Relative equilibria with holes for the surface quasi-geostrophic equations

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RELATIVE EQUILIBRIA WITH HOLES FOR THE SURFACE QUASI-GEOSTROPHIC EQUATIONS

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Abstract. We study the existence of doubly connected rotating patches for the inviscid surface quasi-geostrophic equation left open in [12]. By using the approach proposed by [4] we also prove that close to the annulus the boundaries are actually analytic curves.

1. Introduction

In this paper we investigate the surface quasi-geostrophic (SQG) model which describes the evolution of the potential temperature θ according to the transport equation,

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \ (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ u = -\nabla^\perp(-\Delta)^{-\frac{1}{2}}\theta, \\ \theta_{|t=0} = \theta_0 \end{cases} \tag{1.1}$$

where u refers to the velocity field and $\nabla^\perp = (-\partial_2,\partial_1)$. The operator $(-\Delta)^{-\frac{1}{2}}$ is defined as follows

$$(-\Delta)^{-\frac{1}{2}}\theta(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\theta(y)}{|x-y|}dy.$$

This model is used to study the atmospheric circulations near the tropopause and the ocean dynamics in the upper layers, see for instance [15, 21, 25]. This nonlinear transport equation is more singular than the vorticity equation for the 2D Euler equations where the connection between the velocity and the vorticity is given by the Biot-Savart law

$$u = -\nabla^\perp(-\Delta)^{-1}\theta.$$

Another model appearing in the literature which interpolates between the (SQG) and Euler equations is the $(\mathrm{SQG})_\alpha$ model, see [9], where the velocity is given by

$$u = -\nabla^\perp(-\Delta)^{-1+\frac{\alpha}{2}}\theta, \quad \alpha\in(0,2).$$

These equations have been intensively studied during the past few decades and abundant results have been established in different topics such as the well-posedness problem or the vorticity dynamics. For instance, it is well-known that for Euler equations when the initial data θ_0 belongs to $L^\infty\cap L^1$ then there is a unique global weak solution $\theta\in L^\infty(\mathbb{R}^+;L^\infty\cap L^1)$. This theory fails for $\alpha>0$ due to the singularity of the kernel. However, the local well-posedness can be elaborated in the sub-class of the vortex patches as it was shown in [6] and [14]. Recall that an initial datum is a vortex patch when it takes the form χ_D, which is the characteristic function of a smooth bounded domain D. The solutions keep this structure for a short time, that is, $\theta(t)=\chi_{D_t}$ where D_t is another domain describing the deformation of the initial one in the complex plane. The global existence of these solutions is an outstanding open problem except for Euler equations in which case Chemin proved in [7] the persistence of smooth regularity globally in time. Note that a significant progress towards settling this problem, for α enough close to zero, has been done recently in [24]. Another direction related to the construction of periodic global solutions through the bifurcation theory has been recently investigated. They correspond to rotating patches also called V-states or relative equilibria. In this setting the domain of the patch is explicitly given by a pure rotation with uniform angular velocity, that is, $D_t = R_{x_0,\Omega t}D$ where $R_{x_0,\Omega t}$ is the planar rotation with the center x_0 and the angle Ωt; the

parameter Ω is the angular velocity. The first example of rotating patches goes back for Euler equation to Kirchhoff who discovered that an ellipse of semi-axes a and b rotates uniformly with the angular velocity $\Omega = \frac{ab}{(a^2+b^2)}$; see for instance [1, p304] and [26, p 232]. One century later, Deem and Zabusky gave in [11] numerical evidence of the existence of the V-states with $m-$fold symmetry for each integer $m \in \{2,3,4,5\}$ and afterwards Burbea gave an analytically proof in [2]. The main idea of the demonstration is to reformulate the V-states equations with the contour dynamics equations, using the conformal parametrization Φ, and to implement some bifurcation arguments. The bifurcation from the ellipses to countable curves of non symmetric rotating patches was discussed numerically and analytically in [4, 20, 22]. On the other hand we point out that the extension of this study to the $(\mathrm{SQG})_\alpha$ was successfully carried out in [3, 16]. Moreover the boundary regularity was achieved in [3, 4, 20].

The existence of V-states with one hole, also called doubly connected V-states, has been recently explored in [12, 17]. To fix the terminology, a patch $\theta_0 = \chi_D$ is said to be doubly connected if the domain $D = D_1 \setminus D_2$ with D_1 and D_2 being two simply connected bounded domains such that the closure $\overline{D_2}$ is strictly embedded in D_1. The first result on the existence of m-fold symmetric V-states bifurcating from the annulus $\mathbb{A}_b = \{z; b < |z| < 1\}$ is established in [12]. Roughly speaking, it is shown that for higher modes m there exist two branches of m-fold symmetric doubly connected V-states bifurcating from the annulus at explicit eigenvalues $\Omega_m^{\pm}$. Similar result with more involved computations was obtained for $(\mathrm{SQG})_\alpha$ model with $\alpha \in [0,1)$, see [17]. Actually, it is shown that for given $\alpha \in [0,1)$ and $b \in (0,1)$, there exists $N \in \mathbb{N}$ such that for each $m \geq N$ there exists two curves of m-fold doubly connected V-states bifurcating from the annulus $\mathbb{A}_b$ at the angular velocities

$$\Omega_m^{\pm} = \frac{1}{2}\left((1-b^{-\alpha})S_m + (1-b^2)\Lambda_1(b)\right) \pm \frac{1}{2}\sqrt{\Delta_m(\alpha,b)}$$

with

$$\Delta_m(\alpha,b) = \left((b^{-\alpha}+1)S_m - (1+b^2)\Lambda_1(b)\right)^2 - 4b^2\Lambda_m^2(b),$$

$$\Lambda_m(b) \triangleq \frac{1}{b}\int_0^{+\infty} J_m(bt)J_m(t)\frac{dt}{t^{1-\alpha}}$$

and

$$S_m \triangleq \Lambda_1(1) - \Lambda_m(1).$$

Where J_m refers to the Bessel function of the first kind.

The main goal of this paper is to study the same problem for the SQG equation (1.1) corresponding to $\alpha = 1$. Our aim is twofolds. First we shall establish the existence of doubly connected V-states and second we shall prove that the boundary is analytic. The main result of this paper reads as follows.

Theorem 1.1. *Let $b \in (0,1)$, there exists $N \in \mathbb{N}^\star \setminus \{1\}$ with the following property: For any integer $m \geq N$ there exist two analytic curves of m-fold doubly connected V-states for (1.1) bifurcating from the annulus $\mathbb{A}_b = \{z \in \mathbb{C}, b < |z| < 1\}$ at the angular velocities*

$$\Omega_m^{\pm} = \frac{1}{2}\left[\left(1-\frac{1}{b}\right)S_m + (1-b^2)\Lambda_1(b)\right] \pm \frac{1}{2}\sqrt{\Delta_m(b)} \tag{1.2}$$

where S_m, Λ_m and Δ_m are defined above by taking $\alpha = 1$.

Remarks.

- *For $\alpha = 1$, the expression of S_m can be simplified and takes the form*

$$S_n = \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1}$$

- *As we shall see later in the proofs, the number N is defined as the smallest integer such that*

$$S_N > b\left(\frac{1+b^2}{1+b}\Lambda_1(b) + \frac{2b}{1+b}\Lambda_N(b)\right).$$

- *Our results are in line with results foretold in [12].*

Now we shall sketch the proof of Theorem 1.1 which relies on Crandall-Rabinowitz's theorem applied in suitable Banach spaces that capture the analyticity of the boundary. We mention that these spaces were

introduced in [4] in order to study the simply connected V-states. The first step is to write the boundary equations using the exterior conformal parametrization of the domains D_1 and D_2. These conformal mappings $\Phi_j : \mathbb{D}^c \to D_j^c$ have the following structure

$$\forall |z| > 1,\ \Phi_1(z) = z + \sum_{n\in\mathbb{N}} \frac{a_n}{z^n} \quad \text{and} \quad \Phi_2(z) = bz + \sum_{n\in\mathbb{N}} \frac{c_n}{z^n}.$$

with $\mathbb{D}$ being the unit closed disc. The Fourier coefficients are supposed to be real meaning that we look only for the V-states which are symmetric with respect to the real axis. Notice also that when the boundaries are assumed to be enough smooth then the Φ_j admit unique univalent extension up the boundary. We recall from Section 2 that the boundaries of the V-states are subject to the equations: For $j \in \{1, 2\}$ and $\omega \in \mathbb{T}$

$$\begin{aligned} G_j(\Omega, \Phi_1, \Phi_2)(\omega) &\triangleq \operatorname{Im}\left\{\Big(\Omega\Phi_j(\omega) - S(\Phi_1, \Phi_j)(\omega) + S(\Phi_2, \Phi_j)(\omega)\Big)\overline{\Phi_j'(\omega)\omega}\right\} \\ &= 0 \end{aligned}$$

with

$$S(\Phi_i, \Phi_j)(\omega) = \fint_{\mathbb{T}} \frac{\tau\Phi_i'(\tau) - \omega\Phi_j'(\omega)}{|\Phi_i(\tau) - \Phi_j(\omega)|} \frac{d\tau}{\tau}.$$

To apply the bifurcation arguments we make use of the Banach spaces $X^{k+\log}$ and Y^{k-1} that will be fully described in the subsection 3.2. The main difficulty is to show that the functionals G_j send a small neighborhood in $X^{k+\log}$ of the trivial solution $(\mathrm{Id}, b\mathrm{Id})$ to the space Y^{k-1}. This will be done carefully in Section 4 where additional regularity properties will also be established. The second step is to compute explicitly the linearized operator of the vectorial functional $G = (G_1, G_2)$ at the annular solution $(\mathrm{Id}, b\,\mathrm{Id})$. This part is very computational and after using special structures of the Gauss hypergeometric functions we obtain the following compact expression: Given

$$h_1(\omega) = \sum_{n=1}^{+\infty} a_n\overline{\omega}^n \quad \text{and} \quad h_2(\omega) = \sum_{n=1}^{+\infty} c_n\overline{\omega}^n,\ \omega \in \mathbb{T}$$

we get

$$DG(\Omega, \mathrm{Id}, b\,\mathrm{Id})(h_1, h_2)(\omega) = \frac{i}{2}\sum_{n\geq 1}(n+1)M_{n+1}\begin{pmatrix} a_n \\ c_n \end{pmatrix}\left(\omega^{n+1} - \overline{\omega}^{n+1}\right)$$

where the matrix M_n is given for $n \geq 2$ by

$$M_n \triangleq \begin{pmatrix} \Omega - S_n + b^2\Lambda_1(b) & -b^2\Lambda_n b \\ b\Lambda_n(b) & b\Omega + S_n - b\Lambda_1(b) \end{pmatrix}.$$

With this explicit formula in hand we find the values of Ω leading to a one dimensional kernel operator. We also check the full conditions required by the Crandall-Rabinowitz's theorem. This discussion will be investigated in detail in Section 5.

In what follows, we will need some notations:

- The unit disc and its boundary will be denoted respectively by $\mathbb{D}$ and $\mathbb{T}$.
- The disc of r radius and centered in 0 and its boundary will be denoted by $\mathbb{D}_r$ and $\mathbb{T}_r$.
- We denote by C any positive constant that may change from line to line.
- Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function. We define its mean value by,

$$\fint_{\mathbb{T}} f(\tau)d\tau \triangleq \frac{1}{2i\pi}\int_{\mathbb{T}} f(\tau)d\tau,$$

  where $d\tau$ stands for the complex integration.
- Let be X and Y be two normed spaces. We denote by $\mathcal{L}(X, Y)$ the space of all continuous linear maps $T : X \to Y$ endowed with its usual strong topology.
- Let Y be a vector space and R be a subspace, then Y/R denotes the quotient space.

2. Boundary equations

We intend in this section to write down the equations governing the V-states in the doubly connected case. But before doing that we shall recall the Riemann mapping theorem. To restate this result we need to recall the definition of *simply connected* domains. Let $\widehat{\mathbb{C}} \triangleq \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere, we say that a domain $U \subset \widehat{\mathbb{C}}$ is *simply connected* if the set $\widehat{\mathbb{C}} \setminus U$ is connected.

Theorem 2.1 (Riemann Mapping Theorem)**.** *Let $\mathbb{D}$ denote the unit open ball and $U \subset \mathbb{C}$ be a simply connected bounded domain. Then there is a unique bi-holomorphic map called also conformal, $\Phi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{U}$ taking the form*

$$\Phi(z) = az + \sum_{n\in\mathbb{N}} \frac{a_n}{z^n} \quad \text{with} \quad a > 0.$$

Notice that in this theorem the regularity of the boundary has no effect regarding the existence of the conformal mapping but it contributes in the boundary behavior of the conformal mapping, see for instance [27, 30].

Next, we shall move to the equations governing the boundary of the doubly connected V-states. This can be done in the spirit of the paper [12]. Assume that $\theta_0 = \chi_D$ is a rotating patch with an angular velocity Ω and such that $D = D_1 \setminus D_2$ is a doubly connected domain meaning that D_1 and D_2 are two simply connected bounded domains with $D_2 \subset D_1$. Denote by Γ_1 and Γ_2 their boundaries, respectively. Then following the same lines of [12] we find that the exterior conformal mappings Φ_1 and Φ_2 associated to D_1 and D_2 satisfy the coupled nonlinear equations: For $j \in \{1, 2\}, \omega \in \mathbb{T}$,

$$\begin{aligned} \tilde{G}_j(\Omega, \Phi_1, \Phi_2)(\omega) &\triangleq \operatorname{Im}\left\{\Big(\Omega\Phi_j(\omega) - S(\Phi_1, \Phi_j)(\omega) + S(\Phi_2, \Phi_j)(\omega)\Big)\overline{\Phi_j'(\omega)\overline{\omega}}\right\} \\ &= 0 \end{aligned} \tag{2.1}$$

with

$$S(\Phi_i, \Phi_j)(\omega) = \fint_{\mathbb{T}} \frac{\tau\Phi_i'(\tau) - \omega\Phi_j'(\omega)}{|\Phi_i(\tau) - \Phi_j(\omega)|} \frac{d\tau}{\tau}.$$

Notice that we aim at finding V-states which are small perturbation of the annulus $\mathbb{A}_b$ with $b \in (0, 1)$ and therefore the conformal mappings take the form,

$$\forall |z| \geq 1, \quad \Phi_1(z) = z + f_1(z) = z + \sum_{n=1}^{+\infty} \frac{a_n}{z^n}$$

and

$$\Phi_2(z) = bz + f_2(z) = bz + \sum_{n=1}^{+\infty} \frac{b_n}{z^n}.$$

We shall introduce the functionals

$$G_j(\Omega, f_1, f_2) \triangleq \tilde{G}_j(\Omega, \Phi_1, \Phi_2) \; j = 1, 2. \tag{2.2}$$

Then equations of the V-states become,

$$\forall \omega \in \mathbb{T}, \; G_j(\Omega, f_1, f_2)(\omega) = 0, \; j = 1, 2.$$

Now we can check that the annulus is a rotating patch for any $\Omega \in \mathbb{R}$. Indeed,

$$G_1(\Omega, 0, 0)(\omega) = \operatorname{Im}\left\{-\overline{\omega} \fint_{\mathbb{T}} \frac{\tau - \omega}{|\tau - \omega|} \frac{d\tau}{\tau} + \overline{\omega} \fint_{\mathbb{T}} \frac{b\tau - \omega}{|b\tau - \omega|} \frac{d\tau}{\tau}\right\}.$$

Using the change of variable $\tau = \omega\xi$ in the last equation we obtain:

$$G_1(\Omega, 0, 0)(\omega) = \operatorname{Im}\left\{- \fint_{\mathbb{T}} \frac{\xi - 1}{|\xi - 1|} \frac{d\xi}{\xi} + \fint_{\mathbb{T}} \frac{b\xi - 1}{|b\xi - 1|} \frac{d\xi}{\xi}\right\}.$$

Now we just observe that each integral is real. In fact using the parametrization $\xi = e^{i\eta}$ one gets,

$$\forall a \in (0,1],\ \overline{\fint_{\mathbb{T}} \frac{a\xi - 1}{|a\xi - 1|}\frac{d\xi}{\xi}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{ae^{-i\eta} - 1}{|ae^{-i\eta} - 1|}d\eta.$$

It suffices now to make again the change of variables $\eta \mapsto -\eta$. Hence we find,

$$\forall \omega \in \mathbb{T}, G_1(\Omega, 0, 0)(\omega) = 0.$$

Arguing similarly we also get

$$\forall \omega \in \mathbb{T}, G_2(\Omega, 0, 0)(\omega) = 0.$$

3. Tools

In this section, we shall gather some useful results that we shall use throughout the paper. First, we will recall the Crandall-Rabinowitz's theorem which is the key tool of the proof of our main result. Second, we shall introduce different basic Banach spaces needed in the bifurcation. Last, we shall collect some important properties on special functions and which are helpful in the subsection 5.1 to get compact formula for the linearized operator.

3.1. Crandall-Rabinowitz's theorem.

We intend now to recall Crandall-Rabinowitz's theorem which is an important tool in the bifurcation theory and will be used in the proof of Theorem 1.1. Let $F : \mathbb{R}\times X \to Y$ be a continuous function with X and Y being two Banach spaces. Assume that $F(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}$. Whether or not close to a trivial solution $(\lambda_0, 0)$ one may find a branch of non trivial solutions of the equation $F(\lambda, x) = 0$ is the main concern of the bifurcation theory. The following theorem provides sufficient conditions for the bifurcation based on the structure of the linearized operator at the point $(\lambda_0, 0)$. For more details we refer to [10, 23].

