for the uranium, thorium and potassium contents U_i , Th_i and K_i , the gamma radiation component γ_i , the cosmic radiation component cos_i , the saturation uptake content W_i and the fraction of average water content F_i . These six quantities are subject to systematic and independent errors. The corresponding error model relating these quantities to the total dose-rate \dot{d}_i is then linearized to get a sum of zero-mean gaussian error terms that fully define the covariance matrix Σ involved in our multivariate equivalent dose to age model (Eq 8, 9 and 10). The resulting values for the studied sediments are given in Table 1. The derivation needed to get them are close to the ones used in practice to get $\mu_{\dot{d}}$ and $\sigma_{\dot{d}}$ in the standard approach (Eq. 1) and are detailed in Appendix B.

We also take into account for the error related to the laboratory dose source calibration (with a percent error of 5%). How to modify the covariance matrix to account for this supplemental source of error is detailed in Appendix C.

4.2. Results

4.2.1. Preliminary

As a first step, we study the full model of Section 3.1 when removing the stratigraphic constraints between the five samples. As for the simple model of Section 2.2, without

constraint, we expect the age model not to affect the rest of the inference and particularly the posteriors on the characteristic equivalent doses D_i . Figure 8 displays the posterior distribution for a given age when using 1 and 5 ages (left side) and for a given characteristic dose when using no age model, the age model with 1 age and the age model with the 5 ages (right side). One can observe that as for the non-hierarchical model, the posterior distributions display a coherent behavior: the estimations of the age and equivalent dose remain sensibly unchanged.

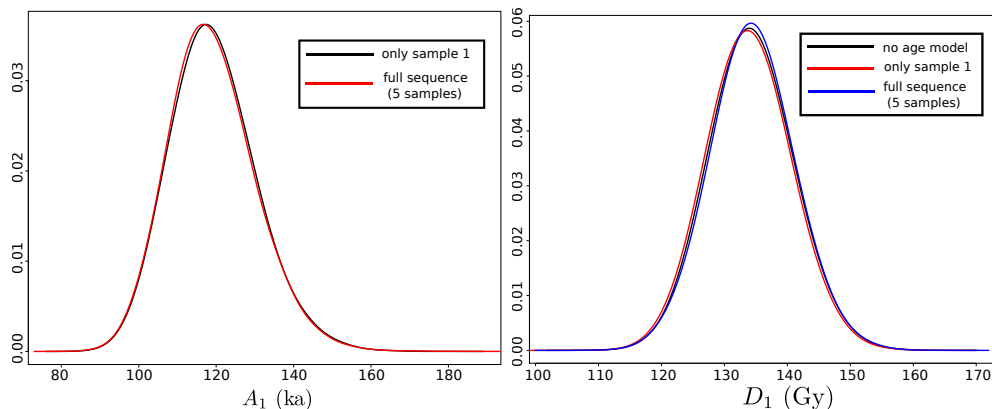


Figure 8: **Marginalized posterior densities for the age A_1 (left) and for the characteristic equivalent dose D_1 (right) when not considering stratigraphic constraints** The black curves consists of the posterior estimated using a single age and the red one of the posterior estimated using 5 ages. The characteristic equivalent dose obtained when considering only the characteristic equivalent dose model is given in blue.

4.2.2. Comparison of the approaches

Table 2 gives a summary of the obtained inferences using the full-model and the two-steps model (using the characteristic equivalent dose posteriors as inputs) when considering the 5 data-sets together or when studying them one by one (so without the stratigraphic constraints). Figure 9 displays the posterior for each age of the sequence with and without using the stratigraphic constraints. These results can be summarized as:

- the stratigraphic constraints correct significantly the datings,
- the posterior standard deviation is slightly reduced when using the stratigraphic constraints,
- the full and the two-steps models give similar inferences for the ages.

This last point is an interesting result suggesting that the two-steps method can be used with confidence when the OSL measurement data are not available. However, the dataset we used here is not so ambiguous and the use of the full-model should be preferred when possible.

