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ON UNCONDITIONAL WELL-POSEDNESS FOR THE PERIODIC MODIFIED KORTEWEG-DE VRIES EQUATION

LUC MOLINET*, DIDIER PILOD† AND STÉPHANE VENTO*

Abstract. We prove that the modified KdV equation is unconditionally well-posed in $H^s(\mathbb{T})$ for $s \geq 1/3$.

1. Introduction

We consider the initial value problem (IVP) associated to the modified KdV (mKdV) equation

$$\partial_t u + \partial_x^3 u \mp \partial_x(u^3) = 0 \tag{1.1}$$

$$u(\cdot, 0) = u_0 \,, \tag{1.2}$$

where $u = u(x,t)$ is a real valued function, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $t \in \mathbb{R}$.

In [1] Bourgain introduced the Fourier restriction norm method and proved that (1.1) is locally well-posed in $H^s(\mathbb{T})$ for $s \geq 1/2$. Note that, by a change of variable, Bourgain substituted the mKdV equation (1.1) by the renormalized mKdV equation

$$v_t + \partial_x^3 v \mp 3(v^2 - P_0(v_0^2))v_x = 0, \quad v(\cdot, 0) = v_0 \,,$$

where $P_0 w$ denotes the mean value of w. This result was then proved to be sharp if one requires moreover the smoothness or the uniform continuity on bounded sets of the solution-map associated with the renormalized equation (see [2], [11], [5]). This obstruction is related to the resonant term $\sum_{k\in\mathbb{Z}} |\widehat{v}(k)|^2 \widehat{v}(k) e^{ikx}$ that appears in the nonlinear part of this equation. However, in [20], Takaoka-Tsutsumi proved that (1.1) is locally well posed in $H^s(\mathbb{T})$ for $s > 3/8$. For this, they first establish a smoothing effect on $|\mathcal{F}_x(v(t))(k)|^2 - |\widehat{v_0}(k)|^2$ and then work in a Bourgain's space depending on the initial data in order to treat the resonant term. This was improved in [19] where the local well-posedness was pushed down to $H^s(\mathbb{T})$ with $s > 1/3$.

The local well-posednesss results proved in these papers mean the following : for any initial data $u_0 \in H^s(\mathbb{T})$ there exists a time $T = T(\|u_0\|_{H^s}) > 0$ only depending on $\|u_0\|_{H^s}$ and a solution u that satisfies the equation at least in some weak sense and is unique in some function space (called resolution space) $X \hookrightarrow C([0]; H^s(\mathbb{T}))$ that can depends on the initial data. Moreover, for any $R > 0$, the flow-map $u_0 \mapsto u$ is continuous from the ball centered at the origin with radius R of $H^s(\mathbb{T})$ into $C([0, T(R)]; H^s(\mathbb{T}))$.

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On the other hand, in [7], Kappeler and Topalov introduced the following notion of solutions which a priori does not always corresponds to the solution in the sense of distribution : *A continuous curve* $\gamma : (a,b) \to H^\beta(\mathbb{T})$ *with* $0 \in (a,b)$ *and* $\gamma(0) = u_0$ *is called a solution of the mKdV equation in* $H^\beta(\mathbb{T})$ *with initial data* u_0 *iff for any* C^∞*-sequence of initial data* $\{u_{0,n}\}$ *converging to* u_0 *in* $H^\beta(\mathbb{T})$ *and for any* $t \in]a,b[$*, the sequence of emanating solutions* $\{u_n\}$ *of the mKdV equation satisfies :* $u_n(t) \to \gamma(t)$ *in* $H^\beta(\mathbb{T})$

Note that a solution in the sense of this definition is necessarily unique. With this notion of solution they proved the global well-posedness of the defocusing mKV equation (with a + sign in front of the nonlinear term) in $H^s(\mathbb{T})$ $s \geq 0$, with a solution-map which is continuous from $L^2(\mathbb{T})$) into $C(\mathbb{R}; L^2(\mathbb{T}))$. Their proof is based on the inverse scattering method and thus depends in a crucial way of the complete integrability of this equation. It is worth noticing that, by Sobolev embedding theorem, their solutions of the defocussing mKdV equation satisfy the equation in the distributional sense as soon as $s \geq 1/6$. In [14] Molinet proved that, actually, the solutions constructed by Kappeler-Topalov always satisfy the equation at least in a weak sense. He also proved that the flow-map cannot be continuously extended in $H^s(\mathbb{T})$ as soon as $s < 0$. Therefore the result of Kappeler-Topalov is in some sense optimal. However it is not known to hold for the focusing equation. Moreover, it uses the integrability of the equation and is thus not suitable to solve perturbations of the defocusing mKdV equation. Also, the question of the existence of a resolution space where the uniqueness holds remains open at this low regularity level.

Another interesting question about uniqueness, even in higher regularity, is to know wether uniqueness also holds in some larger spaces that contains weak solutions. This kind of question was first raised by Kato [8] in the Schrödinger equation context. We refer to such uniqueness in $L^\infty(]0,T[; H^s)$, without intersecting with any auxiliary function space as *unconditional uniqueness*. This ensures the uniqueness of the weak solutions to the equation at the H^s-regularity. This is useful, for instance, to pass to the limit on perturbations of the equation as the perturbative coefficient tends to zero (see for instance [15] for such an application).

Unconditional uniqueness was proved for the mKdV equation to hold in $H^{1/2}(\mathbb{T})$ by Kwon and Oh ([12]) following an approach developed in [3]. In this paper we push down the local well-posedness and the unconditional uniqueness for the mKdV equation to $H^{1/3}(\mathbb{T})$.

To obtain our unconditional uniqueness result we gather the approach developed in [17] based on the construction of modified energies with some ideas of [20] and [19] to derive the smoothing effect. On one side the absence of very small frequencies enables to simplify some estimates on the nonlinear term with respect to [17]. On the other side because of true resonances we need to derive a smoothing effect as in [20]. Actually this is the obtention of the smoothing effect that limits us to the Sobolev index $s \geq 1/3$ (see Remark 4.10). It is also worth noticing that we do not succeed to get an estimate on the $L^\infty_T H^s$-norm of the difference of two solutions with different initial data - this seems to be related to the fact that the flow-map is not Lipschitz below $s = 1/2$. Instead we will establish an a priori estimate in $L^\infty_T H^{s'}$, for some $s' < s$, on the difference of two solutions emanating from the same initial datum. This estimate will lead to the unconditional uniqueness result. It will be also sufficient to prove the well-posedness result thanks to the smoothing

effect which ensures that, given a sequence of solutions $\{u_n\} \subset L^\infty(0,T;H^s(\mathbb{T}))$ to (2.1) associated with a sequence of initial data $\{u_{0,n}\}$ relatively compact in $H^s(\mathbb{T})$, the set $\{u_n(t),\, t \in [0,T]\}$ is relatively compact in $H^s(\mathbb{T})$.

2. Notations, Functions spaces and statement of the result

We will not work directly with the mKdV equation but with the renormalized mKdV equation defined by

$$u_t + \partial_x^3 u \mp \partial_x(u^3 - 3P_0(u^2)u) = 0\,. \tag{2.1}$$

We explain how to come back to the mKdV equation (1.1) in Subsection 6.3. In the sequel of this paper, we choose to the take the sign "+" since this sign will not play any role in our analysis. Let us start by giving our notion of solution.

Definition 2.1. Let $T > 0$ and $s \geq \frac{1}{6}$. We will say that $u \in L^\infty(0,T;H^s(\mathbb{T}))$ is a solution to (1.1) (resp. (2.1)) associated with the initial datum $u_0 \in H^s(\mathbb{T})$ if u satisfies (1.1)-(1.2) (resp. (2.1)-(1.2)) in the distributional sense, i.e. for any test function $\phi \in C_c^\infty(]-T,T[\times\mathbb{T})$, there holds

$$\int_0^\infty \int_{\mathbb{T}} \Big[(\phi_t + \partial_x^3\phi)u + \phi_x F(u)\Big]\, dx\, dt + \int_{\mathbb{T}} \phi(0,\cdot)u_0\, dx = 0 \tag{2.2}$$

where $F(u) = u^3$ (resp. $F(u) = u^3 - 3P_0(u^2)u$).

Remark 2.2. Note that for $u \in L^\infty(0,T;H^s(\mathbb{T}))$, with $s \geq \frac{1}{6}$, u^3 is well-defined and belongs to $L^\infty(0,T;L^1(\mathbb{T}))$. Therefore (2.2) forces $u_t \in L^\infty(0,T;H^{-3}(\mathbb{T}))$ and ensures that (1.1) (resp. (2.1)) is satisfied in $L^\infty(0,T;H^{-3}(\mathbb{T}))$. In particular, $u \in C([0,T];H^{-3}(\mathbb{T}))$ and (2.2) forces the initial condition $u(0) = u_0$. Note that , since $u \in L^\infty(0,T;H^s(\mathbb{T}))$, this actually ensures that $u \in C([0,T];H^{s'}(\mathbb{T}))$ for any $s' < s$. Finally, we notice that this also ensures that u satisfies the Duhamel formula associated with (1.1)-(1.2)) (resp. (2.1)-(1.2)).

Definition 2.3. Let $s \geq \frac{1}{6}$. We will say that the Cauchy problem associated with (1.1) (resp. (2.1)) is unconditionally locally well-posed in $H^s(\mathbb{T})$ if for any initial data $u_0 \in H^s(\mathbb{T})$ there exists $T = T(\|u_0\|_{H^s}) > 0$ and a solution $u \in C([0,T];H^s(\mathbb{T}))$ to (1.1) (resp. (2.1)) emanating from u_0. Moreover, u is the unique solution to (1.1) (resp. (2.1)) associated with u_0 that belongs to $L^\infty(]0,T[;H^s(\mathbb{T}))$. Finally, for any $R > 0$, the solution-map $u_0 \mapsto u$ is continuous from the ball of $H^s(\mathbb{T})$ with radius R centered at the origin into $C([0,T(R)];H^s(\mathbb{T}))$.

Theorem 2.4. *The mKdV equation* (1.1) *and the renormalized mKdV equation* (2.1) *are unconditionally locally well-posed in* $H^s(\mathbb{T})$ *for* $s \geq 1/3$.

2.1. **Notation.** For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c such that $a \leq cb$. We also denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. Moreover, if $\alpha \in \mathbb{R}$, α_+, respectively α_-, will denote a number slightly greater, respectively lesser, than α.

For $a_1,\ a_2,\ a_3 \in \mathbb{R}_+^*$, we define the quantities $a_{max} \geq a_{med} \geq a_{min}$ to be the maximum, sub-maximum and minimum of $a_1,\ a_2$ and a_3. Usually, we use $k_i,\ j_i$ to denote integers and $N_i = 2^{k_i}$, $L_i = 2^{j_i}$ to denote dyadic numbers. For $f = f(x) \in L^2(\mathbb{T})$, we denote its Fourier transform by $\hat{f} : \mathbb{Z} \to \mathbb{C}$ by $\widehat{f}(k) = \int_{\mathbb{T}} e^{-2i\pi kx} f(x)\, dx$ and or any integer $k \in \mathbb{N}$ we set

$$P_k f = \widehat{f}(k)e^{2i\pi kx},\ P_{\sim k} = \sum_{|q|\sim k} \widehat{f}(q)e^{2i\pi kqx} \text{ and } P_{\lesssim k}u = \sum_{|q|\lesssim k} \widehat{f}(q)e^{2i\pi qx}\,.$$

In particular,

$$P_0 f = \widehat{f}(0) = \int_{\mathbb{T}} f(x)\, dx\,.$$

For $u = u(x,t) \in \mathcal{S}'(\mathbb{R}^2)$, $\mathcal{F}_{tx} u = (u)^{\wedge_{tx}}$ will denote its space-time Fourier transform, whereas $\mathcal{F}_x u = \widehat{u}$, respectively $\mathcal{F}_t u = (u)^{\wedge_t}$, will denote its Fourier transform in space, respectively in time. For $s \in \mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, J_x^s and D_x^s, by

$$J_x^s u = \mathcal{F}_x^{-1}\big((1+|k|^2)^{\frac{s}{2}} \mathcal{F}_x u\big) \quad \text{and} \quad D_x^s u = \mathcal{F}_x^{-1}\big(|k|^s \mathcal{F}_x u\big).$$

We also denote by $U(t) = e^{-t\partial_x^3}$ the unitary group associated to the linear part of (1.1), *i.e.*,

$$U(t)u_0 = e^{-t\partial_x^3} u_0 = \mathcal{F}_x^{-1}\big(e^{itk^3} \mathcal{F}_x(u_0)(k)\big)\,.$$

Throughout the paper, we fix a smooth cutoff function χ such that

$$\chi \in C_0^\infty(\mathbb{R}), \quad 0 \le \chi \le 1, \quad \chi_{|_{[-1,1]}} = 1 \quad \text{and} \quad \operatorname{supp}(\chi) \subset [-2,2].$$

We set $\phi(k) := \chi(k) - \chi(2k)$. For $l \in \mathbb{Z}$, we define

$$\phi_{2^l}(k) := \phi(2^{-l}k),$$

and, for $l \in \mathbb{N}^*$,

$$\psi_{2^l}(k,\tau) = \phi_{2^l}(\tau - k^3).$$

By convention, we also denote

$$\phi_0(k) = \chi(2k) \quad \text{and} \quad \psi_0(k,\tau) := \chi(2(\tau - k^3))\,.$$

Any summations over capitalized variables such as N, L, K or M are presumed to be dyadic.We work with non-homogeneous dyadic decompositions *i.e.*, these variables range over numbers of the form $\{2^k : k \in \mathbb{N}\} \cup \{0\}$. We call those numbers *nonhomogeneous dyadic numbers*. Then, we have that $\sum_N \phi_N(k) = 1$

$$\operatorname{supp}(\phi_N) \subset I_N := \{\frac{N}{2} \le |k| \le 2N\},\ N \ge 1, \quad \text{and} \quad \operatorname{supp}(\phi_0) \subset I_0 := \{|k| \le 1\}.$$

Finally, let us define the Littlewood-Paley multipliers P_N, R_K and Q_L by

$$P_N u = \mathcal{F}_x^{-1}\big(\phi_N \mathcal{F}_x u\big), \quad R_K u = \mathcal{F}_t^{-1}\big(\phi_K \mathcal{F}_t u\big) \quad \text{and} \quad Q_L u = \mathcal{F}^{-1}\big(\psi_L \mathcal{F} u\big),$$

$P_{\ge N} := \sum_{K \ge N} P_K$, $P_{\le N} := \sum_{K \le N} P_K$, $Q_{\ge L} := \sum_{K \ge L} Q_K$ and $Q_{\le L} := \sum_{K \le L} Q_K$.

Sometimes, for the sake of simplicity and when there is no risk of confusion, we also denote $u_N = P_N u$.

2.2. **Function spaces.** For $1 \le p \le \infty$, $L^p(\mathbb{T})$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{T})$ denotes the space of all distributions of $(C^\infty(\mathbb{T}))'$ whose usual norm $\|u\|_{H^s} = \|J_x^s u\|_{L^2}$ is finite.

If B is one of the spaces defined above, $1 \le p \le \infty$ and $T > 0$, we define the space-time spaces $L_t^p B_x$, and $L_T^p B_x$ equipped with the norms

$$\|u\|_{L_t^p B_x} = \Big(\int_{\mathbb{R}} \|f(\cdot,t)\|_B^p dt\Big)^{\frac{1}{p}}, \quad \|u\|_{L_T^p B_x} = \Big(\int_0^T \|f(\cdot,t)\|_B^p dt\Big)^{\frac{1}{p}}$$

with obvious modifications for $p=\infty$. For $s,\ b\in\mathbb{R}$, we introduce the Bourgain spaces $X^{s,b}$ related to linear KdV group as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|u\|_{X^{s,b}} := \left(\sum_k \int_{\mathbb{R}} \langle \tau - k^3\rangle^{2b}\langle k\rangle^{2s}|\mathcal{F}_{xt}u(k,\tau)|^2 d\tau\right)^{\frac{1}{2}}, \tag{2.3}$$

where $\langle x\rangle := (1+|x|^2)^{1/2}$. It is easy to see that

$$\|u\|_{X^{s,b}} \sim \|U(-t)u\|_{H^{s,b}_{x,t}} \quad \text{where} \quad \|u\|_{H^{s,b}_{x,t}} = \|J^s_x J^b_t u\|_{L^2_{x,t}}\,. \tag{2.4}$$

We defined the function space Z^s, with $s\geq 0$, by

$$Z^s = X^{s-\frac{11}{10},0}\cap L^\infty_t H^s_x\,. \tag{2.5}$$

Finally, we will also use restriction in time versions of these spaces. Let $T>0$ be a positive time and B be a normed space of space-time functions. The restriction space B_T will be the space of functions $u:\mathbb{R}\times]0,T[\to\mathbb{R}$ or $\mathbb{C}$ satisfying

$$\|u\|_{B_T} := \inf\{\|\tilde{u}\|_B \mid \tilde{u}:\mathbb{R}\times\mathbb{R}\to\mathbb{R} \text{ or } \mathbb{C},\ \tilde{u}|_{\mathbb{R}\times]0,T[} = u\} < \infty\,.$$

3. A priori estimates on solutions

3.1. **Preliminaries.** Let us set

$$m_{min} = \min_{1\leq i\neq j\leq 3}|k_i+k_j| \tag{3.1}$$

and

$$\begin{aligned} A_1 &= \left\{(k_1,k_2,k_3)\in\mathbb{Z}^3 \,:\, |k_2+k_3| = m_{min}\right\},\\ A_2 &= \left\{(k_1,k_2,k_3)\in\mathbb{Z}^3/A_1 \,:\, |k_1+k_3| = m_{min}\right\},\\ A_3 &= \left\{(k_1,k_2,k_3)\in\mathbb{Z}^3/(A_1\cup A_2) \,:\, |k_1+k_2| = m_{min}\right\} = \mathbb{Z}^3/(A_1\cup A_2)\,. \end{aligned} \tag{3.2}$$

Then, it is clear from the definition of those sets that

$$\sum_{j=1}^3 \chi_{A_j}(k_1,k_2,k_3) = 1,\ \forall (k_1,k_2,k_3)\in\mathbb{Z}^3\,. \tag{3.3}$$

Definition 3.1. Let η be a (possibly complex-valued) bounded mesurable function on $\mathbb{R}^3$. We define the *pseudo-product* Π_η (that will also be denoted by Π when there is no risk of confusion) in Fourier variable by

$$\mathcal{F}_x\big(\Pi_\eta(f,g,h)\big)(k) = \sum_{k_1+k_2+k_3=k} \eta(k_1,k_2,k_3)\widehat{f}(k_1)\widehat{g}(k_2)\widehat{h}(k_3)\,. \tag{3.4}$$

Moreover when $(k_1,k_2,k_3)\in A_j$ for some $j=1,2,3$ and $m\sim M$, where M is a dyadic number ≥ 1, we also denote $\Pi^j_{\eta,M}$ (or Π^j_M when there is no risk of confusion) the operator defined in Fourier variable by

$$\mathcal{F}_x\big(\Pi^j_{\eta,M}(f,g,h)\big)(k) = \sum_{k_1+k_2+k_3=k} (\chi_{A_j}\eta)(k_1,k_2,k_3)\phi_M\big(\sum_{q\neq j} k_q\big)\widehat{f}(k_1)\widehat{g}(k_2)\widehat{h}(k_3)\,. \tag{3.5}$$

The following technical lemma corresponds to integration by parts for some pseudo-products (cf. [17]).