Theorem 3.1. *Let X,Y be two Banach spaces, V a neighborhood of 0 in X and let $F : \mathbb{R} \times X \to Y$ with the following properties:*

1 *$F(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}$.*
2 *The partial derivatives F_λ, F_x and $F_{\lambda x}$ exist and are continuous.*
3 *$\mathrm{Ker}(\mathcal{L}_0)$ and $Y/\mathrm{Im}(\mathcal{L}_0)$ are one-dimensional.*
4 *Transversality assumption: $\partial_\lambda\partial_x F(0,0)x_0 \notin \mathrm{Im}(\mathcal{L}_0)$, where*

$$\mathrm{Ker}(\mathcal{L}_0) = span(x_0),\ \mathcal{L}_0 \triangleq \partial_x F(0,0).$$

If Z is any complement of $\mathrm{Ker}(\mathcal{L}_0)$ in X, then there is a neighborhood U of $(0,0)$ in $\mathbb{R}\times X$, an interval $(-a,a)$, and continuous functions $\phi : (-a,a) \to \mathbb{R}$, $\psi : (-a,a) \to Z$ such that $\phi(0) = 0$,$\psi(0) = 0$ and

$$F^{-1}(0)\cap U = \Big\{\big(\phi(\xi), \xi x_0 + \xi\psi(\xi)\big); |\xi| < a\Big\} \cup \Big\{(\lambda, 0); (\lambda, 0) \in U\Big\}$$

3.2. Function spaces.

We shall see later the spaces that we shall introduce in this paragraph will play a central role in the proof of our main theorem. They were first devised in [3] but with a different representation. Let $\varepsilon \in (0,1)$ and introduce the sets

$$C_\varepsilon = \left\{z \in \mathbb{C}|\ \varepsilon < |z| < \frac{1}{\varepsilon}\right\} \quad \text{and} \quad \mathbb{A}_\varepsilon = \Big\{z \in \mathbb{C}|\ \varepsilon < |z|\Big\}.$$

We denote by $\mathcal{A}_\varepsilon$ the set of holomorphic functions h on $\mathbb{A}_\varepsilon$ and such that

$$\forall z \in \mathbb{A}_\varepsilon, \quad h(z) = \sum_{n\geq 1} h_n z^{-n} \quad \text{with} \quad h_n \in \mathbb{R}.$$

For $m \in \mathbb{N}$ we define $\mathcal{A}_\varepsilon^m$ as the set of functions $h \in \mathcal{A}_\varepsilon$ such that

$$\forall z \in \mathbb{A}_\varepsilon, \quad h(z) = \sum_{n \geq 1} h_n z^{-nm+1}.$$

Let $\widehat{\mathcal{A}}_\varepsilon$ be the set of holomorphic functions h on C_ε with the property

$$\forall z \in C_\varepsilon,\, h(z) = i \sum_{n=1}^{+\infty} h_n (z^n - z^{-n}),\, h_n \in \mathbb{R}.$$

For $m \in \mathbb{N}$ we define $\widehat{\mathcal{A}}_\varepsilon^m$ as the set of functions $h \in \widehat{\mathcal{A}}_\varepsilon$ such that,

$$\forall z \in C_\varepsilon,\, h(z) = i \sum_{n=1}^{+\infty} h_n (z^{nm} - z^{-nm}),\, h_n \in \mathbb{R}.$$

Finally we denote by $\tilde{A}_\varepsilon$ the set of holomorphic functions on C_ε and such that,

$$\forall z \in C_\varepsilon,\, h(z) = \sum_{n \in \mathbb{Z}} h_n z^n \text{ with } h_n \in \mathbb{R}.$$

For $k \in \mathbb{N}$ we introduce the spaces,

$$X^{k+\log} = \Bigg\{ h \in \mathcal{A}_\varepsilon, \qquad \int_0^{2\pi} |h(\varepsilon e^{i\theta})|^2 d\theta < +\infty, \int_0^{2\pi} |(\partial_z^k h)(\varepsilon e^{i\theta})|^2 d\theta < +\infty,$$
$$\Big\| \int_{\mathbb{T}} \frac{(\partial_z^k h)(\varepsilon\tau) - (\partial_z^k h)(\varepsilon\cdot)}{|\tau - \cdot|} \frac{d\tau}{\tau} \Big\|_{L^2(\mathbb{T})} < +\infty \Bigg\}$$

and

$$X_m^{k+\log} = X^{k+\log} \cap \mathcal{A}_\varepsilon^m.$$

We also define the spaces,

$$Y^{k-1} = \Bigg\{ h \in \widehat{\mathcal{A}}_\varepsilon,\ \int_0^{2\pi} |h(\varepsilon e^{i\theta})|^2 d\theta < +\infty,\ \int_0^{2\pi} |(\partial_z^{k-1} h)(\varepsilon e^{i\theta})|^2 d\theta < +\infty \Bigg\},$$

$$Y_m^{k-1} = Y^{k-1} \cap \widehat{\mathcal{A}}_\varepsilon^m$$

and

$$\tilde{Y}^{k-1} = \Bigg\{ h \in \tilde{A}_\varepsilon, \qquad \int_0^{2\pi} |h(\varepsilon e^{i\theta})|^2 d\theta < +\infty,\ \int_0^{2\pi} \Big| h(\varepsilon^{-1} e^{i\theta}) \Big|^2 d\theta < +\infty,$$
$$\int_0^{2\pi} \Big| (\partial_z^{k-1} h)(\varepsilon e^{i\theta}) \Big|^2 d\theta < +\infty,\ \int_0^{2\pi} \Big| (\partial_z^{k-1} h)(\varepsilon^{-1} e^{i\theta}) \Big|^2 d\theta < +\infty \Bigg\}.$$

Next we shall be concerned with a characterization of the space $X^{k+\log}$ space in terms of the Fourier coefficients.

Lemma 3.2. *Let $k \in \mathbb{N}$ and $h \in \mathcal{A}_\varepsilon$ with $h(z) = \sum_{n \in \mathbb{N}^*} h_n z^{-n}$. Then $h \in X^{k+\log}$ if and only if*

$$\forall \omega \in \mathbb{T},\, h(\omega) = \sum_{n=1}^{+\infty} h_n \overline{\omega}^n \quad \textit{and} \quad \|h\|^2_{X^{k+\log}} \approx \sum_{n=1}^{+\infty} \frac{h_n^2}{\varepsilon^{2(n+k)}} n^{2k} (1 + \log(n))^2.$$

Proof. It is easy to see that for $z \in \mathbb{A}_\varepsilon$

$$(\partial_z^k h)(z) = \sum_{n=1}^{+\infty} (-1)^k h_n \frac{(n+k-1)!}{(n-1)!} \frac{1}{z^{n+k}}.$$

Hence using the identity (5.1) we get for $\omega \in \mathbb{T}$

$$\fint_{\mathbb{T}} \frac{(\partial_z^k h)(\varepsilon\tau) - (\partial_z^k h)(\varepsilon\omega)}{|\tau-\omega|}\frac{d\tau}{\tau} = \sum_{n=1}^{+\infty}(-1)^k \frac{h_n}{\varepsilon^{n+k}}\frac{(n+k-1)!}{(n-1)!}\fint_{\mathbb{T}}\frac{\overline{\tau}^{n+k}-\overline{\omega}^{n+k}}{|\tau-\omega|}\frac{d\tau}{\tau}$$
$$= \sum_{n=1}^{+\infty}(-1)^k \frac{h_n}{\varepsilon^{n+k}}\frac{(n+k-1)!}{(n-1)!}\overline{\omega}^{n+k}\left[-\frac{2}{\pi} - S_{n+k}\right].$$

Therefore we may obtain the equivalence between the norms since $S_n \sim \log(n)$. □

3.3. **Hypergeometric functions.** We shall give basic results on the Gauss hypergeometric functions. The formulae listed below will be crucial in the computations of the linearized operator associated to the V-state equations. Recall that $\forall (a,b,c) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\setminus(-\mathbb{N})$ the hypergeometric function $z \mapsto F(a,b,c;z)$ is defined on the open unit disc $\mathbb{D}$ by the power series

$$F(a,b,c;z) = \sum_{n=0}^{+\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\ \forall z\in\mathbb{D}.$$

Here, $(x)_n$ is the Pockhhammer symbol defined by,

$$(x)_n = \begin{cases} 1 & n=0 \\ x(x+1)\cdots(x+n-1) & n\geq 1. \end{cases}$$

One may easily see that

$$(x)_n = x(1+x)_{n-1},\ (x)_{n+1} = (x+n)(x)_n.$$

For a future use we recall an integral representation of the hypergeometric function, for instance see [31]. Assume that $c > b > 0$, then

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx,\ \forall z\in\mathbb{D}.$$

The function $\Gamma : \mathbb{C}\setminus(-\mathbb{N}) \to \mathbb{C}$ refers to the gamma function which is an analytic continuation to the negative half plane of the usual gamma function defined on the positive half-place $\{\mathrm{Re}(z) > 0\}$ by the integral representation,

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt.$$

Next, we recall some contiguous functions relations of the hypergeometric series, see [31].

$$cF(a,b,c;z) - cF(a+1,b,c;z) + bzF(a+1,b+1,c+1;z) = 0 \tag{3.1}$$

$$cF(a,b,c;z) - cF(a,b+1,c;z) + azF(a+1,b+1,c+1;z) = 0 \tag{3.2}$$

$$bF(a,b+1,c;z) - aF(a+1,b,c;z) + (a-b)(a,b,c;z) = 0 \tag{3.3}$$

$$cF(a,b,c;z) - (c-b)F(a,b,c+1;z) - bF(a,b+1,c+1;z) = 0 \tag{3.4}$$

We end this discussion with recalling Bessel function J_n of the first kind with $n \in \mathbb{N}$,

$$\forall z\in\mathbb{C},\quad J_n(z) = \sum_{k\geq 0}\frac{(-1)^k}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k+n}.$$

We recall the Sonine-Schafheitlin's formula which hold provided that $0 < b < a$ and the integral is convergent, see for example [31, p. 401],

$$\int_0^{+\infty} \frac{J_\mu(at)J_\nu(bt)}{t^\lambda}dt = \frac{a^{\lambda-\nu-1}b^\nu\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2})}{2^\lambda\Gamma(\nu+1)\Gamma(\frac{1}{2}\mu+\frac{1}{2}\lambda-\frac{1}{2}\nu+\frac{1}{2})} \times F\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}, \nu+1; \frac{b^2}{a^2}\right).$$

4. Regularity of the nonlinear functional

In this section we are going to check that the functionals G_j seen in (2.2) are well-defined and satisfy the regularity assumption required by Crandall-Rabinowitz's theorem. Recall that the exterior domains $\mathbb{C}\backslash D_j$ are parametrized by the conformal mappings Φ_j whose extension to the boundaries enjoy the following structure,

$$\forall\, \omega \in \mathbb{T}, \quad \Phi_1(\omega) = \omega + \sum_{n\in\mathbb{N}^\star} a_n\overline{\omega}^n = \omega + f_1(\omega) \text{ with } a_n \in \mathbb{R}.$$

$$\Phi_2(\omega) = b\omega + \sum_{n\in\mathbb{N}^\star} c_n\overline{\omega}^n = b\omega + f_2(\omega) \text{ with } c_n \in \mathbb{R}.$$

The parameter b belongs to $(0,1)$ which means that we are looking for V-states which are perturbation of the annulus centered at zero and of radius b and 1. Recall that the equations of the V-states are given by,

$$\forall \omega \in \mathbb{T},\ G_j(\Omega, f_1, f_2)(\omega) = 0,\ j = 1, 2,$$

where

$$G_j(\Omega, f_1, f_2)(\omega) = \mathrm{Im}\left\{(\Omega\Phi_j(\omega) - S(\Phi_1, \Phi_j)(\omega) + S(\Phi_2, \Phi_j)(\omega))\,\overline{\Phi_j'(\omega)\overline{\omega}}\right\} \tag{4.1}$$

with

$$S(\Phi_i, \Phi_j)(\omega) = \fint_{\mathbb{T}} \frac{\tau\Phi_i'(\tau) - \omega\Phi_j'(\omega)}{|\Phi_i(\tau) - \Phi_j(\omega)|}\frac{d\tau}{\tau}.$$

The study of the regularity of these functionals will be done in several steps. In the first step we shall analyze the existence of the functionals and in the second one establish some strong regularity.

4.1. Existence.

The main result of this section reads as follows.

Proposition 4.1. *For $j \in \{1,2\}$ and for any $k \geq 3$, there exists $r \in (0,1)$ such that,*

$$\begin{array}{rccc} G_j: & \mathbb{R}\times V_r\times V_r & \longrightarrow & Y^{k-1} \\ & (\Omega, f_1, f_2) & \longmapsto & G_j(\Omega, f_1, f_2) \end{array}$$

is well-defined. Where $V_r = \left\{f \in X^{k+\log},\ \ \|f\|_{X^{k+\log}} \leq r\right\}$.

The proof of this result is postponed later and is founded on the following lemma.

Lemma 4.2. *Let $\varepsilon \in (0,1), j \in \{1,2\}$, $V = b^{j-1}\mathrm{Id} + \tilde{V}$ with $\tilde{V} \in V_r$ and r small enough. Let $h \in V_r$, then the function*

$$K : \omega \in \mathbb{T} \mapsto \fint_{\mathbb{T}} \frac{\tau\partial_\tau h(\tau) - \omega\partial_\tau h(\omega)}{|V(\tau) - V(\omega)|}\frac{d\tau}{\tau}$$

can be extended analytically in C_ε to a function $\tilde{K}$ with $\tilde{K} \in \tilde{Y}^{k-1}$. In addition,

$$\|\tilde{K}\|_{\tilde{Y}^{k-1}} \leq C\Big(\|V\|_{H^k(\varepsilon\mathbb{T})} + \|V\|_{H^k(\varepsilon^{-1}\mathbb{T})}\Big)\|h\|_{X^{k+\log}}.$$

Before giving details of the proof we need to make a comment.

Remark 4.3. *Take*

$$h:\ \begin{array}{ccc}\mathbb{T} & \longrightarrow & \mathbb{C}\\ \tau & \longmapsto & \sum_{n=1}^{+\infty} a_n\overline{\tau}^n\end{array} \quad \text{and} \quad \tilde{h}:\ \begin{array}{ccc}C_\varepsilon & \longrightarrow & \mathbb{C}\\ z & \longmapsto & \sum_{n=1}^{+\infty}\frac{a_n}{z^n}\end{array}$$

then for any $\tau\in\mathbb{T}$, $z\in C_\varepsilon$

$$\partial_\tau h(\tau)=-\sum_{n=1}^{+\infty} n a_n\overline{\tau}^{n+1} \quad \text{and} \quad \partial_z\tilde{h}(z)=-\sum_{n=1}^{+\infty} n\frac{a_n}{z^{n+1}}.$$

Thus,

$$\partial_\tau h=\partial_z\tilde{h}\Big|_{\mathbb{T}}.$$

Proof. By change of variables, we may write

$$K(\omega)=\omega\fint_{\mathbb{T}}\frac{\tau\partial_\tau h(\tau\omega)-\partial_\tau h(\omega)}{|V(\tau\omega)-V(\omega)|}\frac{d\tau}{\tau}.$$

Our next task is to get a holomorphic extension of $\omega\mapsto|V(\tau\omega)-V(\omega)|$. For this aim we write for any $\tau,\omega\in\mathbb{T}$,

$$\begin{aligned}|V(\tau\omega)-V(\omega)|^2&=(V(\tau\omega)-V(\omega))\left(V(\overline{\tau}\omega^{-1})-V(\omega^{-1})\right)\\&=b^{2(j-1)}|\tau-1|^2\,g(\tau,\omega)\,g(\overline{\tau},\omega^{-1}),\end{aligned}$$

where g can be extended in a usual way as follows,

$$\forall z\in\overline{\mathbb{A}_\varepsilon},\quad g(\tau,z)=1+\frac{\tilde{V}(\tau z)-\tilde{V}(z)}{b^{j-1}z(\tau-1)}. \tag{4.2}$$

Therefore we get as a by-product,

$$\exists C>0,\forall\tau\in\mathbb{T},\forall z\in\overline{\mathbb{A}_\varepsilon},\quad C^{-1}\leq|g(\tau,z)|\leq C. \tag{4.3}$$

Now we shall use the following estimate,

$$|\tilde{V}(\tau z)-\tilde{V}(z)|\leq\varepsilon|\tau-1|\|\partial_z\tilde{V}\|_{L^\infty(\varepsilon\mathbb{T})}.$$

This follows from the mean value theorem combined with the maximum principle for holomorphic functions. Indeed, setting $\widehat{V}(z)=\tilde{V}(\frac{1}{z})$, which is holomorphic in the disc $D_{\frac{1}{\varepsilon}}=\left\{z\in\mathbb{C},|z|<\frac{1}{\varepsilon}\right\}$, we deduce by the mean value theorem that for any $z_1,z_2\in D_{\frac{1}{\varepsilon}}$,

$$|\widehat{V}(z_1)-\widehat{V}(z_2)|\leq|z_1-z_2|\|\partial_z\widehat{V}\|_{L^\infty(\overline{D_{\frac{1}{\varepsilon}}})}. \tag{4.4}$$

According to the maximum principle one readily gets

$$\begin{aligned}\|\partial_z\widehat{V}\|_{L^\infty(\overline{D_{\frac{1}{\varepsilon}}})}&=\|\partial_z\widehat{V}\|_{L^\infty(\varepsilon^{-1}\mathbb{T})}\\&=\varepsilon^2\|\partial_z\tilde{V}\|_{L^\infty(\varepsilon\mathbb{T})},\end{aligned}$$

Applying this inequality with $z_1=\frac{1}{\tau z}$ and $z_2=\frac{1}{z}$ for $z\in C_\varepsilon$ we deduce

$$|\tilde{V}(\tau z)-\tilde{V}(z)|\leq\varepsilon|\tau-1|\|\partial_z\tilde{V}\|_{L^\infty(\varepsilon\mathbb{T})}$$

which is the desired inequality. Using Sobolev embedding $X^{k+\log}\hookrightarrow\mathrm{Lip}(\varepsilon\mathbb{T})$ for $k\geq2$ we find

$$|\tilde{V}(\tau z)-\tilde{V}(z)|\leq C|\tau-1|\|\partial_z\tilde{V}\|_{X^{k-1+\log}}\leq C|\tau-1|\|\tilde{V}\|_{X^{k+\log}} \tag{4.5}$$

with C a constant depending on ε.

Consequently, one may find small r such that for $\tilde{V}\in V_r$ the function $z\in C_\varepsilon\mapsto g(\tau,z)g(\tau,z^{-1})$ is holomorphic and does not cross the negative real axis $\mathbb{R}_-$. This allows to define the square root of this latter

function, which remains in turn holomorphic in the same set C_ε. Finally, the holomorphic extension of K to C_ε could be

$$\begin{aligned}\tilde{K}(z) &= z \fint_{\mathbb{T}} \frac{\tau(\partial_z h)(\tau z) - (\partial_z h)(z)}{b^{j-1}|\tau - 1|} g(\tau, z)^{-\frac{1}{2}} g(\overline{\tau}, \frac{1}{z})^{-\frac{1}{2}} \frac{d\tau}{\tau} \\ &\triangleq z \fint_{\mathbb{T}} k(z, \tau) d\tau.\end{aligned}$$

It remains to check the holomorphic structure of this integral with respect to the complex parameter. Observe that for fixed $\tau \in \mathbb{T}\backslash\{1\}$ the function $z \in C_\varepsilon \mapsto k(\tau, z)$ is holomorphic. We also note that the mapping $\tau \in \mathbb{T}\backslash\{1\} \mapsto k(\tau, z)$ is bounded uniformly in $z \in C_\varepsilon$. This follows from the estimate

$$\begin{aligned}|\partial_z h(\tau z) - \partial_z h(z)| &\leq \varepsilon |\tau - 1| \|\partial_z^2 h\|_{L^\infty(\varepsilon\mathbb{T})} \\ &\leq C|\tau - 1| \|h\|_{H^3(\varepsilon\mathbb{T})} \\ &\leq C|\tau - 1| \|h\|_{X^{k+\log}}.\end{aligned}$$

Therefore in view of (4.3), we find a constant C such that for any $(z, \tau) \in \overline{C_\varepsilon} \times \mathbb{T}$

$$|k(z, \tau)| \leq C. \tag{4.6}$$

Consequently $\tilde{K}$ is analytic in the annulus C_ε and therefore it belongs to the class $\tilde{A}_\varepsilon$. Hence, it remains to check that $\tilde{K}$ has finite norm in $\tilde{Y}^{k-1}$. We shall start with the L^2 norm of the inner restriction $\omega \in \mathbb{T} \mapsto \tilde{K}(\varepsilon\omega)$. We observe that

$$\tilde{K}(\varepsilon\omega) = \varepsilon\omega \fint_{\mathbb{T}} k(\varepsilon\omega, \tau) d\tau.$$

It is obvious from (4.6) that

$$\tilde{K}(\varepsilon\cdot) \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$$

with

$$\|\tilde{K}(\varepsilon\cdot)\|_{L^2(\mathbb{T})} \leq C\|V\|_{H^2(\varepsilon\mathbb{T})} \|h\|_{H^3(\varepsilon\mathbb{T})}.$$

As to the estimate over the exterior boundary we proceed in the same way as before and we get

$$\|\tilde{K}(\frac{1}{\varepsilon}\cdot)\|_{L^2(\mathbb{T})} \leq C\|V\|_{H^2(\varepsilon\mathbb{T})} \|h\|_{H^3(\varepsilon\mathbb{T})}.$$

