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# On profinite subgroups of an algebraic group over a local field

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## Abstract

The purpose of this paper is to link anisotropy properties of an algebraic group together with compactness issues in the topological group of its rational points. We find equivalent conditions on a smooth affine algebraic group scheme over a non-Archimedean local field for the associated rational points to admit maximal compact subgroups. We use the structure theory of pseudo-reductive groups provided, whatever the characteristic, by Conrad, Gabber and Prasad. We also investigate thoroughly maximal pro- $p$  subgroups in the semisimple case, using Bruhat-Tits theory.

**Keywords:** Bruhat-Tits buildings, locally compact groups, quasi-reductive groups, integral models.

**MSC classes:** 20G25 (primary), 20G40, 20E42, 22D05 (secondary).

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# 1 Introduction

Given a base field  $k$  and an affine smooth  $k$ -group denoted by  $G$ , we get an abstract group called the group of rational points, denoted by  $G(k)$ . When  $k$  is a topological field, this group inherits a topology from the field. It makes sense to link some algebraic properties of an algebraic  $k$ -group  $G$  and topological properties of its rational points  $G(k)$ . In this article, we consider a non-Archimedean local field  $k$ , hence the topological group  $G(k)$  will be totally disconnected and locally compact. Thus, one can investigate the compact, equivalently profinite, subgroups of  $G(k)$ . In the following, we denote by  $\omega$  the discrete valuation,  $\mathcal{O}_k$  the ring of integers,  $\mathfrak{m}$  its maximal ideal,  $\varpi$  a uniformizer, and  $\kappa = \mathcal{O}_k/\mathfrak{m}$  the residue field.

## 1.1 Existence of maximal compact subgroups

From the algebraic  $k$ -group  $G$ , we deduce the topological group  $G(k)$  thanks to the topology of the base field  $k$ . We would like to get a correspondence between algebraic properties of  $G$  and topological properties of  $G(k)$ . A theorem of Bruhat and Tits makes a link between anisotropy and compactness [BrTi84, 5.1.27] for reductive groups. Another link between algebra and topology is Godement's compactness criterion for arithmetic quotients of non-Archimedean Lie groups, recently extended to positive characteristic by Conrad [Con12, A5]. In the first part, we obtain further results for a general algebraic group over a local field; more precisely, we provide a purely algebraic condition on the  $k$ -group  $G$  for  $G(k)$  to admit maximal compact subgroups. The fact that this condition is non-trivial is roughly explained by the following:

*1.1.1 Examples.* Consider the additive group  $\mathbb{G}_{a,k}$ . Inside the topological group  $(k, +)$ , the subgroups  $\varpi^n \mathcal{O}_k$ , where  $n \in \mathbb{N}$  form a basis of compact open neighbourhoods of the neutral element 0. However,  $k$  is not compact and does not admit a maximal compact subgroup, since  $k$  is the union  $\bigcup_{n \in \mathbb{Z}} \varpi^n \mathcal{O}_k$  of compact subgroups. Moreover,  $(k, +)$  cannot be compactly generated.

On the opposite, consider the multiplicative group  $\mathbb{G}_{m,k}$ . The topological group  $k^\times$  has a unique maximal compact subgroup:  $\mathcal{O}_k^\times$ . Since  $k$  is assumed to be discretely valued by  $\omega : k^\times \rightarrow \mathbb{Z}$ , the topological group  $k^\times$  is compactly generated by  $\mathcal{O}_k^\times$  and an element  $x \in k^\times$  such that  $\omega(x) = 1$ .

In general, maximal compact subgroups of a reductive group are parametrised by its enlarged Bruhat-Tits building [Tit79, 3.2] (the building in [Tit79] corresponds to the enlarged building [BrTi84, 4.2.6]; see [Rou77, II.2] for more details with bounded subgroups).

In fact, the additive group is the prototype of an algebraic group which does not have a maximal compact subgroup in its rational points. More precisely:

**1.1.2 Theorem.** *Let  $k$  be a non-Archimedean local field and  $G$  a connected algebraic  $k$ -group. The topological group  $G(k)$  admits a maximal compact subgroup if, and only if,  $G$  does not contain a non-trivial connected unipotent  $k$ -split normal  $k$ -subgroup.*

*Under these conditions,  $G(k)$  is, moreover, compactly generated.*

---

We will go back to the notion of splitness for unipotent groups; it corresponds to the existence of a filtration with subgroups isomorphic to  $\mathbb{G}_a$ . In characteristic zero, all unipotent groups are split and, in fact, the above algebraic condition amounts to requiring that  $G$  be reductive. In this case, the theorem appears in [PIR94, §3.3]. Here, our theorem covers all cases and the proof, using Bruhat-Tits theory and pseudo-reductive groups, is uniform whatever the characteristic of the local field.

## 1.2 Conjugacy and description of maximal pro- $p$ subgroups

Once we know that an algebraic group  $G$  admits maximal profinite subgroups (which are exactly maximal compact subgroups), we would like to describe them more precisely. In the case of a semisimple  $k$ -group  $G$ , we can deal with integral models of  $G$  and the action of  $G(k)$  on its Bruhat-Tits building  $X(G, k)$ . Unfortunately, there are, in general, several conjugacy classes of maximal profinite subgroups (in the simply connected case, they correspond to the different types of vertices). However, the maximal pro- $p$  subgroups appear, in turn, to take the role of  $p$ -Sylow subgroups, as the following states:

**1.2.1 Theorem.** *Let  $k$  be a non-Archimedean local field of residue characteristic  $p$ . Let  $G$  be a semisimple  $k$ -group. Then,  $G(k)$  admits maximal pro- $p$  subgroups and they are pairwise conjugated.*

*Thanks to geometry of the building, given a suitable integral model  $\mathfrak{G}$  of  $G$ , we can describe one of the maximal pro- $p$  subgroups as  $\pi^{-1}(P)$  where  $\pi : \mathfrak{G}(\mathcal{O}_k) \rightarrow \mathfrak{G}(\kappa)$  comes from the reduction morphism and  $P$  is a  $p$ -Sylow subgroup of the finite group  $\mathfrak{G}(\kappa)$ .*

The choice of the integral model will be specified in Theorem 1.5.3.

## 1.3 Algebraic groups over imperfect fields

As already mentioned, we have to use the notion of a pseudo-reductive group. This notion was first introduced by Borel and Tits in [BoTi78] but was deeply studied only recently, by Conrad, Gabber and Prasad in [CGP15].

If  $k$  is any field, the unipotent radical of a smooth affine algebraic  $k$ -group  $G$ , denoted by  $\mathcal{R}_{u, \bar{k}}(G_{\bar{k}})$ , can fail to descend to a  $k$ -subgroup of  $G$  when  $k$  is imperfect. It has a minimal field of definition which is a finite purely inseparable finite extension of the base field  $k$  [CGP15, 1.1.9]. Hence, we have to replace the unipotent radical  $\mathcal{R}_u(G)_{\bar{k}}$  by the unipotent  $k$ -radical, denoted by  $\mathcal{R}_{u, k}(G)$  and defined as the maximal smooth connected unipotent normal  $k$ -subgroup of  $G$ . However, thanks to the following short exact sequence of algebraic  $k$ -groups:

$$1 \rightarrow \mathcal{R}_{u, k}(G) \rightarrow G \rightarrow G/\mathcal{R}_{u, k}(G) \rightarrow 1$$

we can understand better the algebraic  $k$ -group  $G$ . Of course, when  $k$  is perfect, this is exactly the reductive quotient of  $G$ .

Let  $G$  be a smooth connected affine  $k$ -group. One says that  $G$  is **pseudo-reductive** if  $\mathcal{R}_{u, k}(G)$  is trivial. Over perfect fields, it corresponds to reductivity,

---

but it is far from true in general. We have to face this difficulty because for a local field  $k$  of characteristic  $p$ , we have  $[k : k^p] = p$ .

Thanks to the main structural theorem of Conrad, Gabber and Prasad [CGP15, 5.1.1], we have a deeper understanding of pseudo-reductive groups. Hence, there is some hope to generalise results on reductive groups to pseudo-reductive groups and, by dévissage, to obtain general results on arbitrary connected algebraic groups. Typically, this notion enabled B. Conrad to obtain a Godement compactness criterion in terms of anisotropy for general groups over any local field (note that, until recently, standard references [Mar91] quote this criterion only for reductive groups in positive characteristic, while it was now known to be true without any reductivity condition in characteristic 0).

Thanks to the structure theory of unipotent groups provided by Tits [CGP15, B.2], we have notions of “splitness”, “isotropy” and “anisotropy” for unipotent groups. The most intriguing one is anisotropy, defined as follows.

Let  $U$  be a smooth affine unipotent  $k$ -group. One says that  $U$  is  **$k$ -wound** if there are no nonconstant  $k$ -morphisms to  $U$  from the affine  $k$ -line (where  $U$  and  $\mathbb{A}^1$  are seen as  $k$ -schemes), or equivalently if there is no nontrivial action of  $\mathbb{G}_m$  on  $U$ . Over a perfect base field, such a group has to be trivial; hence, this definition makes sense only for imperfect fields.

We recall the following definition of Bruhat and Tits [BrTi84, 1.1.12], initially introduced in a note of Borel and Tits [BoTi78].

**1.3.1 Definition.** Let  $G$  be a smooth *connected* affine  $k$ -group. One says that  $G$  is **quasi-reductive** if  $\mathcal{R}_{u,k}(G)$  is  $k$ -wound.

*1.3.2 Remark.* Because there is no nontrivial action of  $\mathbb{G}_m$  on  $\mathcal{R}_{u,k}(G)$ , no additional root appears, which preserves symmetries of the set of roots. Hence, it is possible to define a root system of a quasi-reductive group [CGP15, 3.2].

Unless stated otherwise, we assume that (from the least to the most general definition) a semisimple, reductive, pseudo-reductive or quasi-reductive  $k$ -group is connected by definition.

In Theorem 1.1.2, the algebraic  $k$ -group verifying the equivalent conditions are exactly the quasi-reductive ones.

By the same way as in the reductive case [Pra82, BTR theorem], there is a correspondence between compactness and anisotropy for unipotent groups, given by Oesterlé [Oes84, VI.1]: assume that  $k$  is a imperfect local field, then  $U$  is  $k$ -wound if, and only if,  $U(k)$  is compact.

## 1.4 The case of a topological base field

From now on,  $k$  is a local field of residual characteristic  $p$ .

If  $U$  is a connected  $k$ -split unipotent  $k$ -group, we will build, in Lemma 2.4.1 by analogy with the case of  $\mathbb{G}_a$  seen in Example 1.1.1, an exhaustion of the non-compact group  $U(k)$  by (increasing) compact open subgroups. If an algebraic  $k$ -group  $G$  contains such a  $U$  as a normal  $k$ -subgroup, then we will cover, in Proposition 2.4.2, the closed normal subgroup  $U(k)$  by compact open subgroups of  $G(k)$ .

---

Hence, such a  $G$  cannot admit a maximal compact subgroup because such a subgroup would have to contain  $U(k)$  as a closed subgroup.

Conversely, it is well-known that if  $G$  is a semisimple  $k$ -group, then  $G(k)$  has a maximal compact subgroup. Hence, we would like to prove the same fact for any quasi-reductive  $k$ -group. It is natural to exploit properness and finiteness properties of long exact sequences in Galois cohomology attached to some group extensions, but these properties are not satisfied in general. In fact, first Galois cohomology pointed sets of relevant normal subgroups of  $G$  often fail to be finite in positive characteristic (e.g.  $\#H^1(k, Z_G) = \infty$  when  $\text{char}(k) = p > 0$  and  $G = \text{SL}_p$ ; see also [CGP15, 11.3.3] for an example of a unipotent group).

Therefore cohomological methods are not sufficient to conclude. We are using topological properties of rational points. One of them is the following:

**1.4.1 Definition.** A topological group  $G$  is called **Noetherian** if it satisfies the ascending chain condition on open subgroups; this means that any sequence of increasing open subgroups of  $G$  is eventually constant.

*1.4.2 Example.* (1) The discrete abelian group  $(\mathbb{Z}, +)$  is Noetherian since any subgroup of  $\mathbb{Z}$  is an ideal of the Noetherian ring  $\mathbb{Z}$ .

(2) By Example 1.1.1, the additive group of a non-Archimedean local field is not a Noetherian group since it has an infinite strictly increasing sequence of open subgroups, namely  $(\varpi^{-n}\mathcal{O}_k)_{n \in \mathbb{N}}$ .

Because the additive topological group  $(k, +)$  (seen as the group of rational points of the additive group  $\mathbb{G}_a$ ) admits no maximal compact subgroup, there is no hope for a non- $k$ -wound unipotent group  $U$  to have a maximal compact subgroup inside its rational points. Together with Oesterlé's previously mentioned result, this is the heuristics leading to:

**1.4.3 Theorem.** *Let  $k$  be a non-Archimedean local field with residue characteristic  $p$  and  $G$  be a smooth affine  $k$ -group. The following are equivalent:*

- (i) *The identity component  $G^0$  of  $G$  is a quasi-reductive  $k$ -group,*
- (ii)  *$G(k)$  is Noetherian,*
- (iii)  *$G(k)$  admits a maximal compact subgroup,*
- (iv)  *$G(k)$  admits a maximal pro- $p$  subgroup.*

*Moreover, under the above equivalent conditions:*

(1) *Every pro- $p$  (resp. compact) subgroup of  $G(k)$  is contained in a maximal pro- $p$  (resp. compact) subgroup of  $G(k)$ .*

(2) *Every maximal pro- $p$  (resp. compact) subgroup of  $G(k)$  is open.*

**1.4.4 Corollary.** *If  $G$  is a quasi-reductive  $k$ -group, then  $G(k)$  is compactly generated.*

*Proof of corollary.* By [CM13, Lemma 3.22] a locally compact group  $G$  is Noetherian if, and only if, any open subgroup of  $G$  is compactly generated.  $\square$

---

This theorem and its corollary are well-known in the case of a  $p$ -adic field  $k$  (in that case of  $\text{char}(k) = 0$ , quasi-reductivity implies reductivity because all unipotent groups are split) as a proposition of Platonov and Rapinchuk [PIR94, 3.3 Proposition 3.15] and a theorem of Borel and Tits [BoTi65, 13.4]. In nonzero characteristic it is necessary to consider the notion of quasi-reductivity in the statement of the result.