Lemma 3.2. *Let N and M be two homogeneous dyadic numbers satisfying $N \gg 1$. For any real-valued functions $f_1,\ f_2,\ g \in L^2(\mathbb{R})$, we define*

$$T_{M,N}(f_1,f_2,g,g) = \int_{\mathbb{T}} P_N \Pi^3_{1,M}(f_1,f_2,g) P_N \partial_x g\,. \tag{3.6}$$

Then, for $M \ll N$, it holds

$$T_{M,N}(f_1,f_2,g,g) = M \int_{\mathbb{T}} \Pi^3_{\eta_3,M}(f_1,f_2,P_{\sim N} g) P_N g dx, \tag{3.7}$$

where η_3 is a function of (k_1,k_2,k_3) whose $l^\infty-$norm is uniformly bounded in N and M.

Proof. From Plancherel's identity we may rewrite $T_{M,N}$ as

$$\begin{aligned} &T_{M,N}(f_1,f_2,g,g)(k) \\ &\quad = \sum_{k_1+k_2+k_3=k} \chi_{A_3}(k_1,k_2,k_3)\phi_M(k_1+k_2)k\phi_N(k)^2 \widehat{f_1}(k_1)\widehat{f_2}(k_2)\widehat{g}(k_3)\overline{\widehat{g}(k)}\,. \end{aligned}$$

We use that $k = k_1+k_2+k_3$ to decompose $T_{M,N}(f_1,f_2,g,g)$ as follows.

$$\begin{aligned} T_{M,N}(f_1,f_2,g,g) = M &\sum_{\frac{N}{2}\le N_3\le 2N} \int_{\mathbb{T}} \Pi^3_{\tilde\eta_1,M}(f_1,f_2,P_{N_3}g)P_N g\,dx \\ &+ M \sum_{\frac{N}{2}\le N_3\le 2N} \int_{\mathbb{T}} \Pi^3_{\tilde\eta_2,M}(f_1,f_2,P_{N_3}g)P_N g\,dx \\ &+ \widetilde{T}_{M,N}(f_1,f_2,g,g)\,, \end{aligned} \tag{3.8}$$

where

$$\tilde\eta_1(k_1,k_2,k_3) = \phi_N(k)\frac{k_1+k_2}{M}\chi_{\operatorname{supp}\phi_M}(k_1+k_2)\,,$$

$$\tilde\eta_2(k_1,k_2,k_3) = \frac{\phi_N(k)-\phi_N(k_3)}{M}k_3\chi_{\operatorname{supp}\phi_M}(k_1+k_2)\,,$$

and

$$\widetilde{T}_{M,N}(f_1,f_2,g,g) = \sum_{\substack{(k,k_1,k_2,k_3)\in\mathbb{Z}^4\\ k_1+k_2+k_3=k}} \chi_{A_3}(k_1,k_2,k_3)\phi_M(k_1+k_2)k_3\widehat{f_1}(k_1)\widehat{f_2}(k_2)\widehat{g_N}(k_3)\overline{\widehat{g_N}(k)}$$

with the notation $g_N = P_N g$.

First, observe from the mean value theorem and the frequency localization that $\tilde\eta_1$ and $\tilde\eta_2$ are uniformly bounded in M and N.

Next, we deal with $\widetilde{T}_{M,N}(f_1,f_2,g,g)$. By using that $k_3 = k-(k_1+k_2)$ observe that

$$\begin{aligned} \widetilde{T}_{M,N}(f_1,f_2,g,g) = &- \sum_{\substack{(k,k_1,k_2,k_3)\in\mathbb{Z}^4\\ k_1+k_2+k_3=k}} \chi_{A_3}(k_1,k_2,k_3)\phi_M(k_1+k_2)(k_1+k_2)\widehat{f_1}(k_1)\widehat{f_2}(k_2)\widehat{g_N}(k_3)\overline{\widehat{g_N}(k)} \\ &+ S_{M,N}(f_1,f_2,g,g) \end{aligned}$$

with

$$S_{M,N}(f_1,f_2,g,g) = \sum_{\substack{(k,k_1,k_2,k_3)\in\mathbb{Z}^4\\ k_1+k_2+k_3=k}} \chi_{A_3}(k_1,k_2,k_3)\phi_M(k_1+k_2)\widehat{f_1}(k_1)\widehat{f_2}(k_2)\widehat{g_N}(k_3)k\overline{\widehat{g_N}(k)}\,.$$

Since g is real-valued, we have $\overline{\widehat{g_N}(k)} = \widehat{g_N}(-k)$, so that

$$S_{M,N}(f_1,f_2,g,g) = \sum_{\substack{(k,k_1,k_2,k_3)\in\mathbb{Z}^4\\ k_1+k_2+k_3=k}} \chi_{A_3}(k_1,k_2,k_3)\phi_M(k_1+k_2)\widehat{f_1}(k_1)\widehat{f_2}(k_2)\overline{\widehat{g_N}(-k_3)}k\widehat{g_N}(-k)\,.$$

We change variable $\hat{k}_3 = -k = -(k_1+k_2+k_3)$, so that $-k_3 = k_1+k_2+\hat{k}_3$. Thus, $S_{M,N}(f_1,f_2,g,g)$ can be rewritten as

$$-\sum_{\substack{(k,k_1,k_2,k_3)\in\mathbb{Z}^4\\ k_1+k_2+k_3=k}} \chi_{A_3}(k_1,k_2,-k_1-k_2-\hat{k}_3)\phi_M(k_1+k_2)\widehat{f_1}(k_1)\widehat{f_2}(k_2)\hat{k}_3\widehat{g_N}(\hat{k}_3)\overline{\widehat{g_N}(k_1+k_2+\hat{k}_3)}\,.$$

Now, observe that $|k_1+(-k_1-k_2-\hat{k}_3)| = |k_2+\hat{k}_3|$ and $|k_2+(-k_1-k_2-\hat{k}_3)| = |k_1+\hat{k}_3|$. Thus $\chi_{A_3}(k_1,k_2,-k_1-k_2-\hat{k}_3) = \chi_{A_3}(k_1,k_2,\hat{k}_3)$ and we obtain

$$S_{M,N}(f_1,f_2,g,g) = -\widetilde{T}_{M,N}(f_1,f_2,g,g)\,,$$

so that

$$\widetilde{T}_{M,N}(f_1,f_2,g,g) = M\int_{\mathbb{T}} \Pi^3_{\eta_2,M}(f_1,f_2,P_Ng)P_Ng\,dx \tag{3.9}$$

where

$$\eta_2(k_1,k_2,k_3) = -\frac{1}{2}\frac{k_1+k_2}{M}\chi_{\operatorname{supp}\phi_M}(k_1+k_2)$$

is also uniformly bounded function in M and N.

Finally, we define $\eta_1 = \tilde{\eta}_1+\tilde{\eta}_2$ and $\eta_3 = \eta_1+\eta_2$. Therefore the proof of (3.7) follows gathering (3.8) and (3.9). □

The following proposition gives suitable estimates for the pseudo-products Π_M.

Proposition 3.3. *Assume that $m = \min_{1\le i\ne j\le 3}|k_i+k_j| \sim M$ where M is a dyadic number with $M\ge 1$ and η is a bounded mesurable function. Then for all $j = 1,2,3$ and all $f_i \in S(\mathbb{T})$ it holds that*

$$\Big|\int_{\mathbb{T}} \Pi^j_{\eta,M}(f_1,f_2,f_3)f_4dx\Big| \lesssim M\prod_{i=1}^4 \|f_i\|_{L^2}\,, \tag{3.10}$$

where the implicit constant only depends on the L^∞-norm of the function η.

Proof. By symmetry we can assume that $j = 3$. Since the norms in the right-hand side only see the size of the modulus of the Fourier transform, we can assume that all the functions have non negative Fourier transform. By using Plancherel's formula, Hölder and Bernstein inequalities, we get that

$$\begin{aligned}
\Big|\int_{\mathbb{R}} \Pi^3_{\eta,M}(f_1,f_2,f_3)f_4dx\Big| &\lesssim \int_{\mathbb{R}^3} \phi_M(k_1+k_2)|\widehat{f_1}(k_1)||\widehat{f_2}(k_2)||\widehat{f_3}(k-k_1-k_2)||\widehat{f_4}(k)|dk_1dk_2dk\\
&= \int_{\mathbb{R}} P_M(f_1f_2)f_3f_4\,dx\\
&\lesssim \|P_M(f_1f_2)\|_{L^\infty_x}\|f_3\|_{L^2x}\|f_4\|_{L^2_x}\\
&\lesssim M\prod_{i=1}^4 \|f_i\|_{L^2_x}\,.
\end{aligned}$$

□

We are now in position to state the main result of this subsection. In the sequel we set

$$D := \{(k_1,k_2,k_3) \in \mathbb{Z}^3\,,\ (k_1+k_2)(k_1+k_3)(k_2+k_3) \neq 0\}\,, \tag{3.11}$$

$$D^1 = \{(k_1,k_2,k_3) \in D,\ med_{1\le i\neq j\le 3}(|k_i+k_j|) \lesssim 2^{-9}|k_1+k_2+k_3|\} \text{ and } D^2 = D\backslash D^1\,. \tag{3.12}$$

Proposition 3.4. *Assume that $0 < T \le 1$, η is a bounded mesurable function and u_i are functions in $Z^0 := X^{-\frac{11}{10},1} \cap L^\infty_t L^2$. Assume also that $N \gg 1$, $M \ge 1$ and $j \in \{1,2,3\}$. We define*

$$G^T_{\eta,M}(u_1,u_2,u_3,u_4) := \int_{]0,T[\times\mathbb{T}} \Pi^j_{\eta,M}(u_1,u_2,u_3)u_4 dxdt\,. \tag{3.13}$$

Then

$$\left|G^T_{\eta 1_{D^1},M}(u_1,u_2,u_3,P_N u_4)\right| \lesssim T^{1/8}M^{-1}N^{\frac{1}{10}}\prod_{i=1}^4 \|u_i\|_{Z^0_T}\,. \tag{3.14}$$

and

$$\left|G^T_{\eta 1_{D^2},M}(u_1,u_2,u_3,P_N u_4)\right| \lesssim T^{1/8}N^{-\frac{9}{10}}\prod_{i=1}^4 \|u_i\|_{Z^0_T}\,. \tag{3.15}$$

Moreover, the implicit constant in estimates (3.14)-(3.15) *only depend on the L^∞-norm of the function η.*

Remark 3.5. Sometimes, when there is no risk of confusion, we also denote

$$G^T_M(u_1,u_2,u_3,u_4) = G^T_{\eta,M}(u_1,u_2,u_3,u_4)\,.$$

To prove Proposition 3.4, we need the following technical lemmas derived in [16]. For any $0 < T \le 1$, let us denote by 1_T the characteristic function of the interval $]0,T[$. One of the main difficulty in the proof of Proposition 3.4 is that the operator of multiplication by 1_T does not commute with Q_L. To handle this situation, we follow the arguments introduced in [16] and use the decomposition

$$1_T = 1^{low}_{T,R} + 1^{high}_{T,R}\,, \quad \text{with} \quad \mathcal{F}_t\big(1^{low}_{T,R}\big)(\tau) = \chi(\tau/R)\mathcal{F}_t\big(1_T\big)(\tau)\,, \tag{3.16}$$

for some $R > 0$ to be fixed later.

Lemma 3.6. *Let L be a nonhomogeneous dyadic number. Then the operator $Q_{\le L}$ is bounded in $L^\infty_t L^2_x$ uniformly in L. In other words,*

$$\|Q_{\le L}u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{L^\infty_t L^2_x}\,, \tag{3.17}$$

for all $u \in L^\infty_t L^2_x$ and the implicit constant appearing in (3.17) *does not depend on L.*

Lemma 3.7. *For any $R > 0$ and $T > 0$ it holds*

$$\|1^{high}_{T,R}\|_{L^1} \lesssim T \wedge R^{-1} \tag{3.18}$$

and

$$\|1^{high}_{T,R}\|_{L^\infty} + \|1^{low}_{T,R}\|_{L^\infty} \lesssim 1. \tag{3.19}$$

Lemma 3.8. *Assume that $T > 0$, $R > 0$ and $L \gg R$. Then, it holds*

$$\|Q_L(1^{low}_{T,R}u)\|_{L^2} \lesssim \|Q_{\sim L}u\|_{L^2}\,, \tag{3.20}$$

for all $u \in L^2(\mathbb{R}^2)$.

Lemma 3.9. *On D^1, it holds*

$$|k_1| \sim |k_2| \sim |k_3| \sim |k_1+k_2+k_3| \quad \textit{and} \quad \max_{1\le i\neq j\le 3}(|k_i+k_j|) \gtrsim |k_1+k_2+k_3|\,,$$

whereas on D^2 it holds $med_{1\le i\neq j\le 3}(|k_i+k_j|) \gtrsim \max_{1\le i\le 3}|k_i|$.

Proof. To prove the first assertion, we assume without loss of generality that $|k_2+k_3| \ge |k_1+k_3| \ge |k_1+k_2|$. On D^1, this forces $|k_2| \sim |k_3| \sim |k_1+k_2+k_3|$. On one hand $|k_1| \ll |k_1+k_2+k_3|$ would imply $|k_1+k_3| \sim |k_1+k_2+k_3|$ which can not hold. On the other hand, $|k_1| \gg |k_1+k_2+k_3|$ would imply $\max(|k_2|,|k_3|) \sim |k_1| \gg |k_1+k_2+k_3|$ which is in contradiction with the preceding deduction. Therefore $|k_1| \sim |k_1+k_2+k_3|$. Finally, either $k_2k_3 \ge 0$ and then $|k_2+k_3| \gtrsim |k_1+k_2+k_3|$ or $k_2k_3 < 0$ and then $\max(|k_1+k_2|,|k_1+k_3|) \gtrsim |k_1+k_2+k_3|$.

To prove the second assertion, we first notice that this assertion is trivial when $\max_{1\le i\le 3}|k_i| \sim |k|$. We thus can assume that $\max_{1\le i\le 3}|k_i| \gg |k|$. By symmetry, we can assume that $|k_1| \ge |k_2| \ge |k_3|$. This forces $|k_2| \sim |k_1| \gg |k|$. Therefore $|k_1+k_3| = |k-k_2| \sim |k_1|$ and $|k_2+k_3| = |k-k_1| \sim |k_1|$. □

Proof of Proposition 3.4. We define the resonance function of order 3 by

$$\begin{aligned}\Omega^3(k_1,k_2,k_3) &= k_1^3+k_2^3+k_3^3-(k_1+k_2+k_3)^3\\ &= -3(k_1+k_2)(k_1+k_3)(k_2+k_3)\ .\end{aligned} \tag{3.21}$$

We first prove (3.14). According to Lemma 3.9, in the region D^1 it holds $|k_1| \sim |k_2| \sim |k_3| \sim |k|$ and thus

$$|\Omega_3(k_1,k_2,k_3)| \gtrsim M^2N\ . \tag{3.22}$$

We set $R = M^{\frac{16}{7}}$ and decompose 1_T as $1_T = 1_{T,R}^{high} + 1_{T,R}^{low}$. The contribution of the first term in $G_{\eta 1_{D^1},M}(u_1,u_2,u_3,P_Nu_4)$ is easily estimated thanks to Hölder's inequality in time, (3.10), (3.18) and (3.19) by

$$\begin{aligned}\big|G_{\eta 1_{D^1},M}(1_{T,R}^{high}u_1,u_2,u_3,P_Nu_4)\big| &\le T^{1/8}\|1_{T,R}^{high}\|_{L^{8/7}}\Big\|\int_{\mathbb{R}}\Pi_M^j(P_{\sim N}u_1,P_{\sim N}u_2,P_{\sim N}u_3)P_Nu_4dx\Big\|_{L_t^\infty}\\ &\lesssim T^{1/8}R^{-7/8}M\prod_{i=1}^4\|P_{\sim N}u_i\|_{L_t^\infty L^2{}_x}\,,\\ &\lesssim T^{1/8}M^{-1}\prod_{i=1}^4\|P_{\sim N}u_i\|_{L_t^\infty L^2{}_x}\,.\end{aligned}$$

To deal with the contribution of $1_{T,R}^{low}$, we first note that $N \gg 1$ and $M \lesssim N$ so that $R \ll M^2N$ and we are in the hypotheses of Lemma 3.8. Now, thanks to (3.22), we may decompose the contribution of $1_{T,R}^{low}$ as

$$\begin{aligned}G_{\eta 1_{D^1},M}(1_{T,R}^{low}u_1,u_2,u_3,P_Nu_4) \;=\;& G_{\eta 1_{D^1},M}(Q_{\ge M^2N}(1_{T,R}^{low}u_1),u_2,u_3,P_Nu_4)\\ &+G_{\eta 1_{D^1},M}(Q_{<M^2N}(1_{T,R}^{low}u_1),Q_{\ge MN^2}u_2,u_3,P_Nu_4)\\ &+G_{\eta 1_{D^1},M}(Q_{<M^2N}(1_{T,R}^{low}u_1),Q_{<MN^2}u_2,Q_{\ge MN^2}u_3,P_Nu_4)\\ &+G_{\eta 1_{D^1},M}(Q_{<M^2N}(1_{T,R}^{low}u_1),Q_{<MN^2}u_2,Q_{<MN^2}u_3,Q_{\ge MN^2}P_Nu_4)\,.\end{aligned} \tag{3.23}$$

The first term of the right-hand side of the above equality can be estimated thanks to (3.10) and (3.20) by

$$\begin{aligned}\left|G_{\eta 1_{D^1},M}(Q_{\geq M^2N}(1^{low}_{T,R})u_1,u_2,u_3,P_Nu_4)\right| &\lesssim T^{1/2}M\|Q_{\gtrsim NM^2}(1^{low}_{T,R}P_{\sim N}u_1)\|_{L^2_{x,t}}\prod_{j=2}^4\|u_j\|_{L^\infty_TL^2_x}\\ &\lesssim T^{1/2}M^{-1}N^{\frac{1}{10}}\|u_1\|_{X^{-\frac{11}{10},1}}\prod_{i=2}^4\|u_j\|_{L^\infty_TL^2_x}\,.\end{aligned}$$

The other terms can be controlled in exactly the same way. Note that we use (3.17) and not (3.20) for these terms.