Now, we want to control the L^2 norm of $\partial_z^{k-1}\tilde{K}(\varepsilon^{\pm}\cdot)$. In what follows, we just give details about $\partial_z^{k-1}\tilde{K}(\varepsilon\cdot)$, we deal with the other term with similar ideas. The computations are very long and we shall focus only on the leading term of $\partial_z^{k-1}\tilde{K}$. From Leibniz formula we may write

$$\begin{aligned}\partial_z^{k-1}\tilde{K}(z) &= z \fint_{\mathbb{T}} \frac{(\partial_z^k h)(\tau z) - (\partial_z^k h)(z)}{b^{j-1}|\tau - 1|} g(\tau, z)^{-\frac{1}{2}} g(\overline{\tau}, \frac{1}{z})^{-\frac{1}{2}} \frac{d\tau}{\tau} \\ &+ z \fint_{\mathbb{T}} \frac{(\tau^k - 1)(\partial_z^k h)(\tau z)}{b^{j-1}|\tau - 1|} g(\tau, z)^{-\frac{1}{2}} g(\overline{\tau}, \frac{1}{z})^{-\frac{1}{2}} \frac{d\tau}{\tau} \\ &+ z \fint_{\mathbb{T}} \frac{\tau(\partial_z h)(\tau z) - (\partial_z h)(z)}{b^{j-1}|\tau - 1|} \partial_z^{k-1}\Big[g(\tau, z)^{-\frac{1}{2}} g(\overline{\tau}, \frac{1}{z})^{-\frac{1}{2}}\Big] \frac{d\tau}{\tau} + l.o.t. \\ &\triangleq zK_1(z) + zK_2(z) + zK_3(z) + l.o.t.\end{aligned}$$

We shall now check that the terms K_2 and K_3 can actually be included to the low order terms. Indeed, for K_2 we write according to (4.3),

$$\|K_2(\varepsilon\cdot)\|_{L^\infty(\mathbb{T})} \leq C\|\partial_z^k h(\varepsilon\cdot)\|_{L^2(\mathbb{T})}.$$

As to the third term K_3 we shall only extract some significant terms and the other ones are treated in a similar way. First, it is easy to get

$$\partial_z\Big[g(\tau, z)^{-\frac{1}{2}} g(\overline{\tau}, \frac{1}{z})^{-\frac{1}{2}}\Big] = -\frac{1}{2}\partial_z\Big(g(\tau, z) g(\overline{\tau}, \frac{1}{z})\Big) g(\tau, z)^{-\frac{3}{2}} g(\overline{\tau}, \frac{1}{z})^{-\frac{3}{2}}$$

and

$$\partial_z\Big(g(\tau,z)g(\overline{\tau},\frac{1}{z})\Big)=\frac{z(\tau-1)\partial_z\tilde{V}(\tau z)+z\Big(\partial_z\tilde{V}(\tau z)-\partial_z\tilde{V}(z)\Big)-\Big(\tilde{V}(\tau z)-\tilde{V}(z)\Big)}{z^2b^{j-1}(\tau-1)}g(\overline{\tau},\frac{1}{z})$$
$$+\frac{z\Big(\tilde{V}(\frac{\overline{\tau}}{z})-\tilde{V}(\frac{1}{z})\Big)-(\overline{\tau}-1)\partial_z\tilde{V}(\frac{\overline{\tau}}{z})-\Big(\partial_z\tilde{V}(\frac{\overline{\tau}}{z})-\partial_z\tilde{V}(\frac{1}{z})\Big)}{b^{j-1}z(\overline{\tau}-1)}g(\tau,z).$$

Thus

$$\begin{aligned}z\partial_z\Big(g(\tau,z)^{-\frac{1}{2}}g(\overline{\tau},\frac{1}{z})^{-\frac{1}{2}}\Big) &= -\frac{1}{2b^{j-1}}\frac{\partial_z\tilde{V}(\tau z)-\partial_z\tilde{V}(z)}{(\tau-1)}g(\tau,z)^{-\frac{3}{2}}g(\overline{\tau},\frac{1}{z})^{-\frac{1}{2}}\\ &+ \frac{1}{2b^{j-1}}\frac{\partial_z\tilde{V}(\frac{\overline{\tau}}{z})-\partial_z\tilde{V}(\frac{1}{z})}{(\overline{\tau}-1)}g(\tau,z)^{-\frac{1}{2}}g(\overline{\tau},\frac{1}{z})^{-\frac{3}{2}}+l.o.t.\end{aligned}$$

Iterating this procedure we find

$$\begin{aligned}z\partial_z^{k-1}\Big(g(\tau,z)^{-\frac{1}{2}}g(\overline{\tau},\frac{1}{z})^{-\frac{1}{2}}\Big) &= -\frac{1}{2b^{j-1}}\frac{\partial_z^{k-1}\tilde{V}(\tau z)-\partial_z^{k-1}\tilde{V}(z)}{(\tau-1)}g(\tau,z)^{-\frac{3}{2}}g(\overline{\tau},\frac{1}{z})^{-\frac{1}{2}}\\ &+ \frac{1}{2b^{j-1}}\frac{\partial_z^{k-1}\tilde{V}(\frac{\overline{\tau}}{z})-\partial_z^{k-1}\tilde{V}(\frac{1}{z})}{(\overline{\tau}-1)}g(\tau,z)^{-\frac{1}{2}}g(\overline{\tau},\frac{1}{z})^{-\frac{3}{2}}+l.o.t.\end{aligned}$$

It follows that

$$\begin{aligned}K_3(\varepsilon\omega)=&-\frac{1}{2b^{j-1}}\fint_{\mathbb{T}}\frac{\tau\partial_zh(\varepsilon\tau\omega)-\partial_zh(\varepsilon\omega)}{|\tau-1|}\frac{\partial_z^{k-1}\tilde{V}(\varepsilon\tau\omega)-\partial_z^{k-1}\tilde{V}(\varepsilon\omega)}{\tau-1}g(\tau,\varepsilon\omega)^{-\frac{3}{2}}g(\overline{\tau},\frac{1}{\varepsilon\omega})^{-\frac{1}{2}}\frac{d\tau}{\tau}\\ &+\frac{1}{2b^{j-1}}\fint_{\mathbb{T}}\frac{\tau\partial_zh(\varepsilon\tau\omega)-\partial_zh(\varepsilon\omega)}{|\tau-1|}\frac{\partial_z^{k-1}\tilde{V}(\frac{\overline{\tau}}{\varepsilon\omega})-\partial_z^{k-1}\tilde{V}(\frac{1}{\varepsilon\omega})}{\overline{\tau}-1}g(\tau,\varepsilon\omega)^{-\frac{1}{2}}g(\overline{\tau},\frac{1}{\varepsilon\omega})^{-\frac{3}{2}}\frac{d\tau}{\tau}+l.o.t.\end{aligned}$$

By the definition of Hölder spaces

$$|\partial_z^{k-1}\tilde{V}(\varepsilon^{\pm1}\tau\omega)-\partial_z^{k-1}\tilde{V}(\varepsilon^{\pm1}\omega)|\leq\varepsilon^{\pm\frac{1}{2}}\|\partial_z^{k-1}\tilde{V}\|_{C^{\frac{1}{2}}(\varepsilon^{\pm1}\mathbb{T})}|\tau-1|^{\frac{1}{2}}.$$

Thanks to

$$\int_{\mathbb{T}}\frac{1}{|\tau-1|^{\frac{1}{2}}}|d\tau|<+\infty$$

combined with (4.3) we obtain

$$\|K_3(\varepsilon\cdot)\|_{L^\infty(\mathbb{T})}\leq C\Big(\|\partial_zh\|_{L^\infty(\varepsilon\mathbb{T})}+\|\partial_z^2h\|_{L^\infty(\varepsilon\mathbb{T})}\Big)\Big(\|\partial_z^{k-1}\tilde{V}\|_{C^{\frac{1}{2}}(\varepsilon\mathbb{T})}+\|\partial_z^{k-1}\tilde{V}\|_{C^{\frac{1}{2}}(\varepsilon^{-1}\mathbb{T})}\Big)+...$$

Hence, using Sobolev embedding we get

$$\|K_3(\varepsilon\cdot)\|_{L^\infty(\mathbb{T})}\leq C\Big(\|V\|_{H^k(\varepsilon\mathbb{T})}+\|V\|_{H^k(\varepsilon^{-1}\mathbb{T})}\Big)\|h\|_{H^3(\varepsilon\mathbb{T})}.$$

Now let us move to the estimate of the term K_1 which is is the most singular one. For this goal we need the following lemma.

Lemma 4.4. *Let be $\varepsilon\in(0,1)$, $\tilde{V}\in\tilde{V}_r$ and r be small enough. Define for any $\tau\in\mathbb{T}$ and $z\in\varepsilon\mathbb{T}\cup\varepsilon^{-1}\mathbb{T}$*

$$g(\tau,z)=1+\frac{\tilde{V}(\tau z)-\tilde{V}(z)}{b^{j-1}z(\tau-1)}.$$

Then

$$g(\tau,z)^{-\frac{1}{2}}=\left(1+\frac{\partial_z\tilde{V}(z)}{b^{j-1}}\right)^{-\frac{1}{2}}+(\tau-1)H(\tau,z)$$

where $H(\cdot,\varepsilon^{\pm1}\cdot)\in L^\infty(\mathbb{T}\times\mathbb{T})$ and

$$\|H(\cdot,\varepsilon^{\pm1}\cdot)\|_{L^\infty(\mathbb{T}\times\mathbb{T})}\leq C\|\tilde{V}\|_{H^3(\varepsilon^{\pm1}\mathbb{T})}.$$

Proof. We shall only prove the result for $z \in \varepsilon\mathbb{T}$. Similar computations can be done for $z \in \varepsilon^{-1}\mathbb{T}$. From Taylor expansion at the second order we find,

$$\forall z \in \varepsilon\mathbb{T},\ g(\tau,z) = 1 + \frac{\partial_z \tilde{V}(z)}{b^{j-1}} + (\tau-1)H_1(\tau,z),$$

such that

$$|H_1(\tau,z)| \leq C\|\partial_z^2 \tilde{V}\|_{L^\infty(\varepsilon\mathbb{T})}.$$

Using Sobolev embeddings we get for $k \geq 3$ and $(\tau,z) \in \mathbb{T}\times\varepsilon\mathbb{T}$

$$|H_1(\tau,z)| \leq C\|\tilde{V}\|_{X^{k+\log}}.$$

Finally, from standard computations we obtain the identity

$$g(\tau,z)^{-\frac{1}{2}} = \left(1 + \frac{\partial_z \tilde{V}(z)}{b^{j-1}}\right)^{-\frac{1}{2}} + (\tau-1)H(\tau,z)$$

with

$$H(\tau,z) = -\frac{H_1(\tau,z)\left(\sqrt{1+\frac{\partial_z\tilde{V}(z)}{b^{j-1}}} + \sqrt{1+\frac{\partial_z\tilde{V}(z)}{b^{j-1}} + (\tau-1)H_1(\tau,z)}\right)^{-1}}{\sqrt{1+\frac{\partial_z\tilde{V}(z)}{b^{j-1}}}\sqrt{1+\frac{\partial_z\tilde{V}(z)}{b^{j-1}} + (\tau-1)H_1(\tau,z)}}.$$

One may easily check that

$$H(\cdot,\cdot) \in L^\infty(\mathbb{T}\times\varepsilon\mathbb{T})$$

and the desired result follows immediately by choosing the radius r small enough. □

Let us now see how to use the preceding lemma for estimating K_1. According to this lemma one may obtain a constant C depending on ε and b such that

$$\begin{aligned}\|K_1(\varepsilon\cdot)\|_{L^2(\mathbb{T})} \leq C\|(1+\frac{\partial_z\tilde{V}(\varepsilon\cdot)}{b^{j-1}})^{-\frac{1}{2}}(1+\frac{\partial_z\tilde{V}(\frac{1}{\varepsilon\cdot})}{b^{j-1}})^{-\frac{1}{2}}\|_{L^\infty(\mathbb{T})}\Big\|\fint_{\mathbb{T}}\frac{\partial_z^k h(\varepsilon\tau\cdot)-\partial_z^k h(\varepsilon\cdot)}{|\tau-1|}\frac{d\tau}{\tau}\Big\|_{L^2(\mathbb{T})}\\ + C\|\partial_z^k h(\varepsilon\cdot)\|_{L^2(\mathbb{T})} \leq C\|h\|_{X^{k+\log}}.\end{aligned}$$

This concludes the proof of the Lemma 4.2. □

Now, we are in position to give the proof of the Proposition 4.1.

Proof. Note that for any $\omega \in \mathbb{T}$ one has

$$G_j(\Omega,f_1,f_2)(\omega) = \frac{F_j(\omega) - F_j(\frac{1}{\omega})}{2i},$$

with

$$\begin{aligned}F_j(\omega) = \Omega\Phi_j(\omega)\Phi_j'\left(\frac{1}{\omega}\right)\frac{1}{\omega} - \Phi_j'\left(\frac{1}{\omega}\right)\frac{1}{\omega}\fint_{\mathbb{T}}\frac{\tau\partial_\tau\Phi_1(\tau)-\omega\partial_\tau\Phi_j(\omega)}{|\Phi_1(\tau)-\Phi_j(\omega)|}\frac{d\tau}{\tau}\\ + \Phi_j'\left(\frac{1}{\omega}\right)\frac{1}{\omega}\fint_{\mathbb{T}}\frac{\tau\partial_\tau\Phi_2(\tau)-\omega\partial_\tau\Phi_j(\omega)}{|\Phi_2(\tau)-\Phi_j(\omega)|}\frac{d\tau}{\tau}.\end{aligned}$$

We shall prove that F_j belongs to $\tilde{Y}^{k-1}$. The first term of the right-hand side describing the rotation term belongs to that space. The remaining terms are of two kinds: the self-induced terms and the interaction terms. For the first ones we simply use Lemma 4.2 with $h = V = \Phi_j$. As to the interaction terms, the integrand is nowhere singular because the interfaces do not intersect and therefore they are well estimated. We shall briefly give more explanation about this fact. Take the term

$$\widehat{K}(\omega) \triangleq \fint_{\mathbb{T}}\frac{\tau\Phi_2'(\tau)-\omega\Phi_1'(\omega)}{|\Phi_2(\tau)-\Phi_1(\omega)|}\frac{d\tau}{\tau} = \omega\fint_{\mathbb{T}}\frac{\tau\Phi_2'(\tau\omega)-\Phi_1'(\omega)}{|\Phi_2(\tau\omega)-\Phi_1(\omega)|}\frac{d\tau}{\tau}.$$

As before we write, for any $\tau, \omega \in \mathbb{T}$,

$$|\Phi_2(\tau\omega) - \Phi_1(\omega)| = |b\tau - 1|\Big(\tilde{g}(\tau,\omega)\tilde{g}(\overline{\tau},\omega^{-1})\Big)^{\frac{1}{2}}$$

where

$$\forall z \in \overline{\mathbb{A}_\varepsilon},\ \tilde{g}(\tau, z) = 1 + \frac{f_2(\tau z) - f_1(z)}{(b\tau - 1)z}.$$

From the maximum principe,

$$\Big|\frac{f_2(\tau z) - f_1(z)}{(b\tau - 1)z}\Big| \leq \varepsilon \frac{\|f_2(\varepsilon\cdot)\|_{L^\infty(\mathbb{T})} + \|f_1(\varepsilon\cdot)\|_{L^\infty(\mathbb{T})}}{1-b} \leq \varepsilon \frac{2r}{1-b}.$$

Hence

$$\widehat{K}(z) = z \fint_{\mathbb{T}} \frac{\tau\Phi_2'(\tau z) - \Phi_1'(z)}{|b\tau - 1|} \tilde{g}(\tau, z)^{-\frac{1}{2}} \tilde{g}(\overline{\tau}, \frac{1}{z})^{-\frac{1}{2}} \frac{d\tau}{\tau}.$$

Note that

$$0 < 1 - b \leq |b\tau - 1| \leq 1 + b$$

and consequently the integrand is less singular than those of the self-induced terms and thus one can find that $\widehat{K}$ is analytic in C_ε and belongs to $\tilde{Y}^{k-1}$. At this stage we have shown that F_j belongs to the space $\tilde{Y}^{k-1}$ and to achieve the proof of the proposition it remains to check that the Fourier coefficients of $G_j(\Omega, f_1, f_2)$ belong to $i\mathbb{R}$. By the assumptions, the Fourier coefficients of $\Phi_j = b^{j-1}Id + f_j$ are real and thus the coefficient of $\overline{\Phi_j'}$ are real too. From the stability of this property under the multiplication and the conjugation we deduce that the Fourier coefficients of $\omega \mapsto \Omega\Phi_j(\omega)\overline{\Phi_j'(\omega)}\overline{\omega}$ are real. To end the proof we shall check that the Fourier coefficients of $S(\Phi_i, \Phi_j)$ for $i, j \in \{1, 2\}$ are real. We have

$$S(\Phi_i, \Phi_j)(\omega) = \sum_{n\in\mathbb{Z}} a_n\omega^n,\ a_n = \fint_{\mathbb{T}} \frac{S(\Phi_i, \Phi_j)(\omega)}{\omega^{n+1}} d\omega = \fint_{\mathbb{T}}\fint_{\mathbb{T}} \frac{\tau\Phi_i'(\tau) - \omega\Phi_j'(\omega)}{|\Phi_i(\tau) - \Phi_j(\omega)|}\frac{d\tau}{\tau}\frac{d\omega}{\omega^{n+1}}.$$

The coefficient can also be written in the form

$$a_n = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{e^{i\theta}\Phi_i'(e^{i\theta}) - e^{i\eta}\Phi_j'(e^{i\eta})}{|\Phi_i(e^{i\theta}) - \Phi_j(e^{i\eta})|} e^{-in\eta} d\theta d\eta.$$

By taking the conjugate of a_n and using the properties:

$$\overline{\Phi_i(e^{i\theta})} = \Phi_i(e^{-i\theta}),\ \overline{\Phi_i'(e^{i\theta})} = \Phi_i'(e^{-i\theta}) \text{ and } |z| = |\overline{z}|.$$

One may obtain by change of variable

$$\begin{aligned}\overline{a_n} =& \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{e^{-i\theta}\Phi_i'(e^{-i\theta}) - e^{-i\eta}\Phi_j'(e^{-i\eta})}{|\Phi_i(e^{-i\theta}) - \Phi_j(e^{-i\eta})|} e^{in\eta} d\theta d\eta \\ =& \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{e^{i\theta}\Phi_i'(e^{i\theta}) - e^{i\eta}\Phi_j'(e^{i\eta})}{|\Phi_i(e^{i\theta}) - \Phi_j(e^{i\eta})|} e^{-in\eta} d\theta d\eta = a_n.\end{aligned}$$

Consequently the Fourier coefficients of $S(\Phi_i, \Phi_j)$ are real and therefore $G_j(\Omega, \Phi_1, \Phi_2)$ belongs to the space Y^{k-1} and the proof of Proposition 4.1 is now achieved. □

4.2. **Regularity.** The goal of this section is to study the strong regularity of G_j and the main result reads as follows.

Proposition 4.5. *For $j \in \{1, 2\}$ and for any $k \geq 3$, there exists $r \in (0, 1)$ such that,*

$$\begin{array}{rccc} G_j: & \mathbb{R}\times V_r\times V_r & \longrightarrow & Y^{k-1} \\ & (\Omega, f_1, f_2) & \longmapsto & G_j(\Omega, f_1, f_2)\end{array}$$

is of class C^1, where $V_r = \big\{f \in X^{k+\log}|\ \|f\|_{X^{k+\log}} \leq r\big\}$.