For a reductive group  $G$  defined over a  $p$ -adic field, we know moreover that a compact open subgroup is contained in finitely many compact subgroups [PIR94, Proposition 3.16 (1)]. We don't know if this statement is still true for a quasi-reductive group over a local field of positive characteristic. In fact, when  $G(k)$  acts properly on a locally finite affine building, there is a correspondence between its compact open subgroups and the non-empty bounded subsets of the Bruhat-Tits building. In the quasi-reductive case, we have a spherical Tits system by [CGP15, C.2.20] but the existence of an affine Tits system is not yet proven.

## 1.5 Use of buildings and integral models

Though Theorem 1.4.3 gives a good criterion for the existence of maximal compact subgroups, the proof is not constructive in the sense that we do not have any detail about these subgroups. Nevertheless, in the case of a semisimple  $k$ -group  $G$ , denote by  $X(G, k)$  its Bruhat-Tits building. In Proposition 2.2.6, we get a good description of maximal compact subgroups as stabilizers of some points for the continuous action of  $G(k)$  on its Bruhat-Tits building.

As stated in Theorem 1.4.3, for a semisimple  $k$ -group  $G$ , the topological group  $G(k)$  has maximal pro- $p$  subgroups. These groups are a kind of generalisation of Sylow subgroups for a finite group: in the profinite situation, a profinite group has maximal pro- $p$  subgroups and they are pairwise conjugated [Ser94, 1.4 Prop. 3]. By our second main theorem 1.2.1, we know that the (usually non-compact) group  $G(k)$  has maximal pro- $p$  subgroups and that they are pairwise conjugated. The use of Bruhat-Tits buildings and, in particular, of Euclidean buildings associated to pairs  $(G, k)$  allows us to be more precise: we give a useful description of maximal pro- $p$  subgroups by use of a valued root groups datum in the simply-connected case. Thanks to this, in a further work [Loi16], we compute the Frattini subgroup of a maximal pro- $p$  subgroup. There will be a somewhat analogous computation as in [PrR84] where Prasad and Raghunathan compute the commutator subgroup of a parahoric subgroup.

**1.5.1 Theorem.** *Let  $k$  be a non-Archimedean local field and  $G$  a connected semisimple  $k$ -group. If  $P$  is a subgroup of  $G(k)$ , then  $P$  is a maximal pro- $p$  subgroup of  $G(k)$  if, and only if, there exists an alcove  $\mathfrak{c} \subset X(G, k)$  such that  $P$  is a maximal pro- $p$  subgroup of the stabilizer of  $\mathfrak{c}$ .*

*Moreover, such an alcove  $\mathfrak{c}$  is uniquely determined by  $P$  and the set of fixed points by  $P$  in  $X(G, k)$  is contained in the simplicial closure  $cl(\mathfrak{c})$  of  $\mathfrak{c}$ .*

*In particular, there is a natural surjective map from the maximal pro- $p$  subgroups of  $G(k)$  to the alcoves of  $X(G, k)$ . When  $G$  is simply connected, this map is a bijection.*

---

The first part of this theorem is a direct consequence of Proposition 3.1.4 and conjugation of  $p$ -Sylow subgroups in profinite groups since the stabilizer of an alcove is a profinite group by Lemma 2.2.2(3). To get a deeper description of maximal pro- $p$  subgroups, integral models and their reductions are useful.

**1.5.2 Notation.** Let  $\Omega \subset \mathbb{A}$  be a non-empty bounded subset where  $\mathbb{A}$  denotes the standard apartment of the Bruhat-Tits building  $X(G, k)$ . Denote by  $\mathfrak{G}_\Omega$  the corresponding smooth connected affine  $\mathcal{O}_k$ -model of  $G$  (denoted by  $\mathfrak{G}_\Omega^\circ$  in [BrTi84] and by  $\mathfrak{G}_\Omega$  in [Lan96]: they are the same  $\mathcal{O}_k$ -model of  $G$ , up to isomorphism, because they satisfy the same universal property). Denote by  $\mathfrak{G}_\Omega^\dagger$  the (possibly non-connected) smooth affine  $\mathcal{O}_k$ -model defined at [BrTi84, 4.6.18] for the quasi-split case and, by descent, at [BrTi84, 5.1.8] for the general case.

Recall that if  $\Omega$  satisfies a suitable notion of convexity as a subset of a polysimplicial structure (denote by  $cl(\Omega)$  the simplicial closure defined in [BrTi72, 7.1.2]), we assume here that  $\Omega = cl(\Omega)$  and  $G$  is semisimple, then  $\mathfrak{G}_\Omega^\dagger(\mathcal{O}_k)$  is the stabilizer of  $\Omega$  in  $G(k)$  [BrTi84, 4.6.29, 5.1.31]. The group  $\mathfrak{G}_\Omega(\mathcal{O}_k)$  fixes  $\Omega$  pointwise and, when  $G$  is simply-connected we have  $\mathfrak{G}_\Omega = \mathfrak{G}_\Omega^\dagger$  [BrTi84, 5.2.9]. In particular, a simply-connected semisimple  $k$ -group acts on its Bruhat-Tits building by type-preserving isometries.

In part 3.2, we will use  $\mathcal{O}_k$ -models (where  $\mathcal{O}_k$  denotes the ring of integers of  $k$ ) to get the following description:

**1.5.3 Theorem.** *Let  $k$  be a non-Archimedean local field and  $G$  a connected simply connected semisimple  $k$ -group.*

*A maximal pro- $p$  subgroup of  $G(k)$  is conjugated to*

$$P_{\mathbf{c}}^+ = \ker\left(\mathfrak{G}_{\mathbf{c}}(\mathcal{O}_k) \rightarrow \overline{\mathfrak{G}}_{\mathbf{c}}^{\text{red}}(\kappa)\right)$$

where  $\mathbf{c} \subset \mathbb{A}$  denotes an alcove of the standard apartment,  $\kappa$  denotes the residue field of  $k$  and  $\overline{\mathfrak{G}}_{\mathbf{c}}^{\text{red}}$  denotes the reductive quotient of the special fiber of the integral model associated to  $\mathbf{c}$ .

This morphism  $\mathfrak{G}_{\mathbf{c}}(\mathcal{O}_k) \rightarrow \overline{\mathfrak{G}}_{\mathbf{c}}^{\text{red}}(\kappa)$  and its kernel appear in several references like [PrY02] or [Tit79].

## 2 Maximal compact subgroups

### 2.1 Extensions of topological groups

As we consider topological groups, we require that any morphism between such groups be continuous. Recall that the morphism deduced from an algebraic morphism is always continuous for the  $k$ -topology.

#### Noetherian groups

Firstly, let us recall some properties of Noetherian groups (Definition 1.4.1).

**2.1.1 Proposition.** (1) *Any open subgroup of a Noetherian group is Noetherian.*



- 
- (2) *A compact group is Noetherian.*
- (3) *Let  $\varphi : G \rightarrow Q$  a strict (continuous) morphism between topological groups with open image (e.g.  $\varphi$  is an open morphism). If  $Q$  and  $\ker \varphi$  are Noetherian, then so is  $G$ .*
- (4) *Any extension of Noetherian groups is a Noetherian group.*
- (5) *The multiplicative group  $k^\times$  of a non-Archimedean local field  $k$  is Noetherian.*
- (6) *Let  $\psi : H \rightarrow G$  a (continuous) morphism between topological groups. If  $H$  is Noetherian and  $\psi(H)$  is a finite-index normal subgroup of  $G$ , then  $G$  is Noetherian.*

*Proof.* (1) is obvious.

(2) is clear since an open subgroup of a compact group has finite index.

(3) Since  $\text{Im}(\varphi)$  is open in  $Q$ , the subgroup  $\varphi(G)$  is Noetherian by (1). Since  $\varphi$  is a strict morphism, we may and do assume that  $\varphi$  is the quotient map  $G \rightarrow G/H \simeq \varphi(G)$  where  $H = \ker \varphi$ . Let  $(U_n)_n$  an increasing sequence of open subgroups of  $G$ . Since  $H$  is Noetherian, the sequence  $(U_n \cap H)_n$  is eventually constant, say from  $N_1 \in \mathbb{N}$ . Moreover, the sequence  $\varphi(U_n) \simeq U_n H/H$  is eventually constant, say from  $N_2 \geq N_1$ , since  $\varphi(U_n)$  is open in the Noetherian group  $\varphi(G) \simeq G/H$ . We compute  $\varphi(U_n) \simeq U_n/(U_n \cap H) \simeq U_n/(U_{N_1} \cap H) \simeq U_{N_2}/(U_{N_1} \cap H)$  for all  $n \geq N_2$ . Hence  $U_n = U_{N_2}$  for all  $n \geq N_2$ .

(4) By definition, an extension of topological groups is an exact sequence

$$1 \rightarrow H \xrightarrow{j} G \xrightarrow{\pi} Q \rightarrow 1$$

of continuous morphisms which are open on their image. Applying (3) to  $\pi$ , if  $H$  and  $Q$  are Noetherian, then so is  $G$ .

(5) is a consequence of (2) and (4) since  $k^\times$  is an extension of the compact subgroup  $\mathcal{O}_k^\times$  by the Noetherian discrete group  $\omega(k^\times) \simeq \mathbb{Z}$ .

(6) Let  $(U_n)_n$  an increasing sequence of open subgroups of  $G$ . Since  $H$  is Noetherian, the sequence of open subgroups  $\psi^{-1}(U_n)$  is eventually constant, and so is the sequence  $V_n = \psi(\psi^{-1}(U_n)) = U_n \cap \psi(H)$ . The sequence of indices  $[U_n : V_n] = [U_n : U_n \cap \psi(H)]$  is a sequence of integers bounded by the finite index  $[G : \psi(H)]$ . Moreover, since  $U_n$  is an increasing sequence and  $V_n$  is eventually constant, the sequence  $[U_n : V_n]$  is eventually increasing, hence eventually constant. As a consequence, the increasing sequence  $(U_n)_n$  is eventually constant.  $\square$

*2.1.2 Remark.* A motivation to consider the Noetherian property on topological groups is that one can easily prove the existence of maximal subgroups with a given property  $(P)$ , as soon as we know the existence of some open subgroup satisfying the desired property  $(P)$  (like in proof of [2.4.4](#)).

As an example, a Noetherian group with a strict open subgroup has maximal strict open subgroups, and any strict open subgroup is contained in, at least, one of them.

---

### Morphisms of $k$ -scheme and an exact sequence

Secondly, let us recall some properties of algebraic morphisms between topological groups of rational points.

**2.1.3 Lemma.** *Let  $k$  be a non-Archimedean local field. Let  $G$  be a smooth affine algebraic  $k$ -group and  $H$  a normal closed  $k$ -subgroup of  $G$ .*

(a) *There exists a faithfully flat quotient homomorphism  $\pi : G \rightarrow G/H$  where  $G/H$  is a smooth  $k$ -group. Moreover, when  $H$  is smooth,  $\pi$  is smooth.*

*Consider the following exact sequence :*

$$1 \rightarrow H \xrightarrow{j} G \xrightarrow{\pi} G/H \rightarrow 1$$

(b) *The exact sequence (2.1.3) induces an exact sequence of topological groups*

$$1 \rightarrow H(k) \xrightarrow{j_k} G(k) \xrightarrow{\pi_k} (G/H)(k)$$

*Moreover, if  $H$  is smooth, then the continuous morphism  $\pi_k$  is open, and  $j_k$  is a homeomorphism onto its image.*

*Proof.* (a) The quotient morphism exists and is faithfully flat by [SGA3, Exp. VI A Thm 3.2 (iv)]. Hence, the  $k$ -group  $G/H$  is smooth [DG70, II.§5 2.2]. If, moreover  $H$  is smooth, by [DG70, II.§5 5.3 and II.§5 2.2], the morphism  $\pi$  is smooth.

(b) Morphism between  $k$ -schemes of finite type are continuous for the  $k$ -topology, and  $j_k$  is a homeomorphism onto its image by definition of the  $k$ -topology. Since  $\pi$  is smooth, the continuous morphism  $\pi_k$  is open by [GGMB14, lemma 3.1.2 and proposition 3.1.4].

□

### Existence of a pro- $p$ open subgroup

By the Remark 2.1.2, we need and recall the following lemma:

**2.1.4 Lemma.** *Let  $k$  be a non-Archimedean local field of residual characteristic  $p$ . Let  $G$  be a smooth affine  $k$ -group. Then  $G(k)$  contains a pro- $p$  open subgroup.*

*Proof.* Given a closed immersion  $G \rightarrow \mathrm{GL}_{n,k}$  (such an immersion exists [DG70, II.5.5.2]), the topological group  $G(k)$  can be seen as a closed subgroup  $G(k) \subset \mathrm{GL}_n(k)$  endowed with the usual topology. Hence, it is sufficient to prove that  $\mathrm{GL}_n(k)$  contains a pro- $p$  open subgroup  $U$ , since  $U \cap G(k)$  will be a pro- $p$  open subgroup of  $G(k)$ .

The group  $H = \mathrm{GL}_n(\mathcal{O}_k)$  is profinite since it is a totally disconnected compact group. For  $d \in \mathbb{N}^*$ , define

$$H_d = \mathrm{GL}_n(\mathfrak{m}^d) = \left\{ g \in \mathrm{GL}_n(\mathcal{O}_k), g - \mathrm{id} \in \mathfrak{m}_K^d \mathcal{M}_n(\mathcal{O}_k) \right\}$$

The  $H_d$  are normal compact open subgroups of  $\mathrm{GL}_n(\mathcal{O}_k)$ , and form a basis of open neighbourhoods of  $\mathrm{id} \in H$ . Moreover, they are pro- $p$  groups in the same way as [DDMS99, 5.1] for  $\mathrm{GL}_n(\mathbb{Z}_p)$ .