2. Let us now prove (3.15). In the region D^2, Lemma 3.9 ensures that

$$|\Omega_3|\sim M\max(|k_1|,|k_2|,|k_3|)^2\,.$$

By symmetry we can assume that $|k_1|=\max(|k_1|,|k_2|,|k_3|)$. We make a dyadic decomposition in $|k_1|\sim N_1\gtrsim N$. This time we set

$$R=(MN_1^{\frac{9}{10}})^{8/7}\ll MN_1^2\sim|\Omega_3|\,.$$

The contribution of $1^{high}_{T,R}$ in $G_{\eta 1_{D^2},M}(P_{N_1}u_1,P_{\lesssim N_1}u_2,P_{\lesssim N_1}u_3,P_Nu_4)$ is easily estimated thanks to Hölder's inequality in time, (3.10) and (3.18) by

$$\begin{aligned}&\left|G_{\eta 1_{D^2},M}(1^{high}_{T,R}P_{N_1}u_1,P_{\lesssim N_1}u_2,P_{\lesssim N_1}u_3,P_Nu_4)\right|\\ &\qquad\lesssim T^{1/8}\|1^{high}_{T,R}\|_{L^{8/7}}\Big\|\int_{\mathbb{R}}\Pi^j_M(P_{N_1}u_1,P_{\lesssim N_1}u_2,P_{\lesssim N_1}u_3)P_Nu_4dx\Big\|_{L^\infty_t}\\ &\qquad\lesssim T^{1/8}R^{-7/8}M\prod_{i=1}^4\|P_{\lesssim N_1}u_i\|_{L^\infty_tL^2_x}\,,\\ &\qquad\lesssim T^{1/8}N_1^{-\frac{9}{10}}\prod_{i=1}^4\|u_i\|_{L^\infty_tL^2_x}\end{aligned}$$

which yields the desired result by summing over $N_1\gtrsim N$. Finally to estimate the contribution of $1^{low}_{T,R}$ we proceed as in (3.23) by using that this time $\Omega_3\sim MN_1^2$. The contribution of $Q_{\geq MN_1^2}(1^{low}_{T,R}P_{N_1}u_1)$ is estimated thanks to (3.10), (3.17) and (3.20) by

$$\begin{aligned}&\left|G_{\eta 1_{A^c},M}(Q_{\geq MN_1^2}(1^{low}_{T,R}P_{N_1}u_1),P_{\lesssim N_1}u_2,P_{\lesssim N_1}u_3,P_Nu_4)\right|\\ &\qquad\lesssim MT^{1/2}\|P_{N_1}Q_{\gtrsim MN_1^2}(1^{low}_{T,R}u_1)\|_{L^2x,t}\prod_{i=2}^4\|u_i\|_{L^\infty_TL^2x}\\ &\qquad\lesssim T^{1/2}N_1^{-\frac{9}{10}}\|u_1\|_{X^{-\frac{11}{10},1}}\prod_{i=2}^4\|u_i\|_{L^\infty_TL^2x}\,,\end{aligned}$$

Summing over $N_1\gtrsim N$, this is also acceptable. The other regions C_i can be handle in the same way. □

3.2. **Uniform estimates on solutions.** The preceding lemmas enables us to easily get an uniform H^s-bound for solutions to (2.1). This is the aim of this subsection where we do not attempt to get the lowest propagated regularity since we will be forced to take $s\geq 1/3$ in the estimate on the difference.

We first prove refined Strichartz estimates. The following linear estimate in Bourgain's space is established in [1],

$$\|u\|_{L^4(]0,1[\times\mathbb{T})} \lesssim \|u\|_{X^{0,1/3}} \quad \forall u \in X^{0,1/3} .$$

We will make use of the following Strichartz estimate that follows directly from the above estimate (see for instance [6]),

$$\|U(t)\varphi\|_{L^4(]0,T[\times\mathbb{T})} \lesssim T^{\frac{1}{6}}\|\varphi\|_{L^2} , \quad \forall T \in]0,1[, \quad \forall \varphi \in L^2(\mathbb{T}) , \tag{3.24}$$

where the implicit constant does not depend on T.

Lemma 3.10. *Let $0 < T < 1$ and $u \in L^\infty(0,T;H^s(\mathbb{T}))$ be a solution to (2.1) emanating from $u_0 \in H^s(\mathbb{T})$. For $s > \frac{11}{35}$ it holds*

$$\|u\|_{L^4_T L^{20}_x} \lesssim \|u\|_{L^\infty_T H^s}(1 + \|u\|^2_{L^\infty_T H^s}) \tag{3.25}$$

and for $s > \frac{9}{28}$,

$$\|D_x^{\frac{5}{24}} u\|_{L^4_T L^4_x} \lesssim \|u\|_{L^\infty_T H^s}(1 + \|u\|^2_{L^\infty_T H^s}) . \tag{3.26}$$

Proof. Let u be a solution to (2.1) defined on a time interval $[0,T]$. We use a nonhomogeneous Littlewood-Paley decomposition, $u = \sum_N u_N$ where $u_N = P_N u$ and N is a dyadic integer. Since, (3.25) and (3.26) are obvious for $N \lesssim 1$, it suffices to control $\|u_N\|_{L^4_T L^{20}_x}$ for any $N \gg 1$. For such N, Sobolev and Bernstein inequalities lead to

$$\|u_N\|_{L^4_T L^{20}_x} \lesssim N^{\frac{1}{5}}\|u_N\|_{L^4_{Tx}} .$$

Let δ be a nonnegative number to be fixed later. we chop out the interval in small intervals of length $N^{-\delta}$. In other words, we have that $[0,T] = \bigcup_{j\in J} I_j$ where $I_j = [a_j, b_j]$, $|I_j| \sim N^{-\delta}$ and $\#J \sim N^\delta$. Since u_N is a solution to the integral equation

$$u_N(t) = e^{-(t-a_j)\partial_x^3} u_N(a_j) + \int_{a_j}^t e^{-(t-t')\partial_x^3} P_N \partial_x (u^3 - 3P_0(u^2)u)(t')dt'$$

for $t \in I_j$, we deduce from (3.24) that

$$\begin{aligned}
\|u_N\|_{L^4_T L^{20}_x} &\lesssim \Big(\sum_j \|D_x^{-\frac{\delta}{6}+\frac{1}{5}} u_N(a_j)\|^4_{L^2_x}\Big)^{\frac{1}{4}} + \Big(\sum_j \big(\int_{I_j} \|D_x^{-\frac{\delta}{6}+\frac{6}{5}} P_N(u^3 - 3P_0(u^2)u)(t')\|_{L^2_x} dt'\big)^4\Big)^{\frac{1}{4}} \\
&\lesssim N^{\frac{\delta}{4}} \|D_x^{-\frac{\delta}{6}+\frac{1}{5}} u_N\|_{L^\infty_T L^2_x} + \Big(\sum_j |I_j|^3 \int_{I_j} \|D_x^{-\frac{\delta}{6}+\frac{6}{5}} P_N(u^3 - 3P_0(u^2)u)(t')\|^4_{L^2_x} dt'\Big)^{\frac{1}{4}} \\
&\lesssim N^{0-}\Big[\|D_x^{\frac{\delta}{12}+\frac{1}{5}+} u_N\|_{L^\infty_T L^2_x} + \|D_x^{-\frac{11}{12}\delta+\frac{6}{5}+} P_N(u^3 - 3P_0(u^2)u)\|_{L^4_T L^2_x}\Big] .
\end{aligned}$$

Now, to prove (3.25) we notice that Sobolev's inequalities and the fractional Leibniz rule lead to

$$\begin{aligned}
\|D_x^{-\frac{11}{12}\delta+\frac{6}{5}+} P_N(u^3 - 3P_0(u^2)u)\|_{L^4_T L^2_x} &\lesssim \|D_x^{-\frac{11\delta}{12}+1/p+\frac{7}{10}+} P_N(u^3 - 3P_0(u^2)u)\|_{L^4_T L^p_x} \\
&\lesssim \|u\|^2_{L^\infty_T L^q_x} \|D_x^{\kappa+} u\|_{L^\infty_T L^2_x} .
\end{aligned} \tag{3.27}$$

for all $1 \le p \le 2$ and $2 \le q \le \infty$ satisfying $\frac{2}{q} + \frac{1}{2} = \frac{1}{p}$ and $0 < \kappa = -\frac{11\delta}{36} + \frac{7}{10} + \frac{1}{p} < 1$. Thus, the Sobolev embedding yields

$$\|D_x^{-\frac{11}{12}\delta+\frac{6}{5}+} P_N(u^3 - 3P_0(u^2)u)\|_{L^4_T L^2 x} \lesssim \|u\|^3_{L^\infty_T H^{\kappa+}_x} , \tag{3.28}$$

if we choose κ satisfying $\kappa = \frac{1}{2} - \frac{1}{q} = \frac{3}{4} - \frac{1}{2p}$. This implies that

$$\kappa = -\frac{11}{12}\delta + \frac{7}{10} + \frac{1}{p} = -\frac{11}{36}\delta + \frac{11}{5} - 2\kappa \quad \Rightarrow \quad \kappa = -\frac{11}{36}\delta + \frac{11}{15}\,.$$

Then, we choose δ such that $\frac{\delta}{12} + \frac{1}{5} = \kappa$ which leads to

$$\delta = \frac{48}{35}, \quad \kappa = \frac{11}{35}, \quad p = \frac{70}{61} \quad \text{and} \quad q = \frac{70}{13}\,.$$

Therefore, we conclude gathering (3.27)–(**??**) that if u is a solution to (2.1)-(1.2) defined on the time interval $[0,T]$, then for $N \gg 1$,

$$\|P_N u\|_{L^4_T L^{20}_x} \lesssim N^{0-}\Big[\|D_x^{\frac{11}{35}+} P_N u_0\|_{L^2_x} + \|D_x^{\frac{11}{35}+} u\|^3_{L^\infty_T L^2_x}\Big]\,, \tag{3.29}$$

which proves (3.25) with $s > \frac{11}{35}$, by summing over N.

To prove (3.26) we proceed in the same way. We eventually obtain for $N \gg 1$,

$$\|D_x^{\frac{5}{24}} u_N\|_{L^4_T L^4_x} \lesssim \|D_x^{\frac{\delta}{12}+\frac{5}{24}+} u_N\|_{L^\infty_T L^2_x} + \|D_x^{-\frac{11}{12}\delta+\frac{29}{24}+} P_N(u^3 - 3P_0(u^2)u)\|_{L^4_T L^2_x}$$

with

$$\|D_x^{-\frac{11}{12}\delta+\frac{29}{24}+} P_N(u^3 - 3P_0(u^2)u)\|_{L^4_T L^2_x} \lesssim \|u\|^2_{L^q_x} \|D_x^{\kappa+} u\|_{L^\infty_T L^2_x}\,,$$

for all $1 \le p \le 2$ and $2 \le q \le \infty$ satisfying $\frac{2}{q} + \frac{1}{2} = \frac{1}{p}$ and $0 < \kappa = -\frac{11}{12}\delta + \frac{17}{24} + \frac{1}{p} < 1$. Thus, the Sobolev embedding yields

$$\|D_x^{-\frac{11}{12}\delta+\frac{29}{24}+} P_N(u^3 - 3P_0(u^2)u)\|_{L^4_T L^2_x} \lesssim \|u\|^3_{L^\infty_T H^{\kappa+}_x}\,,$$

if we choose κ satisfying $\kappa = \frac{1}{2} - \frac{1}{q} = \frac{3}{4} - \frac{1}{2p}$. This implies that

$$\kappa = -\frac{11}{12}\delta + \frac{17}{24} + \frac{1}{p} = -\frac{11}{12}\delta + \frac{53}{24} - 2\kappa \quad \Rightarrow \quad \kappa = -\frac{11}{36}\delta + \frac{53}{72}\,.$$

Then, choosing δ such that $\frac{\delta}{12} + \frac{5}{24} = \kappa$ which leads to

$$\delta = \frac{19}{14}, \quad \kappa = \frac{9}{28} \quad p = \frac{7}{6} \quad \text{and} \quad q = \frac{28}{5}\,.$$

we obtain (3.26) with $s = \frac{9}{28}+$. □

Lemma 3.11. *Assume that* $0 < T \le 1$, $s > \frac{11}{35}$ *and* $v \in L^\infty(0,T;H^s(\mathbb{T}))$ *is a solution to* (2.1) *emanating from* $u_0 \in H^s(\mathbb{T})$. *Then,*

$$\|v\|_{Z^s_T} \lesssim \|u_0\|_{H^s} + \|v\|^3_{L^\infty_T H^s}(1 + \|v\|_{L^\infty_T H^s})^4 \tag{3.30}$$

Proof. We have to extend the function u from $]0,T[$ to $\mathbb{R}$. For this we follow [13] and introduce the extension operator ρ_T defined by

$$\rho_T u(t) := \eta(t) u(T\mu(t/T))\,, \tag{3.31}$$

where η is the smooth cut-off function defined in Section 2.1 and $\mu(t) = \max(1 - |t-1|, 0)$. ρ_T is a bounded operator from $X^{\theta,b}_T$ into $X^{\theta,b}$ and from $L^p(0;X)$ into $L^p(\mathbb{R};X)$ for any $b \in]-\infty,1]$, $s \in \mathbb{R}$, $p \in [1,\infty]$ and any Banach space X. Moreover, these bounds are uniform for $0 < T < 1$.