Proof. To prove that G_j is of class C^1 we shall first check the existence of its Gâteaux derivative. Second, we will show that this derivative is strongly continuous, and therefore it will necessary coincide with the Fréchet derivative. This will answer the C^1 regularity. We split G_j into two terms, the self-induced term and the interaction term,

$$G_j(\Omega, f_1, f_2) = O_j(\Omega, f_j) + N_j(f_1, f_2),\ j \in \{1, 2\}$$

with

$$\forall \omega \in \mathbb{T},\ O_j(\Omega, f_j)(\omega) \triangleq \mathrm{Im}\left\{ \left(\Omega\Phi_j(\omega) + (-1)^j S(\Phi_j, \Phi_j)(\omega)\right) \overline{\omega}\overline{\Phi_j'(\omega)} \right\}$$

and

$$N_j(f_1, f_2)(\omega) \triangleq (-1)^{j-1}\mathrm{Im}\left\{ S(\Phi_i, \Phi_j)(\omega)\overline{\omega}\overline{\Phi_j'(\omega)} \right\},\ i \neq j.$$

The Gâteaux derivative of G_j at (f_1, f_2) in the direction (h_1, h_2) is given by the formula:

$$\begin{aligned} DG_j(\Omega, f_1, f_2)(h_1, h_2) &= DO_j(\Omega, f_j)h_j + DN_j(f_1, f_2)(h_1, h_2) \\ &\triangleq \lim_{t\to 0}\frac{1}{t}\left[O_j(\Omega, f_j + th_j) - O_j(\Omega, f_j)\right] + \lim_{t\to 0}\frac{1}{t}\left[N_j(f_1 + th_1, f_2 + th_2) - N_j(f_1, f_2)\right] \\ &= \frac{d}{dt}|_{t=0} O_j(\Omega, f_j + th_j) + \left.\frac{d}{dt}\right|_{t=0} N_j(f_1 + th_1, f_2 + th_2), \end{aligned} \tag{4.7}$$

where the limits are taken in the strong topology of Y^{k-1}. Once we have checked the existence of these quantities, it remains to verify that the functions,

$$F_1(t, \omega) \triangleq \frac{1}{t}\Big[O_j(\Omega, f_j + th_j)(\omega) - O_j(\Omega, f_j)(\omega)\Big] - \left.\frac{d}{dt}\right|_{t=0} O_j(\Omega, f_j + th_j)(\omega)$$

and

$$F_2(t, \omega) \triangleq \frac{1}{t}\Big[N_j(\Omega, f_1 + th_1, f_2 + th_2)(\omega) - N_j(\Omega, f_1, f_2)(\omega)\Big] - \left.\frac{d}{dt}\right|_{t=0} N_j(\Omega, f_1 + th_1, f_2 + th_2)(\omega)$$

can be analytically extended on C_ε, and their extension, still denoted by F_j, satisfy

$$\lim_{t\to 0}\|F_j(t)\|_{Y^{k-1}} = 0.$$

The existence of Gâteaux derivative can be done in a straightforward way and one readily gets

$$\begin{aligned} DO_j(\Omega, f_j)h_j(\omega) &= \mathrm{Im}\left\{ \Omega\left(\Phi_j(\omega)\overline{h_j'(\omega)} + \overline{\Phi_j'(\omega)}h_j(\omega)\right)\overline{\omega} + (-1)^j\overline{h_j'(\omega)}\overline{\omega}\oint_{\mathbb{T}} \frac{\tau\Phi_j'(\tau) - \omega\Phi_j'(\omega)}{|\Phi_j(\omega) - \Phi_j(\tau)|}\frac{d\tau}{\tau} \right\} \\ &+ (-1)^{j-1}\mathrm{Im}\left\{ \overline{\Phi_j'(\omega)}\overline{\omega}\oint_{\mathbb{T}} \frac{(\tau\Phi_j'(\tau) - \omega\Phi_j'(\omega))\mathrm{Re}\left(\left(\overline{h_j(\tau) - h_j(\omega)}\right)\left(\Phi_j(\tau) - \Phi_j(\omega)\right)\right)}{|\Phi_j(\tau) - \Phi_j(\omega)|^3}\frac{d\tau}{\tau} \right\} \\ &+ (-1)^j\mathrm{Im}\left\{ \overline{\Phi_j'(\omega)}\overline{\omega}\oint_{\mathbb{T}} \frac{\tau h_j'(\tau) - \omega h_j'(\omega)}{|\Phi_j(\omega) - \Phi_j(\tau)|}\frac{d\tau}{\tau} \right\} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} DN_j(\Omega, f_1, f_2)(h_1, h_2)(\omega) &= (-1)^{j-1}\mathrm{Im}\Bigg\{ \overline{h_j'(\omega)}\overline{\omega}\oint_{\mathbb{T}} \frac{\tau\Phi_i'(\tau) - \omega\Phi_j'(\omega)}{|\Phi_i(\omega) - \Phi_j(\tau)|}\frac{d\tau}{\tau} \\ &+ \overline{\Phi_j'(\omega)}\overline{\omega}\oint_{\mathbb{T}} \frac{\tau h_i'(\tau) - \omega h_j'(\omega)}{|\Phi_i(\omega) - \Phi_j(\tau)|}\frac{d\tau}{\tau} \\ &- \overline{\Phi_j'(\omega)}\overline{\omega}\oint_{\mathbb{T}} \frac{[\tau\Phi_i'(\tau) - \omega\Phi_j'(\omega)]\mathrm{Re}\left(\left(\overline{h_i(\tau) - h_j(\omega)}\right)\left(\Phi_i(\tau) - \Phi_j(\omega)\right)\right)}{|\Phi_i(\tau) - \Phi_j(\omega)|^3}\frac{d\tau}{\tau} \Bigg\}. \end{aligned} \tag{4.9}$$

First we note that $F_1(t,\omega)$ can be written in the form

$$F_1(t,\omega)=\sum_{l=1}^{5}\frac{I_l(t,\omega)-I_l(t,\omega^{-1})}{2i}$$

with

$$\begin{aligned}
I_1(t,\omega) &= \frac{\overline{\Phi_j'(\omega)}\overline{\omega}}{t}\oint_{\mathbb{T}}[\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)]\left[\frac{1}{\Delta^t_{\Phi_j}(h_j)(\tau,\omega)}-\frac{1}{\Delta\Phi_j(\tau,\omega)}\right]\frac{d\tau}{\tau}\\
&\quad+\overline{\Phi_j'(\omega)}\overline{\omega}\oint_{\mathbb{T}}[\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)]\left[\frac{\mathrm{Re}\Big(\Big(\overline{h_j(\tau)-h_j(\omega)}\Big)\Big(\Phi_j(\tau)-\Phi_j(\omega)\Big)\Big)}{[\Delta\Phi_j(\tau,\omega)]^3}\right]\frac{d\tau}{\tau},
\end{aligned}$$

$$\begin{aligned}
I_2(t,\omega) &= \overline{\Phi_j'(\omega)}\overline{\omega}\oint_{\mathbb{T}}[\tau h_j'(\tau)-\omega h_j'(\omega)]\left[\frac{1}{\Delta^t_{\Phi_j}(h_j)(\tau,\omega)}-\frac{1}{\Delta\Phi_j(\tau,\omega)}\right]\frac{d\tau}{\tau},\\
I_3(t,\omega) &= \overline{h_j'(\omega)}\overline{\omega}\oint_{\mathbb{T}}[\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)]\left[\frac{1}{\Delta^t_{\Phi_j}(h_j)(\tau,\omega)}-\frac{1}{\Delta\Phi_j(\tau,\omega)}\right]\frac{d\tau}{\tau},\\
I_4(t,\omega) &= t\overline{h_j'(\omega)}\overline{\omega}\oint_{\mathbb{T}}\frac{\tau h_j'(\tau)-\omega h_j'(\omega)}{\Delta^t_{\Phi_j}(h_j)(\tau,\omega)}\frac{d\tau}{\tau},\\
I_5(t,\omega) &= t\Omega\overline{\omega}\overline{h_j'(\omega)}h_j(\omega).
\end{aligned}$$

We have use the following notations,

$$\Delta\Phi_j(\tau,\omega)=|\Phi_j(\tau)-\Phi_j(\omega)|$$

and

$$\Delta^t_{\Phi_j}(h_j)(\tau,\omega)=|\Phi_j(\tau)+th_j(\tau)-\Phi_j(\omega)-th_j(\omega)|.$$

First, it is not difficult to check the following limit

$$\lim_{t\to 0}\|I_5(t)\|_{\tilde{Y}^{k-1}}=0$$

Moreover, if t is small enought, one may use the Lemma 4.2 with $h=h_j$ and $V=\Phi_j+th_j$ to etablish

$$\lim_{t\to 0}\|I_4(t)\|_{\tilde{Y}^{k-1}}=0$$

We have to rewrite the terms I_1, I_2 and I_3 to compute theirs limits. We begin to rewrite one part of the integrand term:

$$\frac{1}{\Delta^t_{\Phi_j}(h_j)(\tau,\omega)}-\frac{1}{\Delta\Phi_j(\tau,\omega)}=\frac{\Big(\Delta\Phi_j(\tau,\omega)\Big)^2-\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)^2}{\Big(\Delta\Phi_j(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)}. \tag{4.10}$$

Then we display the dependency on t in the numerator

$$\Big(\Delta\Phi_j(\tau,\omega)\Big)^2-\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)^2=-t\left[2\mathrm{Re}\Big(\Big(\overline{h_j(\tau)-h_j(\omega)}\Big)\Big(\Phi_j(\tau)-\Phi_j(\omega)\Big)\Big)\right]-t^2|h_j(\tau)-h_j(\omega)|^2. \tag{4.11}$$

Moreover, straightforward manipulations lead to the following identity usefull for the term I_3

$$\begin{aligned}
&\frac{1}{(\Delta\Phi_j(\tau,\omega))^3}-\frac{2}{(\Delta\Phi_j(\tau,\omega))\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)}=\\
&\frac{\Big[\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big]^2-[\Delta\Phi_j(\tau,\omega)]^2}{[\Delta\Phi_j(\tau,\omega)]^3\Big[\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big]\Big[\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big]}+\frac{\Big[\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big]^2-[\Delta\Phi_j(\tau,\omega)]^2}{\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)(\Delta\Phi_j(\tau,\omega))^2\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)^2}.
\end{aligned} \tag{4.12}$$

Thanks to (4.10),(4.11),we rewrite the terms I_2 and I_3 .

$$
\begin{aligned}
I_2(t,\omega) &= -t^2\overline{\Phi_j'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{\left[\tau h_j'(\tau)-\omega h_j'(\omega)\right]|h_j(\tau)-h_j(\omega)|^2}{\Big(\Delta\Phi_j(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)}\frac{d\tau}{\tau},\\
&\quad -t\overline{\Phi_j'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{\left[\tau h_j'(\tau)-\omega h_j'(\omega)\right]\left[2\mathrm{Re}\Big(\overline{\Big(h_j(\tau)-h_j(\omega)\Big)}\Big(\Phi_j(\tau)-\Phi_j(\omega)\Big)\Big)\right]}{\Big(\Delta\Phi_j(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)}\frac{d\tau}{\tau},\\
I_3(t,\omega) &= -t^2\overline{h_j'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{\left[\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)\right]|h_j(\tau)-h_j(\omega)|^2}{\Big(\Delta\Phi_j(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)}\frac{d\tau}{\tau}\\
&\quad -t\overline{h_j'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{\left[\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)\right]\left[2\mathrm{Re}\Big(\overline{\Big(h_j(\tau)-h_j(\omega)\Big)}\Big(\Phi_j(\tau)-\Phi_j(\omega)\Big)\Big)\right]}{\Big(\Delta\Phi_j(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)}\frac{d\tau}{\tau}
\end{aligned}
$$

Moreover, using in addition 4.12 we can also rewrite I_1.

$$
\begin{aligned}
I_1(t,\omega) &= -t\overline{\Phi_j'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{\left[\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)\right]|h_j(\tau)-h_j(\omega)|^2}{\Big(\Delta\Phi_j(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)}\frac{d\tau}{\tau}\\
&\quad -t^2\overline{\Phi_j'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{\left[\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)\right]\left[\mathrm{Re}\Big(\overline{\Big(h_j(\tau)-h_j(\omega)\Big)}\Big(\Phi_j(\tau)-\Phi_j(\omega)\Big)\Big)\right]|h_j(\tau)-h_j(\omega)|^2}{\Big(\Delta\Phi_j(\tau,\omega)\Big)^3\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)}\frac{d\tau}{\tau}\\
&\quad -t^2\overline{\Phi_j'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{\left[\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)\right]\left[\mathrm{Re}\Big(\overline{\Big(h_j(\tau)-h_j(\omega)\Big)}\Big(\Phi_j(\tau)-\Phi_j(\omega)\Big)\Big)\right]|h_j(\tau)-h_j(\omega)|^2}{\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta\Phi_j(\tau,\omega)\Big)^2\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)^2}\frac{d\tau}{\tau}\\
&\quad -2t\overline{\Phi_j'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{\left[\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)\right]\left[\mathrm{Re}\Big(\overline{\Big(h_j(\tau)-h_j(\omega)\Big)}\Big(\Phi_j(\tau)-\Phi_j(\omega)\Big)\Big)\right]^2}{\Big(\Delta\Phi_j(\tau,\omega)\Big)^3\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)}\frac{d\tau}{\tau}\\
&\quad -2t\overline{\Phi_j'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{\left[\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)\right]\left[\mathrm{Re}\Big(\overline{\Big(h_j(\tau)-h_j(\omega)\Big)}\Big(\Phi_j(\tau)-\Phi_j(\omega)\Big)\Big)\right]^2}{\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)\Big(\Delta\Phi_j(\tau,\omega)\Big)^2\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)^2}\frac{d\tau}{\tau}
\end{aligned}
$$

One may see that we just need to check that the integral term of $I_i(t,\omega)$ belongs to $\tilde{Y}^{k-1}$. We introduce a model integral term, the others term are controled in a similarly way. For any $\omega\in\mathbb{T}$,

$$
\begin{aligned}
P(\omega) &\triangleq \fint_{\mathbb{T}}\frac{\Big(\tau\Phi_j'(\tau)-\omega\Phi_j'(\omega)\Big)\Big(h(\tau)-h(\omega)\Big)\Big(h(\overline{\tau})-h(\overline{\omega})\Big)}{\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)+\Delta\Phi_j(\tau,\omega)\Big)\Big(\Delta\Phi_j(\tau,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau,\omega)\Big)}\frac{d\tau}{\tau}\\
&= \omega\fint_{\mathbb{T}}\frac{\Big((\tau-1)\Phi_j'(\tau\omega)+\Phi_j'(\tau\omega)-\Phi_j'(\omega)\Big)\Big(h_j(\tau\omega)-h_j(\omega)\Big)\Big(h_j(\frac{\overline{\tau}}{\omega})-h_j(\frac{1}{\omega})\Big)}{\Big(\Delta^t_{\Phi_j}(h_j)(\tau\omega,\omega)+\Delta\Phi_j(\tau\omega,\omega)\Big)\Big(\Delta\Phi_j(\tau\omega,\omega)\Big)\Big(\Delta^t_{\Phi_j}(h_j)(\tau\omega,\omega)\Big)}\frac{d\tau}{\tau}
\end{aligned}
$$

Following the same idea of the Lemma 4.2, we can write

$$
\frac{1}{\Delta\Phi_j(\tau\omega,\omega)+\Delta^t_{\Phi_j}(h_j)(\tau\omega,\omega)}=\frac{1}{b^{j-1}|\tau-1|}\frac{1}{\sqrt{g_j(\tau,\omega)g_j(\overline{\tau},\frac{1}{\omega})}+\sqrt{\tilde{g}_j(\tau,\omega)\tilde{g}_j(\overline{\tau},\frac{1}{\omega})}}.
$$

Where g_j and $\tilde{g}_j$ can be extended in the usual ways as follows,

$$\forall z \in \overline{\mathbb{A}_\varepsilon},\ g_j(\tau,z) = 1 + \frac{f_j(\tau z) - f_j(z)}{b^{j-1} z(\tau-1)} \text{ and } \tilde{g}_j(\tau,z) = g_j(\tau,z) + t\frac{h_j(\tau z) - h_j(z)}{b^{j-1} z(\tau-1)}.$$

As before we can extend P analytically in C_ε and control the L^2 norm of the inner restriction $\omega \in \mathbb{T} \mapsto P(\varepsilon^{\pm 1}\omega)$. We just give few details to control the L^2 norm of the leading term of $\partial_z^{k-1}P(\varepsilon\cdot)$, the proof for the control of $\partial_z^{k-1}P(\varepsilon^{-1}\cdot)$ is similar. Using the same arguments than before we may write for $z \in \overline{\mathcal{C}_\varepsilon}$

$$\partial_z^{k-1}P(z) = z \fint_{\mathbb{T}} \frac{\Big((\partial_z^k\Phi_j)(\tau z) - (\partial_z^k\Phi_j)(z)\Big)\Big(h_j(\tau z) - h_j(z)\Big)\Big(h_j(\frac{\overline{\tau}}{z}) - h_j(\frac{1}{z})\Big) g_j(\tau,z)^{-\frac{1}{2}} g_j(\overline{\tau},\frac{1}{z})^{-\frac{1}{2}}}{b^{3(j-1)}|\tau-1|^3\Big(\sqrt{g_j(\tau,z)g_j(\overline{\tau},\frac{1}{z})} + \sqrt{\tilde{g}_j(\tau,z)\tilde{g}_j(\overline{\tau},\frac{1}{z})}\Big)\sqrt{\tilde{g}_j(\tau,z)\tilde{g}_j(\overline{\tau},\frac{1}{z})}} \frac{d\tau}{\tau} + l.o.t$$

Applying the Lemma 4.4 with $\tilde{V}_1 = f_j$ and $\tilde{V}_2 = f_j + th_j$ we can etablish for $z \in \varepsilon\mathbb{T} \cup \varepsilon^{-1}\mathbb{T}$ the following identity

$$\frac{\Big(\sqrt{g(\tau,z)g(\overline{\tau},\frac{1}{z})} + \sqrt{\tilde{g}(\tau,z)\tilde{g}(\overline{\tau},\frac{1}{z})}\Big)^{-1}}{|\tau-1|} = \frac{(\tau-1)}{|\tau-1|}H(\tau,z)$$
$$+ \frac{\Big(\sqrt{(1+\frac{(\partial_z f_j)(z)}{b^{j-1}})(1+\frac{(\partial_z f_j)(\frac{1}{z})}{b^{j-1}})} + \sqrt{(1+\frac{\big(\partial_z(f_j+th_j)\big)(z)}{b^{j-1}})\big(1+\frac{\big(\partial_z(f_j+th_j)\big)(\frac{1}{z})}{b^{j-1}}\big)}\Big)^{-1}}{|\tau-1|}.$$

With $H(\cdot,\varepsilon^{\pm}\cdot) \in L^\infty(\mathbb{T}\times\mathbb{T})$.
Consequently, this identity allows us to deal with the leading term of $\partial_z^{k-1}P$. It follows for $l \in \{1,2,3\}$

$$\lim_{t\to 0}\|I_l\|_{\tilde{Y}^{k-1}} = 0.$$

Eventually, we have proved than F_1 can be extended analytically on C_ε and

$$\lim_{t\to 0}\|F_1\|_{Y^{k-1}} = 0.$$

Moreover, the interaction term F_2 is dealed with the same arguments than the regularity of the interaction term. Our next task is to prove that

$$DG_j : \mathbb{R}\times V_r \times V_r \to \mathcal{L}(\mathbb{R}\times X^{k+\log}\times X^{k+\log}, Y^{k-1})$$

is well-defined and continuous.
For the first part, the non trivial point is that $\forall(\Omega,f_1,f_2) \in \mathbb{R}\times V_r\times V_r$, $DG_j(\Omega,f_1,f_2) \in \mathcal{L}(\mathbb{R}\times X^{k+\log}\times X^{k+\log}, Y^{k-1})$. The linearity is obvious.
As before, we just give details about the continuity of the self-induced term $DO_j(\Omega,f_j)$. To begin we rewrite

$$DO_j(\Omega,f_j)(h_j)(\omega) = \sum_{i=1}^4 \frac{\hat{I}_i(\omega) - \hat{I}_i(\frac{1}{\omega})}{2i}$$

with

$$\hat{I}_1(\omega) = \Omega\frac{1}{\omega}\left[\Phi_j(\omega)h_j'(\frac{1}{\omega}) + h_j(\omega)\Phi_j'(\frac{1}{\omega})\right],$$
$$\hat{I}_2(\omega) = (-1)^j h_j'(\frac{1}{\omega})\frac{1}{\omega}\fint_{\mathbb{T}} \frac{\tau\Phi_j'(\tau) - \omega\Phi_j'(\omega)}{\Delta\Phi_j(\tau,\omega)}\frac{d\tau}{\tau},$$
$$\hat{I}_3(\omega) = (-1)^j \Phi_j'(\frac{1}{\omega})\frac{1}{\omega}\fint_{\mathbb{T}} \frac{\tau h_j'(\tau) - \omega h_j'(\omega)}{\Delta\Phi_j(\tau,\omega)}\frac{d\tau}{\tau},$$

and

$$\hat{I}_4(\omega) = -(-1)^j\Phi_j'(\frac{1}{\omega})\frac{1}{\omega}\fint_{\mathbb{T}} \frac{(\tau\Phi_j'(\tau) - \omega\Phi_j'(\omega))\mathrm{Re}\Big(\Big(\overline{h_j(\tau) - h_j(\omega)}\Big)\Big(\Phi_j(\tau) - \Phi_j(\omega)\Big)\Big)}{\big(\Delta\Phi_j(\tau,\omega)\big)^3}\frac{d\tau}{\tau}.$$