---

**Claim:**  $H_1 = \varprojlim_d H_1/H_d$  is a pro- $p$ -group.

For any  $x \in H_d$ , write  $x = \text{id} + y$  where  $y \in \mathfrak{m}^d \mathcal{M}_n(\mathcal{O}_k)$ . Hence  $x^p = \text{id} + py + \sum_{k=2}^p \binom{k}{p} y^k$ . If  $k \geq 2$ , then  $y^k \in \mathfrak{m}_K^{d+1} \mathcal{M}_n(\mathcal{O}_k)$  because  $d \geq 1$ . If  $\text{char}(k) = p$ , then  $py = 0$ . Else,  $\text{char}(k) = 0$  and  $p \in \mathfrak{m}$ . Hence  $py \in \mathfrak{m}^{d+1} \mathcal{M}_n(\mathcal{O}_k)$ , so  $H_d/H_{d+1}$  is a  $p$ -group.  $\square$

## 2.2 Compact and open subgroups of a semisimple group

In this section, we assume that  $G$  is an affine smooth (connected) semisimple  $k$ -group where  $k$  is a non-Archimedean local field. In Proposition 2.2.6, we describe maximal compact subgroups as stabilizers of, uniquely defined, points of the building. This is still true if we only assume that  $G$  is reductive. We do not assume, in general, that  $G$  is simply connected and some consequences of this additional assumption will be given. Such a group  $G(k)$  acts continuously and strongly transitively on its affine Bruhat-Tits building (with a type-preserving action when  $G$  is, moreover, simply connected). We denote by  $\mathbb{A}$  the standard apartment, by  $\mathfrak{c}$  a chosen alcove in  $\mathbb{A}$  and by  $G^+$  the subgroup of  $G(k)$  consisting of the type-preserving elements.

Define  $B = \text{Stab}_{G(k)}(\mathfrak{c})$  the setwise stabilizer of  $\mathfrak{c}$  and  $B^+ = \text{Stab}_{G^+}(\mathfrak{c})$  the pointwise stabilizer of  $\mathfrak{c}$ . Define  $N = \text{Stab}_{G(k)}(\mathbb{A})$  the setwise stabilizer of  $\mathbb{A}$  in  $G(k)$  and  $N^+ = \text{Stab}_{G^+}(\mathbb{A})$  the setwise stabilizer of  $\mathbb{A}$  in  $G^+$ . Thus,  $(B, N)$  is a generalised BN-pair of  $G(k)$  (see [Gar97, 5.5 and 14.7] for details). Define  $T = B \cap N$  and  $T^+ = B^+ \cap N^+$ , and put  $W^+ = N^+/T^+$ . The set  $\Theta = T/T^+$  is finite [Gar97, 5.5] and we have a Bruhat decomposition  $G(k) = \bigsqcup_{t \in \Theta, w \in W^+} B^+ t w B^+$ . Define the

following bornology on  $G(k)$  by:

**2.2.1 Proposition-definition** ([Gar97, 14.7]). A subset  $H \subset G(k)$  is called **bounded** if  $H$  satisfies the following equivalent properties:

- (i)  $H$  is contained in a finite union of double cosets  $B^+ t w B^+$ , where  $t \in \Theta$  and  $w \in W^+$ ,
- (ii) there exists a point  $x \in X(G, k)$  such that  $H \cdot x \subset X(G, k)$  is bounded,
- (iii) for any bounded subset  $Y \subset X(G)$ , the subset  $H \cdot Y = \{h \cdot y, h \in H \text{ and } y \in Y\} \subset X(G, k)$  is bounded.

**2.2.2 Lemma.** *Under the above assumptions and notations:*

- (1) *The topological group  $N$  acts properly on  $\mathbb{A}$ .*
- (2) *For any non-empty subset  $\Omega \subset \mathbb{A}$ , the pointwise stabilizer of  $\Omega$  in  $G(k)$  is compact.*
- (3) *For any non-empty bounded subset  $\Omega \subset X(G, k)$ , the setwise stabilizer of  $\Omega$  in  $G(k)$  is compact.*

*Proof.* (1) We use the same notation as [Lan96, §1]. In particular, we consider the group  $W = N(k)/Z(k)_b$ , the group  ${}^vW = N(k)/Z(k)$  and the group  $\Lambda = Z(k)/Z(k)_b$ . Denote by  $\pi : N(k) \rightarrow W$  the quotient morphism of topological groups and by  $V = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$  the  $\mathbb{R}$ -vector space generated by the cocharacters. Consider the following commutative diagram [Lan96, 1.6]:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & W & \xrightarrow{\text{pr}} & {}^vW & \longrightarrow & 1 \\ & & \downarrow \nu_1 & & \downarrow \bar{\nu} & & \downarrow j & & \\ 0 & \longrightarrow & V & \longrightarrow & \text{Aff}(\mathbb{A}) & \xrightarrow{\text{pr}} & \text{GL}(V) & \longrightarrow & 1 \end{array}$$

from which we deduce a group action of  $N$  on the affine space  $\mathbb{A}$  [Lan96, 1.8], given by the group homomorphism  $\nu : N(k) \rightarrow \text{Aff}(\mathbb{A}) = \bar{\nu} \circ \pi$ . Let  $x \in \mathbb{A}$ . The stabilizer of  $x$  in  $\text{Aff}(\mathbb{A}) = V \rtimes \text{GL}(V)$  is the set  $\{y \mapsto g(y) - g(x) + x, g \in \text{GL}(V)\}$ . Since the above diagram is commutative, the stabilizer of  $x$  in  $\nu(W)$  is the finite set  $F_x = \{y \mapsto g(y) - g(x) + x, g \in j({}^vW)\}$ . As a consequence, the stabiliser  $\nu^{-1}(F_x)$  of  $x$  in  $N$  is compact because, when  $G$  is semisimple, the kernel of  $\nu$  is the compact subgroup  $Z(k)_b$  [Lan96, 1.2 (ii)].

(2) Using [Lan96, 12.4] notations, denote  $P_x = \langle U_x, N_x \rangle$ . The continuous map  $\mu : N_x \times U_x Z(k)_b \rightarrow P_x$  given by multiplication is a surjective homomorphism [Lan96, 12.6 (ii)]. By (1), the group  $N_x$  is compact and the group  $U_x Z(k)_b$  is compact [Lan96, 12.12 (i)], therefore  $P_x$  is compact. Hence, the pointwise stabilizer of  $\Omega$  written  $P_\Omega = \bigcap_{x \in \Omega} P_x$  [Lan96, 13.3(i) and 12.8] is compact.

(3) If  $x \in X(G, k)$ , then there exists  $g \in G(k)$  such that  $g \cdot x \in \mathbb{A}$  and it gives  $\text{Stab}_{G(k)}(x) = g^{-1} P_{g \cdot x} g$ . This does not depend on the choice of such a  $g \in G(k)$ . Consider  $\Omega \subset X(G, k)$  a non-empty subset. The pointwise stabiliser of  $\Omega$ , denoted by  $G(k)_\Omega$ , is an intersection of subgroups of  $G(k)$  of the form  $g^{-1} P_{g \cdot x} g$ ; so, it is compact by (2). The group  $G(k)_\Omega$  is also the kernel of the action of the setwise stabilizer of  $\Omega$ , denoted by  $\text{Stab}_{G(k)}(\Omega)$ , on the finite polysimplicial sub-complex of  $X(G, k)$  induced by the bounded subset  $\Omega$  (it is finite because  $X(G, k)$  is locally finite). In particular, the quotient group  $\text{Stab}_{G(k)}(\Omega)/G(k)_\Omega$  is finite. The group  $G(k)_\Omega$  is compact, and so is  $\text{Stab}_{G(k)}(\Omega)$ .  $\square$

As a consequence of this lemma, bounded subsets are closely linked to compact subsets.

**2.2.3 Lemma.** *Under the above assumptions and notations:*

- (1) *Every bounded subset of  $G(k)$  is relatively compact.*
- (2) *A subset of  $G(k)$  is compact if, and only if, it is closed and bounded.*
- (3) *Every maximal bounded subgroup of  $G(k)$  is a maximal compact subgroup.*

*Proof.* Recall that  $B^+ = P_{\mathfrak{c}}$  is compact by Lemma 2.2.2 and open in  $G(k)$  by [Lan96, 12.12 (ii)]. Hence, every double coset  $B^+ t w B^+$  is a compact open subset of  $G(k)$ .

---

(1) If  $H \subset G(k)$  is bounded, then by Definition 2.2.1(i)  $H$  is contained in a finite union of double cosets, and this union is a compact subset.

(2) If  $H$  is a compact subset of  $G(k)$ , then  $H$  is closed in  $G(k)$ . The open cover of  $H$  by double cosets has a finite subcovering. By Definition 2.2.1(i),  $H$  is bounded. Conversely, a bounded subset is compact when it is closed, by (1).

(3) If  $H$  is a maximal bounded subgroup, then  $\overline{H}$  is a closed subgroup. It is bounded by Definition 2.2.1(ii) and contains  $H$ . Hence, maximality of  $H$  implies  $H = \overline{H}$  is a maximal compact subgroup, because every compact subgroup is bounded according to (2).  $\square$

Recall that a metric space is said to be CAT(0) if it is geodesic (any two points are connected by a continuous path parametrized by distance) and if any geodesic triangle is at least as thin as in the Euclidean plane (for the same edge lengths). This notion is developed in the book of Bridson and Haefliger [BH99]. The latter condition is a non-positive curvature one (called also (NC) in [Bro89, §VI.3B]), which can also be formulated by requiring the parallelogram inequality [AB08, Prop. 11.4]. We use the following fixed-point theorem to describe compact open subgroups thanks to the metric space  $X(G, k)$ .

**2.2.4 Theorem** (Bruhat-Tits fixed point theorem [Bro89, VI.4]). *Let  $H$  acting isometrically on a complete CAT(0) metric space  $(M, d)$ . If  $M$  has a  $H$ -stable non-empty bounded subset, then  $H$  fixes a point in  $M$ .*

The following corollary is an immediate consequence of the fixed point theorem and the Definition 2.2.1(iii).

**2.2.5 Corollary.** *If  $H$  is a bounded subgroup of  $G(k)$ , then  $H$  fixes a point of  $X(G, k)$ .*

Let us give a proof of the following proposition:

**2.2.6 Proposition.** *Let  $k$  be a non-Archimedean local field and  $G$  a semisimple  $k$ -group. Let  $P$  be a subgroup of  $G(k)$ . The following are equivalent:*

- (i) *the subgroup  $P$  is a maximal compact subgroup of  $G(k)$ ,*
- (ii) *the subgroup  $P$  is the stabilizer of a point  $x \in X(G, k)$ , and if  $P$  fixes a point  $y \in X(G, k)$ , then  $y = x$ .*

*Moreover, if  $G$  is simply connected, such an  $x$  is a vertex in the simplicial complex  $X(G, k)$ .*

*Proof.* (i)  $\Rightarrow$  (ii) If  $P$  is a maximal bounded subgroup, then, by Corollary 2.2.5,  $P$  fixes a point  $x \in X(G, k)$ . Hence,  $P$  is a subgroup of  $\text{Stab}_{G(k)}(x)$  which is a bounded subgroup by 2.2.2(3). We get  $P = \text{Stab}_{G(k)}(x)$  because of the maximality assumption on  $P$ .

We have to show that the maximality of  $P$  implies that  $\text{Stab}_{G(k)}(x)$  does not fix any other point of  $X(G, k)$ . Let  $A$  be an apartment containing  $x$ . Denote by  $\mathcal{H}$  the set of walls in  $A$  and  $d(x)$  the number of walls in  $A$  containing  $x$ . Let us prove

---

by decreasing induction on  $d(x)$  that the maximal bounded subgroup  $\text{Stab}_{G(k)}(x)$  has a unique fixed point in  $A$ .

By transitivity, there exists an element  $g \in G(k)$  such that  $g \cdot A = \mathbb{A}$ . Hence, we have to show that  $g\text{Stab}_{G(k)}(x)g^{-1} = P_{g \cdot x}$  fixes a unique point of  $\mathbb{A}$ . We can and do assume that  $x \in \mathbb{A}$  and  $g = 1$ .

Denote by  $\Phi = \Phi(G, S)$  the relative root system of  $G$ . Consider the maximal case for  $d(x)$ : the point  $x$  is a special vertex of  $\mathbb{A}$  and then  $d(x) = d_{\max} = \frac{1}{2}\text{Card}(\Phi)$ . For every affine root  $a+l$  such that the associated wall contains  $x$ , the set of points of  $\mathbb{A}$  fixed by the root group  $U_{a,x}$  is exactly a half-apartment [Lan96, 13.3 (ii)]. Hence for every relative root  $a \in \Phi$ , the set of points in  $A$  fixed by  $U_{a,x}$  and  $U_{-a,x}$  is the wall associated to  $a+l$  and  $-a-l$ . The set of fixed points by  $P_x$  in  $\mathbb{A}$  is  $\{x\}$  because  $x$  is a special vertex and  $P$  contains  $U_{a,x}$  for every relative root  $a \in \Phi$  [Lan96, § 10].

Now assume that  $d < d_{\max}$ . By contradiction, assume that  $P_x$  fixes another point  $y \in \mathbb{A}$ . The action being isometric and  $[x, y]$  being metrically characterized [AB08, Prop. 11.5],  $P_x$  fixes the line segment  $[x, y]$ . If  $[x, y]$  cross a wall, we get a point  $z \in \mathbb{A}$  fixed by  $P_x$  such that  $d(z) > d(x)$ . By induction,  $P_z$  has a unique fixed point, so  $P_z$  contains strictly  $P_x$  and this contradicts the maximality of  $P_x$ . Hence,  $y$  and  $x$  are on the same facet  $F$ . If  $F \neq \{x\}$  and  $x \neq y$ , since the action is continuous and preserves the polysimplicial structure, the group  $P_x$  fixes  $\overline{F} \cap (x, y)$ . Hence  $P$  fixes a point  $z \in \overline{F} \setminus F$ . We get again  $d(z) > d(x)$  and this will contradict the maximality of  $P_x$ .