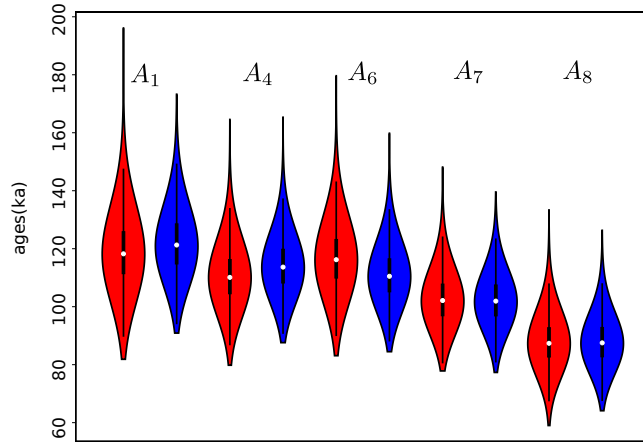


Figure 9: Violin plot from the marginalized posterior densities for the ages with stratigraphic constraints (blue) and without (red).

	.	A_8	A_7	A_6	A_4	A_1
no strati	full	88.03 ± 7.78	102.79 ± 8.32	117.38 ± 10.14	110.68 ± 9.09	118.86 ± 11.01
	two-steps	88.19 ± 7.82	102.03 ± 8.22	117.92 ± 10.07	110.86 ± 9.24	119.67 ± 11.05
strati	full	88.05 ± 7.62	102.43 ± 8.03	111.12 ± 8.65	114.31 ± 8.87	122.10 ± 10.46
	two-steps	88.45 ± 7.78	102.30 ± 8.17	111.99 ± 8.95	115.39 ± 9.25	123.01 ± 10.65

Table 2: Posterior mean and standard deviation for the 5 ages for the full and the two-steps model by setting or not the stratigraphic constraints.

5. Conclusion

In this work, we considered a multivariate model with multiplicative Gaussian error under a Bayesian framework. This model was then used as a component of an OSL dating model but also to convert directly a set of equivalent doses to the resulting ages. As mentioned in introduction, such a multiplicative model is not specific to OSL but related to any paleo-dosimetric dating method (*e.g.* Thermo-luminescence (TL), Electron Spin Resonance (ESR), Infrared stimulated luminescence (IRSL) datings) and we hope this work to be useful to these other techniques. Notice that interestingly, based on a more intuitive reasoning than the derivation we provide here, the use of the $1/A$ prior has already been evoked by Millard [13] in a general discussion about prior distributions for ages and dates in chronometric modeling. It should be mentioned here that using an uniform prior for A in this setting can provide counter-intuitive results.

The two proposed dating models were then applied to a simple real dataset to illustrate the interest of using the stratigraphic constraints when available. The inferences given by the full and the two-steps approaches are similar on this example. This is a positive result suggesting that the two-steps method can be used when the measurement data are not available. Notice that the dataset we used is unfortunately free of external dating allowing to correct the systematic components of the error. However, some data-sets would greatly benefit from this possibility (*e.g.* [14]). The two-steps model will be made available in a dedicated software to allow users to incorporate OSL datings in addition to other datings to build chronologies. The full model will also be made available to explore deeply its added-value.

To our opinion, the main limitation of our approach consists on the assumption of gaussianity of the distributions associated to the dosimetry errors. While, there is strictly speaking no reason to assume a gaussian shape for the dosimetry error, we think that this limitation is in practice a minor point. First, one can visualize the true error model on the dosimetry errors by simulation and then check that a gaussian shape is a good approximation (which has been observed to be the case [10, 21] and which is, in most cases, performed in the standard approach). Second, one must remind here that some error terms involved in the dose rate estimation are not well characterized and that in practice, only coarse (and generally highly conservative [14]) estimations are available.

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Appendix A. Construction of non-informative priors

Appendix A.1. Jeffreys prior for (θ) with $Y \sim N(d(\theta) \cdot \alpha, d(\theta) \cdot \Sigma \cdot d(\theta))$ with $\alpha = [\alpha_1, \dots, \alpha_N]^T$ and Σ positive definite known

We denote $d(\mathbf{v})$ as the diagonal matrix whose diagonal entries are v_1, \dots, v_N .

Using the fact that $d(\boldsymbol{\theta})$ is diagonal, a few algebra leads to rewrite the conditional density as:

$$f_Y(\mathbf{y}|\boldsymbol{\theta}) \propto \frac{1}{\prod_i \theta_i} \exp([d(\boldsymbol{\theta})^{-1} \Sigma^{-1} \boldsymbol{\alpha}, \mathbf{v}_{\boldsymbol{\theta}}]^T [\mathbf{y}, \mathbf{v}_{\mathbf{y}}]),$$