Using the Lemma 4.2 and an adaptation, one may find a constant C such that for $p \in \{1,2,3,4\}$ the following estimate is checked

$$\|\hat{I}_p\|_{\tilde{Y}^{k-1}} \leq C\|\Phi_j\|_{X^{k+\log}}\|h_j\|_{X^{k+\log}}.$$

Consequently, DG_j is well-defined. The continuity of DG_j is the final point of the proof. We just explain the continuity of $DO_j(\Omega,\cdot)$. Let be $f_j, \tilde{f}_j \in V_r \times V_r$ and $h_j \in X^{k+\log}$ with $\|h_j\|_{X^{k+\log}} = 1$, we have for any $\omega \in \mathbb{T}$

$$DO_j(\Omega, f_j)(h_j)(\omega) - DO_j(\Omega, \tilde{f}_j)(h_j)(\omega) = \sum_{i=1}^{9} \frac{\tilde{I}_i(\omega) - \tilde{I}_i(\frac{1}{\omega})}{2i}$$

with

$$\begin{aligned}
\tilde{I}_1(\omega) &= \frac{\Omega}{\omega}\Big(\Phi_j(\omega) - \tilde{\Phi}_j(\omega)\Big)h_j'(\frac{1}{\omega}) + \frac{\Omega}{\omega}\Big(\Phi_j'(\frac{1}{\omega}) - \tilde{\Phi}_j'(\frac{1}{\omega})\Big)h_j(\omega),\\
\tilde{I}_2(\omega) &= \frac{(-1)^j}{\omega}h_j'(\frac{1}{\omega}) \fint_{\mathbb{T}} \frac{\tau\Big(\Phi_j'(\tau) - \tilde{\Phi}_j'(\tau)\Big) - \omega\Big(\Phi_j'(\omega) - \tilde{\Phi}_j'(\omega)\Big)}{\Delta\Phi_j(\tau,\omega)}\frac{d\tau}{\tau},\\
\tilde{I}_3(\omega) &= \frac{(-1)^j}{\omega}h_j'(\frac{1}{\omega}) \fint_{\mathbb{T}} \Big[\tau\tilde{\Phi}_j'(\tau) - \omega\tilde{\Phi}_j'(\omega)\Big]\left[\frac{1}{\Delta\Phi_j(\tau,\omega)} - \frac{1}{\Delta\tilde{\Phi}_j(\tau,\omega)}\right]\frac{d\tau}{\tau},\\
\tilde{I}_4(\omega) &= \frac{(-1)^j}{\omega}\Big(\Phi_j'(\frac{1}{\omega}) - \tilde{\Phi}_j'(\frac{1}{\omega})\Big) \fint_{\mathbb{T}} \frac{\tau h_j'(\tau) - \omega h_j'(\omega)}{\Delta\Phi_j(\tau,\omega)}\frac{d\tau}{\tau},\\
\tilde{I}_5(\omega) &= \frac{(-1)^j}{\omega}\tilde{\Phi}_j'(\frac{1}{\omega}) \fint_{\mathbb{T}} \Big(\tau h_j'(\tau) - \omega h_j'(\omega)\Big)\left[\frac{1}{\Delta\Phi_j(\tau,\omega)} - \frac{1}{\Delta\tilde{\Phi}_j(\tau,\omega)}\right]\frac{d\tau}{\tau},
\end{aligned}$$

$$\tilde{I}_6(\omega) = \frac{(-1)^j}{\omega}\Big(\tilde{\Phi}_j'(\frac{1}{\omega}) - \Phi_j'(\frac{1}{\omega})\Big) \fint_{\mathbb{T}} \frac{\Big(\tau\tilde{\Phi}_j'(\tau) - \omega\tilde{\Phi}_j'(\omega)\Big)\mathrm{Re}\Big(\Big(\overline{h_j(\omega) - h_j(\tau)}\Big)\Big(\tilde{\Phi}_j(\omega) - \tilde{\Phi}_j(\tau)\Big)\Big)}{\big(\Delta\tilde{\Phi}_j(\tau,\omega)\big)^3}\frac{d\tau}{\tau},$$

$$\tilde{I}_7(\omega) = \frac{(-1)^j}{\omega}\Phi_j'(\frac{1}{\omega}) \fint_{\mathbb{T}} \frac{\Big(\tau\Big(\tilde{\Phi}_j'(\tau) - \Phi_j'(\tau)\Big) - \omega\Big(\tilde{\Phi}_j'(\omega) - \Phi_j'(\omega)\Big)\Big)\mathrm{Re}\Big(\Big(\overline{h_j(\omega) - h_j(\tau)}\Big)\Big(\tilde{\Phi}_j(\omega) - \tilde{\Phi}_j(\tau)\Big)\Big)}{\big(\Delta\tilde{\Phi}_j(\tau,\omega)\big)^3}\frac{d\tau}{\tau},$$

$$\tilde{I}_8(\omega) = \frac{(-1)^j}{\omega}\Phi_j'(\frac{1}{\omega}) \fint_{\mathbb{T}} \frac{\Big(\tau\Phi_j'(\tau) - \omega\Phi_j'(\omega)\Big)\mathrm{Re}\Big(\Big(\overline{h_j(\omega) - h_j(\tau)}\Big)\Big(\big(\tilde{\Phi}_j(\omega) - \Phi_j(\omega)\big) - \big(\tilde{\Phi}_j(\tau) - \Phi_j(\tau)\big)\Big)\Big)}{\big(\Delta\tilde{\Phi}_j(\tau,\omega)\big)^3}\frac{d\tau}{\tau}$$

and

$$\begin{aligned}
\tilde{I}_9(\omega) = \frac{(-1)^j}{\omega}\Phi_j'(\frac{1}{\omega}) \fint_{\mathbb{T}} \big(\tau\Phi_j'(\tau) - \omega\Phi_j'(\omega)\big)\mathrm{Re}\Big(\Big(\overline{h_j(\omega) - h_j(\tau)}\Big)\Big(\Phi_j(\omega) - \Phi_j(\tau)\Big)\Big)\times\\
\left[\frac{1}{\big(\Delta\tilde{\Phi}_j(\tau,\omega)\big)^3} - \frac{1}{\big(\Delta\Phi_j(\tau,\omega)\big)^3}\right]\frac{d\tau}{\tau}.
\end{aligned}$$

For $p \in \{1,2,4,6,7,8\}$, one may extend $\tilde{I}_p$ as before. The control of the $\tilde{Y}^{k-1}$ norm leads on the lemma 4.2 and an adaptation, we can find a constant C such that

$$\|\tilde{I}_p\|_{\tilde{Y}^{k-1}} \leq C\|\Phi_j - \tilde{\Phi}_j\|_{X^{k+\log}}.$$

We give few details for the integral term of $\tilde{I}_3$. As

$$\frac{1}{\Delta\Phi_j(\tau,\omega)}-\frac{1}{\Delta\tilde{\Phi}_j(\tau,\omega)}=\frac{\Big(\tilde{\Phi}_j(\tau)-\tilde{\Phi}_j(\omega)\Big)\Big(\Big(\tilde{\Phi}_j(\overline{\tau})-\Phi_j(\overline{\tau})\Big)-\Big(\tilde{\Phi}_j(\frac{1}{\omega})-\Phi_j(\frac{1}{\omega})\Big)\Big)}{\big(\Delta\Phi_j(\tau,\omega)\big)\big(\Delta\tilde{\Phi}_j(\tau,\omega)\big)\big(\Delta\Phi_j(\tau,\omega)+\Delta\tilde{\Phi}_j(\tau,\omega)\big)}$$
$$+\frac{\Big(\Phi_j(\overline{\tau})-\Phi_j(\frac{1}{\omega})\Big)\Big(\tilde{\Phi}_j(\tau)-\Phi_j(\tau)-\tilde{\Phi}_j(\omega)+\Phi_j(\omega)\Big)}{\big(\Delta\Phi_j(\tau,\omega)\big)\big(\Delta\tilde{\Phi}_j(\tau,\omega)\big)\big(\Delta\Phi_j(\tau,\omega)+\Delta\tilde{\Phi}_j(\tau,\omega)\big)}.$$

The integral can be split in two terms and we just give few details for one. We deal with the other in the same way. After a change of variabe, we shall extend and control the term

$$\tilde{\tilde{I}}_3(\omega)\triangleq\omega\fint_{\mathbb{T}}\frac{\Big(\tau\tilde{\Phi}_j'(\tau\omega)-\tilde{\Phi}_j'(\omega)\Big)\Big(\tilde{\Phi}_j(\tau\omega)-\tilde{\Phi}_j(\omega)\Big)\Big(\Big(\tilde{\Phi}_j(\frac{\overline{\tau}}{\omega})-\Phi_j(\frac{\overline{\tau}}{\omega})\Big)-\Big(\tilde{\Phi}_j(\frac{1}{\omega})-\Phi_j(\frac{1}{\omega})\Big)\Big)}{\big(\Delta\Phi_j(\tau\omega,\omega)\big)\Big(\Delta\tilde{\Phi}_j(\tau\omega,\omega)\Big)\Big(\Delta\Phi_j(\tau\omega,\omega)+\Delta\tilde{\Phi}_j(\tau\omega,\omega)\Big)}\frac{d\tau}{\tau}.$$

As before, we can write

$$\Delta\Phi_j(\tau\omega,\omega)=b^{j-1}|\tau-1|\sqrt{g(\tau,\omega)g(\overline{\tau},\frac{1}{\omega})},$$
$$\Delta\tilde{\Phi}_j(\tau\omega,\omega)=b^{j-1}|\tau-1|\sqrt{\tilde{g}(\tau,\omega)\tilde{g}(\overline{\tau},\frac{1}{\omega})},$$

and

$$\Delta\Phi_j(\tau\omega,\omega)+\Delta\tilde{\Phi}_j(\tau\omega,\omega)=b^{j-1}|\tau-1|\Big(\sqrt{g(\tau,\omega)g(\overline{\tau},\frac{1}{\omega})}+\sqrt{\tilde{g}(\tau,\omega)\tilde{g}(\overline{\tau},\frac{1}{\omega})}\Big).$$

where we can extend g and $\tilde{g}$ as usual,

$$\forall z\in\overline{\mathbb{A}_\varepsilon},\ g(\tau,z)=1+\frac{f_j(\tau z)-f_j(z)}{b^{j-1}\omega(\tau-1)}\text{ and }\tilde{g}(\tau,z)=1+\frac{\tilde{f}_j(\tau z)-\tilde{f}_j(z)}{b^{j-1}\omega(\tau-1)}.$$

Thus the holomorpic extension of $\tilde{\tilde{I}}_3$ on C_ε is given by

$$\tilde{\tilde{I}}_3(z)=z\fint_{\mathbb{T}}\frac{\Big(\tau\partial_z\tilde{\Phi}_j(\tau z)-\partial\tilde{\Phi}_j(z)\Big)\Big(\tilde{\Phi}_j(\tau z)-\tilde{\Phi}_j(z)\Big)\Big(\tilde{\Phi}_j(\frac{\overline{\tau}}{z})-\Phi_j(\frac{\overline{\tau}}{z})-\tilde{\Phi}_j(\frac{1}{z})+\Phi_j(\frac{1}{z})\Big)}{b^{3(j-1)}|\tau-1|^3\sqrt{\tilde{g}(\tau,z)\tilde{g}(\overline{\tau},\frac{1}{z})}\sqrt{g(\tau,z)g(\overline{\tau},\frac{1}{z})}\Big(\sqrt{g(\tau,z)g(\overline{\tau},\frac{1}{z})}+\sqrt{\tilde{g}(\tau,z)\tilde{g}(\overline{\tau},\frac{1}{z})}\Big)}\frac{d\tau}{\tau}.$$

Concerning the $\tilde{Y}^{k-1}$ norm, we just give some details for the L^2 norm of the leading term of $\partial_z^{k-1}\tilde{\tilde{I}}_3(\varepsilon\cdot)$. For $z\in\overline{\mathcal{C}_\varepsilon}$, one may write

$$\partial_z^{k-1}\tilde{\tilde{I}}_3(z)=\fint_{\mathbb{T}}\frac{\Big(\partial_z^k\tilde{\Phi}_j(\tau z)-\partial_z^k\tilde{\Phi}_j(z)\Big)\Big(\tilde{\Phi}_j(\tau z)-\tilde{\Phi}_j(z)\Big)\Big(\tilde{\Phi}_j(\frac{\overline{\tau}}{z})-\Phi_j(\frac{\overline{\tau}}{z})-\tilde{\Phi}_j(\frac{1}{z})+\Phi_j(\frac{1}{z})\Big)}{|\tau-1|^3\sqrt{\tilde{g}(\tau,z)\tilde{g}(\overline{\tau},\frac{1}{z})}\sqrt{g(\tau,z)g(\overline{\tau},\frac{1}{z})}\Big(\sqrt{g(\tau,z)g(\overline{\tau},\frac{1}{z})}+\sqrt{\tilde{g}(\tau,z)\tilde{g}(\overline{\tau},\frac{1}{z})}\Big)}\frac{d\tau}{\tau}+l.o.t$$

The control of the L^2 norm of the inner restriction $(\omega\mapsto\partial_z^{k-1}\tilde{\tilde{I}}_3(\varepsilon\omega))$ must ensure the continuity. First, one may obtain the following identity for $z\in\varepsilon\mathbb{T}\cup\varepsilon^{-1}\mathbb{T}$:

$$\frac{\Big(\tilde{\Phi}_j(\tau z)-\tilde{\Phi}_j(z)\Big)\Big(\tilde{\Phi}_j(\frac{\overline{\tau}}{z})-\Phi_j(\frac{\overline{\tau}}{z})-\tilde{\Phi}_j(\frac{1}{z})+\Phi_j(\frac{1}{z})\Big)}{|\tau-1|^3\sqrt{\tilde{g}(\tau,z)\tilde{g}(\overline{\tau},\frac{1}{z})}\sqrt{g(\tau,z)g(\overline{\tau},\frac{1}{z})}\Big(\sqrt{g(\tau,z)g(\overline{\tau},\frac{1}{z})}+\sqrt{\tilde{g}(\tau,z)\tilde{g}(\overline{\tau},\frac{1}{z})}\Big)}=\frac{K(z)}{|\tau-1|}+\frac{(\tau-1)}{|\tau-1|}H(\tau,z)$$

with the estimations

$$\|K\|_{L^\infty(\varepsilon^{\pm}\mathbb{T})}\leq C\|\Phi_j-\tilde{\Phi}_j\|_{X^{k+\log}},\ \|H\|_{L^\infty(\mathbb{T}\times\varepsilon^{\pm}\mathbb{T})}\leq C\|\Phi_j-\tilde{\Phi}_j\|_{X^{k+\log}}.\tag{4.13}$$

The proof leads on the lemma 4.4 and an adaptation. Hence, we can estimate for $p\in\{3,5\}$:

$$\|\tilde{I}_p\|_{\tilde{Y}^{k-1}}\leq C\|\Phi_j-\tilde{\Phi}_j\|_{X^{k+\log}}$$

For $\tilde{I}_9$ we just need to notice this decomposition

$$\frac{1}{\big(\Delta\Phi_j(\tau,\omega)\big)^3}-\frac{1}{\big(\Delta\Phi_j(\tau,\omega)\big)^3}=\frac{1}{\big(\Delta\Phi_j(\tau,\omega)\big)^2}\left[\frac{1}{\Delta\Phi_j(\tau,\omega)}-\frac{1}{\Delta\tilde{\Phi}_j(\tau,\omega)}\right]$$
$$+\frac{1}{\big(\Delta\Phi_j(\tau,\omega)\big)\big(\Delta\tilde{\Phi}_j(\tau,\omega)\big)}\left[\frac{1}{\Delta\Phi_j(\tau,\omega)}-\frac{1}{\Delta\tilde{\Phi}_j(\tau,\omega)}\right]$$
$$+\frac{1}{\big(\Delta\tilde{\Phi}_j(\tau,\omega)\big)^2}\left[\frac{1}{\Delta\Phi_j(\tau,\omega)}-\frac{1}{\Delta\tilde{\Phi}_j(\tau,\omega)}\right].$$

With this writting and the same arguments than for $\tilde{I}_3$, we get

$$\|\tilde{I}_9\|_{\tilde{Y}^{k-1}}\leq C\|\Phi_j-\tilde{\Phi}_j\|_{X^{k+\log}}.$$

Finally, DG_j is continuous. □

5. Study of the linearized operator

The main task of this section is to perform a spectral study of the linearized operator of the functional G introduced in (2.2) at the annular solution $(\mathrm{Id}, b\mathrm{Id})$. The first subsection is dedicated to an explicit computation of this operator and to get a more user-friendly expression through some basic identities on hypergeometric functions. In the second part, we want to find the values of Ω leading to a one-dimensional kernel for the linearized operator. We show that for each frequency mode this study reduces to a second degree equation on the variable Ω. The dimension of the kernel is achieved through the strict monotonicity of the eigenvalues with respect to the frequency. Lastly, we check the full assumptions of the Crandall-Rabinowitz's theorem especially the transversality condition which holds only when the eigenvalues are simple.