(ii)  $\Rightarrow$  (i) Conversely, let  $x \in X(G, k)$  be such that the group  $P = \text{Stab}_{G(k)}(x)$  has a unique fixed point. If  $P'$  is a bounded subgroup containing  $P$ , and  $y \in X(G, k)$  a point fixed by  $P'$ , then  $P$  fixes  $y$  and  $y = x$  because of uniqueness. Hence  $P' \subset \text{Stab}_{G(k)}(x) = P$ .

Moreover, if  $G$  is simply connected, the stabilizer of a facet fixes it pointwise [BrTi84, 5.2.9]. Because of the above equivalence, a maximal bounded subgroup is exactly the stabilizer of a vertex of  $X(G, k)$ .  $\square$

*2.2.7 Remark.* By uniqueness of the fixed point, we get an injective map from the set of maximal bounded subgroups of  $G(k)$  to the set of points in  $X(G, k)$ . Denote by  $X(G)_{\max}$  the image of this map. It is easy to remark that  $X(G, k)_{\max}$  contains the vertices of the polysimplicial complex  $X(G, k)$ .

Moreover, it is easy to see that every  $x \in X(G, k)_{\max}$  is the isobarycentre of its facet  $F$ , because the stabilizer in  $G(k)$  of  $x$  acts by isometries on  $F$  and  $x$  is the only fixed point. Be careful that the converse is not true: the stabilizer of the isobarycentre of a facet is not a maximal bounded subgroup in general.

*2.2.8 Remark.* Using the proof of Proposition 2.2.6, it is not hard to see that a compact subgroup  $H \subset G(k)$  is always contained in a maximal one. Consider a fixed point  $x \in X(G, k)$  by  $H$  of maximal degree  $d(x)$  (this does not depends on the choice of an apartment). Hence,  $H$  is contained in  $\text{Stab}_{G(k)}(x)$ . Claim:  $\text{Stab}_{G(k)}(x)$  is a maximal compact subgroup. By contradiction, if  $\text{Stab}_{G(k)}(x)$  is not, then it fixes a second point  $y$ , and then one can find a fixed point on the line  $(x, y)$  of higher degree: this contradicts the maximality of  $d(x)$ .

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Now, we need further investigation on compact open subgroups to prove Noetherianity for absolutely simple semisimple groups.

**2.2.9 Lemma.** *Let  $U$  be a compact subgroup of  $G(k)$  and denote by  $\Omega = X(G, k)^U$  the non-empty subset of points fixed by  $U$ . If  $U$  is open, then  $\Omega$  is a bounded subset of  $X(G, k)$ .*

*Proof.* By contradiction, assume that  $\Omega$  is not bounded. Let  $x_0 \in \Omega$ . Since  $\Omega$  is not bounded, one can choose a sequence  $x_n \in \Omega$  such that  $d(x_n, x_0) \geq n$ . Let  $\overline{X(G, k)}$  be a compactification of  $X(G, k)$ , defined in [RTW10]. Let  $x \in \overline{X(G, k)}$  be a limit point of  $(x_n)_n$  (it exists because  $\overline{X(G, k)}$  is a compact space by [RTW10, 3.34]).

Because  $\begin{array}{ccc} X(G, k) & \rightarrow & \mathbb{R} \\ y & \mapsto & d(x_0, y) \end{array}$  is continuous,  $x \notin X(G, k)$ . By [RTW10, 4.20 (i)], there exists a maximal  $k$ -split torus  $S'$  such that  $x_0, x \in \overline{\mathbb{A}(S', k)}$ , and one can assume that  $S' = S$ .

The group  $U$  is open in the subgroup  $\text{Stab}_{G(k)}(x_0)$ , which is compact by 2.2.2. Hence, for every relative root  $a \in \Phi$ , the intersection  $U_a(k) \cap U$  has finite index in the subgroup  $U_{a, x_0}$ . Hence,  $U$  contains  $\overline{U_{a, l_a}}$  for some  $l_a \in [f_{x_0}(a), +\infty[$ .

Because  $G(k)$  acts continuously on  $\overline{X(G, k)}$ , the point  $x \in \overline{X(G, k)}$  is fixed by  $U$ . Because  $x \notin \mathbb{A}(S, k)$ , there exists a root  $a \in \Phi$  such that  $U_{a, x} = \text{Stab}_{G(k)}(x) \cap U_a(k) = \{1\}$  by description of this stabilizer [RTW10, 4.14]. But  $U_{a, x} \supset U \cap U_a(k) \supset U_{a, l_a}$ , and we get a contradiction.  $\square$

**2.2.10 Proposition.** *Every compact open subgroup of  $G(k)$  is contained in finitely many compact (open) subgroups of  $G(k)$ .*

*Proof.* Consider a compact open subgroup  $U \subset G(k)$ . By Lemma 2.2.9, the set  $\Omega = X(G)^U$  is non-empty and bounded. By Remark 2.2.8,  $U$  is contained in a maximal compact subgroup. Since  $X(G, k)$  is locally finite (because  $k$  is a local field), by remark 2.2.7  $U$  is contained in finitely many maximal compact subgroups. Since  $U$  is open, it has finite index in any maximal compact subgroup. Hence, the set of compact subgroups containing  $U$  is finite.  $\square$

We now obtain the first step of the main theorem 1.4.3 by the following:

**2.2.11 Proposition.** *Let  $k$  be a non-Archimedean local field and  $G$  be an almost  $k$ -simple, semisimple group. Then  $G(k)$  is Noetherian.*

*Proof.* Let  $(U_n)_{n \in \mathbb{N}}$  an increasing sequence of open subgroups of  $G(k)$ . Denote by  $G(k)^+$  the normal subgroup of  $G(k)$  generated by rational points of the unipotent radical of minimal parabolic  $k$ -subgroups of  $G$  [BoTi73, 6.2].

Let us start with statements on open subgroups of  $G(k)$ . Consider an open subgroup  $U$  of  $G(k)$ .

**Claim:** If  $U \cap G(k)^+$  is bounded, then  $U$  is compact.

Indeed, assume  $U \cap G(k)^+$  is bounded. Since  $U$  is open, it is closed. The group  $G(k)^+$  is closed according to [BoTi73, 6.14]. Thus,  $U \cap G(k)^+$  is compact by 2.2.3(2). By [BoTi73, 6.14], the quotient group  $G(k)/G(k)^+$  is compact. Hence  $UG(k)^+/G(k)^+$  is a compact open subgroup of  $G(k)/G(k)^+$ . The natural bijective

continuous homomorphism  $U/(U \cap G(k)^+) \rightarrow UG(k)^+/G(k)^+$  is open and hence a homeomorphism, so  $U/(U \cap G(k)^+)$  is compact. It follows that  $U$  is compact.

**Claim:** If  $U$  is not bounded, then  $U$  contains  $G(k)^+$

Indeed, a non-bounded open subgroup is not compact. Hence,  $U \cap G(k)^+$  is a non-bounded subgroup of  $G(k)^+$ . By a theorem of Prasad, attributed to Tits [Pra82, Theorem (T)], we get  $U \supset G(k)^+$ .

Let us now finish the proof by distinguishing two cases.

**First case:**  $U_n$  is bounded (hence compact) for all  $n \in \mathbb{N}$ .

By Proposition 2.2.10,  $U_0$  is contained in finitely many compact subgroups. Hence, the increasing sequence of compact open subgroups  $(U_n)_{n \in \mathbb{N}}$  is eventually constant.

**Second case:**  $U_N$  is not bounded for some  $N \in \mathbb{N}$ .

Hence, for all  $n \geq N$ , the group  $U_n$  is not bounded and contains  $G(k)^+$ . The open subgroup  $U_N/G(k)^+$  of the compact group  $G(k)/G(k)^+$  has finite index. Hence, the sequence  $(U_n)_{n \in \mathbb{N}}$  is eventually constant.  $\square$

## 2.3 Quasi-reductive groups

### The case of a commutative quasi-reductive group

**2.3.1 Proposition.** *Let  $k$  be a non-Archimedean local field. If  $C$  is a smooth connected commutative quasi-reductive  $k$ -group, then  $C(k)$  is Noetherian.*

*Proof.* This proof follows the beginning of the proof of [Con12, 4.1.5].

Let  $S$  be the maximal  $k$ -split torus of  $C$  (it is unique by  $k$ -rational conjugacy [CGP15, C.2.3]). Consider the smooth quotient of algebraic  $k$ -groups:

$$1 \longrightarrow S \xrightarrow{j} C \xrightarrow{\pi} C/S \longrightarrow 1$$

**Claim:** The connected smooth abelian  $k$ -group  $C/S$  does not contain any subgroup isomorphic to  $\mathbb{G}_a$  or  $\mathbb{G}_m$ .

Applying [SGA3, Exp. XVII 6.1.1(A)(ii)] to the preimage in  $C$  of a subgroup isomorphic to  $\mathbb{G}_a$  (see [Con12, 4.1.4] for a more direct proof), we get a contradiction with quasi-reductiveness of  $C$ . Applying [Bor91, 8.14 Cor.] to the preimage in  $C$  of a subgroup isomorphic to  $\mathbb{G}_m$ , we get a contradiction with maximality of  $S$ .

By Lemma 2.1.3(b) and Hilbert 90 theorem, we get a short exact sequence of topological groups:

$$1 \longrightarrow S(k) \xrightarrow{j_k} C(k) \xrightarrow{\pi_k} (C/S)(k) \longrightarrow 1$$

where  $\pi_k$  is a surjective open morphism.

By [Con12, A.5.7], the topological group  $(C/S)(k)$  is compact, hence it is Noetherian by Proposition 2.1.1(2) (In this commutative case, we also have a direct proof considering the smooth quotient of  $C/S$  by its maximal  $k$ -torus, which is anisotropic). By Proposition 2.1.1(4) and (5), the topological group  $S(k) \simeq (k^\times)^n$  (where  $n = \dim S$ ) is Noetherian. Applying Proposition 2.1.1(3) to  $\pi_k$ , the topological group  $C(k)$  is Noetherian.  $\square$



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## The case of a pseudo-reductive group

Thanks to [CGP15], we have structure theorems on pseudo-reductive groups, well summarized in [Con12, §2]. In particular, there is a lot of flexibility in the choice of a (generalised) standard presentation, so that we can reduce the question of Noetherianity from pseudo-reductive groups to semisimple groups and commutative quasi-reductive groups.

**2.3.2 Lemma.** *Let  $k$  be a non-Archimedean local field and  $k'$  a nonzero finite reduced  $k$ -algebra, and write  $k' = \prod_{i \in I} k'_i$  where  $k'_i/k$  are extensions of local fields of finite degree (but possibly non-separable). Let  $G'$  be a smooth connected  $k'$ -group and denote by  $G'_i$  its fiber over the factor field  $k'_i$ . Consider the smooth connected  $k$ -group  $G = R_{k'/k}(G')$ . If each fiber  $G'_i$  is either an absolutely simple semisimple  $k'_i$ -group or a basic exotic pseudo-reductive  $k'_i$ -group, then the topological group  $G(k)$  is Noetherian.*

*Proof.* Write  $R_{k'/k}(G') = \prod_{i \in I} R_{k'_i/k}(G'_i)$  [CGP15, A.5.1]. There is a topological isomorphism  $R_{k'/k}(G')(k) \simeq \prod_{i \in I} G'_i(k'_i)$ . If each factor  $G'_i(k'_i)$  is Noetherian, then so is  $G(k)$  by Proposition 2.1.1(4).

From now on, assume that  $k'/k$  is a finite extension of local fields. It is sufficient to show that  $G'(k')$  is Noetherian.

If  $G'$  is an absolutely simple semisimple  $k'$ -group, then by Proposition 2.2.11 the topological group  $G'(k')$  is Noetherian.

Otherwise,  $G'$  is a basic exotic pseudo-reductive  $k'$ -group (see [CGP15, 7.2] or [Con12, 2.3.1] for a convenient definition). Hence we are in the case of a field with  $\text{char}(k') \in \{2, 3\}$ . Then, by [CGP15, 7.3.3, 7.3.5],  $G(k')$  is topologically isomorphic to  $\overline{G}(k')$  where  $\overline{G}$  is an absolutely simple semisimple  $k'$ -group. Hence,  $G(k')$  is Noetherian again by Proposition 2.2.11.  $\square$

**2.3.3 Proposition.** *Let  $k$  be a non-Archimedean local field and  $G$  a pseudo-reductive group. Then  $G(k)$  is Noetherian.*

*Proof.* This proof almost follows the proof of [Con12, 4.1.9], based on structure theorem of pseudo-reductive groups over a local field. Let us recall the main steps of this proof.

If  $k$  is any field of characteristic  $p \neq 2, 3$ , then a pseudo-reductive  $k$ -group is always standard according to [CGP15, 5.1.1].

If  $k$  is a local field of characteristic  $p \in \{2, 3\}$ , then we are in the convenient case of a base field  $k$  with  $[k : k^p] = p$ . Hence, by theorem [CGP15, 10.2.1],  $G$  is the direct product  $G_1 \times G_2$  of a generalised standard pseudo-reductive  $k$ -group  $G_1$  and a totally non-reduced pseudo-reductive  $k$ -group  $G_2$ . Moreover, the  $k$ -group  $G_2$  is always trivial when  $p \neq 2$ .

**First step:** Assume  $G_2$  is not trivial (hence  $\text{char}(k) = 2$ ). By [CGP15, 9.9.4], the topological group  $H(k)$ , deduced from a basic non-reduced pseudo-simple  $k$ -group  $H$  (see definition [CGP15, 10.1.2]) is topologically isomorphic to  $\text{Sp}_{2n}(K)$  for some  $n$  and an extension of local fields  $K/k$ . By Proposition 2.2.11,  $\text{Sp}_{2n}(K)$  is Noetherian, hence so is  $H(k)$ . By [CGP15, 10.1.4], the totally non-reduced  $k$ -group  $G_2$  is isomorphic to a Weil restriction  $R_{k'/k}(G'_2)$  where  $k'$  is a nonzero finite

reduced  $k$ -algebra and fibers of  $G'_2$  are basic non-reduced pseudo-simple  $k$ -groups. By Lemma 2.3.2,  $G_2(k)$  is Noetherian.