where

$$\mathbf{v}_{\boldsymbol{\theta}} = -0.5[\Sigma_{1,1}^{-1} 1/\theta_1^2, 2\Sigma_{1,2}^{-1} 1/\theta_1 1/\theta_2, \dots, 2\Sigma_{1,N}^{-1} 1/\theta_1 1/\theta_N, \Sigma_{2,2}^{-1} 1/\theta_2^2, \dots, 2\Sigma_{2,N}^{-1} 1/\theta_2 1/\theta_N, \dots, \Sigma_{N-1,N-1}^{-1} 1/\theta_{N-1}^2, \Sigma_{N-1,N}^{-1} 1/\theta_{N-1} 1/\theta_N, \Sigma_{N,N}^{-1} 1/\theta_N^2],$$

and

$$\mathbf{v}_{\mathbf{y}} = [y_1^2, y_1 y_2, \dots, y_1 y_N, y_2^2, y_2 y_3, \dots, y_2 y_N, \dots, y_{N-1}^2, y_{N-1} y_N, y_N^2].$$

It thus constitutes an exponential family of dimension $\frac{N(N+3)}{2}$.

By denoting \mathbf{w} the vector composed of $[1/\theta_1, \dots, 1/\theta_N]$, one can write the log-density as:

$$lf(\mathbf{y}) \propto \sum_{i=1}^N \log(w_i) - \frac{1}{2}(\mathbf{w}^T d(\mathbf{y}) \Sigma^{-1} d(\mathbf{y}) \mathbf{w}) + \boldsymbol{\alpha}^T \Sigma^{-1} d(\mathbf{y}) \mathbf{w},$$

The Hessian matrix of the log-density lf equals:

$$\frac{\partial^2 lf(\mathbf{y})}{\partial^2 \mathbf{w}} = -d(\mathbf{y}) \Sigma^{-1} d(\mathbf{y}) - d([\frac{1}{w_1^2}, \dots, \frac{1}{w_N^2}]).$$

After integration, we obtain

$$E\left[\frac{\partial^2 l f}{\partial^2 \mathbf{w}} \mid \mathbf{w}\right]_{i,j} = \begin{cases} -\frac{1}{w_i w_j} [\Sigma_{i,j}^{-1} (\Sigma_{i,j} + \alpha_i \alpha_j)], & \text{if } i \neq j \\ -\frac{1}{w_i^2} [\Sigma_{i,i}^{-1} (\Sigma_{i,i} + \alpha_i^2) + 1], & \text{if } i = j. \end{cases} \quad (\text{A.1})$$

Then,

$$I(\mathbf{w}) = -\det(E[\frac{\partial^2 l f}{\partial^2 \mathbf{w}} \mid \mathbf{w}]) = \det(d(\frac{1}{w_i}) \cdot C \cdot d(\frac{1}{w_i})) \propto \prod_{i=1}^N \frac{1}{w_i^2}, \quad (\text{A.2})$$

where C is a $N \times N$ matrix with only positive entries and that does not depend from \mathbf{w} . Finally, one gets the density in θ by :

$$p_{\Theta}(\boldsymbol{\theta}) \propto p_W([1/\theta_1, \dots, 1/\theta_N]) \prod_{i=1}^N \frac{1}{\theta_i^2} \propto \prod_{i=1}^N |\theta_i| \prod_{i=1}^N \frac{1}{\theta_i^2} = \prod_{i=1}^N \frac{1}{|\theta_i|}. \quad (\text{A.3})$$

Appendix A.2. Jeffreys prior for θ with $Y \sim N(\alpha\theta, \beta^2\theta^2 + \sigma^2)$ with α, β and σ known

The likelihood function can be write of the form

$$f_Y(y|\theta) \propto \frac{e^{-(\alpha^2\theta^2)/(2(\beta^2\theta^2+\sigma^2))}}{\sqrt{\beta^2\theta^2 + \sigma^2}} e^{(y, -y^2)^T((\alpha\theta)/(\beta^2\theta^2+\sigma^2), 1/(2(\beta^2\theta^2+\sigma^2)))},$$

so this model constitutes an exponential family of dimension 2. The log density is:

$$l f(y) = -\frac{1}{2} \log(\beta^2\theta^2 + \sigma^2) - \frac{1}{2} \log(2\pi) - \frac{1}{2} \frac{(y - \alpha\theta)^2}{\beta^2\theta^2 + \sigma^2}, \quad (\text{A.4})$$

and

$$\frac{\partial^2 l f}{\partial \theta^2} = \frac{-\beta^2 - \alpha^2}{(\sigma^2 + \theta^2\beta^2)} + \frac{2\theta^2\beta^4}{(\sigma^2 + \theta^2\beta^2)^2} + (\theta\alpha - y) \frac{4\theta\alpha\beta^2}{(\sigma^2 + \theta^2\beta^2)^2} \quad (\text{A.5})$$

$$+ (\theta\alpha - y)^2 \left(\frac{\beta^2}{(\sigma^2 + \theta^2\beta^2)^2} - \frac{4\theta^2\beta^4}{(\sigma^2 + \theta^2\beta^2)^3} \right). \quad (\text{A.6})$$