5.1. Linearized operator.

The primary purpose of this section is to compute the linearized operator of G at the trivial solution $(\mathrm{Id}, b\mathrm{Id})$ and to reach a more simplified and compact expression. Since $G=(G_1,G_2)$ then for given $(h_1,h_2)\in X^{k+\log}\times X^{k+\log}$, we have

$$DG(\Omega,0,0)(h_1,h_2)=\begin{pmatrix} D_{f_1}G_1(\Omega,0,0)h_1+D_{f_2}G_1(\Omega,0,0)h_2\\ D_{f_1}G_2(\Omega,0,0)h_1+D_{f_2}G_2(\Omega,0,0)h_2\end{pmatrix}.$$

Replacing in (4.7), (4.8) and (4.9) Φ_1 by Id and Φ_2 by $b\mathrm{Id}$ yields

$$DG_1(\Omega,0,0)(h_1,h_2)(\omega)=\Omega\,\mathcal{L}_0(h_1)(\omega)+\mathcal{L}_1(h_1)(\omega)+\mathcal{L}_2(h_1,h_2)(\omega),$$
$$DG_2(\Omega,0,0)(h_1,h_2)(\omega)=\Omega\,b\mathcal{L}_0(h_2)(\omega)+\mathcal{L}_1(h_2)(\omega)+\mathcal{L}_3(h_1,h_2)(\omega)$$

with

$$\mathcal{L}_0(h_j)(\omega)=\mathrm{Im}\left\{h_j'(\overline{\omega})+h_j(\omega)\overline{\omega}\right\},$$

$$\mathcal{L}_1(h_j)(\omega)=\mathrm{Im}\left\{(-1)^j\overline{h_j'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{\tau-\omega}{|\omega-\tau|}\frac{d\tau}{\tau}-(-1)^j\overline{\omega}\fint_{\mathbb{T}}\frac{(\tau-\omega)\mathrm{Re}\Big((\overline{h_j(\tau)-h_j(\omega)})(\tau-\omega)\Big)}{|\tau-\omega|^3}\frac{d\tau}{\tau}\right\}$$
$$+\mathrm{Im}\left\{(-1)^j\overline{\omega}\fint_{\mathbb{T}}\frac{\tau h_j'(\tau)-\omega h_j'(\omega)}{|\omega-\tau|}\frac{d\tau}{\tau}\right\},$$

$$\begin{aligned}\mathcal{L}_2(h_1,h_2)(\omega) &= \mathrm{Im}\left\{\overline{h_1'(\omega)}\overline{\omega}\fint_{\mathbb{T}}\frac{b\tau-\omega}{|b\tau-\omega|}\frac{d\tau}{\tau}\right\}-\mathrm{Im}\left\{\overline{\omega}\fint_{\mathbb{T}}\frac{\omega h_1'(\omega)-\tau h_2'(\tau)}{|\omega-b\tau|}\frac{d\tau}{\tau}\right\}\\ &+ \mathrm{Im}\left\{-\overline{\omega}\fint_{\mathbb{T}}\frac{(b\tau-\omega)\mathrm{Re}\Big((h_1(\omega)-h_2(\tau))\overline{(\omega-b\tau)}\Big)}{|b\tau-\omega|^3}\frac{d\tau}{\tau}\right\}\end{aligned}$$

and

$$\mathcal{L}_3(h_1,h_2)(\omega)=\mathrm{Im}\left(-\overline{\omega}h_2'(\overline{\omega})\fint_{\mathbb{T}}\frac{\tau-b\omega}{|\tau-b\omega|}\frac{d\tau}{\tau}+\overline{\omega}b\fint_{\mathbb{T}}\frac{\omega h_2'(\omega)-\tau h_1'(\tau)}{|b\omega-\tau|}\frac{d\tau}{\tau}\right)$$
$$+\mathrm{Im}\left(b\overline{\omega}\fint_{\mathbb{T}}\frac{(\tau-b\omega)\mathrm{Re}\Big((h_1(\tau)-h_2(\omega))(\overline{\tau-b\omega})\Big)}{|\tau-b\omega|^3}\frac{d\tau}{\tau}\right).$$

We shall now compute the Fourier series of the mapping $\omega\mapsto DG(\Omega,0,0)(h_1,h_2)(\omega)$ with

$$h_1(\omega)=\sum_{n=1}^{+\infty}a_n\overline{\omega}^n \text{ and } h_2(\omega)=\sum_{n=1}^{+\infty}c_n\overline{\omega}^n,\ \omega\in\mathbb{T}$$

where a_n and c_n are real for all the values $n\in\mathbb{N}^\star$. This is summarized in the following proposition.

Proposition 5.1. *Let $b\in(0,1)$, $n\in\mathbb{N}^\star$ and define*

$$\Lambda_n(b)\triangleq\frac{1}{b}\int_0^{+\infty}J_n(bt)J_n(t)dt,\quad S_n\triangleq\frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1}$$

with J_n refers to the Bessel function of the first kind. Then, we have

$$DG(\Omega,0,0)(h_1,h_2)(\omega)=\frac{i}{2}\sum_{n\geq1}(n+1)M_{n+1}\begin{pmatrix}a_n\\c_n\end{pmatrix}\left(\omega^{n+1}-\overline{\omega}^{n+1}\right),\ \forall\omega\in\mathbb{T}$$

where the matrix M_n is given for $n\geq2$ by:

$$M_n=\begin{pmatrix}\Omega-S_n+b^2\Lambda_1(b) & -b^2\Lambda_n(b)\\ b\Lambda_n(b) & b\Omega+S_n-b\Lambda_1(b)\end{pmatrix}.$$

Proof. We begin with the easier term $\mathcal{L}_0(h_j)(\omega)$. Thus by straightforward computations we obtain

$$\mathcal{L}_0(h_1)(\omega)=\frac{i}{2}\sum_{n=1}^{+\infty}(n+1)a_n(\omega^{n+1}-\overline{\omega}^{n+1}).$$

and

$$\mathcal{L}_0(h_2)(\omega)=\frac{i}{2}\sum_{n=1}^{+\infty}(n+1)c_n(\omega^{n+1}-\overline{\omega}^{n+1}).$$

The computation of $\mathcal{L}_1(h_1)(\omega)$ lies on the following identities whose proofs can be found in [16]. Let $n\in\mathbb{N}^\star$ and $\omega\in\mathbb{T}$ then

$$\fint_{\mathbb{T}}\frac{\tau^n-\omega^n}{|\omega-\tau|}\frac{d\tau}{\tau}=-\frac{2\omega^n}{\pi}\sum_{k=0}^{n-1}\frac{1}{2k+1}\tag{5.1}$$

and

$$\fint_{\mathbb{T}}\frac{(\tau-\omega)^2(\tau^n-\omega^n)}{|\omega-\tau|^3}\frac{d\tau}{\tau}=\frac{2\omega^{n+2}}{\pi}\sum_{k=1}^{n}\frac{1}{2k+1}.\tag{5.2}$$

Performing straightforward computations we obtain the result

$$\mathcal{L}_1(h_1)(\omega)=\mathrm{Im}\left\{\sum_{n=1}^{+\infty}na_n\omega^n\fint_{\mathbb{T}}\frac{\tau-\omega}{|\tau-\omega|}\frac{d\tau}{\tau}+\frac{1}{2\omega}\sum_{n=1}^{+\infty}a_n\fint_{\mathbb{T}}\frac{(\tau^n-\omega^n)(\tau-\omega)^2}{|\tau-\omega|^3}\frac{d\tau}{\tau}\right\}$$
$$+\mathrm{Im}\left\{\overline{\omega}\sum_{n=1}^{+\infty}na_n\fint_{\mathbb{T}}\frac{\overline{\tau}^n-\overline{\omega}^n}{|\tau-\omega|}\frac{d\tau}{\tau}+\frac{1}{2\omega}\sum_{n=1}^{+\infty}a_n\fint_{\mathbb{T}}\frac{\overline{\tau}^n-\overline{\omega}^n}{|\tau-\omega|}\frac{d\tau}{\tau}\right\}.$$

Noticing the following equality

$$\forall\tau\in\mathbb{T},\forall\omega\in\mathbb{T},\fint_{\mathbb{T}}\frac{\overline{\tau}^n-\overline{\omega}^n}{|\omega-\tau|}\frac{d\tau}{\tau}=\overline{\fint_{\mathbb{T}}\frac{\tau^n-\omega^n}{|\omega-\tau|}\frac{d\tau}{\tau}},$$

we obtain thanks to (5.1) and (5.2) the following identity

$$\mathcal{L}_1(h_1)(\omega) = -\sum_{n=1}^{+\infty} \alpha_n a_n \mathrm{Im}\left\{\overline{\omega}^{n+1}\right\} + \sum_{n=1}^{+\infty} \beta_n a_n \mathrm{Im}\left\{\omega^{n+1}\right\}$$

where

$$\alpha_n \triangleq \frac{2n+1}{\pi} \sum_{k=0}^{n-1} \frac{1}{2k+1}$$

and

$$\beta_n \triangleq -\frac{2n}{\pi} + \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{2k+1}.$$

As

$$\alpha_n + \beta_n = \frac{2(n+1)}{\pi} \sum_{k=1}^{n} \frac{1}{2k+1}.$$

Finally we get

$$\mathcal{L}_1(h_1)(\omega) = -\frac{i}{2} \sum_{n=1}^{+\infty} a_n (n+1) S_{n+1} (\omega^{n+1} - \overline{\omega}^{n+1}).$$

In the same way we obtain

$$\mathcal{L}_1(h_2)(\omega) = \frac{i}{2} \sum_{n=1}^{+\infty} c_n (n+1) S_{n+1} (\omega^{n+1} - \overline{\omega}^{n+1}).$$

To compute $\mathcal{L}_2(h_1, h_2)(\omega)$ we begin to rewrite

$$-\overline{\omega} \fint_{\mathbb{T}} \frac{(b\tau - \omega)\mathrm{Re}\Big(h_1(\omega) - h_2(\tau))\overline{(\omega - b\tau)}\Big)}{|b\tau - \omega|^3} \frac{d\tau}{\tau} = \frac{\overline{\omega}}{2} \fint_{\mathbb{T}} \frac{h_1(\omega) - h_2(\tau)}{|b\tau - \omega|} \frac{d\tau}{\tau} + \frac{\overline{\omega}}{2} \fint_{\mathbb{T}} \frac{(b\tau - \omega)^2 \overline{(h_1(\omega) - h_2(\tau))}}{|b\tau - \omega|^3} \frac{d\tau}{\tau}.$$

Replacing h_j and h_j' by their expressions, we obtain the following identity

$$\begin{aligned} \mathcal{L}_2(h_1, h_2)(\omega) &= \mathrm{Im}\left\{ -\sum_{n=1}^{+\infty} n a_n \fint_{\mathbb{T}} \frac{b\tau - \omega}{|b\tau - \omega|} \frac{d\tau}{\tau} + \sum_{n=1}^{+\infty} n\overline{\omega} \fint_{\mathbb{T}} \frac{a_n \overline{\omega}^n - c_n \overline{\tau}^n}{|\omega - b\tau|} \frac{d\tau}{\tau} + \frac{1}{2} \sum_{n=1}^{+\infty} \overline{\omega} \fint_{\mathbb{T}} \frac{a_n \overline{\omega}^n - \overline{\tau}^n}{|b\tau - \omega|} \frac{d\tau}{\tau} \right\} \\ &+ \mathrm{Im}\left\{ \frac{b}{2} \sum_{n=1}^{+\infty} \overline{\omega} \frac{(b\omega - \tau)(a_n \omega^n - c_n \tau^n)}{|b\tau - \omega|^3} d\tau - \frac{1}{2} \sum_{n=1}^{+\infty} \frac{(b - \omega\overline{\tau})(a_n \omega^n - c_n \tau^n)}{|b\tau - \omega|^3} d\tau \right\}. \end{aligned}$$

To compute these terms, we will use the identities proved in [12]: Let $b \in (0,1)$ and $n \in \mathbb{N}$, then for any $\omega \in \mathbb{T}$ we have

$$\fint_{\mathbb{T}} \frac{\tau^{n-1}}{|b\tau - \omega|} d\tau = \omega^n b^n \frac{(\frac{1}{2})_n}{n!} F\left(\frac{1}{2}, n + \frac{1}{2}, n+1; b^2\right), \tag{5.3}$$

$$\begin{aligned} \fint_{\mathbb{T}} \frac{\overline{\tau}^{n+1}}{|b\tau - \omega|} d\tau &= \overline{\fint_{\mathbb{T}} \frac{\tau^{n-1}}{|b\tau - \omega|} d\tau} \\ &= \overline{\omega}^n b^n \frac{(\frac{1}{2})_n}{n!} F\left(\frac{1}{2}, n + \frac{1}{2}, n+1; b^2\right), \end{aligned} \tag{5.4}$$

$$\fint_{\mathbb{T}} \frac{\overline{\tau}^{n+1}}{|\omega - b\tau|^3} d\tau = \omega^n b^n \frac{(\frac{3}{2})_n}{n!} F\left(\frac{3}{2}, n + \frac{3}{2}, n+1; b^2\right), \tag{5.5}$$

$$\fint_{\mathbb{T}} \frac{\tau^{n-1}}{|\omega - b\tau|^3} d\tau = \omega^n b^n \frac{(\frac{3}{2})_n}{n!} F\left(\frac{3}{2}, n + \frac{3}{2}, n+1; b^2\right), \tag{5.6}$$

$$\oint_{\mathbb{T}} \frac{(b\tau-\omega)(a\omega^n-c\tau^n)}{|\omega-b\tau|^3}d\tau = -\omega^{n+2}b\left[\frac{3}{2}aF\left(\frac{1}{2},\frac{5}{2},2;b^2\right) - cb^n\frac{\left(\frac{3}{2}\right)_{n+1}}{(n+1)!}F\left(\frac{1}{2},n+\frac{5}{2},n+2;b^2\right)\right], \tag{5.7}$$

and

$$\oint_{\mathbb{T}} \frac{(b\omega-\tau)(c\omega^n-a\tau^n)}{|\omega-b\tau|^3}d\tau = -\omega^{n+2}b^2\left[\frac{3}{8}cF\left(\frac{3}{2},\frac{5}{2},3;b^2\right) - ab^n\frac{\left(\frac{1}{2}\right)_{n+2}}{(n+2)!}F\left(\frac{3}{2},n+\frac{5}{2},n+3;b^2\right)\right]. \tag{5.8}$$

We shall split the computation in many parts. By using (5.3) and (5.4) we find

$$-\sum_{n=1}^{+\infty} na_n \oint_{\mathbb{T}} \frac{b\tau-\omega}{|b\tau-\omega|}\frac{d\tau}{\tau} = \sum_{n=1}^{+\infty} na_n\left[F\left(\frac{1}{2},\frac{1}{2},1;b^2\right) - \frac{b^2}{2}F\left(\frac{1}{2},\frac{3}{2},2;b^2\right)\right]\omega^{n+1}.$$

Moreover

$$\sum_{n=1}^{+\infty}\left(n+\frac{1}{2}\right)\overline{\omega}\oint_{\mathbb{T}}\frac{a_n\overline{\omega}^n-c_n\overline{\tau}^n}{|\omega-b\tau|}\frac{d\tau}{\tau} = \sum_{n=1}^{+\infty}\left(n+\frac{1}{2}\right)\left[a_nF\left(\frac{1}{2},\frac{1}{2},1;b^2\right) - c_nb^n\frac{\left(\frac{1}{2}\right)_n}{n!}F\left(\frac{1}{2},n+\frac{1}{2},n+1;b^2\right)\right]\overline{\omega}^{n+1}.$$

Now, using (5.7) we obtain

$$\frac{b}{2}\sum_{n=1}^{+\infty}\overline{\omega}\oint_{\mathbb{T}}\frac{(b\omega-\tau)(a_n\omega^n-c_n\tau^n)}{|b\tau-\omega|^3}d\tau = \frac{b^2}{2}\left[c_nb^n\frac{\left(\frac{3}{2}\right)_{n+1}}{(n+1)!}F\left(\frac{1}{2},n+\frac{5}{2},n+2;b^2\right) - \frac{3}{2}a_nF\left(\frac{1}{2},\frac{5}{2},2;b^2\right)\right]\omega^{n+1}.$$

For the last term of $\mathcal{L}_2(h_1,h_2)$ we use (5.5) and (5.6)

$$\begin{aligned}-\frac{1}{2}\sum_{n=1}^{+\infty}\oint_{\mathbb{T}}\frac{(b-\omega\overline{\tau})(a_n\omega^n-c_n\tau^n)}{|b\tau-\omega|^3}d\tau &= -\frac{1}{2}\sum_{n=1}^{+\infty}a_n\left[\frac{3}{2}b^2F\left(\frac{3}{2},\frac{5}{2},2;b^2\right) - F\left(\frac{3}{2},\frac{3}{2},1;b^2\right)\right]\omega^{n+1}\\ &+\frac{1}{2}\sum_{n=1}^{+\infty}c_n\frac{b^n}{n!}\left(\frac{3}{2}\right)_n\left[\frac{(n+\frac{3}{2})}{(n+1)}b^2F\left(\frac{3}{2},n+\frac{5}{2},n+2;b^2\right) - F\left(\frac{3}{2},n+\frac{3}{2},n+1;b^2\right)\right]\omega^{n+1}.\end{aligned}$$

Now we shall apply (3.1) with $a=\frac{1}{2}$, $b=\tilde{n}+\frac{3}{2}$, $c=\tilde{n}+1$ and $z=b^2$ where $\tilde{n}\in\{0,n\}$

$$-\frac{1}{2}\sum_{n=1}^{+\infty}\oint_{\mathbb{T}}\frac{(b-\omega\overline{\tau})(a_n\omega^n-c_n\tau^n)}{|b\tau-\omega|^3}d\tau = \frac{1}{2}\sum_{n=1}^{+\infty}\left[a_nF\left(\frac{1}{2},\frac{3}{2},1;b^2\right) - c_n\frac{b^n}{n!}\left(\frac{3}{2}\right)_n F\left(\frac{1}{2},n+\frac{3}{2},1;b^2\right)\right]\omega^{n+1}.$$

Finally we get

$$\mathcal{L}_2(h_1,h_2)(\omega) = \frac{i}{2}\sum_{n=1}^{+\infty}\left[a_n(\tilde{\gamma_n}-\gamma_n) + (\tilde{\beta_n}-\beta_n)c_n\right](\omega^{n+1}-\overline{\omega}^{n+1})$$

with

$$\begin{aligned}\gamma_n =& n\left[F\left(\frac{1}{2},\frac{1}{2},1;b^2\right) - \frac{b^2}{2}F\left(\frac{1}{2},\frac{3}{2},2;b^2\right)\right] - \frac{3}{4}b^2F\left(\frac{1}{2},\frac{5}{2},2;b^2\right) + \frac{1}{2}F\left(\frac{1}{2},\frac{3}{2},1;b^2\right),\\ \tilde{\gamma_n} =& (n+\frac{1}{2})F\left(\frac{1}{2},\frac{1}{2},1;b^2\right),\\ \beta_n =& \frac{b^{n+2}}{2}\frac{(\frac{3}{2})_{n+1}}{(n+1)!}F\left(\frac{1}{2},n+\frac{5}{2},n+2;b^2\right) - \frac{b^n}{2}\frac{(\frac{3}{2})_n}{n!}F\left(\frac{1}{2},n+\frac{3}{2},n+1;b^2\right)\end{aligned}$$

and

$$\tilde{\beta_n} = -(n+\frac{1}{2})b^n\frac{(\frac{1}{2})_n}{n!}F\left(\frac{1}{2},n+\frac{1}{2},n+1;b^2\right).$$

Now, we want to simplify the expression of $\mathcal{L}_2(h_1,h_2)$ through the use of the identities (3.1)-(3.4). We begin with (3.2) with $a=\frac{1}{2}$, $b=\tilde{n}+\frac{1}{2}$, $c=\tilde{n}+1$ and $z=b^2$ where $\tilde{n}\in\{0,n\}$ which implies

$$\gamma_n-\tilde{\gamma_n} = -n\frac{b^2}{2}F\left(\frac{1}{2},\frac{3}{2},2;b^2\right) - \frac{3}{4}b^2F\left(\frac{1}{2},\frac{5}{2},2;b^2\right) + \frac{b^2}{4}F\left(\frac{3}{2},\frac{3}{2},2;b^2\right)$$

and

$$\beta_n - \tilde{\beta}_n = \frac{b^{n+2}}{2(n+1)!}\left(\frac{3}{2}\right)_n \left[\left(n+\frac{3}{2}\right)F\left(\frac{1}{2}, n+\frac{5}{2}, n+2; b^2\right) - \frac{1}{2}F\left(\frac{1}{2}, n+\frac{3}{2}, n+1; b^2\right)\right].$$

Thus using (3.3), one may check the following expression

$$\begin{aligned}\mathcal{L}_2(h_1,h_2)(\omega) &= \frac{i}{2}\sum_{n=1}^{+\infty}(n+1)\frac{b^2}{2}F\left(\frac{1}{2},\frac{3}{2},2;b^2\right)a_n(\omega^{n+1}-\overline{\omega}^{n+1}) \\ &\quad - \frac{i}{2}\sum_{n=1}^{+\infty}\frac{b^{n+2}}{n!}(\frac{1}{2})_{n+1}F\left(\frac{1}{2},n+\frac{3}{2},n+2;b^2\right)c_n(\omega^{n+1}-\overline{\omega}^{n+1}).\end{aligned}$$