**Second step:** From now on, we can assume that  $G = G_1$  is a generalised standard pseudo-reductive  $k$ -group, together with a generalised standard presentation  $(G', k'/k, T', C)$  and  $C' = \mathcal{Z}_{G'}(T')$  where  $k'$  is a nonzero finite reduced  $k$ -algebra,  $T'$  is a maximal  $k'$ -torus of  $G'$  and  $C$  is a Cartan  $k$ -subgroup of  $G$ . Write  $k' = \prod_{i \in I} k'_i$  where  $k'_i/k$  are finite extensions of local fields. By definition of a generalised standard presentation,  $G'$  is a  $k'$ -group whose fibers, denoted by  $G'_i$ , are absolutely simple simply connected semisimple or basic exotic pseudo-reductive. Hence, by Lemma 2.3.2, the topological group  $R_{k'/k}(G')(k)$ , which is topologically isomorphic to  $\prod_{i \in I} G'_i(k'_i)$ , is Noetherian. Moreover, by Propositions 2.3.1 and 2.1.1(4), the topological group  $(R_{k'/k}(H') \rtimes C)(k)$  is Noetherian.

By definition of a standard presentation,  $G \simeq (R_{k'/k}(H') \rtimes C)/R_{k'/k}(C')$ . According to the proof of [Con12, 4.1.9] (since  $H^1(k, R_{k'/k}(C'))$  is finite), the continuous morphism between topological groups  $\pi_k : (R_{k'/k}(H') \rtimes C)(k) \rightarrow G(k)$  is open with a normal image which has finite index [Con12, 4.1.9 (4.1.2)]. Hence, by 2.1.1(6) applied to this morphism  $\pi_k$ , the group  $G(k)$  is Noetherian.  $\square$

## General case

**2.3.4 Proposition.** *Let  $k$  be a non-Archimedean local field and  $G$  be a quasi-reductive group. Then  $G(k)$  is Noetherian.*

*Proof.* Consider the pseudo-reductive quotient of  $G$  :

$$1 \longrightarrow \mathcal{R}_{u,k}(G) \longrightarrow G \xrightarrow{\pi} G/\mathcal{R}_{u,k}(G) \longrightarrow 1$$

By Lemma 2.1.3(b) one has the following exact sequence of topological groups:

$$1 \longrightarrow \mathcal{R}_{u,k}(G)(k) \longrightarrow G \xrightarrow{\pi_k} (G/\mathcal{R}_{u,k}(G))(k)$$

where the homomorphism  $\pi_k$  is open because  $\mathcal{R}_{u,k}(G)$  is smooth.

Applying [Oes84, VI.1] to the  $k$ -wound unipotent group  $\mathcal{R}_{u,k}(G)$ , the topological group  $\mathcal{R}_{u,k}(G)(k)$  is compact, hence it is Noetherian by Proposition 2.1.1(2). Applying Proposition 2.3.3 to the pseudo-reductive  $k$ -group  $G/\mathcal{R}_{u,k}(G)$ , we get that the topological group  $(G/\mathcal{R}_{u,k}(G))(k)$  is Noetherian. Hence, by Proposition 2.1.1(3), the topological group  $G(k)$  is Noetherian.  $\square$

## 2.4 Proof of the equivalence theorem

Now, there are no extra difficulties to prove Theorem 1.4.3 giving an equivalence between an algebraic property and topological ones. We prove successively (iii) or (iv)  $\Rightarrow$  (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) and (iv).

Let us prove (iii) or (iv)  $\Rightarrow$  (i).

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**2.4.1 Lemma.** *Let  $k$  be a non-Archimedean local field. If  $U$  is a smooth connected affine unipotent  $k$ -group, then  $U(k)$  is the union of an increasing sequence, indexed by  $\mathbb{Z}$ , of pro- $p$  open subgroups  $(U_n)_{n \in \mathbb{Z}}$  whose intersection is trivial.*

*Moreover, when  $U$  is not  $k$ -wound, one can assume that  $U_n$  is strictly increasing.*

*Proof.* Denote by  $\varpi$  a uniformizer of  $\mathcal{O}_k$  and, for all  $n \in \mathbb{Z}$ , denote  $\mathfrak{m}^n = \varpi^n \mathcal{O}_k \subset k$ . Denote by  $\mathbb{U}_m$  the smooth connected unipotent  $k$ -split  $k$ -group of upper triangular unipotent matrices. For  $n \in \mathbb{Z}$ , define

$$P_n = \left\{ (x_{i,j})_{1 \leq i, j \leq m}, \begin{array}{ll} x_{i,j} = 0 & \text{if } i > j \\ x_{i,j} = 1 & \text{if } i = j \\ x_{i,j} \in \mathfrak{m}^{n(i-j)} & \text{if } i < j \end{array} \right\} \subset \mathbb{U}_m(k)$$

The sequence  $(P_n)_{n \in \mathbb{Z}}$  is an increasing sequence of groups whose intersection is trivial and union is equal to  $\mathbb{U}_m(k)$ . For all  $n$ , the subgroup  $P_n$  of  $\mathbb{U}_m(k)$  is open since it contains the open neighbourhood of identity  $(1 + \mathfrak{m}^{|n|(m-1)} \mathcal{M}_m(k)) \cap \mathbb{U}_m(k)$ . And it is a pro- $p$ -group since every  $P_{n+1}$  is a normal subgroup of  $P_n$  such that the quotient  $P_n/P_{n+1}$  is a  $p$ -group.

By [Bor91, 15.5(ii)], there is a closed immersion  $U \rightarrow \mathbb{U}_m$  for some  $m \in \mathbb{N}$ . Define  $U_n = P_n \cap U(k)$ . The sequence of subgroups  $(U_n)_{n \in \mathbb{Z}}$  is increasing, the intersection  $\bigcap_{n \in \mathbb{Z}} U_n$  is trivial and the union is  $\bigcup_{n \in \mathbb{Z}} U_n = U(k)$ , because the same holds for  $(P_n)$  and  $\mathbb{U}_m(k)$ . Every  $U_n$  is a pro- $p$  subgroup of  $U(k)$  because  $U(k) \subset \mathbb{U}_m(k)$  is closed, and it is an open subgroup of  $U(k)$  because  $P_n$  is open in  $\mathbb{U}_m(k)$ .

Now, assume that  $U$  is not  $k$ -wound. Since  $U(k)$  is not compact by [Oes84, VI.1], every  $U_n$  is distinct from  $U(k)$ . Moreover,  $U_n$  is never trivial because it is open in  $U(k)$ . Hence, one can extract a strictly increasing sequence  $(U'_{\varpi(n)})_{n \in \mathbb{Z}}$  with the same properties as before.  $\square$

**2.4.2 Proposition.** *Let  $k$  be a non-Archimedean local field and  $G$  a smooth connected affine  $k$ -group. Assume that the topological group  $G(k)$  contains either a maximal pro- $p$  subgroup, or a maximal compact subgroup. Then,  $G^0$  is a quasi-reductive  $k$ -group.*

*Proof.* Denote by  $U = \mathcal{R}_{u,k}(G^0)$  the unipotent  $k$ -radical of  $G$  and by  $H$  a maximal compact or pro- $p$  subgroup of  $G(k)$ . By contradiction, let us prove that  $U$  is  $k$ -wound.

If it is not, denote by  $\mathcal{Z}_U$  the maximal smooth central  $k$ -subgroup of  $U$  (it exists, built as the maximal smooth closed  $k$ -subgroup [CGP15, C.4.2] of the center of  $U$ ). By proposition [CGP15, B.3.2], the closed  $k$ -subgroup  $\mathcal{Z}_U$  contains a closed  $k$ -subgroup  $k$ -isomorphic to  $\mathbb{G}_a$ , hence  $\mathcal{Z}_U$  is a non-trivial non- $k$ -wound  $k$ -group. By [Oes84, VI.1], the topological group  $\mathcal{Z}_U(k)$  is not compact.

Since  $\mathcal{Z}_U(k)$  is a characteristic subgroup of the normal subgroup  $U(k)$  of  $G(k)$ , it is normalised by  $H$ . By Lemma 2.4.1,  $\mathcal{Z}_U(k)$  is covered by an increasing sequence indexed by  $\mathbb{Z}$  of pro- $p$  open subgroups  $(Z_n)_{n \in \mathbb{Z}}$ . For all  $n \in \mathbb{Z}$ , define the subset  $C_n = \bigcup_{h \in H} hZ_n h^{-1}$  of  $\mathcal{Z}_U(k)$  normalised by  $H$ . The subset  $C_n$  is compact as the image of the compact set  $H \times Z_n$  by the continuous map

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$(g, h) \mapsto ghg^{-1}$ . Since  $(C_n \cap Z_m)_{m \in \mathbb{Z}}$  is an open covering of  $C_n$  by an increasing sequence, there exists some  $m_n \in \mathbb{Z}$  such that  $C_n \subset Z_{m_n}$ . Define  $P_n$  to be the closure in  $G(k)$  of the subgroup generated by  $C_n$ . It is a closed subgroup of the pro- $p$  group  $Z_{m_n}$ , hence it is a pro- $p$  group normalised by  $H$ . Hence, the subgroup  $H_n \cdot H$  of  $G(k)$ , directly generated by  $H$  and  $H_n$ , is a pro- $p$  group (as the image of a semi-direct product of pro- $p$  groups  $H_n \rtimes H$  by the surjective morphism  $H_n \rtimes H \rightarrow H_n \cdot H$  induced by multiplication [RZ10, 2.2.1(e)] and [Ser94, 1.4 Prop.4(b)]) containing  $H$ . Hence,  $H_n \subset H$  by maximality of  $H$  as a compact or pro- $p$  subgroup of  $G(k)$ . As a consequence  $H$  contains the union

$$\bigcup_{n \in \mathbb{Z}} H_n = \bigcup_{n \in \mathbb{Z}} \bigcup_{h \in H} hZ_n h^{-1} = \bigcup_{h \in H} h \left( \bigcup_{n \in \mathbb{Z}} Z_n \right) h^{-1} = \bigcup_{h \in H} h\mathcal{Z}_U(k)h^{-1} = \mathcal{Z}_U(k).$$

Since  $\mathcal{Z}_U(k)$  is a non-compact closed subgroup of  $H$ , we get a contradiction with compactness of  $H$ .  $\square$

Let us prove (i)  $\Rightarrow$  (ii).

**2.4.3 Proposition.** *Let  $k$  be a non-Archimedean local field and  $G$  a smooth affine  $k$ -group. If  $G^0$  is a quasi-reductive  $k$ -group, then  $G(k)$  is Noetherian.*

*Proof.* The identity component  $G^0$  of  $G$  is a smooth normal  $k$ -subgroup of  $G$  [DG70, II.§5 1.1 and 2.1], and the quotient  $F = G/G^0$  is a (smooth) finite  $k$ -group [DG70, II.§5 1.10].

By Lemma 2.1.3(b), we have an exact sequence of topological groups

$$1 \rightarrow G^0(k) \rightarrow G(k) \xrightarrow{\pi_k} F(k)$$

where  $\pi_k$  is an open morphism.

By Proposition 2.3.4, the topological group  $G^0(k)$  is Noetherian and  $F(k)$  is Noetherian because it is finite. As a consequence, by Proposition 2.1.1(3), the topological group  $G(k)$  is Noetherian.  $\square$

To conclude, let us finish the proof by showing that (ii)  $\Rightarrow$  (iii) and (iv).

**2.4.4 Proposition.** *Let  $k$  be a non-Archimedean local field and  $G$  a smooth affine  $k$ -group. If  $G(k)$  is Noetherian, then  $G(k)$  admits a maximal compact subgroup and a maximal pro- $p$  subgroup.*

*Proof.* By contradiction, assume that  $G(k)$  does not contain a maximal pro- $p$  (resp. compact) subgroup.

By induction, it is possible to define a strictly increasing sequence of pro- $p$  (resp. compact) open subgroups. Basis of the induction is given by Lemma 2.1.4. Induction step: since  $G(k)$  does not admit a maximal pro- $p$  (resp. compact) subgroup, given a pro- $p$  (resp. compact) open subgroup  $U_n$ , there exists a pro- $p$  (resp. compact) subgroup  $U_{n+1}$  containing  $U_n$  strictly. The group  $U_{n+1}$  is open since it contains  $U_n$ .

Such a sequence cannot exist since  $G(k)$  is Noetherian: there is a contradiction.  $\square$

Let us now prove the second part of Theorem 1.4.3

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**2.4.5 Lemma.** *Let  $k$  be a non-Archimedean local field and  $G$  a smooth affine  $k$ -group. If  $P$  is a pro- $p$  (resp. compact) subgroup of  $G(k)$ , then  $P$  is contained in a pro- $p$  (resp. compact) open subgroup of  $G(k)$ .*

*Proof.* Denote by  $U$  the pro- $p$  open subgroup of  $G(k)$  given by Lemma 2.1.4. The index  $[P : U \cap P]$  is finite since  $P$  is compact and  $U \cap P$  is open in  $P$ . Hence, the set  $\{x^{-1}Ux, x \in P\}$  is finite. Define  $U_0 = \bigcap_{x \in P} x^{-1}Ux$ . It is an open pro- $p$  subgroup of  $G(k)$  normalised by  $P$ . Hence the group  $P_0 = P \cdot U_0$  is an open subgroup of  $G(k)$ . It is compact as the image of  $P \times U_0$  by the continuous multiplication map  $G(k) \times G(k) \rightarrow G(k)$ . When, moreover,  $P$  is pro- $p$ , the group  $P_0$  is pro- $p$  as the image of the pro- $p$  group  $P \times U_0$  by the surjective multiplication homomorphism  $P \times U_0 \rightarrow P \cdot U_0$ .  $\square$

*Proof of second part of Theorem 1.4.3.* Using the same construction by induction as in proof of 2.4.4, statements (1) and (2) are a direct result from Noetherianity and Lemma 2.4.5.  $\square$

### 3 Maximal pro- $p$ subgroups of a semisimple group

Failure to compactness of maximal bounded subgroups in the group of rational points of a non-semisimple  $k$ -group involves extra difficulties to use profinite subgroups results. As an example of bad behaviour of non-semisimple groups, the maximal pro- $p$  subgroup of  $\mathbb{G}_m(k) = k^\times$  is not finitely generated when  $k = \mathbb{F}_q((t))$ . From now on, we reduce our study to the case of a semisimple  $k$ -group  $G$  and we only consider smooth affine  $k$ -groups, that we will call algebraic  $k$ -group.