By using this extension operator, it is clear that we only have to estimate the

$X^{s-\frac{11}{10},1}$-norm of $\rho_T u$ to prove (3.30). Now, using the Duhamel formula associated to (2.1), the standard linear estimates in Bourgain's spaces and the fractional Leibniz rule (*c.f.* Theorem A.12 in [10]), we have that

$$\begin{aligned}\|u\|_{X_T^{s-\frac{11}{10},1}} &\lesssim \|u_0\|_{H^s} + \|\partial_x(u^3)\|_{X_T^{s-\frac{11}{10},0}} + \|P_0(u^2)u_x\|_{X_T^{s-\frac{11}{10},0}} \\ &\lesssim \|u_0\|_{H^s} + \|J_x^{s-\frac{1}{10}}(u^3)\|_{L^2_T L^2_x} + \|u\|^2_{L^\infty_T L^2_x}\|u\|_{L^2_T H_x^{s-\frac{1}{10}}} \\ &\lesssim \|u_0\|_{H^s} + \|J_x^s(u^3)\|_{L^2_T L_x^{\frac{5}{3}}} + \|u\|^2_{L^\infty_T L^2_x}\|u\|_{L^2_T H^s_x} \\ &\lesssim \|u_0\|_{H^s} + \|u\|^2_{L^4_T L^{20}_x}\|J_x^s u\|_{L^\infty_T L^2_x} + \|u\|^2_{L^\infty_T L^2_x}\|u\|_{L^2_T H^s_x}\,,\end{aligned} \tag{3.32}$$

which leads to (3.30) thanks to (3.25). □

Proposition 3.12. *Assume that* $0 < T \leq 1$, $s > 2/9$ *and that* $u \in Z^s$ *is a solution to* (2.1) *with initial data* $u_0 \in H^s(\mathbb{T})$. *Then,*

$$\|u\|^2_{L^\infty_T H^s} \lesssim \|u_0\|^2_{H^s} + T^{1/8}\|u\|^4_{Z^s}\,. \tag{3.33}$$

Proof. By using (1.1), we have

$$\frac{1}{2}\frac{d}{dt}\|P_N u(\cdot,t)\|^2_{L^2_x} = -\Re\Big[\int_{\mathbb{T}} \partial_x P_N\Big(u^3 - 3P_0(u^2)u\Big)P_N u\,dx\Big]\,.$$

which yields after integration in time between 0 and t and summation over N

$$\sum_N \|P_N u(t)\|^2_{H^s_x} \lesssim \|u_0\|^2_{H^s} + \sum_N N^{2s}\Big|\Re\Big[\int_{\mathbb{T}\times[0,t]} \partial_x P_N\Big(u^3 - 3P_0(u^2)u\Big)P_N u dxds\Big]\Big|\,. \tag{3.34}$$

In the case where $N \lesssim 1$, we easily get

$$\Big|\sum_N N\int_{\mathbb{T}\times[0,t]} \partial_x P_N\Big(u^3 - 3P_0(u^2)u\Big)P_N u\,dxds\Big| \lesssim \|u\|^4_{L^4_T L^4_x} \lesssim \|u\|^4_{L^\infty_T H^{\frac{1}{4}}_x} \lesssim \|u\|^4_{Z^0_T}\,. \tag{3.35}$$

In the following, we can then assume that $N \gg 1$ and we use the classical decomposition of $N(u) := \partial_x(u^3 - 3P_0(u^2)u)$ in a resonant and a non resonant part by writing :

$$\begin{aligned}\mathcal{F}_x[N(u)](k) &= ik\Big[\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_1+k_3)(k_2+k_3)\neq 0}} \hat{u}(k_1)\hat{u}(k_2)\hat{u}(k_3) \\ &\quad +3\hat{u}(k)\hat{u}(k)\hat{u}(-k)\Big] \\ &:= \Big(\mathcal{F}_x\Big[A(u,u,u)\Big](k) + \mathcal{F}_x\Big[B(u,u,u)\Big](k)\Big)\Big]\end{aligned}$$

i.e.

$$\partial_x(u^3 - 3P_0(u^2)u) = \partial_x\Big(A(u,u,u) + B(u,u,u)\Big)\,. \tag{3.36}$$

Now, we notice that, since u is real-valued, we have

$$\int_{\mathbb{T}} \partial_x P_N B(u,u,u)P_N u = ik\sum_{k\in\mathbb{Z}} |\hat{u}(k)|^2|\varphi_N(k)\hat{u}(k)|^2 \in i\mathbb{R}\,. \tag{3.37}$$

Therefore (3.34) and (3.36) lead to

$$\begin{aligned}\sum_{N\gg 1}\|P_N u(t)\|_{H^s_x}^2 &\lesssim \|u_0\|_{H^s}^2+\Big|\Re\Big[\sum_{N\gg 1}N\int_{\mathbb{T}\times[0,t]}\partial_x P_N(A(u,u,u))P_N u\,dxds\Big]\Big|\\ &\lesssim \|u_0\|_{H^s}^2+\sum_{N\gg 1}J_N(u)\,.\end{aligned}$$

By using the decomposition in (3.3), we get that $J_N(u)=\sum_{l=1}^3 J_N^l(u)$ with

$$J_N^l(u)=N^{2s}\sum_{M\geq 1}\int_{[0,t]\times\mathbb{T}}\partial_x P_N\Pi^l_{1_D,M}(u,u,u)P_N u\,,$$

where D is defined in (3.11) and where we performed a dyadic decomposition in $1\leq m\sim M$. Thus, by symmetry, it is enough to estimate $J_N^3(u)$ that still will be denoted $J_N(u)$ for sake of simplicity. We rewrite $J_N^3(u)$ as

$$\begin{aligned}N^{2s}\sum_{M\geq 1}\Big(\sum_{(k_1,k_2,k_3)\in D^1}+\sum_{(k_1,k_2,k_3)\in D^2}\Big)\int_{]0,t[}T_{M,N}(u,u,u,u)dt\\ =: I_N^1(u)+I_N^2(u)\,,\end{aligned}\tag{3.38}$$

where $T_{M,N}(u,u,u,u)$ is defined in (3.6) and $\mathcal{A}$ is defined in (3.12).

Estimate for $I_N^2(u)$.
1. $N_{med}\geq N^{1/2}$. Then (3.15) yields

$$\begin{aligned}|I_N^2| &\lesssim T^{1/8}\sum_{N_{max}\gtrsim N}\sum_{N^{1/2}\lesssim N_{med}\lesssim N_{max}}\sum_{1\leq M\lesssim N_{med}}\frac{N^{2s}N}{N^{\frac{9}{10}}}N_{max}^{-s}N^{-s}N_{med}^{-s}\\ &\qquad\|P_{N_{max}}u\|_{Z^s}\|P_{N_{med}}u\|_{Z^s}\|P_N u\|_{Z^s}\|u\|_{Z^0}\\ &\lesssim T^{1/8}N^{-s/2+\frac{1}{10}}\|u\|_{Z^s}^4\end{aligned}$$

which is acceptable for $s>1/5$.
2. $N_{med}<N^{1/2}$. Then $N_{max}\sim N$ and $M_{min}\lesssim N^{1/2}$. According to (3.7) we thus can write

$$I_N^2=\sum_{1\leq M\lesssim N^{1/2}}MN^{2s}\int_{\mathbb{T}}\Pi^3_{\eta_3,M}(u,u,u_{\sim N})P_N u dx,$$

where η_3 is a function of (k_1,k_2,k_3) whose $L^\infty-$norm is uniformly bounded in N and M. Therefore, by (3.15), we get

$$\begin{aligned}|I_N^2| &\lesssim T^{1/8}\sum_{1\leq M\lesssim N^{1/2}}\frac{N^{2s}M}{N^{\frac{9}{10}}}N^{-2s}\|P_{\sim N}u\|_{Z^s}^2\|u\|_{Z^0}^2\\ &\lesssim T^{1/8}N^{-\frac{2}{5}}\|u\|_{Z^s}^4\,.\end{aligned}$$

Estimate for I_N^1. According to (3.7) we have

$$I_N^1=\sum_{1\leq M\ll N}MN^{2s}\int_{\mathbb{R}}\Pi^3_{\eta_3,M}(u,u,u_{\sim N})P_N u dx,$$

where η_3 is a function of (k_1, k_2, k_3) whose $L^\infty-$norm is uniformly bounded in N and M. According to (3.14), we thus have

$$\begin{aligned} \left|I_N^1\right| &\lesssim T^{1/8} \sum_{1\leq M\ll N} N^{-2s+\frac{1}{10}} \|u_{\sim N}\|_{Z^s}^4 \\ &\lesssim T^{1/8} N^{-2s+\frac{1}{10}+} \|u\|_{Z^s}^4 \ , \end{aligned}$$

which is acceptable for $s > \frac{1}{20}$. This concludes the proof of the proposition. □

Combining Lemma 3.11 and Proposition 3.12 we can easily get an a priori estimate on the $H^{\frac{11}{35}}(\mathbb{T})$-norm of smooth solutions to (2.1). This will be done in Section 6.

4. The smoothing estimate

The aim of this section is to prove the proposition below that show a kind of smoothing effect first observed in [20]. This smoothing effect is the only way we know to treat some resonant terms involving B (see (3.36)) when estimating the difference of two solutions. Note that, by symmetry, the terms involving B do cancel in the proof of the energy estimate (3.33).

Theorem 4.1. *Let $s \geq 1/3$ be fixed. For any solution $u \in Z_T^s$ of (2.1)-(1.2) and any $k \in \mathbb{Z}$ it holds*

$$\sup_{t\in]0,T[} k\Big|\widehat{u}(t,k)|^2 - |\widehat{u_0}(k)|^2\Big| \lesssim \sup_{N\gtrsim k}\Big[\Big(\frac{k}{N}\Big)^{s-} \|P_{\leq N}u\|_{Z_T^s}^4 (1+\|P_{\leq N}u\|_{Z_T^s}^4)\Big] \tag{4.1}$$

where the implicit constant does not depend on k.

4.1. **Notations.** In this section we will widely use the following notations : $\vec{k}_{(3)} = (k_1, k_2, k_3)$. Let D, D^1 and D^2 be defined as in (3.11)-(3.12). We set

$$\Gamma_3(k) = \{\vec{k}_{(3)} \in \mathbb{Z}^3,\ k_1 + k_2 + k_3 = k\}\,,$$

$$D(k) := \Gamma_3(k) \cap D, \quad D^1(k) = D(k) \cap D^1 \text{ and } D^2(k) = D(k) \cap D^2\ ,$$

$m_1 = |k_2 + k_3|$, $m_2 = |k_1 + k_3|$, $m_3 = |k_1 + k_2|$, $m_{min} = \min(m_1, m_2, m_3)$ and m_{med} =med(m_1, m_2, m_3).

$$D_M^i(k) = \{\vec{k}_{(3)} \in D^i(k)\,,\ m_{min} \sim M\}, \quad i = 1, 2\,.$$

M_1, M_2, M_3, M_{min} and M_{med} are the dyadic integers associated with respectively m_1, m_2, m_3, m_{min} and m_{med}.
For $i \in \{1, 2, 3, 4\}$, we set $\vec{k}_{i(3)} = (k_{i1}, k_{i2}, k_{i3})$,

$$m_{i,min} = \min(|k_{i1}+k_{i2}|, |k_{i1}+k_{i3}|, |k_{i2}+k_{i3}|),\ m_{i,med} = \text{med}\{|k_{i1}+k_{i2}|, |k_{i1}+k_{i3}|, |k_{i2}+k_{i3}|\}$$

$M_{i,min}$ and $M_{i,med}$ are the dyadic numbers associated with respectively $m_{i,min}$ and $m_{i,med}$.

4.2. **Multilinear space-time estimates.**

4.2.1. *Trilinear estimates.*

Lemma 4.2. *Let $f_j \in l^2(\mathbb{Z})$, $j = 1, ..., 4$. Then it holds*

$$\begin{aligned} \mathcal{J}_k^3 &:= \sum_{\vec{k}_{(3)} \in \Gamma^3(k),\, k_4 = -k} \phi_M(k_1 + k_2) \prod_{j=1}^{4} |f_j(k_j)| \\ &\lesssim [(M^{\frac{1}{2}} \|f_3\|_{l^2}) \wedge (M \|f_3\|_{l^\infty})] \|f_4\|_{l^\infty} \prod_{j=1}^{2} \|f_j\|_{l^2} \,. \end{aligned} \tag{4.2}$$

Proof. We can assume without loss of generality that $f_i \geq 0$ for $i = 1, \cdots, 4$. Then, we can write

$$\begin{aligned} \mathcal{J}_k^3 &\leq |f_4(k)| \|\phi_M(f_1 * f_2)\|_{l^2(\mathbb{Z})} \|f_3\|_{l^2(\mathbb{Z})} \\ &\leq M^{1/2} \|f_1 * f_2\|_{l^\infty(\mathbb{Z})} \|f_3\|_{l^2(\mathbb{Z})} \|f_4\|_{l^\infty(\mathbb{Z})} \\ &\leq M^{\frac{1}{2}} \prod_{j=1}^{3} \|f_j\|_{l^2} \|f_4\|_{l^\infty(\mathbb{Z})} \,, \end{aligned}$$

or

$$\begin{aligned} \mathcal{J}_k^3 &\leq |f_4(k)| \|\phi_M(f_1 * f_2)\|_{l^1(\mathbb{Z})} \|f_3\|_{l^\infty(\mathbb{Z})} \\ &\leq M \|f_1 * f_2\|_{l^\infty(\mathbb{Z})} \|f_3\|_{l^\infty(\mathbb{Z})} \|f_4\|_{l^\infty(\mathbb{Z})} \\ &\leq M \prod_{j=1}^{2} \|f_j\|_{l^2} \prod_{i=3}^{4} \|f_i\|_{l^\infty(\mathbb{Z})} \,, \end{aligned}$$

which proves (4.2). □

Proposition 4.3. *Assume that $0 < T \leq 1$, η is a bounded measurable function and u_i are functions in $Z_T^0 := X_T^{-\frac{10}{9}+,1} \cap L_T^\infty L^2$. Assume also that $k \geq 2^9$, $M \geq 1$ and $j \in \{1, 2, 3\}$. We define*

$$J_{\eta,M}^{3,k,T}(u_1, u_2, u_3, u_4) := \int_{[0,T]\times\mathbb{T}} \Pi_{\eta,M}^j(u_1, u_2, u_3) P_k u_4 dx dt \,. \tag{4.3}$$

Then

$$\left| J_{\eta 1_{D^1},M}^{3,k,T}(u_1, u_2, u_3, u_4) \right| \lesssim \frac{k^{\frac{1}{10}}}{M^{3/2}} \prod_{i=1}^{4} \|u_i\|_{Z_T^0} \,. \tag{4.4}$$

and

$$\left| J_{\eta 1_{D^2},M}^{3,k,T}(u_1, u_2, u_3, u_4) \right| \lesssim M^{-1/2} k^{-\frac{9}{10}} \prod_{i=1}^{4} \|u_i\|_{Z_T^0} \,. \tag{4.5}$$

Moreover, the implicit constant in estimate (3.14) *only depends on the L^∞-norm of the function η.*

Proof. We proceed exactly as in the proof of Proposition 3.4 but with the help of (4.2) instead of (3.10). □

4.2.2. *Quintic linear estimates.* We use the following notations : $\vec{k}_{(5)} = (k_1, k_2, .., k_6) \in \mathbb{Z}^6$ and for any $k \in \mathbb{Z}$,

$$\Gamma_5(k) = \{\vec{k}_{(5)} \in \mathbb{Z}^6, \sum_{i=1}^{6} k_i = k\}\,.$$

Before stating our quintic space-time estimates, let us define the resonance function of order 5 for $\vec{k}_{(5)} = (k_1, \cdots, k_6) \in \Gamma^5(0)$ by

$$\Omega^5(\vec{k}_{(5)}) = k_1^3 + k_2^3 + k_3^3 + k_4^3 + k_5^3 + k_6^3\,. \tag{4.6}$$

It is worth noticing that a direct calculus leads to

$$\Omega^5(\vec{k}_{(5)}) = \Omega^3(k_1, k_2, k_3) + \Omega^3(k_4, k_5, k_6)\,. \tag{4.7}$$

In the sequel we set

$$\vec{k}_{1(3)} = (k_{11}, k_{12}, k_{13})\,.$$

Lemma 4.4. *Let $f_j \in l^2(\mathbb{Z})$, $j = 1, ..., 6$. Then it holds that*

(4.8)

$$\mathcal{J}_k^{5,1} := \sum_{\vec{k}_{(5)} \in \Gamma^5(k),\, k_6 = -k} \phi_M(k_1+k_2)\phi_{M'}(k_4+k_5) \prod_{j=1}^{6} |f_j(k_j)| \lesssim [(M^{\frac{1}{2}} M') \wedge (M M'^{\frac{1}{2}})] \prod_{j=1}^{6} \|f_j\|_{l^2}$$

and

(4.9)

$$\mathcal{J}_k^{5,1} \lesssim \|f_6\|_{l^2} \min\Big(M' \prod_{i=1}^{2} \|\langle\cdot\rangle^{1/4} f_i\|_{l^2} \prod_{j=3}^{5} \|f_j\|_{l^2}, M \prod_{j=1}^{3} \|f_j\|_{l^2} \prod_{i=4}^{5} \|\langle\cdot\rangle^{1/4} f_i\|_{l^2} \Big)\,.$$

(4.10)

$$\mathcal{J}_k^{5,2} := \sum_{\vec{k}_{(5)} \in \Gamma^5(k),\, k_6 = -k} \phi_M((k_1+k_2+k_3)+k_4)\phi_{M'}(k_1+k_2) \prod_{j=1}^{6} |f_j(k_j)| \lesssim M^{\frac{1}{2}} M' \prod_{j=1}^{6} \|f_j\|_{l^2}\,.$$

$$\tilde{\mathcal{J}}_k^{5} := \sum_{\substack{\vec{k}_{(3)} \in \Gamma^3(k) \\ (k_4, k_5, k_6) \in \Gamma^3(-k)}} \phi_M(k_1+k_2)\phi_{M'}(k_4+k_5) \prod_{j=1}^{6} |f_j(k_j)| \lesssim M^{\frac{1}{2}} M'^{1/2} \prod_{j=1}^{6} \|f_j\|_{l^2}\,. \tag{4.11}$$

Proof. We can assume without loss of generality that $f_i \geq 0$ for $i = 1, \cdots, 6$. Proceeding in the same way as in the proof of Lemma 4.2, we get

$$\begin{aligned}
\mathcal{J}_k^{5,1} &\leq |f_6(k_6)| \|f_3\|_{l^2(\mathbb{Z})} \min\Big(\|\phi_M(f_1 * f_2)\|_{l^2(\mathbb{Z})} \|\phi_{M'}(f_4 * f_5)\|_{l^1(\mathbb{Z})}, \|\phi_M(f_1 * f_2)\|_{l^1(\mathbb{Z})} \|\phi_{M'}(f_4 * f_5)\|_{l^2(\mathbb{Z})} \Big) \\
&\leq [(M^{\frac{1}{2}} M') \wedge (M M'^{1/2})] \prod_{j=1}^{6} \|f_j\|_{l^2}\,.
\end{aligned}$$

In the same way,

$$\begin{aligned}
\mathcal{J}_k^{5,1} &\leq |f_6(k)| \min\Big(\|f_1 * f_2 * f_3\|_{l^\infty(\mathbb{Z})} \|\phi_{M'}(f_4 * f_5)\|_{l^1(\mathbb{Z})}, \|\phi_M(f_1 * f_2)\|_{l^1(\mathbb{Z})} \|f_3 * f_4 * f_5\|_{l^1(\mathbb{Z})} \Big) \\
&\lesssim |f_6(k)| \min\Big(\|f_1 * f_2\|_{l^2(\mathbb{Z})} \|M' \prod_{i=3}^{5} \|f_i\|_{l^2(\mathbb{Z})}\,,\ \|f_4 * f_5\|_{l^2(\mathbb{Z})} M' \prod_{i=1}^{3} \|f_i\|_{l^2(\mathbb{Z})} \Big)
\end{aligned}$$

which leads to the desired result by using that

$$\|f_1 * f_2\|_{l^2(\mathbb{Z})} \lesssim \prod_{i=1}^{2} \|\langle\cdot\rangle^{1/4} f_i\|_{l^2} \ ,.$$