Now we focus on $\mathcal{L}_3(h_1,h_2)$ given by

$$\begin{aligned}\mathcal{L}_3(h_1,h_2)(\omega) &= \mathrm{Im}\left\{-\overline{\omega}h_2'(\overline{\omega})\fint_{\mathbb{T}}\frac{\tau-b\omega}{|\tau-b\omega|}\frac{d\tau}{\tau} + \overline{\omega}b\fint_{\mathbb{T}}\frac{\omega h_2'(\omega)-\tau h_1'(\tau)}{|b\omega-\tau|}\frac{d\tau}{\tau}\right\} \\ &\quad + \mathrm{Im}\left\{b\overline{\omega}\fint_{\mathbb{T}}\frac{(\tau-b\omega)\mathrm{Re}\Big(\big(h_1(\tau)-h_2(\omega)\big)\big(\overline{\tau-b\omega}\big)\Big)}{|\tau-b\omega|^3}\frac{d\tau}{\tau}\right\}.\end{aligned}$$

Observe that

$$\begin{aligned}b\overline{\omega}\fint_{\mathbb{T}}\frac{(\tau-b\omega)\mathrm{Re}\Big(\big(h_1(\tau)-h_2(\omega)\big)\big(\overline{\tau-b\omega}\big)\Big)}{|\tau-b\omega|^3}\frac{d\tau}{\tau} &= \frac{b}{2}\overline{\omega}\fint_{\mathbb{T}}\frac{h_1(\tau)-h_2(\omega)}{|\tau-b\omega|}\frac{d\tau}{\tau} \\ &\quad + \frac{b}{2}\overline{\omega}\fint_{\mathbb{T}}\frac{\big(\tau-b\omega\big)^2\Big(\overline{h_1(\tau)-h_2(\omega)}\Big)}{|\tau-b\omega|^3}\frac{d\tau}{\tau}.\end{aligned}$$

Replacing h_j and h_j' by their expressions we get

$$\begin{aligned}\mathcal{L}_3(h_1,h_2)(\omega) &= \mathrm{Im}\left\{\sum_{n=1}^{+\infty}nc_n\omega^n\fint_{\mathbb{T}}\frac{\tau-b\omega}{|\tau-b\omega|}\frac{d\tau}{\tau} - \overline{\omega}b\sum_{n=1}^{+\infty}\left(n+\frac{1}{2}\right)\fint_{\mathbb{T}}\frac{c_n\overline{\omega}^n-a_n\overline{\tau}^n}{|b\omega-\tau|}\frac{d\tau}{\tau}\right\} \\ &\quad + \mathrm{Im}\left\{\frac{b}{2}\overline{\omega}\sum_{n=1}^{+\infty}\fint_{\mathbb{T}}\frac{\Big(b\omega-\tau\Big)\Big(c_n\omega^n-a_n\tau^n\Big)}{|\tau-b\omega|^3}d\tau + \frac{b^2}{2}\sum_{n=1}^{+\infty}\fint_{\mathbb{T}}\frac{\Big(1-b\omega\overline{\tau}\Big)\Big(c_n\omega^n-a_n\tau^n\Big)}{|\tau-b\omega|^3}d\tau\right\}.\end{aligned}$$

As before, we shall split the computations in many parts. Thanks to (5.3) and (5.4), the first term takes the form

$$\begin{aligned}\sum_{n=1}^{+\infty}nc_n\omega^n\fint_{\mathbb{T}}\frac{\tau-b\omega}{|\tau-b\omega|}\frac{d\tau}{\tau} &= \sum_{n=1}^{+\infty}nc_n\omega^n\fint_{\mathbb{T}}\frac{\tau-b\omega}{|\tau b-\omega|}\frac{d\tau}{\tau} \\ &= \sum_{n=1}^{+\infty}nbc_n\Big[\frac{1}{2}F\left(\frac{1}{2},\frac{3}{2},2;b^2\right) - F\left(\frac{1}{2},\frac{1}{2},1;b^2\right)\Big]\omega^{n+1}.\end{aligned}$$

Note that we have used in the first line the identity

$$\forall \tau,\omega\in\mathbb{T},\quad |\tau-b\omega| = |b\tau-\omega|$$

For the second term, we use (5.3) and (5.4) to obtain

$$\begin{aligned}-\overline{\omega}b\sum_{n=1}^{+\infty}\left(n+\frac{1}{2}\right)\fint_{\mathbb{T}}\frac{c_n\overline{\omega}^n-a_n\overline{\tau}^n}{|b\omega-\tau|}\frac{d\tau}{\tau} &= -\sum_{n=1}^{+\infty}\left(n+\frac{1}{2}\right)bc_nF\left(\frac{1}{2},\frac{1}{2},1;b^2\right)\overline{\omega}^{n+1} \\ &\quad + \sum_{n=1}^{+\infty}\left(n+\frac{1}{2}\right)a_n\frac{b^{n+1}}{n!}\left(\frac{1}{2}\right)_n F\left(\frac{1}{2},n+\frac{1}{2},n+1;b^2\right)\overline{\omega}^{n+1}.\end{aligned}$$

The computation of the third term can be done in view of (5.8),

$$\frac{b}{2}\overline{\omega}\sum_{n=1}^{+\infty}\oint_{\mathbb{T}}\frac{\Big(b\omega-\tau\Big)\Big(c_n\omega^n-a_n\tau^n\Big)}{|\tau-b\omega|^3}d\tau=\sum_{n=1}^{+\infty}\frac{a_n}{2}\frac{b^{n+3}}{(n+2)!}\left(\frac{1}{2}\right)_{n+2}F\left(\frac{3}{2},n+\frac{5}{2},n+3;b^2\right)\omega^{n+1}$$
$$-\sum_{n=1}^{+\infty}\frac{3b^3}{16}c_nF\left(\frac{3}{2},\frac{5}{2},3;b^2\right)\omega^{n+1}.$$

For the last term, we use (5.5) and (5.6)

$$\frac{b^2}{2}\sum_{n=1}^{+\infty}\oint_{\mathbb{T}}\frac{\Big(1-b\omega\overline{\tau}\Big)\Big(c_n\omega^n-a_n\tau^n\Big)}{|\tau-b\omega|^3}d\tau=\sum_{n=1}^{+\infty}\frac{b^3}{2}c_n\left[\frac{3}{2}F\left(\frac{3}{2},\frac{5}{2},2;b^2\right)-F\left(\frac{3}{2},\frac{3}{2},1;b^2\right)\right]\omega^{n+1}$$
$$-\sum_{n=1}^{+\infty}\frac{b^{n+3}}{2}a_n\left[\frac{\left(\frac{3}{2}\right)_{n+1}}{(n+1)!}F\left(\frac{3}{2},n+\frac{5}{2},n+2;b^2\right)-\frac{\left(\frac{3}{2}\right)_n}{n!}F\left(\frac{3}{2},n+\frac{3}{2},n+1;b^2\right)\right]\omega^{n+1}$$

Finally, we obtain the following expression

$$\mathcal{L}_3(h_1,h_2)(\omega)=\frac{i}{2}\sum_{n=1}^{+\infty}\Big[a_n(\tilde{\Delta}_n-\Delta_n)+c_n(\tilde{\theta}_n-\theta_n)\Big]\left(\omega^{n+1}-\overline{\omega}^{n+1}\right)$$

with

$$\begin{aligned}
\Delta_n&=\frac{b^{n+3}}{2}\frac{(\frac{1}{2})_{n+2}}{(n+2)!}F\left(\frac{3}{2},n+\frac{5}{2},n+3;b^2\right)-\frac{b^{n+3}}{2}\frac{(\frac{3}{2})_{n+1}}{(n+1)!}F\left(\frac{3}{2},n+\frac{5}{2},n+2;b^2\right)\\
&\quad+\frac{b^{n+3}}{2}\frac{(\frac{3}{2})_n}{n!}F\left(\frac{3}{2},n+\frac{3}{2},n+1;b^2\right),\\
\tilde{\Delta}_n&=\left(n+\frac{1}{2}\right)b^{n+1}\frac{(\frac{1}{2})_n}{n!}F\left(\frac{1}{2},n+\frac{1}{2},n+1;b^2\right),\\
\theta_n&=bn\left[\frac{1}{2}F\left(\frac{1}{2},\frac{3}{2},2;b^2\right)-F\left(\frac{1}{2},\frac{1}{2},1;b^2\right)\right]-\frac{3b^3}{16}F\left(\frac{3}{2},\frac{5}{2},3;b^2\right)\\
&\quad+\frac{b^3}{2}\left[\frac{3}{2}F\left(\frac{3}{2},\frac{5}{2},2;b^2\right)-F\left(\frac{3}{2},\frac{3}{2},1;b^2\right)\right]
\end{aligned}$$

and

$$\tilde{\theta}_n=-\left(n+\frac{1}{2}\right)bF\left(\frac{1}{2},\frac{1}{2},1;b^2\right).$$

We want to simplify the expression of $\mathcal{L}_3(h_1,h_2)$. First we note that

$$\theta_n-\tilde{\theta}_n=\frac{nb}{2}F\left(\frac{1}{2},\frac{3}{2},2;b^2\right)+\frac{b}{2}F\left(\frac{1}{2},\frac{1}{2},1;b^2\right)-\frac{3b^3}{16}F\left(\frac{3}{2},\frac{5}{2},3;b^2\right)$$
$$+\frac{b^3}{2}\left[\frac{3}{2}F\left(\frac{3}{2},\frac{5}{2},2;b^2\right)-F\left(\frac{3}{2},\frac{3}{2},1;b^2\right)\right].$$

By using (3.2) we get

$$\theta_n-\tilde{\theta}_n=\frac{b}{2}(n+1)F\left(\frac{1}{2},\frac{3}{2},2;b^2\right)+\frac{b}{2}F\left(\frac{1}{2},\frac{1}{2},1;b^2\right)-\frac{b}{2}F\left(\frac{1}{2},\frac{3}{2},1;b^2\right)$$
$$+\frac{b^3}{2}\left[\frac{3}{2}F\left(\frac{3}{2},\frac{5}{2},2;b^2\right)-F\left(\frac{3}{2},\frac{3}{2},1;b^2\right)\right].$$

Now combining (3.2) with $a=b=\frac{1}{2}$, $c=1$ and $z=b^2$ as well as (3.4) with $a=b=\frac{3}{2}$, $c=1$ and $z=b^2$, one may obtain the following identity

$$F\left(\frac{1}{2},\frac{1}{2},1;b^2\right)-F\left(\frac{1}{2},\frac{3}{2},1;b^2\right)+\frac{3}{2}b^2F\left(\frac{3}{2},\frac{5}{2},2;b^2\right)-b^2F\left(\frac{3}{2},\frac{3}{2},1;b^2\right)=0.$$

Consequently,

$$\theta_n-\tilde{\theta}_n=\frac{b}{2}(n+1)F\left(\frac{1}{2},\frac{3}{2},2;b^2\right).$$

On the other hand

$$\begin{aligned}\Delta_n-\tilde{\Delta}_n=b^{n+1}\frac{(\frac{1}{2})_{n+1}}{(n+1)!}\Bigg[&\frac{b^2}{2}\frac{(n+\frac{3}{2})}{(n+2)}F\left(\frac{3}{2},n+\frac{5}{2},n+3;b^2\right)-b^2\left(n+\frac{3}{2}\right)F\left(\frac{3}{2},n+\frac{5}{2},n+2;b^2\right)\\&+b^2(n+1)F\left(\frac{3}{2},n+\frac{3}{2},n+1;b^2\right)-(n+1)F\left(\frac{1}{2},n+\frac{1}{2},n+1;b^2\right)\Bigg].\end{aligned}$$

Applying (3.2) with $a=\frac{1}{2},b=n+\frac{1}{2},c=n+1$ and $z=b^2$ one gets,

$$\begin{aligned}\Delta_n-\tilde{\Delta}_n=b^{n+1}\frac{(\frac{1}{2})_{n+1}}{(n+1)!}\Bigg[&\frac{b^2}{2}\frac{(n+\frac{3}{2})}{(n+2)}F\left(\frac{3}{2},n+\frac{5}{2},n+3;b^2\right)-b^2\left(n+\frac{3}{2}\right)F\left(\frac{3}{2},n+\frac{5}{2},n+2;b^2\right)\\&+b^2(n+1)F\left(\frac{3}{2},n+\frac{3}{2},n+1;b^2\right)-(n+1)F\left(\frac{1}{2},n+\frac{3}{2},n+1;b^2\right)+\frac{b^2}{2}F\left(\frac{3}{2},n+\frac{3}{2},n+2;b^2\right)\Bigg].\end{aligned}$$

Again applying (3.2) with $a=\frac{1}{2},b=n+\frac{3}{2},c=n+2$ and $z=b^2$, we deduce

$$\begin{aligned}\Delta_n-\tilde{\Delta}_n=b^{n+1}\frac{(\frac{1}{2})_{n+1}}{(n+1)!}\Bigg[&(n+\frac{3}{2})F\left(\frac{1}{2},n+\frac{5}{2},n+2;b^2\right)-(n+\frac{3}{2})F\left(\frac{1}{2},n+\frac{3}{2},n+2;b^2\right)\\&-b^2\left(n+\frac{3}{2}\right)F\left(\frac{3}{2},n+\frac{5}{2},n+2;b^2\right)+b^2(n+1)F\left(\frac{3}{2},n+\frac{3}{2},n+1;b^2\right)\\&-(n+1)F\left(\frac{1}{2},n+\frac{3}{2},n+1;b^2\right)+\frac{b^2}{2}F\left(\frac{3}{2},n+\frac{3}{2},n+2;b^2\right)\Bigg].\end{aligned}$$

We use (3.4) with $a=\frac{3}{2},b=n+\frac{3}{2},c=n+1$ and $z=b^2$ to cancel some terms

$$\begin{aligned}\Delta_n-\tilde{\Delta}_n=b^{n+1}\frac{(\frac{1}{2})_{n+1}}{(n+1)!}\Bigg[&(n+\frac{3}{2})F\left(\frac{1}{2},n+\frac{5}{2},n+2;b^2\right)-(n+\frac{3}{2})F\left(\frac{1}{2},n+\frac{3}{2},n+2;b^2\right)\\&-(n+1)F\left(\frac{1}{2},n+\frac{3}{2},n+1;b^2\right)\Bigg].\end{aligned}$$

Finally, using (3.4) with $a=\frac{1}{2},b=n+\frac{3}{2},c=n+1$ and $z=b^2$, we obtain

$$\Delta_n-\tilde{\Delta}_n=-\frac{b^{n+1}}{(n+1)!}\left(\frac{1}{2}\right)_{n+1}(n+1)F\left(\frac{1}{2},n+\frac{3}{2},n+2;b^2\right).$$

Consequently, we have

$$\begin{aligned}\mathcal{L}_3(h_1,h_2)(\omega)=&\frac{i}{2}\sum_{n=1}^{+\infty}\frac{b^{n+1}}{(n+1)!}\left(\frac{1}{2}\right)_{n+1}(n+1)F\left(\frac{1}{2},n+\frac{3}{2},n+2;b^2\right)a_n(\omega^{n+1}-\overline{\omega}^{n+1})\\&-\frac{i}{2}\sum_{n=1}^{+\infty}\frac{b}{2}(n+1)F\left(\frac{1}{2},\frac{3}{2},2;b^2\right)c_n(\omega^{n+1}-\overline{\omega}^{n+1}).\end{aligned}$$

As we have (see [7])

$$\Lambda_n(b)=\frac{(\frac{1}{2})_n}{n!}b^{n-1}F\left(\frac{1}{2},n+\frac{1}{2},n+1,b^2\right).$$

the proof of the proposition is now achieved. □

5.2. **Monotonicity of the eigenvalues.** In what follows we shall use the variable $\lambda \triangleq 1 - 2\Omega$ instead of Ω. The main task is to list the suitable conditions on the used parameters in order to guarantee a one-dimensional kernel. Recall from Proposition 5.1 that the operator $DG(\Omega, 0, 0)$ acts as a Fourier matrix multiplier and the determinant of each matrix M_n is given by

$$\det(M_n) = \frac{b}{4}(\lambda^2 - 2C_n\lambda + D_n) \tag{5.9}$$

with

$$\left\{ \begin{array}{rcl} C_n & = & 1 + \left(\frac{1}{b} - 1\right) S_n - (1 - b^2)\Lambda_1(b) \\ D_n & = & -\frac{4}{b}S_n^2 + 2\left[\frac{1}{b} - 1 + 2(1+b)\Lambda_1(b)\right] S_n - 4b^2\left(\Lambda_1^2(b) - \Lambda_n^2(b)\right) - 2(1-b^2)\Lambda_1(b) + 1 \end{array} \right. .$$

From that proposition one can easily see that the kernel of $DG(\Omega, 0, 0)$ is non trivial if and only if

$$\left\{\exists n \geq 2,\ \det(M_n) = 0\right\}$$

Therefore the dimension of the kernel is related to the structure of the eigenvalues and to how they depend on the frequency modes. Observe that $\lambda \mapsto \det(M_n)$ is a second order polynomial and the roots structure depends on the reduced discriminant which is given by

$$\Delta_n = \left(\frac{1}{b} + 1\right) S_n - (1 + b^2)\Lambda_1(b)^2 - 4b^2\Lambda_n^2(b).$$

We shall prove the following proposition.

Proposition 5.2. (1) *For any $n \in N^*$ we have $\Lambda_n(b) \geq 0$, $n \mapsto S_n$ is a strictly increasing sequence, $n \mapsto \Lambda_n(b)$ is a strictly decreasing sequence and $b \mapsto \Lambda_n(b)$ is a strictly increasing function.*

(2) *There exists $N \geq 2$ such that for any $n \geq N$ we get $\Delta_n > 0$ and the equation $\det(M_n) = 0$ admits two different real solutions given by*

$$\lambda_n^{\pm} = C_n \pm \sqrt{\Delta_n}.$$

(3) *The sequences $(\Delta_n)_{n\geq N}$ and $(\lambda_n^+)_{n\geq N}$ are strictly increasing and $(\lambda_n^-)_{n\geq N}$ is strictly decreasing.*

(4) *$\forall m > n > N$ we have*

$$\lambda_m^- < \lambda_n^- < \lambda_n^+ < \lambda_m^+.$$

Proof. (1) The positivity and the monotonicity of $\Lambda_n(b)$ follow easily from the integral representation

$$\Lambda_n(b) = \frac{b^{n-1}}{\Gamma^2(\frac{1}{2})} \int_0^1 x^{n-\frac{1}{2}}(1-x)^{-\frac{1}{2}}(1-b^2x)^{-\frac{1}{2}}dx, \text{ for } b \in (0,1).$$

As to the monotonicity of S_n it is obvious.

(2) We write $\Delta_n(b) = E_n(b)F_n(b)$ with

$$\left\{ \begin{array}{rcl} E_n(b) & = & \left(\frac{1}{b} + 1\right) S_n - (1 + b^2)\Lambda_1(b) - 2b\Lambda_n(b) \\ F_n(b) & = & \left(\frac{1}{b} + 1\right) S_n - (1 + b^2)\Lambda_1(b) + 2b\Lambda_n(b) \end{array} \right. .$$

We remark that

$$\Delta_n(b) > 0 \text{ if and only if } E_n(b) > 0 \text{ or } F_n(b) < 0.$$

Using the strictly monotonicity of the sequences $(\Lambda_n)_{n\in\mathbb{N}*}$ and $(S_n)_{n\in\mathbb{N}*}$ we get

$$E_{n+1}(b) - E_n(b) = \left(1 + \frac{1}{b}\right)(S_{n+1} - S_n) - 2b\big(\Lambda_{n+1}(b) - \Lambda_n(b)\big) > 0.$$

Therefore $(E_n(b))_{n\in\mathbb{N}*}$ is a strictly increasing. As $\lim\limits_{n\mapsto+\infty} E_n(b) = +\infty$ and

$$E_1(b) = -(1+b)^2\Lambda_1(b),$$

we obtain that

$$\exists N \in \mathbb{N}^\star \text{such that } \forall n \geq N,\ E_n(b) > 0.$$

This implies the assertion (2).