Finally,

$$I(\theta) = -E\left[\frac{\partial^2 l f}{\partial \theta^2} \mid \theta\right] = \frac{2\beta^4\theta^2}{(\beta^2\theta^2 + \sigma^2)^2} + \frac{\alpha^2}{(\beta^2\theta^2 + \sigma^2)}, \quad (\text{A.7})$$

and we obtain:

$$p(\theta) \propto \sqrt{\frac{\theta^2(2\beta^4 + \alpha^2\beta^2) + \sigma^2\alpha^2}{(\theta^2\beta^2 + \sigma^2)^2}}.$$

When setting $\sigma = 0$, one recovers $p(\theta) \propto \frac{1}{|\theta|}$.

Appendix B. Building the covariances

In this section, we show how to build the covariance matrix for a specific real example. As performed in the original publication of the studied data-set [18], we consider the dose-rate \dot{d}_i as the sum of its β , γ and cosmic components (the α component is supposed to be negligible):

$$\dot{d}_i = \dot{d}_{\beta,i} + \dot{d}_{\gamma,i} + \dot{d}_{cos,i} \quad (\text{B.1})$$

In practice, these three quantities are estimated through the combination of several physical quantities that are themselves not completely known. More specifically, for each sample i , one assesses values for the uranium, thorium and potassium contents U_i , Th_i and K_i , the gamma radiation component γ_i , the cosmic radiation component cos_i , the saturation uptake content W_i and the fraction of average water content F_i . These 6 quantities are subject to the following errors:

- Measuring each of the U_i , Th_i , K_i , γ and cosmic components of the dosimetry is subject to a zero-mean gaussian noise. Noises are independent between measurements. The corresponding standard deviations are known and called respectively, for each sample i , $\sigma_{U,i}$, $\sigma_{Th,i}$, $\sigma_{K,i}$ and $\sigma_{cos,i}$.
- Measuring the U, Th, K and γ components of the dosimetry is subject to systematic percentage errors $\epsilon_{U,c}$, $\epsilon_{Th,c}$, $\epsilon_{K,c}$ and $\epsilon_{\gamma,c}$ of same variance $\sigma_{d,c}^2$ related to the calibration of the measurement device (typically $\sigma_{d,c} = 0.1$).
- The assessment of the product $W_i F_i$ is subject to an uncertainty modeled by a zero-mean gaussian noise $\epsilon_{WF,i}$ of given standard deviation $\sigma_{WF,i}$ which is independent across samples.

Other sources of errors are considered as negligible.

The equations relating all these quantities to the β , γ and cosmic dosimetries are the following [1]:

$$\forall i \in [1, N], \dot{d}_{\beta,i} = \frac{1}{1 + \psi_{\beta}(W_i F_i + \epsilon_{WF,i})} (F_{\beta,U} S_{\beta,U} (U_i + \epsilon_{U,i})(1 + \epsilon_{U,c}) + F_{\beta,Th} S_{\beta,Th} (Th_i + \epsilon_{Th,i})(1 + \epsilon_{Th,c}) + F_{\beta,K} S_{\beta,K} (K_i + \epsilon_{K,i})(1 + \epsilon_{K,c})) \quad (\text{B.2})$$

where ψ_{β} is the absorption of the β radiation by water, $F_{\beta,U}$, $F_{\beta,Th}$ and $F_{\beta,K}$ are the β attenuation coefficients, $S_{\beta,U}$, $S_{\beta,Th}$ and $S_{\beta,K}$ are the specific beta dose rates,

$$\dot{d}_{\gamma,i} = (\gamma_i + \epsilon_{\gamma,i})(1 + \epsilon_{\gamma,c}) \frac{1 + \psi_{\gamma} W_i^c F_i^c}{1 + \psi_{\gamma}(W_i F_i + \epsilon_{WF,i})} \quad (\text{B.3})$$

where ψ_{γ} is the absorption of the γ radiation by water and W_i^c and F_i^c are respectively the saturation water uptake and the fractional average water at time of sampling,