To derive (4.10), we notice that

$$\begin{aligned} \mathcal{J}_k^{5,2} &\leq |f_6(k_6)| \|f_5\|_{l^2(\mathbb{Z})} \Big\| \Big[\phi_M \Big([\phi_{M'}(f_1 * f_2)] * f_3 \Big) \Big] * f_4 \Big\|_{l^2(\mathbb{Z})} \\ &\lesssim M^{1/2} |f_6(k_6)| \|f_5\|_{l^2(\mathbb{Z})} \|f_4\|_{l^2(\mathbb{Z})} \|[\phi_{M'}(f_1 * f_2)] * f_3\|_{l^2(\mathbb{Z})} \end{aligned}$$

which yields the desired estimate. Finally,

$$\begin{aligned} \tilde{\mathcal{J}}_k^{5} &\leq \|\Phi_M(f_1 * f_2)\|_{l^2(\mathbb{Z})} \|f_3\|_{l^2(\mathbb{Z})} \|[\Phi_{M'}(f_4 * f_5)] * f_6\|_{l^\infty(\mathbb{Z})} \\ &\leq M^{1/2} \|f_1 * f_2\|_{l^\infty(\mathbb{Z})} \|f_3\|_{l^2(\mathbb{Z})} \|\Phi_{M'}(f_4 * f_5)\|_{l^2(\mathbb{Z})} \|f_6\|_{l^2(\mathbb{Z})} \\ &\leq M^{\frac{1}{2}} M'^{1/2} \prod_{j=1}^{6} \|f_j\|_{l^2} \ , \end{aligned}$$

□

Proposition 4.5. *Assume that $0 < T \leq 1$, $\eta : \mathbb{Z}^5 \to \mathbb{C}$ is a bounded measurable function and u_i are functions in $Z_T^0 := X_T^{-\frac{11}{10},1} \cap L_T^\infty L_x^2$. Assume also that $k \geq 2^9$, $M \geq 1$ and $K \gg 1$. We define*

$$\begin{aligned} &J_{\eta,M}^{5,k,T}(\vec{u}_{1(3)}, u_2, u_3, u_4) \\ &\quad := \sum_{\vec{k}_{(3)} \in D_M^1(k)} \sum_{\substack{\vec{k}_{1(3)} \in D^2(k_1) \\ |\Omega_3(\vec{k}_{1(3)})| \not\sim |\Omega_3(k_1,k_2,k_3)|}} \int_{[0,T]} \eta(k_{1(3)}, k_2, k_3) \prod_{j=1}^{3} \widehat{u}_{1j}(k_{1j}) \prod_{i=2}^{4} \widehat{u}_i(k_i) \end{aligned}$$

and

$$\begin{aligned} &\widetilde{J}_{\eta,M,K}^{5,k,T}(u_{1(3)}, u_2, u_3, u_4) \\ &\quad := \sum_{\substack{\vec{k}_{(3)} \in D_M^2(k) \\ |k_1| \gg |k_2| \vee |k_3|}} \sum_{\substack{\vec{k}_{1(3)} \in D(k_1) \\ |\Omega_5(k_{1(3)}, k_2, k_3, k_4)| \gtrsim K}} \int_{[0,T]} \eta(\vec{k}_{1(3)}, k_2, k_3) \prod_{j=1}^{3} \widehat{u}_{1j}(k_{1j}) \prod_{i=2}^{4} \widehat{u}_i(k_i) \end{aligned}$$

with $k_4 = -k$ and where $\vec{u}_{1(3)} := (u_{11}, u_{12}, u_{13})$.
Then

$$\left| J_{\eta,M}^{5,k,T}(\vec{u}_{1(3)}, u_2, u_3, u_4) \right| \lesssim M^{1/2} k^{-\frac{9}{10}+} \prod_{j=1}^{3} \|u_{1j}\|_{Z_T^0} \prod_{i=2}^{4} \|u_i\|_{Z_T^0} \tag{4.12}$$

and

$$\begin{aligned} &\left| \widetilde{J}_{\eta,M,K}^{5,k,T}(P_{N_{11}} u_{11}, P_{N_{12}} u_{12}, P_{N_{13}} u_{13}, u_2, u_3, u_4) \right| \\ &\qquad \lesssim \frac{M}{K} N_{1,max}^{\frac{11}{10}} N_{1,min}^{\frac{1}{2}} \prod_{j=1}^{3} \|u_{1j}\|_{Z_T^0} \prod_{i=2}^{4} \|u_i\|_{Z_T^0} \ , \end{aligned} \tag{4.13}$$

for any K satisfying $K \gg N_{1,max}^{\frac{11}{20}}$.
Moreover, the implicit constant in estimate (3.14) only depends on the L^∞-norm of the function η.

Proof. We first notice that (4.7) ensures that $|\Omega_5(\vec{k}_{1(3)}, k_2, k_3, k_4))| \gtrsim |\Omega_3(\vec{k}_{1(3)})|$ on the support of $J^{5,k,T}_{\eta,M}(u_{1(3)}, u_2, u_3, u_4)$. According to this we may bound $J^{5,k,T}_{\eta,M}(u_{1(3)}, u_2, u_3, u_4)$ by

$$I := \sum_{N_{11}\geq 1} \sum_{1\leq M'\lesssim N_{11}} \sum_{\vec{k}_{(3)}\in D^1_M(k)} \sum_{\substack{\vec{k_{1(3)}}\in D^2_{M'}(k_1)\\ |\Omega_5(k_{1(3)},k_2,k_3,k_4))|\gtrsim M'N_{11}^2}}$$
$$\Big|\int_{[0,T]} \eta(k_{1(3)}, k_2, k_3)\widehat{P_{N_{11}}u}_{11}(k_{11}) \prod_{j=2}^{3} \widehat{P_{\lesssim N_{11}}u}_{1j}(k_{1j}) \prod_{i=2}^{4} \widehat{u}_i(k_i)\Big|$$

We proceed as in the proof of Proposition 3.4 to bound I. Setting $R = M'^{1+}k^{\frac{9}{10}}N_{11}^{0+} \ll M'N_{11}^2$ and using (4.8), we can easily estimate the contribution of $1^{low}_{T,R}P_{N_{11}}u_{11}$ to I by

$$\sum_{N_{11}\geq k} \sum_{1\leq M'\lesssim N_{11}} \sum_{\vec{k}_{(3)}\in D^1_M(k)} \sum_{\substack{\vec{k_{1(3)}}\in D^2_{M'}(k_1)\\ |\Omega_5(k_{1(3)},k_2,k_3,k_4)|\gtrsim M'N_{11}^2}}$$
$$\|1_{R,T}\|_{L^1_T}\Big\|\eta(k_{1(3)}, k_2, k_3)\widehat{P_{N_{11}}u}_{1j}(k_{1j}) \prod_{j=2}^{3} \widehat{P_{\lesssim N_{11}}u}_{1j}(k_{1j}) \prod_{i=2}^{4} |\widehat{u}_i(k_i)\Big\|_{L^\infty_T}$$
$$\lesssim \sum_{N_{11}\geq 1} \sum_{1\leq M'\lesssim N_{11}} (M'^{1+}k^{\frac{9}{10}}N_{11}^{0+})^{-1}M^{1/2}M' \prod_{j=1}^{3} \|u_{1j}\|_{L^\infty_T L^2_x} \prod_{i=2}^{4} \|u_i\|_{L^\infty_T L^2_x}$$
$$\lesssim M^{1/2}k^{-\frac{9}{10}} \prod_{j=1}^{3} \|u_{1j}\|_{L^\infty_T L^2_x} \prod_{i=2}^{4} \|u_i\|_{L^\infty_T L^2_x} \ .$$

Then we decompose the contribution of $1^{high}_{T,R}P_{N_{11}}u_{11}$ in the same way as in (3.23). The contribution of $Q_{\gtrsim M'N_{11}^2}1^{high}_{T,R}P_{N_{11}}u_{11}$ can be estimated thanks to (4.8) by

$$\sum_{N_{11}\geq k} \sum_{1\leq M'\lesssim N_{11}} M^{1/2}M'(M'N_{11}^2)^{-1}N_{11}^{\frac{11}{10}}\|u_{11}\|_{X_T^{-\frac{11}{10},1}} \prod_{j=1}^{2} \|u_{1j}\|_{L^\infty_T L^2_x} \prod_{i=2}^{4} \|u_i\|_{L^\infty_T L^2_x}$$
$$\lesssim M^{1/2}k^{-\frac{9}{10}+} \prod_{j=1}^{3} \|u_{1j}\|_{Z^0_T} \prod_{i=2}^{4} \|u_i\|_{Z^0_T}$$

and the other contributions can be estimated in the same way.

Now to prove (4.13) we use (4.2) instead of (4.8). Actually, since $|k_1| \gg |k_2|\vee|k_3|$ on the support of $\widetilde{J}^{5,k,T}_{\eta,M}$, we know that $m_{min} = |k_2 + k_3|$ and $|k_1| \sim k$. Therefore

(4.2) and Bernstein inequalities lead to

$$\Big|\sum_{\substack{\vec{k}_{(3)}\in D^2_M(k)\\ |k_1|\gg|k_2|\vee|k_3|}}\sum_{\substack{\vec{k}_{1(3)}\in D(k_1)\\ |\Omega_5(\vec{k}_{1(3)},k_2,k_3,k_4)|\gtrsim K}}\eta(\vec{k}_{1(3)},k_2,k_3)\prod_{j=1}^{3}\widehat{P_{N_{1j}}u_{1j}}(k_{1j})\prod_{i=2}^{4}\widehat{u_i}(k_i)\Big|$$

$$\lesssim M\Big\|\prod_{j=1}^{3}P_{N_{1j}}u_{1j}\Big\|_{L^1_x}\prod_{i=2}^{4}\|u_i\|_{L^2_x}$$

$$\lesssim MN_{1,min}^{\frac12}\prod_{j=1}^{3}\|P_{N_{1j}}u_{1j}\|_{L^2_x}\prod_{i=2}^{4}\|u_i\|_{L^2_x}\ .$$

With this estimate in hand, (4.13) follows from the same considerations as (4.12) by taking $R=N_{1,max}^{\frac{11}{10}}/K$ and using that $\Omega_5\geq K\gg N_{1,max}^{\frac{11}{20}}$ ensures that $R\ll\Omega_5$. □

4.2.3. 7-*linear estimates.* We use the following notations : $\vec{k}_{(7)}=(k_1,k_2,..,k_8)\in\mathbb{Z}^8$ and for any $k\in\mathbb{Z}$,

$$\Gamma_7(k)=\{\vec{k}_{(7)}\in\mathbb{Z}^8,\sum_{i=1}^{8}k_i=k\}\ .$$

The proof of the following lemma follows from exactly the same considerations as the ones used in the proof of Lemma 4.4.

Lemma 4.6. *Let $f_j\in l^2(\mathbb{Z})$, $j=1,...,8$. Setting*

$$\mathcal{J}_k^{7,1}:=\sum_{\vec{k}_{(7)}\in\Gamma^7(k),k_8=-k}\phi_M\Big(\sum_{q=1}^{6}k_q\Big)\phi_{M_1}(k_1+k_2)\phi_{M_2}(k_4+k_5)\prod_{j=1}^{8}|f_j(k_j)|$$

and

$$\mathcal{J}_k^{7,2}:=\sum_{\vec{k}_{(7)}\in\Gamma^7(k),k_8=-k}\phi_M(\sum_{q=4}^{7}k_q)\phi_{M_1}(k_1+k_2)\phi_{M_2}(k_4+k_5)\prod_{j=1}^{8}|f_j(k_j)|$$

it holds

(4.14)

$$\mathcal{J}_k^{7,1}+\mathcal{J}_k^{7,2}\lesssim\min\Big(M^{\frac12}M_1M_2\prod_{j=1}^{8}\|f_j\|_{l^2}\ ,\ M^{\frac12}M_1M_2^{1/2}\prod_{j=4}^{5}\|\langle\xi\rangle^{1/4}\ f_j\|_{l^2}\prod_{\substack{j=1\\ j\notin\{4,5\}}}^{8}\|f_j\|_{l^2}\Big)\ .$$

Proposition 4.7. *Assume that $0<T\leq1$, $\eta:\mathbb{Z}^7\to\mathbb{C}$ is a bounded measurable function. Assume also that u_{1i}, u_{2i} with $i=1,2,3$, and u_3, u_4 are functions in $Z_T^0:=X_T^{-\frac{11}{10},1}\cap L^\infty_TL^2_x$ and that $k\geq2^9$, $M\geq1$ and $M_1\geq1$. For $i=1,2$, we set $\vec{u}_{i(3)}:=(u_{i1},u_{i2},u_{i3})$ and define*

$$J^{7,k,T,i}_{\eta,M,M_1}(\vec{u}_{1(3)},\vec{u}_{2(3)},u_3,u_4):=\sum_{\vec{k}_{(3)}\in D^1_M(k)}\sum_{\vec{k}_{1(3)}\in D^1_{M_1}(k_1)}\sum_{\substack{\vec{k}_{2(3)}\in D^i(k_2)\\ \Omega_5(\vec{k}_{1(3)},k_2,k_3,k_4)\not\sim\Omega_3(k_{2(3)})}}$$

$$\int_{[0,T]}\eta(\vec{k}_{1(3)},\vec{k}_{2(3)},k_3)\prod_{j=1}^{3}\widehat{u}_{1j}(k_{1j})\prod_{m=1}^{3}\widehat{u}_{2m}(k_{2m})\prod_{q=3}^{4}\widehat{u}_q(k_q)$$

with $k_4 = -k$ and where .
Then
(4.15)

$$\big|J^{7,k,T,1}_{\eta,M,M_1}(\vec{u}_{1(3)},\vec{u}_{2(3)},u_3,u_4)\big| \lesssim M^{1/2}M_1k^{\frac{1}{10}}\prod_{j=1}^{3}\|u_{1j}\|_{Z^0_T}\|u_{2j}\|_{Z^0_T}\prod_{q=3}^{4}\|u_q\|_{Z^0_T}$$

and
(4.16)

$$\big|J^{7,k,T,2}_{\eta,M,M_1}(\vec{u}_{1(3)},\vec{u}_{2(3)},u_3,u_4)\big| \lesssim M^{1/2}M_1k^{-\frac{9}{10}}\prod_{j=1}^{3}\|u_{1j}\|_{Z^0_T}\|u_{2j}\|_{Z^0_T}\prod_{q=3}^{4}\|u_q\|_{Z^0_T}\,.$$

Moreover, the implicit constant in estimate (3.14) *only depends on the L^∞-norm of the function η.*

Proof. We define the resonance function of order 7 for $\vec{k}_{(7)} = (k_1,\cdots,k_8) \in \Gamma^7(0)$ by

$$\Omega^7(\vec{k}_{(7)}) = \sum_{i=1}^{8} k_i^3\,. \tag{4.17}$$

Again a direct calculation shows that

$$\Omega_7(\vec{k}_{1(3)},\vec{k}_{2(3)},k_3,k_4) = \Omega_5(\vec{k}_{1(3)},k_2,k_3,k_4) + \Omega_3(\vec{k}_{2(3)}) \tag{4.18}$$

and thus $|\Omega_7(\vec{k}_{1(3)},\vec{k}_{2(3)},k_3,k_4) \gtrsim |\Omega_3(\vec{k}_{2(3)})|$ on the support of $J^{7,k,T,i}_{\eta,M,M_1}$ with $i \in \{1,2\}$. (4.15)-(4.16) follow then from the same considerations as in the proof of Propositions 4.3-4.5 by making use of (4.14). □

4.3. **Proof of Theorem 4.1.** To prove Theorem 4.1 we construct a modified energy in the same way as in [17]. Note that this way of construction of modified energies has much in common with the I-method [4].

Let us first notice that (4.1) is obviously satisfied for $k \lesssim 1$. Theorem 4.1 will then be a direct consequence of Lemma 4.8 and Proposition 4.9 below.

Definition of the modified energy : For $t \geq 0$, we define the modified energy at the mode $k > 2^9$ by

$$\mathcal{E}_k(t) = \mathcal{E}_k(u(t)) = \frac{k}{2}|\widehat{u}(t,k)|^2 + \alpha\mathcal{E}^{3,1}_k(t) + \beta\mathcal{E}^{3,2}_k(t) + \gamma\mathcal{E}^5_k(t) \tag{4.19}$$

where α, γ and β are real constants to be determined later.
In the sequel of this subsection, to simplify the formula, we set $k_4 = -k$.
$\mathcal{E}^{3,1}_k$, $\mathcal{E}^{3,2}_k$, $\mathcal{E}^5_k$ are then defined as follows :

$$\mathcal{E}^{3,1}_k(u) = k^2\Re\Big[\sum_{M<k^{2/3-}}\sum_{\vec{k}_{(3)}\in D^1_M(k)}\frac{1}{\Omega^3(\vec{k}_{(3)})}\prod_{j=1}^{4}\widehat{u}(k_j)\Big]\,,$$

$$\mathcal{E}^{3,2}_N(u) = k^2\Re\Big[\sum_{\substack{\vec{k}_{(3)}\in D^2(k)\\ k_{med}<k^{\frac{2}{3}}}}\frac{1}{\Omega^3(\vec{k}_{(3)})}\prod_{j=1}^{4}\widehat{u}(k_j)\Big]$$

where $\vec{k}_{(3)} = (k_1, k_2, k_3)$ and the dyadic decompositions in N_j are nonhomogeneous,

(4.20)

$$\mathcal{E}_k^5(u) = k^2 \Re\Big[\sum_{i=1}^{4} \sum_{M\geq 1} \sum_{\vec{k}_{(3)}\in D_M^1(k)} \sum_{\substack{\vec{k}_{i(3)}\in D^1(k_i)\\ \Omega^3(\vec{k}_{(3)})\ll\Omega^3(\vec{k}_{i(3)})}} \frac{k_i}{\Omega^3(\vec{k}_{(3)})\Omega^5(\vec{k}_{i(5)})} \prod_{\substack{j=1\\ j\neq i}}^{4} \widehat{u}(k_j) \prod_{q=1}^{3} \widehat{u}(k_{i,q})\Big] .$$

with the notation

$$\vec{k}_{j(5)} = (\vec{\tilde{k}}_{j(3)}, \vec{k}_{j(3)}) \in \Gamma^5$$

where $\vec{\tilde{k}}_{j(3)}$ is defined by

$$\vec{\tilde{k}}_{1(3)} = (k_2, k_3, k_4),\ \vec{\tilde{k}}_{2(3)} = (k_1, k_3, k_4),\ \vec{\tilde{k}}_{3(3)} = (k_1, k_2, k_4),\ \vec{\tilde{k}}_{4(3)} = (k_1, k_2, k_3) .$$

Next, we show that if $s > \frac{1}{4}$, then the non quadratic part of $\mathcal{E}_k(u)$ is controlled by the H^s-norm of u.