(3) Straightforward computations yield

$$\begin{aligned}
\Delta_{n+1}-\Delta_n &= \left(1+\frac{1}{b}\right)^2 (S_{n+1}^2-S_n^2) - 4b^2\left(\Lambda_{n+1}^2(b)-\Lambda_n^2(b)\right) - 2(1+b^2)\left(1+\frac{1}{b}\right)\Lambda_1(b)(S_{n+1}-S_n)\\
&= \left(1+\frac{1}{b}\right)(S_{n+1}-S_n)\left[\left(1+\frac{1}{b}\right)(S_{n+1}+S_n) - 2(1+b^2)\Lambda_1(b)\right] - 4b^2\left(\Lambda_{n+1}^2(b)-\Lambda_n^2(b)\right)\\
&> \left(1+\frac{1}{b}\right)(S_{n+1}-S_n)\left[\left(1+\frac{1}{b}\right)(S_{n+1}+S_n) - 2(1+b^2)\Lambda_1(b)\right]\\
&> \left(1+\frac{1}{b}\right)(S_{n+1}-S_n)\left[\left(1+\frac{1}{b}\right)(S_{n+1}+S_n) - 2(1+b^2)\Lambda_1(b) - 2b\Big[\Lambda_{n+1}(b)+\Lambda_n(b)\Big]\right]\\
&> \left(1+\frac{1}{b}\right)(S_{n+1}-S_n)\big(E_{n+1}(b)+E_n(b)\big) > 0.
\end{aligned}$$

As $(\Delta_n)_{n\geq n}$ and $(S_n)_{n\in\mathbb{N}^\star}$ are strictly increasing, $(\lambda_n^+)_{n\geq N}$ is also strictly increasing. We focus now on λ_n^-:

$$\begin{aligned}
\lambda_{n+1}^- - \lambda_n^- &= \left(\frac{1}{b}-1\right)(S_{n+1}-S_n) - \big[\sqrt{\Delta_{n+1}}-\sqrt{\Delta_n}\big]\\
&= \left(\frac{1}{b}-1\right)(S_{n+1}-S_n) - \frac{\Delta_{n+1}-\Delta_n}{\sqrt{\Delta_{n+1}}+\sqrt{\Delta_n}}\\
&= -\frac{1}{\sqrt{\Delta_{n+1}}+\sqrt{\Delta_n}}\left[\left(1+\frac{1}{b}\right)(S_{n+1}-S_n)\left[\left(1+\frac{1}{b}\right)(S_{n+1}+S_n) - 2(1+b^2)\Lambda_1(b)\right]\right]\\
&+ \frac{4b^2(\Lambda_{n+1}^2(b)-\Lambda_n^2(b))}{\sqrt{\Delta_{n+1}}+\sqrt{\Delta_n}} + \left(\frac{1}{b}-1\right)(S_{n+1}-S_n)\\
&< -\frac{1}{\sqrt{\Delta_{n+1}}+\sqrt{\Delta_n}}\left[\left(1+\frac{1}{b}\right)(S_{n+1}-S_n)\left[\left(1+\frac{1}{b}\right)(S_{n+1}+S_n) - 2(1+b^2)\Lambda_1(b)\right]\right]\\
&+ \left(\frac{1}{b}-1\right)(S_{n+1}-S_n)\\
&< \frac{1}{b}\left[1-\frac{1}{\sqrt{\Delta_{n+1}}+\sqrt{\Delta_n}}\left[\left(1+\frac{1}{b}\right)(S_{n+1}+S_n) - 2(1+b^2)\Lambda_1(b)\right]\right](S_{n+1}-S_n)\\
&- \left[1+\frac{1}{\sqrt{\Delta_{n+1}}+\sqrt{\Delta_n}}\left[\left(1+\frac{1}{b}\right)(S_{n+1}+S_n) - 2(1+b^2)\Lambda_1(b)\right]\right](S_{n+1}-S_n)\\
&< 0
\end{aligned}$$

because $\big[\left(1+\frac{1}{b}\right)(S_{n+1}+S_n) - 2(1+b^2)\Lambda_1(b)\big] > E_{n+1}(b)+E_n(b) > 0$ and $\sqrt{\Delta_n} < \left(1+\frac{1}{b}\right)S_n - (1+b^2)\Lambda_1(b)$.

Consequently the sequence $(\lambda_n^-)_{n\geq N}$ is strictly decreasing.

(4) It is obvious and follows from (2) and (3). □

5.3. **Proof of Theorem 1.1.** This section is dedicated to the proof of the main result of this paper which is deeply related to the spectral study developed in the preceding section combined with Crandall-Rabinowitz's theorem. To proceed, fix $b \in (0,1)$ and $m \geq N$, where N was defined in Proposition 5.2. Set,

$$X_m^{k+\log} = X^{k+\log} \cap \mathcal{A}_\varepsilon^m.$$

We define the ball of radius $r \in (0,1)$ by

$$B_r^m = \Big\{ f \in X_m^{k+\log}, \|f\|_{X_m^{k+\log}} \leq r \Big\}$$

and we introduce the neighborhood of the trivial solution $(0,0)$,

$$V_{m,r} \triangleq B_r^m \times B_r^m.$$

The set $V_{m,r}$ is endowed with the induced topology of the product spaces. Take $(f_1, f_2) \in V_{m,r}$ then the expansions of the associated conformal mappings Φ_1, Φ_2 in $\mathbb{A}_\varepsilon$ are given successively by

$$\Phi_1(z) = z + f_1(z) = z\left(1 + \sum_{n=1}^{+\infty} \frac{a_n}{z^{nm}}\right)$$

and

$$\Phi_2(z) = bz + f_2(z) = z\left(b + \sum_{n=1}^{+\infty} \frac{c_n}{z^{nm}}\right).$$

Consequently for any $z \in \mathbb{A}_\varepsilon$

$$\Phi_j(e^{\frac{2i\pi}{m}} z) = e^{\frac{2i\pi}{m}} \Phi_j(z),\ j = 1,2 \text{ and } |z| > \varepsilon. \tag{5.10}$$

From Proposition 5.2 recall the definition of the eigenvalues $\lambda_m^\pm$ and the associated angular velocities are

$$\begin{aligned}\Omega_m^\pm &= \frac{1}{2} - \frac{1}{2}\lambda_m^\pm \\ &= \frac{1}{2}\tilde{C}_m \pm \frac{1}{2}\sqrt{\Delta_m}\end{aligned}$$

with

$$\Delta_m = \left(\left(\frac{1}{b} + 1\right)S_m - (1+b^2)\Lambda_1(b)\right)^2 - 4b^2\Lambda_m^2(b)$$

and

$$\tilde{C}_m = \left(1 - \frac{1}{b}\right)S_m + (1-b^2)\Lambda_1(b).$$

Note that S_m and $\Lambda_m(b)$ were introduced in Proposition 5.1. The V-states equations are described in (4.1) and (2.2) which we restate here, for $j \in \{1,2\}$,

$$\tilde{G}(\Omega, \Phi_1, \Phi_2) \triangleq G(\Omega, f_1, f_2) \text{ and } G = (G_1, G_2)$$

with

$$\tilde{G}_j(\Omega, \Phi_1, \Phi_2)(\omega) = \operatorname{Im}\left\{\left(\Omega\Phi_j(\omega) - \fint_{\mathbb{T}} \frac{\tau\Phi_1'(\tau) - \omega\Phi_j'(\omega)}{|\Phi_1(\tau) - \Phi_j(\omega)|}\frac{d\tau}{\tau} + \fint_{\mathbb{T}} \frac{\tau\Phi_2'(\tau) - \omega\Phi_j'(\omega)}{|\Phi_2(\tau) - \Phi_j(\omega)|}\frac{d\tau}{\tau}\right)\overline{\Phi_j'(\omega)\overline{\omega}}\right\}.$$

The following result is more precise than Theorem 1.1.

Theorem 5.3. *Let $k \geq 3, N$ be as in the Proposition 5.2, $m \geq N$, and take $\Omega \in \{\Omega_m^\pm\}$. Then, the following assertions hold true.*

1. *There exists $r > 0$ such that $G : \mathbb{R} \times V_{m,r} \mapsto Y_m^{k-1} \times Y_m^{k-1}$ is well- defined and is of class C^1.*
2. *The kernel of $DG(\Omega, 0, 0)$ is one dimensional and generated by*

$$v_{0,m} : \omega \in T \mapsto \begin{pmatrix} \Omega + \frac{S_m}{b} - \Lambda_1(b) \\ -\Lambda_m(b) \end{pmatrix} \overline{\omega}^{m-1}.$$

3. *The range of $DG(\Omega, 0, 0)$ is closed and is of co-dimension one in $Y_m^{k-1} \times Y_m^{k-1}$.*
4. *Transversality assumption: If Ω is a simple eigenvalue ($\Delta_m > 0$) then*

$$\partial_\Omega DG(\Omega_m^\pm, 0, 0)v_{0,m} \notin \operatorname{Im}\left(DG(\Omega_m^\pm, 0, 0)\right).$$

Proof. (1) Compared to Theorems 4.1 and 4.5, we just need to check that $G = (G_1, G_2)$ preserves the m-fold symmetry and maps $X_m^{k+\log} \times X_m^{k+\log}$ into $Y_m^{k-1} \times Y_m^{k-1}$. To this end, it is sufficient to check that for

given $(f_1, f_2) \in X_m^{k+\log} \times X_m^{k+\log}$ the Fourier coefficients of $\tilde{G}_j(\Omega, \Phi_1, \Phi_2)$ vanish at frequencies which are not integer multiple of m. This amounts to proving that,

$$\tilde{G}_j(\Omega, \Phi_1, \Phi_2)(e^{i\frac{2\pi}{m}}\omega) = \tilde{G}_j(\Omega, \Phi_1, \Phi_2)(\omega),\ \forall \omega \in \mathbb{T},\ j = 1, 2.$$

As

$$\Phi_j'(e^{\frac{2i\pi}{m}}\omega) = \Phi_j'(\omega), \tag{5.11}$$

the property is obvious for the first term $\mathrm{Im}\{\Omega\overline{\omega}\overline{\Phi_j'(\omega)}\Phi_j(\omega)\}$. For the two last terms of $\tilde{G}_j$ it is enough to check the identity,

$$\forall \omega \in \mathbb{T},\ S(\Phi_i, \Phi_j)(e^{\frac{2i\pi}{m}}\omega) = e^{\frac{2i\pi}{m}} S(\Phi_i, \Phi_j)(\omega).$$

This follows easily by making the change of variables $\tau = e^{\frac{2i\pi}{m}}\xi$ and from (5.10) and (5.11),

$$\begin{aligned} S(\Phi_i, \Phi_j)(e^{\frac{2i\pi}{m}}\omega) &= \fint_{\mathbb{T}} \frac{e^{\frac{2i\pi}{m}}\xi\Phi_i'(e^{\frac{2i\pi}{m}}\xi) - e^{\frac{2i\pi}{m}}\omega\Phi_j'(e^{\frac{2i\pi}{m}}\omega)}{|\Phi_i(e^{\frac{2i\pi}{m}}\xi) - \Phi_j(e^{\frac{2i\pi}{m}}\omega)|} \frac{d\xi}{\xi} \\ &= e^{\frac{2i\pi}{m}} \fint_{\mathbb{T}} \frac{\xi\Phi_i'(\xi) - \omega\Phi_j'(\omega)}{|\Phi_i(\xi) - \Phi_j(\omega)|} \frac{d\xi}{\xi} \\ &= e^{\frac{2i\pi}{m}} S(\Phi_i, \Phi_j)(\omega). \end{aligned}$$

This concludes the proof of the following statement,

$$\forall\, (f_1, f_2) \in V_{m,r}, \quad G(\Omega, f_1, f_2) \in Y_m^{k-1} \times Y_m^{k-1}.$$

(2) We shall describe the kernel of linear operator $DG(\Omega_m^{\pm}, 0, 0)$ and show that it is one-dimensional. Let h_1, h_2 be two functions in $X_m^{k+\log}$ such that

$$h_1(\omega) = \sum_{n=1}^{+\infty} a_n \overline{\omega}^{nm-1} \text{ and } h_2(\omega) = \sum_{n=1}^{+\infty} c_n \overline{\omega}^{nm-1}. \tag{5.12}$$

Recall from Proposition 5.1 the following expression,

$$DG(\Omega, 0, 0)(h_1, h_2) = \frac{i}{2} \sum_{n \geq 1} nm M_{nm} \begin{pmatrix} a_n \\ c_n \end{pmatrix} (\omega^{nm} - \overline{\omega}^{nm}) \tag{5.13}$$

where the matrice M_n is given for $n \geq 2$ by :

$$M_n = \begin{pmatrix} \Omega - S_n + b^2\Lambda_1(b) & -b^2\Lambda_n(b) \\ b\Lambda_n(b) & b\Omega + S_n - b\Lambda_1(b) \end{pmatrix}.$$

Now if $\Omega \in \{\Omega_m^{\pm}\}$ then

$$\det(M_m) = 0.$$

Thus, the kernel of $DG(\Omega, 0, 0)$ is non trivial and is one-dimensional if and only if:

$$\det(M_{nm}) \neq 0,\ \forall n \geq 2.$$

This condition is ensured by Proposition 5.2. Hence we have the equivalence:

$$(h_1, h_2) \in \mathrm{Ker}(DG(\Omega, 0, 0)) \quad \text{if and only if} \quad a_n = c_n = 0\ \forall n \geq 2 \text{ and } (a_1, c_1) \in \mathrm{Ker}(M_m) \tag{5.14}$$

Hence, a generator of $\mathrm{Ker}\,(DG(\Omega, 0, 0))$ can be chosen as the pair of functions

$$\omega \in \mathbb{T} \mapsto \begin{pmatrix} \Omega + \frac{S_m}{b} - \Lambda_1(b) \\ -\Lambda_m(b) \end{pmatrix} \overline{\omega}^{m-1}.$$

(3) We introduce

$$\begin{aligned} Z_m = \Bigg\{ & g = (g_1, g_2) \in Y_m^{k-1} \times Y_m^{k-1} | g(\omega) = \sum_{n \geq 1} \begin{pmatrix} A_n \\ C_n \end{pmatrix} (\omega^{nm} - \overline{\omega}^{nm}),\ \forall \omega \in \mathbb{T} \\ & \text{s.t. } (A_n, C_n) \in \mathbb{R}^2\ \forall n \geq 2 \text{ and } \exists (a_1, c_1) \in \mathbb{R}^2 \text{ with } M_m \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} \Bigg\}. \end{aligned}$$

Z_m is closed and of codimension 1 in $Y_m^{k-1}\times Y_m^{k-1}$. The following inclusion is obvious

$$\mathrm{Im}(DG(\Omega,0,0))\subset Z_m.$$

Therefore it remains just to check the converse. Let $(g_1,g_2)\in Z_m$, we shall prove that the equation :

$$DG(\Omega,0,0)(h_1,h_2)=(g_1,g_2)$$

admits a solution $(h_1,h_2)\in X_m^{k+\log}\times X_m^{k+\log}$ where the Fourier expansions of these functions are given in (5.12). According to (4.7), the preceding equation is equivalent to

$$nmM_{nm}\left(\begin{array}{c}a_n\\ c_n\end{array}\right)=\left(\begin{array}{c}A_n\\ C_n\end{array}\right),\ \forall n\in\mathbb{N}^\star.$$

For $n=1$, the existence follows from the condition of space Z_m and therefore we shall only focus on $n\geq 2$. Owing to (5.14) the sequences $(a_n)_{n\geq 2}$ and $(c_n)_{n\geq 2}$ are uniquely determined by the formula

$$\left(\begin{array}{c}a_n\\ c_n\end{array}\right)=\frac{1}{nm}M_{nm}^{-1}\left(\begin{array}{c}A_n\\ C_n\end{array}\right),\ \forall n\geq 2.$$

By computing the matrix M_{nm}^{-1} we deduce that for all $n\geq 2$,

$$\left\{\begin{array}{rcl}a_n&=&\frac{b(\Omega+\frac{1}{b}S_{nm}-\Lambda_1(b))}{mn\det(M_{nm})}A_n+\frac{b^2\Lambda_{nm}(b)}{mn\det(M_{nm})}C_n\\ c_n&=&-\frac{b\Lambda_{nm}(b)}{mn\det(M_{nm})}A_n+\frac{(\Omega-bS_{nm}+b^2\Lambda_1(b))}{mn\det(M_{nm})}C_n\end{array}\right..$$

We just need to check that $(h_1,h_2)\in X_m^{k+\log}\times X_m^{k+\log}$. We shall develop the computations only for h_1 since the same analysis can be applied to h_2. By using the characterization given by Lemma 3.2 one writes

$$\begin{aligned}\|h_1\|^2_{X^{k+\log}}&\approx|a_1|^2+\sum_{n=2}^{+\infty}\frac{(mn)^{2k}}{\varepsilon^{2(nm+k-1)}}\left(1+\log(nm)\right)^2\left[\frac{b\left(\Omega+\frac{1}{b}S_{nm}-\Lambda_1(b)\right)}{mn\det(M_{nm})}A_n+\frac{b^2\Lambda_{nm}(b)}{mn\det(M_{nm})}C_n\right]^2\\&\lesssim|a_1|^2+\sum_{n=2}^{+\infty}\frac{(mn)^{2(k-1)}}{\varepsilon^{2(nm+k-1)}}\frac{\left(1+\log(nm)\right)^2}{\det(M_{nm})^2}\left[S_{nm}^2A_n^2+\Lambda_{nm}(b)^2C_n^2\right]\\&\lesssim|a_1|^2+\sum_{n=2}^{+\infty}\frac{(mn)^{2(k-1)}}{\varepsilon^{2(nm+k-1)}}\left(A_n^2+C_n^2\right)\\&\lesssim\|g_1\|_{Y_m^{k-1}}+\|g_2\|_{Y_m^{k-1}}.\end{aligned}$$

We have used the asymptotics $S_{nm}\sim\log(nm)$ and $|\det(M_{nm})|\sim S_{nm}^2$.
(4) We have

$$\partial_\Omega DG(\Omega_m^\pm,0,0)v_{0,m}=\frac{im}{2}\left(\begin{array}{c}\Omega+\frac{S_m}{b}-\Lambda_1(b)\\ -b\Lambda_m(b)\end{array}\right)(\omega^m-\overline{\omega}^m).$$

We resort to reductio ad absurdum and we suppose that

$$\partial_\Omega DG(\Omega_m^\pm,0,0)v_{0,m}\in\mathrm{Im}(DG(\Omega_m^\pm,0,0)).$$

Then there exists $(a_1,c_1)\in\mathbb{R}^2$ such that

$$\left(\begin{array}{c}\Omega+\frac{S_m}{b}-\Lambda_1(b)\\ -b\Lambda_m(b)\end{array}\right)=M_m\left(\begin{array}{c}a_1\\ c_1\end{array}\right).$$

As M_m has a one-dimension kernel, $\left(\begin{array}{c}\Omega+\frac{S_m}{b}-\Lambda_1(b)\\ -\Lambda_m(b)\end{array}\right)$ will be a scalar multiple of one column of the matrix M_m which happens if and only if

$$\left(\Omega+S_m-\Lambda_1(b)\right)^2-b^2\Lambda_m(b)^2=0. \tag{5.15}$$

Combining this equation with $\det(M_m)=0$, we get

$$\left(\Omega-S_m+b^2\Lambda_1(b)\right)\left(\Omega+\frac{S_m}{b}-\Lambda_1(b)\right)+\left(\Omega+\frac{S_m}{b}-\Lambda_1(b)\right)^2=0.$$

This yields

$$\left(\Omega+\frac{S_m}{b}-\Lambda_1(b)\right)\left(2\Omega+(b^2-1)\Lambda_1(b)+\left(-1+\frac{1}{b}\right)S_m\right)=0$$

which is equivalent to

$$\Omega+\frac{S_m}{b}-\Lambda_1(b)=0 \text{ ou } \Omega=\frac{1}{2}\left((1-b^2)\Lambda_1(b)+(1-\frac{1}{b})S_m\right).$$

This first possibility is excluded by (5.15) because $\Lambda_m(b)\neq 0$ and the second one is also impossible because it corresponds to a double eigenvalue which is not also the case here. We obtain an absurdity and this concludes the proof of Theorem 5.3. □

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