$$\dot{d}_{cos,i} = cos_i + \epsilon_{cos,i}. \quad (\text{B.4})$$

The noise terms $\epsilon_{U,c}$, $\epsilon_{Th,c}$, $\epsilon_{K,c}$ and $\epsilon_{\gamma,c}$ are common to the N equations. Similarly to the common practice to compute \dot{d} and its associated standard error, one then considers all error terms as independent and linearizes the previous error models to get the following age to characteristic equivalent-dose model:

$$\forall i \in [1, N], D_i = A_i(\dot{d}_i + \epsilon_{\dot{d},i} + \alpha_i \epsilon_{\dot{d},c})$$

with

$$\begin{aligned} \dot{d}_i &= \frac{1}{1 + \psi_\beta W_i F_i} (r_U U_i + r_{Th} Th_i + r_K K_i) \\ &\quad \frac{1 + \psi_\gamma W_i^c F_i^c}{1 + \psi_\gamma W_i F_i} \gamma_i + \cos_i \end{aligned} \quad (\text{B.5})$$

calling $r_U = F_{\beta,U} S_{\beta,U}$, $r_{Th} = F_{\beta,Th} S_{\beta,Th}$ and $r_K = F_{\beta,K} S_{\beta,K}$ and $\epsilon_{\dot{d},i} \sim N(0, \sigma_{\dot{d},i}^2)$ with:

$$\sigma_{\dot{d},i}^2 = \frac{r_U^2 \sigma_{i,U}^2 + r_{Th}^2 \sigma_{i,Th}^2 + r_K^2 \sigma_{i,K}^2}{(1 + \psi_\beta W_i F_i)^2} + \frac{(1 + \psi_\gamma W_i^c F_i^c)^2 \sigma_{i,\gamma}^2}{(1 + \psi_\gamma W_i F_i)^2} + \quad (\text{B.6})$$

$$\sigma_{W_i F_i}^2 \left(\frac{\psi_\beta}{(1 + \psi_\beta W_i F_i)^2} (F_{\beta,U} S_{\beta,U} U_i + F_{\beta,Th} S_{\beta,Th} Th_i + F_{\beta,K} S_{\beta,K} K_i) + \quad (\text{B.7})$$

$$\begin{aligned} &\frac{\gamma_i \psi_\gamma}{(1 + \psi_\gamma W_i F_i)^2} (1 + W_i^c F_i^c) \Big)^2 \\ &+ \sigma_{\cos,i}^2 \end{aligned} \quad (\text{B.8})$$

and

$$\alpha_{\dot{d},i}^2 = \frac{F_{\beta,U}^2 S_{\beta,U}^2 U_i^2 + F_{\beta,Th}^2 S_{\beta,Th}^2 Th_i^2 + F_{\beta,K}^2 S_{\beta,K}^2 K_i^2}{(1 + \psi_\beta W_i F_i)^2} + \frac{((1 + \psi_\gamma W_i^c F_i^c) \gamma_i)^2}{(1 + \psi_\gamma W_i F_i)^2}, \quad (\text{B.9})$$

which completely defines the covariance matrix Σ .

Appendix C. Accounting for uncertainties due to the laboratory dose source

As for now, we focused on accounting for uncertainties on the natural annual dose-rate \dot{d} . Another source of uncertainty that is commonly considered to achieve an age estimation from an equivalent dose consists of the uncertainty over the laboratory source used for administering regenerative and test doses needed to assess each characteristic equivalent dose D_i . This source is only known with a given calibration error of known standard error σ_{lab} (which considering that all the samples have been processed using the same experimental device, is a systematic error). The associated error is often called a percent error and impacts D_i as a percentage of itself [1]. A convenient way to take into account for this supplemental source of error consists in simply modifying the covariance matrix as:

$$\begin{aligned} \Sigma_{i,i} &= A_i^2 (\sigma_{\dot{d},i}^2 + \alpha_i^2 \sigma_{\dot{d},c}^2 + \dot{d}_i^2 \sigma_{lab}^2), \\ \text{for } i \neq j, \Sigma_{i,j} &= A_i A_j (\alpha_i \alpha_j \sigma_{\dot{d},c}^2 + \dot{d}_i \dot{d}_j \sigma_{lab}^2). \end{aligned}$$

Acknowledgements

The authors thank Guillaume Guérin, Chantal Tribolo and Norbert Mercier for introducing us the problem and providing and discussing the data of [Section 4](#).