Lemma 4.8. *Let $s > 1/4$ For any $u \in H^s(\mathbb{T})$ it holds*

$$|\mathcal{E}_k^{3,1}(u)| + |\mathcal{E}_k^{3,2}(u)| \lesssim \|P_{\lesssim k} u\|_{H^s}^4 \tag{4.21}$$

and

$$|\mathcal{E}_k^5(u)| \lesssim \|P_{\lesssim k} u\|_{H^s}^6 \tag{4.22}$$

Proof. Since $|\Omega_3(\vec{k}_{(3)})| \geq M_{min} M_{med} k$ on $D^1(k)$, (4.2) leads to

$$|\mathcal{E}_k^{3,1}(u)| \lesssim \sum_{1\leq M_{min}\leq M_{med}} \frac{k^2 M_{min}^{1/2} k^{-4s}}{M_{min} M_{med} k} \|P_{\sim k} u\|_{L_t^\infty H_x^s}^4 \lesssim k^{1-4s} \|P_{\lesssim k} u\|_{H_x^s}^4$$

which is acceptable. In the same way, on $D^2(k)$, it holds $|\Omega_3(\vec{k}_{(3)})| \geq M_{min} k^2$ and thus (4.2) leads this time to

$$|\mathcal{E}_k^{3,2}(u)| \lesssim \sum_{M\geq 1} \frac{k^2 M^{1/2} k^{-2s}}{M k^2} \|P_{\sim k} u\|_{L_t^\infty H_x^s}^2 \|P_{\lesssim k} u\|_{L_t^\infty L_x^2}^2 \lesssim k^{-2s} \|P_{\lesssim k} u\|_{H_x^s}^4 .$$

Now, fix $i \in \{1, 2, 3, 4\}$. For $\vec{k}_{(3)} \in D^1(k)$ and $\vec{k}_{i(3)} \in D^1(k_i)$ we have

$$|k_1| \sim |k_2| \sim |k_3| \sim |k_4| \sim |k_{i,1}| \sim |k_{i,2}| \sim |k_{i,3}| \sim k .$$

Moreover, for $\vec{k}_{i(5)}$ such that $\Omega^3(\vec{k}_{(3)}) \ll \Omega^3(\vec{k}_{i(3)})$ we have

$$|\Omega^5(\vec{k}_{i(5)})| = |\Omega^3(\vec{k}_{(3)}) + \Omega^3(\vec{k}_{i(3)})| \sim |\Omega^3(\vec{k}_{i(3)})| \gg |\Omega^3(\vec{k}_{(3)})| > 0 .$$

Therefore $\mathcal{E}_k^5(u)$ is well defined and according to (4.8)-(4.11),

$$\begin{aligned} |\mathcal{E}_k^5(u)| &\lesssim \sum_{i=1}^{4} \sum_{1\leq M_{min}\leq M_{med}} \sum_{1\leq M_{i,min}\leq M_{i,med}} \frac{k^3 M_{min}^{1/2} M_{i,min} k^{-6s}}{M_{min} M_{med} M_{i,min} M_{i,med} k^2} \|P_{\sim k} u\|_{H_x^s}^6 \\ &\lesssim k^{1-6s} \|P_{\lesssim k} u\|_{H_x^s}^6 \end{aligned}$$

which is acceptable. □

Proposition 4.9. *Let $s \geq 1/3$. Then for $k > 2^9$,*

$$|\mathcal{E}_k(t) - \mathcal{E}_k(0)| \lesssim \sup_{N\gtrsim k} \Big[\Big(\frac{k}{N}\Big)^{s-} \|P_{\lesssim N} u\|_{Z_T^s}^4 (1 + \|P_{\lesssim N} u\|_{Z_T^s}^4)\Big] \tag{4.23}$$

Proof. Since u is real valued we can restrict ourself to positive k. As above, we differentiate $\mathcal{E}_k$ with respect to time and then integrate between 0 and t to get

$$\begin{aligned}\mathcal{E}_k(t) &= \mathcal{E}_k(0) - k\Re\Big[\int_0^t P_k\partial_x(A(u,u,u)-B(u,u,u))P_ku\Big] + \alpha\int_0^t \frac{d}{dt}\mathcal{E}_k^{3,1}(t')dt' \\ &\quad + \beta\int_0^t \frac{d}{dt}\mathcal{E}_k^{3,2}(t')dt' + \gamma\int_0^t \frac{d}{dt}\mathcal{E}_k^{5}(t')dt' \\ &=: k^2\mathcal{E}_k(0) + I_k + \alpha J_k^1 + \beta J_k^2 + \gamma K_k\,. \qquad (4.24)\end{aligned}$$

As in the preceding section, since u is real-valued, the contribution of $\partial_x B(u,u,u)$ is purely imaginary and thus vanishes. Recalling that we set $k_4=-k$, we can thus rewrite I_k in Fourier variables as

$$I_k = k^2\Im\Big[\int_0^t \sum_{\vec{k}_{(3)}\in D(k)} \widehat{u}(k_1)\widehat{u}(k_2)\widehat{u}(k_3)\widehat{u}_4(k_4)\Big]$$

with $D(k)$ defined as in the beginning of this section. We denote by I_k^1 and I_k^2 the contributions to I_k of respectively $D^1(k)$ and $D^2(k)$. Finally, we decompose I_k^1 and I_k^2 in the following way :

$$\begin{aligned}I_k^1 &= (\sum_{M\geq k^{\frac{7}{12}}} + \sum_{M<k^{\frac{7}{12}}})k^2\Im\Big[\int_0^t \sum_{\vec{k}_{(3)}\in D^1_M(k)} \widehat{u}(k_1)\widehat{u}(k_2)\widehat{u}(k_3)\widehat{u}(k_4)\Big] \\ &= I_k^{1,high} + I_k^{1,low}\end{aligned}$$

and

$$\begin{aligned}I_k^2 &= (\sum_{k_{med}\geq k^{\frac{2}{3}}} + \sum_{k_{med}<k^{\frac{2}{3}}})k^2\Im\Big[\int_0^t \sum_{\vec{k}_{(3)}\in D^2(k)} \widehat{u}(k_1)\widehat{u}(k_2)\widehat{u}(k_3)\widehat{u}(k_4)\Big] \\ &= I_k^{2,high} + I_k^{2,low}\,.\end{aligned}$$

• **Estimate on $I_k^{1,high}$** According to (4.4) we have

$$\begin{aligned}|I_k^{1,high}| &\lesssim k^2 \sum_{M\geq k^{\frac{7}{12}}} \frac{k^{\frac{1}{10}}}{M^{3/2}} k^{-4s}\prod_{i=1}^4 \|P_{\sim k}u\|_{Z^s} \\ &\lesssim k^{-4s+\frac{49}{40}}\|P_{\lesssim k}u\|_{Z^s}^4\,,\end{aligned}$$

which is acceptable for $s>\frac{49}{160}$.

• **Estimate on $I_k^{1,low}+\alpha J_k$.**

By (2.1), we can rewrite $\frac{d}{dt}\mathcal{E}_k^{3,1}$ as the sum of the "linear" contribution[1]

$$\sum_{M<k^{\frac{7}{12}}} k^2\Re\Big[\sum_{\vec{k}_{(3)}\in D^1_M(k)} \frac{i(k_1^3+k_2^3+k_3^3+k_4^3)}{\Omega^3(\vec{k}_{(3)})}\prod_{j=1}^4 \widehat{u}(k_j)\Big]$$

[1]By "linear" contribution, we mean the contribution of the linear part when substituting u_t thanks to the equation

and the "nonlinear" contribution

$$\sum_{i=1}^{4}\sum_{M<k^{\frac{7}{12}}} k^2\Re\Big[\sum_{\vec{k}_{(3)}\in D^1_M(k)}\frac{-ik_i}{\Omega^3(\vec{k}_{(3)})}\prod_{\substack{j=1\\ j\neq i}}^{4}\widehat{u}(k_j)\Big(3|\widehat{u}(k_i)|^2\widehat{u}(k_i)+\sum_{\vec{k_i}_{(3)}\in D(k_i)}\prod_{q=1}^{3}\widehat{u}(k_{i,q})\Big)\Big].$$

Using the resonance relation (3.21), we see by choosing $\alpha=1$ that $I_k^{1,low}$ is canceled out by the linear contribution of $\int_0^t \frac{d}{dt}\mathcal{E}_N^3$. Hence,

$$I_k^{1,low}+J_k=\sum_{j=0}^{1}c_jA_k^j\,,$$

where, by symmetry,

$$\begin{aligned}\tilde{A}_k^0 &= \sum_{M<k^{\frac{7}{12}}} k^3\Im\Big[\int_0^t\sum_{\vec{k}_{(3)}\in D^1_M(k)}\frac{1}{\Omega^3(\vec{k}_{(3)})}\prod_{j=1}^{3}\widehat{u}(k_j)\\ &\qquad\qquad \Big(3|\widehat{u}(-k)|^2\widehat{u}(-k)+\sum_{\vec{k_4}_{(3)}\in D(-k)}\prod_{q=1}^{3}\widehat{u}(k_{4,q})\Big)\Big]\\ &= A_k^{0,0}+A_k^0\end{aligned}$$

and

$$\begin{aligned}\tilde{A}_k^1 &= 3\sum_{M<k^{\frac{7}{12}}} k^2\Im\Big[\int_0^t\sum_{\vec{k}_{(3)}\in D^1_M(k)}\frac{k_1}{\Omega^3(\vec{k}_{(3)})}\prod_{j=2}^{4}\widehat{u}(k_j)\\ &\qquad\qquad \Big(3|\widehat{u}(k_1)|^2\widehat{u}(k_1)+\sum_{\vec{k_1}_{(3)}\in D(k_1)}\prod_{q=1}^{3}\widehat{u}(k_{1,q})\Big)\Big]\\ &= A_k^{1,0}+A_k^1\,.\end{aligned}$$

It thus remains to treat the terms $\tilde{A}_k^j$ corresponding to the nonlinear contribution of $\frac{d}{dt}\mathcal{E}_N^3$. Since $|k_i|\sim k$, A_k^0 and A_k^1 can be treated almost in the same way. Actually, some estimates on A_k^0 are easier thanks to (4.11). We thus only consider $\tilde{A}_k^1$. First, thanks to (4.2) , $A_k^{1,0}$ can be easily estimated by

$$\begin{aligned}|A_k^{1,0}| &\lesssim \sum_{1\le M_{min}\le M_{med}}\frac{k^3M_{min}^{1/2}}{M_{min}M_{med}k}k^{-6s}\|P_{\sim k}u\|^6_{L^\infty_tH^s_x}\\ &\lesssim k^{2-6s}\|P_{\lesssim k}u\|^6_{L^\infty_tH^s_x}\end{aligned}$$

which is acceptable for $s\ge 1/3$.
By symmetry we can assume that $M_{11}\le M_{12}\le M_{13}$. We set $N_{1,max}=\max(N_{11},N_{12},N_{13})$.
2.1 $|\Omega_3(\vec{k}_{(3)})|\gtrsim|\Omega_3(\vec{k}_{1(3)})|$. Then we must have

$$M_{11}\lesssim\frac{M_{min}M_{med}k}{M_{12}M_{13}}\,.$$

2.1.1 $\vec{k}_{1(3)} \in D^2(k_1)$. In this region it holds $M_{12} \gtrsim k$ and thus $M_{11} \lesssim \frac{M_{min}M_{med}}{k}$. On account of (4.8)-(4.10) we get

$$\begin{aligned}
|A_k^1| &\lesssim \sum_{1\le M_{min}\le \min(M_{med},k^{\frac{7}{12}})} \sum_{1\le M_{11}\lesssim \frac{M_{min}M_{med}}{k}} \sum_{N_{1,max}\gtrsim k} \frac{k^3 M_{min}^{1/2} M_{11}}{M_{min}M_{med}k} k^{-3s} N_{1,max}^{-s} \\
&\quad \|P_{\sim k}u\|^3_{L_t^\infty H_x^s} \|P_{N_{1,max}}u\|_{L_t^\infty H_x^s} \|P_{\lesssim N_{1,max}}u\|^2_{L_t^\infty L_x^2} \\
&\lesssim \sum_{N_{1,max}\gtrsim k} \sum_{1\le M_{min}<k^{\frac{7}{12}}} M_{min}^{1/2} k^{1-4s} (\frac{k}{N_{1,max}})^s \|P_{\lesssim N_{1,max}}u\|^6_{L_t^\infty H_x^s} \\
&\lesssim \sum_{N_{1,max}\gtrsim k} \sum_{1\le M_{min}<k^{\frac{7}{12}}} (\frac{M_{min}}{k^{\frac{7}{12}}})^{1/2} k^{\frac{31}{24}-4s} (\frac{k}{N_{1,max}})^s \|P_{\lesssim N_{1,max}}u\|^6_{L_t^\infty H_x^s} \\
&\lesssim \sum_{N_{1,max}\gtrsim k} k^{-4s+\frac{31}{24}} (\frac{k}{N_{1,max}})^s \|P_{\lesssim N_{1,max}}u\|^6_{L_t^\infty H_x^s}
\end{aligned}$$

which is acceptable for $s \ge \frac{31}{96}$.

2.1.2 $\vec{k}_{1(3)} \in D^1(k_1)$. Then it holds

$$|k_{11}| \sim |k_{12}| \sim |k_{13}| \sim k\,.$$

Since $|\Omega_3(\vec{k}_{(3)})| \gtrsim |\Omega_3(\vec{k}_{1(3)})|$, we must have $M_{min}M_{med} \gtrsim M_{11}^2$. Therefore (4.8)-(4.10) lead to

$$\begin{aligned}
|A_k^1| &\lesssim \sum_{1\le M_{min}\le M_{med}} \frac{k^3 M_{min}^{1/2} (M_{min}M_{med})^{1/2}\, k^{-6s}}{M_{min}M_{med}k} \|P_{\sim k}u\|^6_{L_t^\infty H_x^s} \\
&\lesssim k^{2-6s} \|P_{\sim k}u\|^6_{L_t^\infty H_x^s}
\end{aligned}$$

which is acceptable for $s \ge 1/3$.

2.2. $|\Omega_3(\vec{k}_{(3)})| \ll |\Omega_3(\vec{k}_{1(3)})|$. Then, by (4.7), $|\Omega_5| \sim |\Omega_3(\vec{k}_{1(3)})| \gg |\Omega_3(\vec{k}_{(3)})|$.

2.2.1. $\vec{k}_{1(3)} \in D^2(k_1)$. According to (4.12) we then obtain

$$\begin{aligned}
|A_k^1| &\lesssim \sum_{N_{1,max}\gtrsim k} \sum_{1\le M_{min}\le M_{med}} \frac{k^3 M_{min}^{1/2} k^{-\frac{9}{10}+}}{M_{min}M_{med}k} k^{-3s} N_{1,max}^{-s} \|P_{\sim k}u\|^3_{Z^s} \|P_{N_{1,max}}u\|_{Z^s} \|P_{\lesssim N_{1,max}}u\|^2_{Z^0} \\
&\lesssim \sum_{N_{1,max}\gtrsim k} k^{-4s+\frac{11}{10}-} (\frac{k}{N_{1,max}})^s \|P_{\lesssim N_{1,max}}u\|^6_{Z^s} \qquad (4.25)
\end{aligned}$$

which is acceptable for $s > \frac{11}{40}$.