The conjugacy theorem 1.2.1 is the generalisation to arbitrary characteristic of [PIR94, Theorem 3.10], which Platonov and Rapinchuk prove in characteristic 0 and attribute to Matsumoto. The proof is given in part 3.1, using Bruhat-Tits buildings instead of maximal orders.

Furthermore, as we obtained a description of maximal profinite subgroups of  $G(k)$  in Proposition 2.2.6, Theorem 1.5.1 establishes an analogous description of maximal pro- $p$  subgroups. It is proven in part 3.3. In practice, description by integral models established in Theorem 1.5.3 are more convenient; it is proven in part 3.2.

#### 3.1 Proof of the conjugacy theorem

Let us first investigate the case of an algebraic group defined over a finite field. This case corresponds to special fibers of integral  $\mathcal{O}_k$ -models (these models are useful in order to make a description of profinite subgroups).

**3.1.1 Lemma.** *Let  $k$  be a finite field of characteristic  $p$ . Let  $H$  be a connected algebraic  $k$ -group. Thus,  $H$  has Borel subgroups defined over  $k$  (Lang's theorem). The  $p$ -Sylow subgroups of the finite group  $H(k)$  are exactly the groups  $B_u(k)$  where  $B$  is a Borel subgroup of  $H$  defined over  $k$  and  $B_u$  is the unipotent radical of  $B$ .*

*Moreover, the map  $B \mapsto B_u(k)$  is a bijection between the set of Borel  $k$ -subgroups of  $H$  and the set of  $p$ -Sylow subgroups of  $H(k)$ .*

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*Proof.* Denote by  $q$  the cardinal of  $k$ . Let  $P$  be a  $p$ -Sylow subgroup of  $H(k)$ . Let  $g \in P$  and  $g = g_s \cdot g_u$  the Jordan decomposition of  $g$ . Since  $H$  is affine, there exists an integer  $n \in \mathbb{N}^*$  and a faithful linear representation  $\rho : H \hookrightarrow GL_{n,k}$  [Bor91, 5.1] such that  $\rho(g_s) = \rho(g)_s$ . Hence, the order of this element divides  $(q-1)^n$ , so it is prime to  $p$ . As a consequence  $g = g_u$ . Hence  $P$  consists in unipotent elements of  $H(k)$ . Since  $k$  is perfect and  $H$  is connected, by [BoTi71, 3.7], there exists a Borel  $k$ -subgroup  $B$  such that  $P$  is contained in the group of rational points of the unipotent radical of  $B$ , denoted by  $B_u(k)$ . Since  $k$  is perfect,  $B_u$  is  $k$ -split. Hence,  $B_u(k)$  is a  $p$ -group. Since  $P$  is a  $p$ -Sylow subgroup of  $H(k)$ , so  $P = B_u(k)$  by maximality.

Since the Borel subgroups are  $H(k)$ -conjugated [Bor91, 16.6], and since the  $p$ -Sylow subgroups of the finite group  $H(k)$  are  $H(k)$ -conjugated, we obtain a surjective map  $\Psi : B \mapsto B_u(k)$  between Borel  $k$ -subgroups of  $H$  and  $p$ -Sylow subgroups of  $H(k)$ . Let us show that it is a bijective map.

Fix  $B$  a Borel  $k$ -subgroup of  $H$  and  $S$  a maximal  $k$ -split torus of  $B$ , hence of  $H$ . Define  $T = \mathcal{Z}_H(S)$ , it is a maximal torus of  $H$  defined over  $k$  since an algebraic group over a finite field is quasi-split. Since  $k$  is perfect, the unipotent radical of  $B$  is  $k$ -split [BrTi84, 1.1.11]. The  $k$ -group  $B$  has a Levi decomposition  $B = T \cdot B_u$  [CGP15, C.2.4].

On the one hand, since  $H(k)$  acts by conjugation on the set of Borel  $k$ -subgroups of  $H$ , the number of Borel  $k$ -subgroups is equal to the cardinal of  $H(k)/\mathcal{N}_{H(k)}(B)$ . By a theorem due to Chevalley [Bor91, 11.16], a Borel subgroup of  $H$  is equal to its normalizer, hence  $\mathcal{N}_{H(k)}(B) = B(k)$ . On the other hand, since  $H(k)$  acts by conjugation on the set of its  $p$ -Sylow subgroups, the number of its  $p$ -Sylow subgroups is equal to the cardinal of  $H(k)/\mathcal{N}_{H(k)}(B_u(k))$ .

Hence, it suffices to show  $\mathcal{N}_{H(k)}(B_u(k)) = B(k)$ . Denote by  $N = \mathcal{N}_H(S)$  the normalizer of  $S$  in  $H$ . Since  $N$  normalises  $T$ , we get that  $N(k)$  normalises  $T(k)$ . Since  $B(k) = T(k)B_u(k) = B_u(k)T(k)$ , by [CGP15, C.2.8], we get  $G(k) = B_u(k)N(k)B_u(k)$ . Let  $g \in \mathcal{N}_{H(k)}(B_u(k)) \subset H(k)$ . Write  $g = unu'$  with  $u, u' \in B_u(k)$  and  $n \in N(k)$ . By contradiction, suppose that  $n \notin T(k)$ . Thus the Weyl group  ${}_k W = N(k)/T(k)$  is not trivial, hence the group  $H$  is not solvable and admits opposite root subgroups [Spr98, 7.1.3, 7.1.5 and 7.2], which are  $k$ -split since  $k$  is perfect [Bor91, 15.5 (ii)]. Hence there exists  $u \in B_u(k)$  such that  $n^{-1}un \notin B_u(k)$ . This contradicts  $n \in \mathcal{N}_{H(k)}(B_u(k))$ . Hence  $\mathcal{N}_{H(k)}(B_u(k)) \subset B(k)$ .

As a consequence, the equality  $\mathcal{N}_{H(k)}(B_u(k)) = \mathcal{N}_{H(k)}(B) = B(k)$  completes the proof.  $\square$

*3.1.2 Remark.* The bijective correspondence between Borel  $k$ -subgroups of  $H$  and  $p$ -Sylow subgroups of  $H(k)$  is useless in what follows. We only need to know that the number of Borel  $k$ -subgroups is prime to  $p$  (that is also a consequence of Bruhat decomposition).

More precisely, from this proof we get that the normalizer of a  $p$ -Sylow subgroup of  $H(k)$  is exactly  $B(k)$ . Over a local field instead of a finite field, this will be generalised by Proposition 3.3.3 with a simple connectedness assumption: normalizers of a maximal pro- $p$  subgroups are exactly Iwahori subgroups.

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When a  $p$ -group acts on a finite set of cardinal prime to  $p$ , orbit-stabilizer theorem gives the existence of a fixed point. This statement can be generalised to the action of a pro- $p$  group.

**3.1.3 Lemma.** *Let  $p$  be a prime and  $X$  a finite set of cardinal prime to  $p$ . If  $G$  is a pro- $p$  group acting continuously on  $X$ , then  $G$  fixes an element of  $X$ .*

*Proof.* For all  $x \in X$ , denote by  $G_x$  the stabilizer of  $x$ . Since  $X$  is finite,  $G_x$  is open. Let  $H = G_X = \bigcap_{x \in X} G_x$  be the subgroup of  $G$  fixing  $X$  pointwise. Then  $H$  is a normal open subgroup of  $G$ . Hence  $G/H$  is a  $p$ -group acting on  $X$ . By the orbit-stabilizer theorem,  $G/H$  fixes an element  $x \in X$ . Hence  $G$  fixes  $x$ .  $\square$

Since a profinite subgroup is compact, by Bruhat-Tits fixed point theorem, such a subgroup of  $G(k)$  fixes a point  $x_0 \in X(G, k)$ . Since the action of  $G(k)$  preserves the structure of the simplicial complex, we get an action on the star of  $x_0$ , that means an action on the set of facets whose closure contains  $x_0$ . Showing that the subset of alcoves of this set is a finite set of cardinal prime to  $p$ , we will get the following:

**3.1.4 Proposition.** *A pro- $p$  subgroup of  $G(k)$  setwise stabilises an alcove of  $X(G, k)$ .*

*Proof.* Let  $U$  be a pro- $p$  subgroup of  $G(k)$ . By Proposition 2.2.6, there exists a point  $y \in X(G, k)$  such that  $\text{Stab}_{G(k)}(y)$  is a maximal compact subgroup of  $G(k)$  containing  $U$ . Consider the (non-empty) set  $\mathcal{C}_y$  of alcoves of  $X(G, k)$  whose closure contains  $y$ . Be careful that we forget the Euclidean structure provided by  $X(G, k)$  and we only look at  $\mathcal{C}_y$  as a discrete set.

Denote by  $F$  the facet of  $X(G, k)$  containing  $y$ . By conjugation, assume that  $F \subset \mathbb{A}$ . Define the star of  $F$ , denoted by  $X(G, k)_F$ , as the set of facets  $F'$  of  $X(G, k)$  such that  $F \subset \overline{F'}$ . We endow this set with the partial order  $F' \leq F'' \Leftrightarrow F' \subset \overline{F''}$ . Denote by  $\mathfrak{G}_F$  the connected integral model of  $G$  associated to  $F$  (see definition in chapters [BrTi84, 4.6 and 5.1]). Denote by  $\kappa$  the residue field of  $k$  and consider  $\mathcal{P}_F$  the set of  $\kappa$ -parabolic subgroups of  $\overline{\mathfrak{G}_F}$  ordered by the inverse of the inclusion. There is an isomorphism of ordered sets between  $X(G)_F$  and  $\mathcal{P}_F$  [BrTi84, 4.6.32 et 5.1.32 (i)] such that maximal simplices of  $X(G)_F$  are exactly the elements of  $\mathcal{C}_y$ , and the minimal parabolic  $\kappa$ -subgroups of  $\overline{\mathfrak{G}_F}$  correspond to them bijectively. By Lang's theorem [Bor91, 16.6], the minimal parabolic  $\kappa$ -subgroups of  $\overline{\mathfrak{G}_F}$  are exactly its Borel  $\kappa$ -subgroups. By Lemma 3.1.1, we obtain a bijection between  $\mathcal{C}_y$  and the set of  $p$ -Sylow subgroups of  $\mathfrak{G}_F(\kappa)$ .

Since  $G(k)$  preserves the poly-simplicial structure of  $X(G, k)$  and  $U$  fixes  $y$ , the group  $U$  acts on  $\mathcal{C}_y$ . For all  $\mathbf{c}, \mathbf{c}' \in \mathcal{C}_y$ , by continuity of the action  $G(k) \times X(G, k) \rightarrow X(G, k)$ , the subset  $\{g \in U, g \cdot \overline{\mathbf{c}} = \overline{\mathbf{c}'}\}$  is closed in  $U$ . As a consequence,  $U$  acts continuously on the finite set  $\mathcal{C}_y$ , whose cardinal is congruent to 1 modulo  $p$ . By Lemma 3.1.3,  $U$  fixes an alcove  $\mathbf{c} \in \mathcal{C}_y$ , hence  $U$  setwise stabilises it in  $X(G, k)$ .  $\square$

We now can give a proof of conjugation of maximal pro- $p$  subgroup theorem.

---

*Proof of Theorem 1.2.1.* Let  $U, U'$  be two maximal pro- $p$  subgroups of  $G(k)$ . Let  $\mathbf{c}, \mathbf{c}'$  be alcoves stabilized by the action of  $U$  and  $U'$  respectively (they exist by Proposition 3.1.4). Since  $G(k)$  acts transitively on the set of alcoves of  $X(G, k)$ , there exists an element  $g \in G(k)$  such that  $g \cdot \mathbf{c}' = \mathbf{c}$ . Hence  $gU'g^{-1}$  stabilises  $\mathbf{c}$ . As a consequence,  $U$  and  $gU'g^{-1}$  are two maximal pro- $p$  subgroups of  $P = \text{Stab}_{G(k)}(\mathbf{c})$  which is compact by Lemma 2.2.2(3). Hence,  $U$  et  $gU'g^{-1}$  are two  $p$ -Sylow subgroups of the profinite group  $P$ . By conjugation of  $p$ -Sylow subgroups theorem [Ser94, 1.4 Prop. 3],  $U$  and  $gU'g^{-1}$  are conjugated in  $P$ , so  $U$  and  $U'$  are conjugated in  $G(k)$ .  $\square$

We now need to use root groups and integral models to prove the uniqueness of the alcove setwise stabilized by a given maximal pro- $p$  subgroup. Theorem 1.5.1 will be proven in part 3.3.

## 3.2 Integral models

In the proof of Proposition 3.1.4, integral models were used; here, we will make a more systematic use of them.

Let  $\Omega$  a non-empty bounded subset of the standard apartment  $\mathbb{A}$ . Denote by  $\pi_\kappa : \mathfrak{G}_\Omega^\dagger(\mathcal{O}_\kappa) \rightarrow \mathfrak{G}_\Omega^\dagger(\kappa)$  the canonical reduction map. Denote by  $\overline{\mathfrak{G}}_\Omega = \left( \mathfrak{G}_\Omega^\dagger \right)_\kappa$  the special fiber. Denote by  $(\overline{\mathfrak{G}}_\Omega)^\circ$  the identity component of the  $\kappa$ -group  $\overline{\mathfrak{G}}_\Omega$ , and by  $R_u(\overline{\mathfrak{G}}_\Omega)$  its unipotent radical, defined over  $\kappa$  because  $\kappa$  is perfect [BoTi65, 0.7]. Denote by  $\overline{\mathfrak{G}}_\Omega^{\text{red}} = \overline{\mathfrak{G}}_\Omega / R_u(\overline{\mathfrak{G}}_\Omega)$  the quotient  $\kappa$ -group (possibly non-connected since  $\overline{\mathfrak{G}}_\Omega$  may be not connected). The root system of its identity component is the set  $\Phi_\Omega$  of roots  $a \in \Phi$ , where  $\Phi$  denotes the relative root system of  $G$ , such that the root  $a$  seen as an affine map is constant over  $\Omega$  and has values in the set  $\Gamma'_a$  [Lan96, 10.36]. Note that, when  $\Omega$  contains an alcove, no root of  $\Phi$  is constant on  $\Omega$  since an alcove of  $\mathbb{A}$  is open in  $\mathbb{A}$ , hence  $\Phi_\Omega$  is empty.