2.2.2. $\vec{k}_{1(3)} \in D^1(k_1)$. Then we must have

$$|k_{11}| \sim |k_{12}| \sim |k_{13}| \sim |k_1| \sim |k_2| \sim |k_3| \sim k\,. \qquad (4.26)$$

and

$$\Omega^3(\vec{k}_{1(3)}) \sim M_{1,min}M_{1,med}k\,.$$

We call $A_k^{1,low}$ this contribution to A_k^1 and $A_k^{0,low}$ the same contribution to A_k^0. Using the equation and the resonance relation (4.6), we can rewrite $K_k := \int_0^t \frac{d}{dt}\mathcal{E}_k^5$

as

$$\Re\Big[\int_0^t \sum_{i=1}^4 \sum_{M\geq 1} k^2 \sum_{\vec{k}_{(3)}\in D^1_M(k)} \sum_{\substack{\vec{k_i}_{(3)}\in D^1(k_i)\\ \Omega^3(\vec{k}_{(3)})\ll\Omega^3(\vec{k_i}_{(3)})}} \frac{ik_i}{\Omega^3(\vec{k}_{(3)})}\int_0^t \prod_{\substack{j=1\\ j\neq i}}^4 \widehat{u}(k_j)\prod_{q=1}^3 \widehat{u}(k_{iq})$$

$$+\int_0^t \sum_{i=1}^4 \sum_{M\geq 1} k^2 \sum_{\vec{k}_{(3)}\in D^1_M(k)} \sum_{\substack{\vec{k_i}_{(3)}\in D^1(k_i)\\ \Omega^3(\vec{k}_{(3)})\ll\Omega^3(\vec{k_i}_{(3)})}} \frac{k_i}{\Omega^3(\vec{k}_{(3)})\Omega^5(\vec{k_i}_{(5)})}\prod_{q=1}^3 \widehat{u}(k_{iq})$$

$$\sum_{\substack{m=1\\ m\neq i}}^4 \prod_{\substack{j=1\\ j\neq i,m}}^4 \widehat{u}(k_j)(-ik_j)\Big(3|\widehat{u}(k_j)|^2\widehat{u}(k_j)+\sum_{\vec{k}_{j(3)}\in D(k_j)}\prod_{p=1}^3 \widehat{u}(k_{jp})\Big)$$

$$+\int_0^t \sum_{i=1}^4 \sum_{M\geq 1} k^2 \sum_{\vec{k}_{(3)}\in D^1_M(k)} \sum_{\substack{\vec{k_i}_{(3)}\in D^1(k_i)\\ \Omega^3(\vec{k}_{(3)})\ll\Omega^3(\vec{k_i}_{(3)})}} \frac{k_i}{\Omega^3(\vec{k}_{(3)})\Omega^5(\vec{k_i}_{(5)})}\prod_{\substack{j=1\\ j\neq i}}^4 \widehat{u}(k_j)$$

$$\sum_{m=1}^3 \prod_{\substack{q=1\\ q\neq m}}^3 \widehat{u}(k_{i,q})(-ik_{im})\Big(3|\widehat{u}(k_{im})|^2\widehat{u}(k_{im})+\sum_{\vec{k}_{i,m(3)}\in D(k_{i,m})}\prod_{p=1}^3 \widehat{u}(k_{i,m,p})\Big)\Big]$$

$$:= H_k^1+\widetilde{H}_k^2+\widetilde{H}_k^3.$$

• **Estimate on $A_k^{1,low}+A_k^{0,low}+K_k$.**
By choosing $\gamma=1$, the above calculations lead to

$$A_k^{1,low}+A_k^{0,low}+K_k=\widetilde{H}_k^2+\widetilde{H}_k^3\,. \tag{4.27}$$

Because of (4.26), $\widetilde{H}_k^2$ and $\widetilde{H}_k^3$ can be estimated in the same way. We thus only consider $\widetilde{H}_k^2$. It thus suffices to $\widetilde{H}_k^2$ of any fixed couple $(i,m)\in\{1,2,3,4\}^2$ with $i\neq m$. By symmetry, we can restrict ourselves to $(i,m)\in\{(1,2),(1,4)\}$. Since the case $m=4$ is easier (see (4.11)), we only consider the case $(i,m)=(1,2)$. We thus have to bound

$$I_k = k^2\Im\Big[\int_0^t \sum_{M\geq 1}\sum_{\vec{k}_{(3)}\in D^1_M(k)} \sum_{\substack{\vec{k_1}_{(3)}\in D^1(k_1)\\ \Omega^3(\vec{k}_{(3)})\ll\Omega^3(\vec{k_1}_{(3)})}} \frac{k_1k_2}{\Omega^3(\vec{k}_{(3)})\Omega^5(\vec{k_1}_{(5)})}\prod_{j=3}^4 \widehat{u}(k_j)$$

$$\prod_{\substack{q=1\\ q\neq 1}}^3 \widehat{u}(k_{1,q})\Big(3|\widehat{u}(k_2)|^2\widehat{u}(k_2)+\sum_{\vec{k}_{2(3)}\in D(k_2)}\prod_{p=1}^3 \widehat{u}(k_{2p})\Big)\Big]$$

$$= H_k^{3,0}+H_k^3\,.$$

First by (4.8)-(4.10), we easily get

$$|H^{3,0}|\lesssim \sum_{M\geq 1}\sum_{M_{1,min}\geq 1}\frac{k^4}{M^2kM_{1,min}^2k}M_{1,min}k^{-8s}\|P_{\sim k}u\|_{H^s}^8$$
$$\lesssim k^{2-8s}\|P_{\sim k}u\|_{H^s}^8$$

which is acceptable for $s\geq 1/4$.
Now, to bound H_k^3 we separate different contributions.
2.2.2.1 $|\Omega_5(k_{1(5)})|\gtrsim|\Omega_3(k_{2(3)})|$. Then we must have $|\Omega_3(k_{2(3)})|\lesssim|\Omega_3(k_{1(3)})|$ since

$|\Omega_5(k_{1(5)})| \sim |\Omega_3(k_{1(3)})|$. This forces $M_{2,min} \lesssim M_{1,med}$.
2.2.2.1.1 $N_{2,med} \sim N_{2,max}$. Then (4.14) leads to

$$\begin{aligned} |H_k^3| &\lesssim \sum_{\substack{1\le M\le M_{med}\\ 1\le M_{1,min}\le M_{1,med}\lesssim k}} \sum_{N_{2,max}\gtrsim k} \frac{k^4 M^{1/2} M_{1,min} M_{1,med}}{M M_{med} k M_{1,min} M_{1,med} k} \\ & \quad k^{-5s} N_{2,max}^{-2s} \|P_{\sim k} u\|^5_{L_t^\infty H_x^s} \|P_{\sim N_{2,max}} u\|^2_{L_t^\infty H_x^s} \|P_{\lesssim N_{2,max}} u\|_{L_t^\infty L_x^2} \\ &\lesssim \sum_{N_{2,max}\gtrsim k} k^{-7s+2+} (\frac{k}{N_{2,max}})^{2s} \|P_{\lesssim N_{2,max}} u\|^8_{L_t^\infty H_x^s} \end{aligned}$$

which is acceptable for $s > 2/7$.
2.2.2.1.2 $N_{2,med} \ll N_{2,max}$. On account of (4.26), it holds $N_{2,max} \sim k$ and $|\Omega_3(k_{2(3)})| \sim M_{2,min} k^2$. The inequality $|\Omega_3(k_{2(3)})| \lesssim |\Omega_3(k_{1(3)})|$ then ensures that

$$M_{2,min} \lesssim \frac{M_{1,min} M_{1,med}}{k} .$$

(4.14) thus leads for $s \ge 1/4$ to

$$\begin{aligned} |H_k^3| &\lesssim \sum_{\substack{1\le M\le M_{med}\\ 1\le M_{1,min}\le M_{1,med}\lesssim k}} \frac{k^4 M^{1/2} M_{1,min} (M_{1,min} M_{1,med} k^{-1})^{1/2}}{M M_{med} k M_{1,min} M_{1,med} k} k^{-6s} \\ & \quad \|P_{\sim k} u\|^6_{L_t^\infty H_x^s} \|P_{\lesssim k} u\|^2_{L_t^\infty H_x^{1/4}} \\ &\lesssim \sum_{\substack{1\le M\le M_{med}\\ 1\le M_{1,min}\le M_{1,med}\lesssim k}} \frac{k^{3/2} k^{-6s}}{M^{1/2} M_{med}} \left(\frac{M_{1,min}}{M_{1,med}}\right)^{1/2} \|P_{\lesssim k} u\|^8_{L_t^\infty H_x^s} \\ &\lesssim k^{-6s+\frac{3}{2}+} \|P_{\lesssim k} u\|^8_{L_t^\infty H_x^s} \end{aligned}$$

which is acceptable for $s > 1/4$.
2.2.2.2 $|\Omega_5(k_{1(5)})| \ll |\Omega_3(k_{2(3)}|$**.** Then on account of (4.18) it holds $|\Omega_7| \sim |\Omega_3(k_{2(3)}|$.
2.2.2.2.1 $\vec{k}_{2(3)} \in D^2(k_2)$. Then (4.16) gives

$$\begin{aligned} |H_k^3| &\lesssim \sum_{\substack{M\ge 1\\ M_{1,min}\ge 1}} \sum_{N_{2,max}\gtrsim k} \frac{k^4 M^{1/2} M_{1,min}}{M^2 k M_{1,min}^2 k} k^{-5s} N_{2,max}^{-s} k^{-\frac{9}{10}} \|P_{\sim k} u\|^5_{Z^s} \|P_{N_{2,max}} u\|_{Z^s} \|P_{\lesssim N_{2,max}} u\|^2_{Z^0} \\ &\lesssim \sum_{N_{2,max}\gtrsim k} k^{-6s+\frac{11}{10}} (\frac{k}{N_{2,max}})^s \|P_{\lesssim N_{2,max}} u\|^6_{Z^s} \end{aligned}$$

which is acceptable for $s \ge \frac{11}{60}$.
2.2.2.2.2 $\vec{k}_{2(3)} \in D^1(k_2)$**.** Then we have

$$|k_{21}| \sim |k_{22}| \sim |k_{23}| \sim k .$$

Therefore (4.15) leads to

$$\begin{aligned} |H_k^3| &\lesssim \sum_{\substack{M\ge 1\\ M_{1,min}\ge 1}} \frac{k^4 M^{1/2} M_{1,min}}{M^2 k M_{1,min}^2 k} k^{-8s} k^{\frac{1}{10}} \|P_{\sim k} u\|^8_{Z^s} \\ &\lesssim k^{\frac{21}{10}-8s} \|P_{\sim k} u\|^8_{Z^s} \end{aligned}$$

which is acceptable for $s \geq \frac{21}{88}$.

• **Estimate on $I_k^{2,high}$**

By symmetry, in the sequel we assume $|k_1| \geq |k_2| \geq |k_3|$. We note that on $D^2(k)$ it holds $|\Omega_3(\vec{k}_{(3)})| \gtrsim M_{min} k_1^2$.

1. $|k_1| \sim |k_2| \geq |k_3|$**.**

1.1. $|k_3| \gtrsim k$. Then by (4.5)

$$\begin{aligned} I_k^{2,high} &\lesssim \sum_{N_1 \gtrsim k} \sum_{M \geq 1} \frac{k^2 k^{-\frac{9}{10}}}{M^{1/2}} k^{-2s} N_1^{-2s} \|P_k u\|_{Z^s} \|P_{\gtrsim k} P_{\lesssim N_1} u\|_{Z^s} \|P_{\sim N_1} u\|_{Z^s}^2 \\ &\lesssim \sum_{N_1 \gtrsim k} k^{\frac{11}{10}-4s} (\frac{k}{N_1})^{2s} \|P_k u\|_{Z^s} P_{\lesssim N_1} u\|_{Z^s}^3 \ , \end{aligned}$$

which is acceptable for $s \geq \frac{11}{40}$.

1.2. $|k_3| \ll k$. It forces $m_{min} = |k_1 + k_2| \sim k$ and thus (4.5) leads to

$$\begin{aligned} I_k^{2,high} &\lesssim \sum_{N_1 \gtrsim k} \sum_{M \sim k} \frac{k^2 k^{-\frac{9}{10}}}{M^{1/2}} k^{-s} N_1^{-2s} \|P_k u\|_{Z^s} \|P_{\sim N_1} u\|_{Z^s}^2 \|P_{\lesssim N_1} u\|_{Z^0} \\ &\lesssim \sum_{N_1 \gtrsim k} k^{\frac{3}{5}-3s} (\frac{k}{N_1})^{2s} \|P_{\lesssim N_1} u\|_{Z^s}^5 \end{aligned}$$

which is acceptable for $s \geq \frac{1}{5}$.

2. $|k_1| \gg |k_2| \geq |k_3|$. Then $|k_1| \sim k$ and $m_{min} = |k_2 + k_3|$. In this region we will make use of the fact that $|k_2| \geq k^{\frac{2}{3}}$.

2.1. $|k_3| \ll |k_2|$. Then it holds $m_{min} = |k_2 + k_3| \sim |k_2| \geq k^{\frac{2}{3}}$ and thus by (4.5),

$$\begin{aligned} I_k^{2,high} &\lesssim \sum_{M \geq k^{\frac{2}{3}}} \frac{k^2 k^{-\frac{9}{10}}}{M^{1/2}} k^{-2s} k^{-\frac{2s}{3}} \|P_{\sim k} u\|_{Z^s}^2 \|P_{\gtrsim k^{\frac{2}{3}}} P_{\lesssim k} u\|_{Z^s} \|P_{\lesssim k} u\|_{Z^0} \\ &\lesssim k^{(\frac{23}{30} - \frac{8s}{3})} \|P_{\lesssim k} u\|_{Z^s}^4 \ , \end{aligned}$$

which is acceptable for $s > \frac{23}{80}$.

2.2. $|k_2| \sim |k_3|$. Then (4.5) leads to

$$\begin{aligned} I_k^{2,high} &\lesssim \sum_{M \geq 1} \frac{k^2 k^{-\frac{9}{10}}}{M^{1/2}} k^{-2s} k^{-\frac{4s}{3}} \|P_{\sim k} u\|_{Z^s}^2 \|P_{\gtrsim k^{\frac{2}{3}}} P_{\lesssim k} u\|_{Z^s}^2 \\ &\lesssim k^{\frac{11}{10} - \frac{10s}{3}} \|P_{\lesssim k} u\|_{Z^s}^4 \ , \end{aligned}$$

which is acceptable for $s > \frac{33}{100}$.

Estimate on $I_k^{2,low} + \beta J_k^2$. By (2.1), we can rewrite $\frac{d}{dt}\mathcal{E}_k^{3,2}$ as the sum of the "linear" contribution

$$k^2 \Re \Big[\sum_{\substack{\vec{k}_{(3)} \in D^2(k) \\ |k_{med}| < k^{\frac{2}{3}}}} \frac{i(k_1^3 + k_2^3 + k_3^3 + k_4^3)}{\Omega^3(\vec{k}_{(3)})} \prod_{j=1}^4 \widehat{u}(k_j) \Big]$$

and the "nonlinear" contribution

$$k^2\Re\Big[\sum_{i=1}^{4}\sum_{\substack{\vec{k}_{(3)}\in D^2(k)\\ |k_{med}|<k^{\frac{2}{3}}}}\frac{-ik_i}{\Omega^3(\vec{k}_{(3)})}\prod_{\substack{j=1\\ j\neq i}}^{4}\widehat{u}(k_j)\Big(3|\widehat{u}(k_i)|^2\widehat{u}(k_i)+\sum_{\vec{k_i}_{(3)}\in D(k_i)}\prod_{q=1}^{3}\widehat{u}(k_{i,q})\Big)\Big]$$

$$=k^2\Im\Big[\sum_{i=1}^{4}\sum_{\substack{\vec{k}_{(3)}\in D^2(k)\\ |k_{med}|<k^{\frac{2}{3}}}}\frac{k_i}{\Omega^3(\vec{k}_{(3)})}\prod_{\substack{j=1\\ j\neq i}}^{4}\widehat{u}(k_j)\Big(3|\widehat{u}(k_i)|^2\widehat{u}(k_i)+\sum_{\vec{k_i}_{(3)}\in D(k_i)}\prod_{q=1}^{3}\widehat{u}(k_{i,q})\Big)\Big]$$

$$=\sum_{j=1}^{4}(B_k^{i,0}+B_k^i)\,.$$

Using the resonance relation (3.21), we see by choosing $\alpha=1$ that $I_k^{2,low}$ is canceled out by the linear contribution of $\int_0^t\frac{d}{dt}\mathcal{E}_k^{3,2}$. Hence,

$$I_k^{2,low}+J_k^2=\sum_{i=1}^{4}(B_k^{i,0}+B_k^i)$$

Note that since $|k_{med}|<k^{\frac{2}{3}}$ we must have $|k_{max}|\sim k$. In the sequel, by symmetry we assume that $k_{max}=k_1$. This forces $|k_1|\sim k$ and $M_{min}=M_1\lesssim k^{\frac{2}{3}}$.

Estimate on $B_k^{i,0}$, $j=1,2,3,4$

Let N_i be the dyadic variable associated to the dyadic decomposition with respect to k_i. Note that $N_i\lesssim k$. According to (4.2) it holds

$$\begin{aligned}|B_k^{i,0}| &\lesssim \sum_{M\geq 1}\sum_{1\leq N_i\leq k}\frac{k^2M^{1/2}N_i}{Mk^2}k^{-s}N_i^{-3s}\|P_{\sim k}u\|_{L_t^\infty H_x^s}\|P_{N_i}u\|^3_{L_t^\infty H_x^s}\|P_{\lesssim k}u\|^2_{L_T^\infty L_x^2}\\ &\lesssim k^{-s}k^{\max(0,1-3s)}\|P_{\lesssim k}u\|^6_{L_t^\infty H_x^s}\end{aligned}$$

which is acceptable for $s\geq 1/4$.

Estimate on B_k^2 and B_k^3

By symmetry, these two contributions can be treated in exactly the same way. So we only consider B_k^2. By symmetry we can assume that $|k_{21}|\geq|k_{22}|\geq|k_{23}|$. For $i=1,2,3$, let N_{2i} be the dyadic variable associated to the dyadic decomposition with respect to k_{2i}. Recall also that $|k_2|<k^{\frac{2}{3}}$ on the contribution of B_k^2.