Denote by  $\pi_q : \overline{\mathfrak{G}}_\Omega \rightarrow \overline{\mathfrak{G}}_\Omega^{\text{red}}$  the quotient  $\kappa$ -morphism of algebraic  $\kappa$ -groups, and, by notation abuse,  $\pi_q : \mathfrak{G}_\Omega^\dagger(\kappa) \rightarrow \overline{\mathfrak{G}}_\Omega^{\text{red}}(\kappa)$  the homomorphism of abstract groups deduced from  $\pi_q$ . It will be clear from the context which of these two morphisms will be considered.

**3.2.1 Notation.** Identifying the abstract groups  $\mathfrak{G}_\Omega^\dagger(\kappa) = \overline{\mathfrak{G}}_\Omega(\kappa)$ , we can define the composite morphism  $\pi_\Omega = \pi_q \circ \pi_\kappa$ . Denote by  $P_\Omega^+$  the kernel of  $\pi_\Omega$ .

More specifically, if  $F$  is a facet of the building  $X(G, k)$ , by transitivity, there exists an element  $g \in G(k)$  such that  $g \cdot F \subset \mathbb{A}$ . Denote  $P_F^+ = g^{-1}P_{g \cdot F}^+g$ . This group does not depend on the choice of such a  $g$ .

The goal is to show that, when  $G$  is simply connected,  $P_F^+$  is a maximal pro- $p$  subgroup of the profinite (by Lemma 2.2.2(3)) subgroup  $\text{Stab}_{G(k)}(F)$ . Note that with this notation, it is not required that the facet  $F$  be contained in the standard apartment  $\mathbb{A}$ .

**3.2.2 Lemma.** *The morphism  $\pi_\kappa$  is a surjective group homomorphism and its kernel  $\ker \pi_\kappa$  is a pro- $p$  group.*



*Proof.* Surjectivity of  $\pi_\kappa$  is a consequence of smoothness of the  $\mathcal{O}_k$ -model  $\mathfrak{G}_\Omega^\dagger$  [BLR90, 2.3 Prop. 5].

The smooth affine  $\mathcal{O}_k$ -group of finite type  $\mathfrak{G}_\Omega^\dagger$  has a faithful linear representation, that means a closed immersion,  $\rho : \mathfrak{G}_\Omega^\dagger \rightarrow \mathcal{GL}_{n, \mathcal{O}_k}$  for which it corresponds a surjective Hopf  $\mathcal{O}_k$ -algebras homomorphism  $\varphi : A \twoheadrightarrow B$  where  $A$  and  $B$  denote respectively the  $\mathcal{O}_k$ -Hopf algebras of  $\mathcal{GL}_{n, \mathcal{O}_k}$  and  $\mathfrak{G}_\Omega^\dagger$ . Denote by  $\widetilde{\pi}_\kappa : \mathcal{GL}_{n, \mathcal{O}_k}(\mathcal{O}_k) \rightarrow \mathcal{GL}_{n, \mathcal{O}_k}(\kappa)$  the canonical surjective homomorphism (defined as  $\pi_\kappa$  above). Hence  $\ker \pi_\kappa = \{f : B \rightarrow \mathcal{O}_k, f \otimes 1 = \varepsilon \otimes 1\}$  and  $\ker \widetilde{\pi}_\kappa = \{f : A \rightarrow \mathcal{O}_k, f \otimes 1 = \widetilde{\varepsilon} \otimes 1\}$  where  $\varepsilon$  (resp.  $\widetilde{\varepsilon}$ ) is the co-unit of  $B$  (resp.  $A$ ).

On  $\mathcal{O}_k$  points, we have  $\ker \widetilde{\pi}_\kappa = \mathrm{GL}_n(\mathfrak{m})$ , according to notation of the proof of Lemma 2.1.4. Since  $\widetilde{\varepsilon} = \varphi^* \varepsilon$ , we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \pi_\kappa & \xrightarrow{\subset} & \mathfrak{G}_\Omega^\dagger(\mathcal{O}_k) & \xrightarrow{\pi_\kappa} & \mathfrak{G}_\Omega^\dagger(\kappa) \longrightarrow 1 \\ & & \downarrow & & \downarrow \rho_{\mathcal{O}_k} & & \downarrow \rho_\kappa \\ 0 & \longrightarrow & \ker \widetilde{\pi}_\kappa & \xrightarrow{\subset} & \mathrm{GL}_n(\mathcal{O}_k) & \xrightarrow{\widetilde{\pi}_\kappa} & \mathrm{GL}_n(\kappa) \longrightarrow 1 \end{array}$$

Hence  $\ker \pi_\kappa$  is isomorphic to a closed subgroup of  $\ker \widetilde{\pi}_\kappa$ , so it is a pro- $p$  group.  $\square$

**3.2.3 Proposition.** *The group  $P_\Omega^+$  is a normal pro- $p$  subgroup of  $\mathfrak{G}_\Omega^\dagger(\mathcal{O}_k)$ .*

*Proof.* By Lemma 2.1.3(b), we have  $\ker \pi_q = R_u(\overline{\mathfrak{G}_\Omega})(\kappa)$ , hence it is a  $p$ -group as a group of rational points of a unipotent  $\kappa$ -group. The following sequence of group homomorphism  $1 \longrightarrow \ker \pi_\kappa \xrightarrow{\subset} \ker(\pi_q \circ \pi_\kappa) \xrightarrow{\pi_\kappa} \ker \pi_q \xrightarrow{\pi_q} 1$  is exact. Indeed, check that  $\pi_\kappa(\ker \pi_q \circ \pi_\kappa) = \ker \pi_q$ .

If  $g \in \pi_\kappa(\ker \pi_q \circ \pi_\kappa)$ , then there exists  $h \in \ker \pi_q \circ \pi_\kappa$  such that  $g = \pi_\kappa(h)$ . hence  $\pi_q(g) = \pi_q \circ \pi_\kappa(h) = 1$ , and so  $g \in \ker \pi_q$ .

Conversely, if  $g \in \ker \pi_q$ , by surjectivity of  $\pi_\kappa$  (given by Lemma 3.2.2), there exists  $h \in \mathfrak{G}_\Omega(\mathcal{O}_k)$  such that  $\pi_\kappa(h) = g$ . Hence  $\pi_q \circ \pi_\kappa(h) = \pi_q(g) = 1$ , and so  $h \in \ker(\pi_q \circ \pi_\kappa)$ . Hence  $g \in \pi_\kappa(\ker(\pi_q \circ \pi_\kappa))$ .

As a consequence,  $P_\Omega^+ = \ker \pi_\Omega$  is a pro- $p$  group.  $\square$

**3.2.4 Lemma.** *Let  $k$  be a finite field of characteristic  $p$ . If  $H$  is a reductive  $k$ -group, then  $H(k)$  does not have a non-trivial normal  $p$ -subgroup.*

*Proof.* Let  $P$  be a normal  $p$ -subgroup of  $H(k)$ . It is a subgroup of a  $p$ -Sylow subgroup of  $H(k)$ . By Lemma 3.1.1, there exists a Borel  $k$ -subgroup  $B$  such that  $P \subset \mathcal{R}_u(B)(k)$ .

Let  $S$  be a maximal  $k$ -split torus of  $H$ . Denote  $T = \mathcal{Z}_H(S)$ , it is a maximal torus of  $H$  defined over  $k$  and contained in  $B$ . Let  $n \in \mathcal{N}_H(T)(k)$  such that  $B$  and  $nBn^{-1}$  are opposite Borel  $k$ -subgroups. Hence,  $B \cap nBn^{-1} = T$  [Bor91, 14.1] is a torus. We have  $nPn^{-1} = P$  because  $P$  is normal in  $H(k)$ . Hence,  $P$  is a subgroup of  $T(k)$  and  $\#T(k)$  is prime to  $p$ . As a consequence  $P \subset T(k)$  is trivial.  $\square$

To obtain results about maximality of  $\ker \pi_\Omega$ , we require that  $\pi_\Omega$  is surjective.

---

**3.2.5 Lemma.** *The morphism of abstract groups  $\pi_\Omega$  is surjective.*

*In particular, if  $Q$  is a  $p$ -Sylow subgroup of  $\overline{\mathfrak{G}}_\Omega^{\text{red}}(\kappa)$ , then  $\pi_\Omega^{-1}(Q)$  is a maximal pro- $p$  subgroup of  $\mathfrak{G}_\Omega^\dagger(\mathcal{O}_k)$ .*

*Proof.* A finite field is perfect, hence by [Ser94, III.2.1 Prop. 6] applied to the connected ( $\kappa$ -split) unipotent  $\kappa$ -group  $U = R_u(\overline{\mathfrak{G}}_\kappa)$ , we have  $H^1(\kappa, U) = 0$ . Hence by [Ser94, I.5.5 Prop.38] the morphism of abstract groups  $\pi_q$  is surjective. According to Lemma 3.2.2, the composite morphism  $\pi_\Omega$  is surjective.

By Proposition 3.2.3, the surjective morphism  $\pi_\Omega$  has a pro- $p$  kernel. Hence, for every  $p$ -subgroup  $Q$  of  $\overline{\mathfrak{G}}_\Omega^{\text{red}}(\kappa)$ , the group  $\pi_\Omega^{-1}(Q)$  is pro- $p$  (as an extension of such groups). Hence, if  $Q$  is a  $p$ -Sylow subgroup, then  $\pi_\Omega^{-1}(Q)$  is a maximal pro- $p$  subgroup.  $\square$

**3.2.6 Proposition.** *If  $\overline{\mathfrak{G}}_\Omega$  is connected, then the kernel  $P_\Omega^+$  is a maximal normal pro- $p$  subgroup of  $\mathfrak{G}_\Omega^\dagger(\mathcal{O}_k)$ .*

*Proof.* Let  $\tilde{P}$  be a normal pro- $p$  subgroup of  $\mathfrak{G}_\Omega(\mathcal{O}_k)$  containing  $P_\Omega^+$ . By [Ser94, I.1.4 Prop.4], its image by the surjective morphism  $\pi_\Omega$  (see Lemma 3.2.5) is a normal  $p$ -subgroup of  $\overline{\mathfrak{G}}_\Omega^{\text{red}}(\kappa)$ .

When  $\overline{\mathfrak{G}}_\Omega$  is connected, the quotient  $\overline{\mathfrak{G}}_\Omega^{\text{red}}$  is a connected reductive  $\kappa$ -group. Hence, by Lemma 3.2.4,  $\pi(\tilde{P})$  is trivial and  $\tilde{P} = P_\Omega^+$ .  $\square$

### Under simple connectedness assumption

From now on, assume that the semisimple  $k$ -group  $G$  is simply connected. Hence  $\mathfrak{G}_\Omega^\dagger = \mathfrak{G}_\Omega$  [BrTi84, 4.6.32 and 5.1.31].

**3.2.7 Proposition.** *Assume  $\Omega = \mathfrak{c} \subset \mathbb{A}$  is an alcove and  $G$  is simply connected. Then  $P_\Omega^+$  is a maximal pro- $p$  subgroup of  $\mathfrak{G}_\Omega(\mathcal{O}_k)$ .*

First, recall the following fact, given by Tits [Tit79, 3.5.2]:

**3.2.8 Lemma.** *Under above assumptions and notations, the algebraic  $\kappa$ -group  $\overline{\mathfrak{G}}_\Omega$  is connected.*

*Proof of Proposition 3.2.7.* Since  $\Omega$  is an alcove, the root system of  $\overline{\mathfrak{G}}_\Omega/R_u(\overline{\mathfrak{G}}_\Omega)$  is empty [BrTi84, 4.6.12(i), 5.1.31]. By Lemma 3.2.8,  $\overline{\mathfrak{G}}_\Omega^{\text{red}}$  is a connected reductive quasi-split  $\kappa$ -group with a trivial root system. Hence, it is a  $\kappa$ -torus and so, does not have a non-trivial  $p$ -subgroup. Hence, for every pro- $p$  subgroup  $P$  of  $\mathfrak{G}_\Omega^\dagger(\mathcal{O}_k) = \mathfrak{G}_\Omega(\mathcal{O}_k)$ , the image  $\pi_\Omega(P)$  by the surjective morphism  $\pi_\Omega$  (Lemma 3.2.5) is a  $p$ -group [Ser94, 1.4 Prop.4], hence trivial. As a consequence, the kernel  $P_\Omega^+$  is the (unique) maximal pro- $p$  subgroup of  $\mathfrak{G}_\Omega(\mathcal{O}_k)$ .  $\square$

Now, one can give a proof of Theorem 1.5.3.

*Proof of Theorem 1.5.3.* Let  $P$  a maximal pro- $p$  subgroup. By Proposition 3.1.4, we have  $P \subset \text{Stab}_{G(k)}(\mathfrak{c})$ . Let  $\mathfrak{c}_0 \subset \mathbb{A}$ . By strong transitivity of  $G(k)$  on the building  $X(G, k)$ , there exists  $g \in G(k)$  such that  $g\mathfrak{c}_0 = \mathfrak{c}$ . Hence,  $g^{-1}Pg$  is a maximal pro- $p$  subgroup of  $\mathfrak{G}_{\mathfrak{c}_0}(\mathcal{O}_k)$ . By Proposition 3.2.7, we have  $P = gP_{\mathfrak{c}_0}^+g^{-1}$ .  $\square$

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### Valued root group datum in the quasi-split simply connected case

To conclude in the simply connected case, let us interpret this group in terms of a valued root group datum. This could be a bit tricky in the general case and, in the two next propositions, we assume that  $G$  is, moreover, a quasi-split semisimple  $k$ -group. In a further work [Loi16], we compute the Frattini subgroup of a maximal pro- $p$ -subgroup by the explicit decomposition of Proposition 3.2.9.

Denote by  $S$  the maximal  $k$ -split torus chosen in the construction of the building and by  $T = \mathcal{Z}_G(S)$  the associated maximal  $k$ -torus. Denote by  $T(k)_b$  the (unique) maximal profinite subgroup of  $T(k)$  and by  $T(k)_b^+$  its (unique) maximal pro- $p$  subgroup.

**3.2.9 Proposition.** *The group  $P_{\mathbf{c}}^+$  admits the following directly generated product structure:*

$$P_{\mathbf{c}}^+ = \left( \prod_{a \in \Phi_{\text{nd}}^+} U_{-a, f_{\mathbf{c}}(-a)} \right) \cdot T(k)_b^+ \cdot \left( \prod_{a \in \Phi_{\text{nd}}^+} U_{a, f_{\mathbf{c}}(a)} \right)$$

where  $\Phi_{\text{nd}}$  denotes the non-divisible roots of the relative  $k$ -root system  $\Phi(G, S)$ .

In particular,  $T(k)_b^+ = P_{\mathbf{c}}^+ \cap T(k)_b$ .

*Proof.* By the simple connectedness assumption, proposition [Lan96, 3.5] gives  $\mathfrak{T}(\mathcal{O}_k) = T(k)_b$  where  $\mathfrak{T}$  denotes the integral model of  $T$  defined in [Lan96, §3].

Since  $\mathbf{c}$  is an alcove, for any relative root  $a \in \Phi$ , we have  $f_{\mathbf{c}}(a) + f_{\mathbf{c}}(-a) > 0$ . By computation in [Lan96, 5.9, 5.12, 6.5] and axioms of a valued root group datum, the directly generated product  $\mathfrak{U}_{-\Phi^+, \mathbf{c}}(\mathcal{O}_k) \cdot \mathfrak{T}(\mathcal{O}_k) \cdot \mathfrak{U}_{\Phi^+, \mathbf{c}}(\mathcal{O}_k)$  is a group, hence it is equal to  $\mathfrak{G}_{\mathbf{c}}(\mathcal{O}_k)$  by [BrTi84, 4.6.6].

In the proof of Proposition 3.2.7, we have seen that  $\overline{\mathfrak{G}}_{\mathbf{c}}^{\text{red}}(\kappa)$  does not have a non-trivial  $p$ -subgroup. Hence  $\mathfrak{U}_{\pm\Phi^+, \mathbf{c}}(\mathcal{O}_k) \subset \ker \pi_{\mathbf{c}} = P_{\mathbf{c}}^+$  since the image of a pro- $p$  group by a surjective continuous morphism is a pro- $p$  group. Thus, we obtain the equality (3.2.9).  $\square$

By quasi-splitness and simple connectedness, the maximal  $k$ -torus  $T$  is an induced torus [BrTi84, 4.4.16], generated by coroots, and we can be more precise about the above description by root group datum:

**3.2.10 Proposition.** *There is the following isomorphism of topological groups:*

$$\begin{aligned} \prod_{a \in \Delta} \hat{a}^{\vee} : \prod_{a \in \Delta} (1 + \mathfrak{m}_{L_a}) &\rightarrow T(k)_b^+ \\ (t_a)_{a \in \Delta} &\mapsto \prod_{a \in \Delta} \hat{a}^{\vee}(t_a) \end{aligned} \quad (1)$$

where  $\hat{a} = 2a$  if  $2a \in \Phi$ , and  $\hat{a} = a$  otherwise;  $L_a$  denotes the minimal field of definition of the root  $a$  (defined in [BrTi84, 4.1.3]) and  $\mathfrak{m}_{L_a}$  denotes the maximal ideal of its ring of integers.

*Proof.* Since  $G$  is a simply connected quasi-split semisimple  $k$ -group, by [BrTi84, 4.4.16],  $T$  is an induced torus and, more precisely, there is the following isomorphism  $\prod_{a \in \Delta} \hat{a}^{\vee} : \prod_{a \in \Delta} R_{L_a/K}(\mathbb{G}_{m, L_a}) \simeq T$ , where  $\Delta$  denotes a basis of the relative root system  $\Phi$ . By uniqueness, up to isomorphism, of the  $\mathcal{O}_k$ -model,  $\mathfrak{T}$  is  $\mathcal{O}_k$ -isomorphic

to  $\prod_{a \in \Delta} R_{\mathcal{O}_{L_a}/\mathcal{O}_k}(\mathbb{G}_{m, \mathcal{O}_{L_a}})$ . Hence, there is a natural isomorphism  $\prod_{a \in \Delta} \mathcal{O}_{L_a}^\times \simeq \mathfrak{T}(\mathcal{O}_k) = T(k)_b$  of topological abelian groups, and the maximal pro- $p$  subgroup is isomorphic to the direct product  $\prod_{a \in \Delta} (1 + \mathfrak{m}_{L_a})$ .  $\square$

### 3.3 Description using the action on a building

We now can derive the useful description of a maximal pro- $p$  subgroup of  $G(k)$ , as a pro- $p$ -Sylow of the setwise stabilizer of a suitable alcove. To prove Theorem 1.5.1, it suffices to show that every maximal pro- $p$  subgroup of  $G(k)$  can be realised as such a group.

*Proof of Theorem 1.5.1.* Let  $P$  be a maximal pro- $p$  subgroup of  $G(k)$ . By Proposition 3.1.4, there exists an alcove  $\mathfrak{c}$  such that  $P$  setwise stabilizes  $\mathfrak{c}$ . By strong transitivity, we can and do assume that  $\mathfrak{c} \subset \mathbb{A}$ . In particular,  $P$  is a maximal pro- $p$  subgroup of  $\mathfrak{G}_{\mathfrak{c}}^\dagger(\mathcal{O}_k)$ .

Firstly, we show the uniqueness of such an alcove  $\mathfrak{c}$ . By Lemma 2.2.2, the topological group  $\mathfrak{G}_{\mathfrak{c}}^\dagger(\mathcal{O}_k)$  is compact, hence profinite. By Sylow theorem for profinite groups [Ser94, 1.4 Prop.3 et 4 (a)], there exists  $g_0 \in \mathfrak{G}_{\mathfrak{c}}^\dagger(\mathcal{O}_k)$  such that  $P$  contains  $g_0 P_{\mathfrak{c}}^+ g_0^{-1} = P_{\mathfrak{c}}^+$ . It suffices to show that  $P_{\mathfrak{c}}^+$  does not stabilises any alcove of  $X(G, k)$  different from  $\mathfrak{c}$ .

For all  $a \in \Phi$ , the image by  $\pi_{\mathfrak{c}}$  of the root group  $U_{a, \mathfrak{c}}(\mathcal{O}_k)$  is trivial because  $\overline{\mathfrak{U}}_{a, \mathfrak{c}}$  is a root group of  $\overline{\mathfrak{G}}_{\mathfrak{c}}^{red}$  [Lan96, 10.34], hence trivial because  $\mathfrak{c}$  is an alcove [Lan96, 10.36]. Hence  $P_{\mathfrak{c}}^+$  contains the subgroup  $U_{\mathfrak{c}}$  of  $G(k)$  generated by  $U_{a, \mathfrak{c}}$  for every  $a \in \Phi$ . The group  $P_{\mathfrak{c}}^+$  acts on the set of all facets of  $X(G, k)$  not contained in  $cl(\mathfrak{c})$  since it setwise stabilizes  $cl(\mathfrak{c})$  and preserves the simplicial structure of  $X(G, k)$ . Let  $F$  be such a facet. Let  $A'$  be an apartment containing  $\mathfrak{c}$  and  $F$ . Let  $A''$  be an apartment containing  $\mathfrak{c}$  but not  $F$ . Since the group  $U_{\mathfrak{c}}$  acts transitively on the set of apartments containing  $\mathfrak{c}$  [Lan96, 13.7], there exists  $u \in U_{\mathfrak{c}} \subset P_{\mathfrak{c}}^+$  such that  $u \cdot A' = A''$ . Hence  $P_{\mathfrak{c}}^+$  does not stabilize  $F$ .

Conversely, let  $\mathfrak{c}$  be an alcove of  $X(G)$  and  $P$  be a maximal pro- $p$  subgroup of  $\text{Stab}_{G(k)}(\mathfrak{c})$ . Let  $P'$  be a maximal pro- $p$  subgroup of  $G(k)$  containing  $P$ . Such a  $P'$  exists by Lemma 2.4.5 and Proposition 2.2.10. Let  $\mathfrak{c}'$  be the unique alcove stabilized by  $P'$ , hence by  $P$ . Since  $P$  contains  $P_{\mathfrak{c}}^+$  according to Lemma 3.2.5, it does not stabilize any facet of  $X(G, k)$  out from  $cl(\mathfrak{c})$ . Hence  $\mathfrak{c} = \mathfrak{c}'$  and  $P'$  is a maximal pro- $p$  subgroup of  $\text{Stab}_{G(k)}(\mathfrak{c})$ . By maximality of  $P$ , we have  $P' = P$ .  $\square$

**3.3.1 Corollary.** *If  $G$  is a simply connected semisimple  $k$ -group, then  $P$  is a maximal pro- $p$  subgroup of  $G(k)$  if, and only if, there exists an alcove  $\mathfrak{c}$  of  $X(G, k)$  such that  $P = P_{\mathfrak{c}}^+$ . Moreover, such an alcove  $\mathfrak{c}$  is uniquely determined by  $P$  and the set of fixed points by  $P$  in  $X(G, k)$  is exactly the simplicial closure  $cl(\mathfrak{c})$ .*

*Proof.* The first part is a consequence of Proposition 3.2.7 and of the first part of Theorem 1.5.3.

When  $G$  is simply connected, the stabilizer of an alcove is also its pointwise stabilizer [BrTi84, 5.2.9]. This and Theorem 1.5.3 gives the second part.  $\square$

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## Iwahori subgroups in the simply connected case

Recall the following definitions [BrTi84, 5.2]

### 3.3.2 Definition.

(1) Given a facet  $F$  of  $X(G, k)$ , call **connected pointwise stabilizer** of  $F$  the subgroup  $\mathfrak{G}_F(\mathcal{O}_k)$  of  $G(k)$ .

(2) A subgroup of  $G(k)$  is called a **parahoric** (resp. **Iwahori**) subgroup if, and only if, it is the connected pointwise stabilizer of a facet (reps. an alcove) of  $X(G, k)$ .

To conclude this study of pro- $p$  subgroups, the following well-known proposition is a kind of generalisation of Lemma 3.1.1.

**3.3.3 Proposition.** *Assume that  $G$  is simply connected. A subgroup of  $G(k)$  is an Iwahori subgroup if, and only if, it is the normalizer in  $G(k)$  of a maximal pro- $p$  subgroup of  $G(k)$ .*

*Proof.* Let  $\mathfrak{c}$  be an alcove of  $\mathbb{A}$ , let  $g \in G(k)$  an element and  $H$  the stabilizer of  $g \cdot \mathfrak{c}$ . Since the semisimple  $k$ -group  $G$  is simply connected, the stabilizer  $H$  is in fact an Iwahori subgroup [BrTi84, 5.2.9]. By Proposition 3.2.3,  $gP_{\mathfrak{c}}^+g^{-1}$  is a normal pro- $p$  subgroup of  $H$ . Hence  $H \subset \mathcal{N}_{G(k)}(gP_{\mathfrak{c}}^+g^{-1})$ . For every element  $h \in \mathcal{N}_{G(k)}(gP_{\mathfrak{c}}^+g^{-1})$ , every  $u \in P_{\mathfrak{c}}^+$  and  $x \in \mathfrak{c}$ , one has  $h^{-1}uh \cdot x = x$  because  $gP_{\mathfrak{c}}^+g^{-1}$  fixes  $g \cdot \mathfrak{c}$  pointwise. Hence  $h \cdot x$  is a point in  $X(G, k)$  fixed by  $P_{\mathfrak{c}}^+$ , so  $h \cdot x \in \mathfrak{c}$  since it cannot be contained on the boundary of  $\mathfrak{c}$ . Since the action of  $G(k)$  preserves the simplicial structure of  $X(G, k)$ , the element  $h$  stabilises  $\mathfrak{c}$ . Hence  $\mathcal{N}_{G(k)}(gP_{\mathfrak{c}}^+g^{-1}) = H$ . By Theorem 1.5.3, it gives the first implication.

Conversely, let  $U$  be a maximal pro- $p$  subgroup of  $G(k)$ . Define  $H = \mathcal{N}_{G(k)}(U)$ . Denote by  $\mathfrak{c}$  be the unique alcove fixed by  $U$  given by Theorem 1.5.1. By uniqueness of  $\mathfrak{c}$ , the subgroup  $H$  stabilises  $\mathfrak{c}$ . By Proposition 3.2.7 (and conjugation),  $U$  is a normal subgroup of  $\text{Stab}_{G(k)}(\mathfrak{c})$ . Hence  $H = \mathcal{N}_{G(k)}(U) = \text{Stab}_{G(k)}(\mathfrak{c})$  is an Iwahori subgroup of  $G(k)$ .  $\square$

**3.3.4 Corollary.** *Iwahori subgroups of  $G(k)$  are  $G(k)$ -conjugated.*

*Proof.* This is [Tit79, 3.7]. It is immediate by Theorem 1.2.1 and Proposition 3.3.3.  $\square$

An interest of Proposition 3.3.3 is to have an “intrinsic” definition (from the group theory point of view, in other words a description not using the action on the Bruhat-Tits building) of Iwahori subgroups in good cases (e.g. a simply connected group over a local field). This provides a quick way to describe the affine Tits system in purely group-theoretic terms.

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