1. $|k_{21}|\geq k$. Then we must have $N_{22}\sim N_{21}$. On account of (4.2) and Hölder and Sobolev inequalities, we have for $s\geq 1/4$,

$$\begin{aligned}B_k^2 &\lesssim \sum_{M\geq 1}\sum_{N_{21}\geq k}\frac{k^2k^{\frac{2}{3}}M^{1/2}}{Mk^2}k^{-2s}N_{21}^{-2s}\\ &\qquad\|P_{\sim k}u\|^2_{L_T^\infty H_x^s}\|P_{N_{21}}u\|^2_{L_T^\infty H_x^s}\|P_{\lesssim k^{\frac{2}{3}}}u\|^2_{L_T^\infty L_x^\infty}\\ &\lesssim \sum_{M\geq 1}\sum_{N_{21}\geq k}\frac{k^{\frac{2}{3}}}{M^{1/2}}k^{-2s}N_{21}^{-2s}k^{\frac{1}{3}}\\ &\qquad\|P_{\sim k}u\|^2_{L_T^\infty H_x^s}\|P_{N_{21}}u\|^2_{L_T^\infty H_x^s}\|P_{\lesssim k^{\frac{2}{3}}}u\|^2_{L_T^\infty H_x^{1/4}}\\ &\lesssim \sum_{N_{21}\geq k}k^{4(-s+\frac{1}{4})}(\frac{k}{N_{21}})^{2s}\|P_{\lesssim N_{21}}u\|^6_{L_T^\infty H_x^s}\end{aligned}$$

which is acceptable for $s \geq 1/4$.
2. $|k_{21}| < k$**.**
2.1 $|k_2| \gg |k_3|$**.** Then we must have $m_{min} = |k_2 + k_3| \sim |k_2|$. Therefore, using similar considerations as in (4.2), Hölder and Sobolev inequalities, we get for $s \geq 1/4$,

$$\begin{aligned} B_k^2 &\lesssim \sum_{M \sim N_2 \lesssim k^{\frac{2}{3}}} \frac{k^2 N_2 M^{1/2}}{M k^2} k^{-2s} \|P_{\sim k} u\|^2_{L^\infty_T H^s_x} \|P_{\lesssim k} u\|^4_{L^\infty_T H^{1/4}_x} \\ &\lesssim k^{(-2s+\frac{1}{3})} \|P_{\lesssim k} u\|^6_{L^\infty_T H^s} \ , \end{aligned}$$

which is acceptable for $s > \frac{1}{6}$.
2.2 $|k_2| \sim |k_3|$**.** Then, for $s \geq 1/4$, we have

$$\begin{aligned} B_k^2 &\lesssim \sum_{M \geq 1} \sum_{N_2 \lesssim k^{\frac{2}{3}}} \frac{k^2 N_2 M^{1/2}}{M k^2} k^{-2s} N_2^{-2s} N_2^{1/2} \|P_{\sim k} u\|^2_{L^\infty_T H^s_x} \|P_{\sim N_2} u\|^2_{L^\infty_T H^s_x} \|P_{\lesssim N_2} u\|^2_{L^\infty_T H^{1/4}_x} \\ &\lesssim k^{-2s} k^{\max(0,(1-\frac{4}{3}s)} \|P_{\lesssim k} u\|^6_{L^\infty_T H^s} \ , \end{aligned}$$

which is acceptable for $s \geq \frac{3}{10}$.
Estimate on B_k^1
By symmetry we can assume that

$$|k_{11}| \geq |k_{12}| \geq |k_{13}| \, .$$

For $i = 1, 2, 3$, let $N_{1,i}$ be the dyadic variable associated to the dyadic decomposition with respect to $k_{1,i}$.
1. $|\Omega_3(\vec{k}_{(3)})| \not\sim |\Omega_3(\vec{k}_{1(3)})|$**.**
1.1. $M_{1,med} \geq 2^{-6}|k_{11}|$**.** Then $|\Omega_5| \gtrsim |\Omega_3(\vec{k}_{1(3)})| \gtrsim k_{11}^2$ and (4.13) leads, for $s > 1/4$ to

$$\begin{aligned} B_k^1 &\lesssim \sum_{M \lesssim k} \sum_{N_{11} \gtrsim k} \sum_{N_{13} \leq N_{12} \leq N_{11}} \frac{k^2 k M N_{13}^{1/2-s}}{M k^2 N_{11}^2} k^{-s} N_{11}^{-s} N_{12}^{-s} N_{11}^{\frac{11}{10}} \\ &\qquad \|P_k u\|_{Z^s} \|P_{N_{11}} u\|_{Z^s} \|P_{N_{12}} u\|_{Z^s} \|P_{N_{13}} u\|_{Z^s} \|P_{\lesssim k} u\|^2_{Z^0} \\ &\lesssim \sum_{N_{11} \gtrsim k} k^{-2s+\frac{1}{10}+} (\frac{k}{N_{11}})^{-s-\frac{9}{10}} \|P_{\lesssim N_{11}} u\|^6_{Z^s} \end{aligned}$$

which acceptable for $s > \frac{1}{20}$.
1.2. $M_{1,med} \leq 2^{-6}|k_{11}|$**.** Then $|k_{13}| \sim |k_{12}| \sim |k_{11}| \sim k$ and (4.8) leads to

$$\begin{aligned} B_k^1 &\lesssim \sum_{M \geq 1} \sum_{M_{1,min} \geq 1} \frac{k^2 k M^{1/2} M_{1,min}}{M k^2 M_{1,min}^2 k} k^{-4s} k^{\frac{10}{9}} \|P_{\sim k} u\|^4_{Z^s} \|u\|^2_{Z^0} \\ &\lesssim k^{\frac{10}{9}-4s} \|P_{\sim k} u\|^4_{Z^s} \|u\|^2_{Z^0} \end{aligned}$$

which acceptable for $s > \frac{5}{18}$.
2. $|\Omega_3(\vec{k}_{(3)})| \sim |\Omega_3(\vec{k}_{1(3)})|$**.**
2.1 $|k_{11}| \sim |k_{12}|$**.** Then we claim that $|k_{13}| \gtrsim k$. Indeed, recalling that $|k_{11} + k_{12} + k_{13}| \sim k$, $N_{13} \ll k$ would imply that $|k_{11} + k_{12}| \sim k$ and thus $|\Omega_3(\vec{k}_{1(3)})| \sim k k_{11}^2 \gtrsim$

$k^3 \gg |\Omega_3(\vec{k}_{(3)})|$ since $|\Omega_3(\vec{k}_{(3)})| \leq k^{\frac{8}{3}}$. Therefore, noticing that $M_{1,min}k_{11}^2 \sim M_1 k^2$ forces $M_{1,min} \lesssim M_1 \lesssim |k_2| \lesssim k^{\frac{2}{3}}$), (4.8) leads, for $s \geq 1/4$, to

$$\begin{aligned} B_k^1 &\lesssim \sum_{1\leq M_{1,min}\lesssim M_1 \lesssim N_2 \lesssim k^{\frac{2}{3}}} \sum_{N_{11}\geq N_{13}\gtrsim k} \sum \frac{k^2 k M_{1,min}^{1/2} M_1}{M_1 k^2} k^{-s} N_{13}^{-s} N_{11}^{-2s} N_2^{-1/4} \\ &\quad \|P_k u\|_{L_t^\infty H_x^s} \|P_{\sim N_{11}} u\|_{L_t^\infty H_x^s}^2 \|P_{N_{13}} u\|_{L_t^\infty H_x^s} \|P_{N_2} u\|_{L_T^\infty H_x^{1/4}} \|P_{\lesssim k^{\frac{2}{3}-}} u\|_{L_T^\infty L_x^2} \\ &\lesssim k^{-4s+\frac{7}{6}} (\frac{k}{N_{11}})^{2s} \|P_{\lesssim N_{11}} u\|_{L_t^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s \geq \frac{7}{24}$.

2.2 $|k_{11}| \gg |k_{12}| \geq |k_{13}|$. Then $|k_{11}| \sim k$, $M_{1,min} = M_{11}$ and $|\Omega_3(\vec{k}_{1(3)})| \sim M_{11}k^2$. Therefore, $|\Omega_3(\vec{k}_{(3)})| \sim |\Omega_3(\vec{k}_{1(3)})|$ forces $M_1 \sim M_{11}$.

2.2.1 $k_{11} \neq k$. Let $k' \in \{k_{12}, k_{13}, k_2, k_3\}$ such that $|k'| = \max(|k_{12}|, |k_{13}|, |k_2|, |k_3|)$. Then we have

$$|\Omega_3(k_{11}, k, k')| \gtrsim k^2 \,. \tag{4.28}$$

2.2.1.1 $\Omega_5 \gtrsim k^2$. Then we have for $1/4 \leq s \leq 1/2$, (4.13) leads to

$$\begin{aligned} B_k^1 &\lesssim \sum_{M_1 \lesssim k^{\frac{2}{3}}} \sum_{1\leq N_{13}\leq N_{12}\lesssim k} \frac{k^3 M_1 N_{13}^{\frac{1}{2}-s}}{M_1 k^4} k^{-2s} N_{12}^{-s} k^{\frac{11}{10}} \\ &\quad \|P_{\sim k} u\|_{Z^s}^2 \|P_{N_{12}} u\|_{Z^s} \|P_{N_{13}} u\|_{Z^s} \|P_{\lesssim k} u\|_{Z^0}^2 \\ &\lesssim k^{-2s+\frac{1}{10}+} \|P_{\lesssim k} u\|_{Z^s}^6 \,, \end{aligned}$$

which is acceptable for $s \geq \frac{1}{20}$.

2.2.1.2 $\Omega_5 \ll k^2$. Then we have $\Omega_3(k_{(3)}) \sim \Omega_3(k_{1(3)})$ and thus $M_1 \sim M_{1,1}$. Moreover, let (z_1, z_2, z_3) be such that $\{z_1, z_2, z_3\} := \{k_{12}, k_{13}, k_2, k_3\} - \{k'\}$. It follows from (4.28) that $\Omega_3(z_1, z_2, z_3) \gtrsim k^2$. Since $|k_2| \vee |k_3| < k^{\frac{2}{3}}$ this forces $k' \in \{k_{12}, k_{13}\}$ and $|k_{12}| \wedge |k_{13}| \gtrsim k^{\frac{2}{3}}$. For $s \geq 1/4$, (4.9) thus yields

$$\begin{aligned} B_k^1 &\lesssim \sum_{M_{1,1}\sim M_1 \lesssim k^{\frac{2}{3}}} \frac{k^3 M_{1,1}}{M_1 k^2} k^{-2s} k^{-\frac{4s}{3}} \\ &\quad \|P_{\sim k} u\|_{L_t^\infty H_x^s}^2 \|P_{\gtrsim k^{\frac{2}{3}}} P_{\lesssim k} u\|_{L_T^\infty H_x^s}^2 \|P_{\lesssim k^{\frac{2}{3}}} u\|_{L_T^\infty H_x^{1/4}}^2 \\ &\lesssim k^{-\frac{10}{3}s+1+} \|P_{\lesssim k} u\|_{L_T^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s > \frac{3}{10}$.

2.2.2 $k_{11} = k$. This is the more complicated case. Following [20] we first notice that

$$\Big|\frac{k^2}{(k_1+k_2)(k_1+k_3)} - 1\Big| = \Big|\frac{(k_2+k_3)k - k_2 k_3}{(k_1+k_2)(k_1+k_3)}\Big| \lesssim \frac{|k_2| \vee |k_3|}{|k|} \,. \tag{4.29}$$

We decompose the contribution of this region to B_k^1 as

$$\begin{aligned} B_k^1 &= \Im\Big[\sum_{|k_2|\vee|k_3|\leq k^{\frac{2}{3}}} \Big[\Big(\frac{k^2}{(k_1+k_2)(k_1+k_3)} - 1\Big) + 1\Big] \frac{k_1}{(k_2+k_3)} \widehat{u}(k)\widehat{u}(k_{12})\widehat{u}(k_{13})\widehat{u}(k_2)\widehat{u}(k_3)\widehat{u}(-k)\Big] \\ &= C_k^1 + C_k^2 \,. \end{aligned}$$

It is also worth noticing that since $k_{12}+k_{13}+k_2+k_3=0$, in this region we must have

$$(k_{12}+k_{13}) = -(k_2+k_3) \Longrightarrow M_{1,1} = M_1\,. \tag{4.30}$$

Estimate on C_k^1
1. $|k_2|\vee|k_3| \gg |k_2|\wedge|k_3|$**.** By symmetry we can assume that $|k_2|\gg|k_3|$ which forces $M_1\sim|k_2|$. According to (4.29) and (4.9), C_k^1 can be easily estimated for $s\geq 1/4$ by

$$\begin{aligned}|C_k^1| &\lesssim \sum_{N_2\sim M_1\lesssim k^{\frac{2}{3}}} \frac{k}{M_1}\frac{N_2}{k} M_1 k^{-2s} N_2^{-s} \|P_k u\|^2_{L^\infty_t H^s_x} \|P_{N_2}u\|_{L^\infty_T H^s_x} \|P_{\lesssim k}u\|^2_{L^\infty_T H^{1/4}_x} \|P_{\lesssim k}u\|_{L^\infty_T L^2_x}\\ &\lesssim k^{-\frac{8}{3}s+\frac{2}{3}} \|P_{\lesssim k}u\|^6_{L^\infty_t H^s_x}\end{aligned}$$

which is acceptable for $s\geq\frac{1}{4}$.
2. $|k_2|\sim|k_3|$**.** Then $|k_2|\wedge|k_3|\gtrsim M_1$ and since $M_1=M_{1,1}$ we also have $|k_{12}|\gtrsim M_1$. Therefore, According to (4.29)-(4.30) and (4.9), C_k^1 can be easily estimated for $s<1/2$ by

$$\begin{aligned}|C_k^1| &\lesssim \sum_{M_1\lesssim N_2, N_{1,2}\lesssim k^{\frac{2}{3}}} \frac{k}{M_1}\frac{N_2}{k} M_1^{1/2} M_1 k^{-2s} N_2^{-2s} N_{12}^{-s}\\ &\qquad \|P_k u\|^2_{L^\infty_t H^s_x} \|P_{N_2}u\|^2_{L^\infty_T H^s_x} \|P_{N_{12}}u\|_{L^\infty_T H^s_x} \|P_{\lesssim k}u\|_{L^\infty_T L^2_x}\\ &\lesssim \sum_{M_1\lesssim N_2\lesssim k^{\frac{2}{3}}} M_1^{\frac{1}{2}-s} N_2^{1-2s} k^{-2s} \|P_{\lesssim k}u\|^6_{L^\infty_t H^s_x}\\ &\lesssim k^{1-4s}\|P_{\lesssim k}u\|^6_{L^\infty_t H^s_x}\end{aligned}$$

which is acceptable for $s>1/4$.
Estimate on C_k^2
Rewriting k_1 as $k_1=k_{11}+k_{12}+k_{13}$ we decompose C_k^2 as the sum of three terms $C_k^{21}+C_k^{22}+C_k^{23}$.
Estimate on C_k^{22} and C_k^{23}
We only consider C_k^{22} which is the contribution of k_{12} since C_k^{23} can be treated in exactly the same way. We proceed as for C_k^1.
1. $|k_{12}|\gg|k_{13}|$. This forces $M_{11}\sim|k_{12}|$. According to (4.30) and (4.9), C_k^1 can be easily estimated for $s\geq 1/4$ by

$$\begin{aligned}|C_k^{22}| &\lesssim \sum_{N_{12}\sim M_1\lesssim k^{\frac{2}{3}}} \frac{N_{12}}{M_1} M_1^{1/2} k^{-2s} N_{12}^{-s} \|P_k u\|^2_{L^\infty_t H^s_x} \|P_{N_{12}}u\|_{L^\infty_T H^s_x} \|P_{\lesssim k}u\|^2_{L^\infty_T H^{1/4}_x} \|P_{\lesssim k}u\|_{L^\infty_T L^2_x}\\ &\lesssim k^{-\frac{8}{3}s+\frac{2}{3}} \|P_{\lesssim k}u\|^6_{L^\infty_t H^s_x}\end{aligned}$$

which is acceptable for $s>\frac{1}{4}$.
2. $|k_{12}|\sim|k_{13}|$**.** Then $|k_{12}|\wedge|k_{13}|\gtrsim M_1$ and since $M_1=M_{1,1}$ we also have $|k_2|\vee|k_3|\gtrsim M_1$. Therefore, according to (4.9), C_k^1 can be easily estimated for

$s<1/2$ by

$$\begin{aligned}|C_k^{22}| &\lesssim \sum_{M_1\lesssim N_2, N_{12}\lesssim k^{\frac{2}{3}}} \frac{N_{12}}{M_1} M_1^{1/2} M_1 k^{-2s} N_2^{-s} N_{12}^{-2s} \\ &\quad \|P_k u\|^2_{L_t^\infty H_x^s} \|P_{N_2} u\|_{L_T^\infty H_x^s} \|P_{N_{12}} u\|^2_{L_T^\infty H_x^s} \|P_{\lesssim k} u\|_{L_T^\infty L_x^2} \\ &\lesssim \sum_{M_1\lesssim N_{12}\lesssim k^{\frac{2}{3}}} M_1^{\frac{1}{2}-s} N_{12}^{1-2s} k^{-2s} \|P_{\lesssim k} u\|^6_{L_t^\infty H_x^s} \\ &\lesssim k^{1-4s} \|P_{\lesssim k} u\|^6_{L_t^\infty H_x^s}\end{aligned}$$

which is acceptable for $s>1/4$.

Estimate on C_k^{21} We first notice that since $|k_1|\gg |k_2|\vee|k_3|$ and $|k_{11}|\gg|k_{12}|\vee|k_{13}|$, $(k_1+k_2)(k_1+k_3)(k_2+k_3)\neq 0$ if and only if $k_2+k_3\neq 0$ and similarly, $(k_{11}+k_{12})(k_{11}+k_{13})(k_{12}+k_{13})\neq 0$ if and only if $k_{12}+k_{13}=-(k_2+k_3)\neq 0$. We can thus rewrite C_k^{21} as

$$C_k^{21} = k|u(k)|^2 \Im\Big[\sum_{\substack{k_1+k_2+k_3=k,\, |k_2|\vee|k_3|\leq k^{\frac{2}{3}},\, k_2+k_3\neq 0\\ k+k_{12}+k_{13}=k_1,\, |k_{12}|\vee|k_{13}|\ll k}} \frac{1}{k_2+k_3} \widehat{u}(k_{12})\widehat{u}(k_{13})\widehat{u}(k_2)\widehat{u}(k_3)\Big]$$

We now separate the contributions $C_k^{21,low}$ and $C_k^{21,high}$ of the regions $|k_{12}|\vee|k_{13}|\leq k^{\frac{2}{3}}$ and $|k_{12}|\vee|k_{13}|>k^{\frac{2}{3}}$. Let us start by bounding $C_k^{21,high}$. In the region $|k_{13}|\sim|k_{12}|$, it can be bounded for $s\geq 1/4$ by

$$\begin{aligned}|C_k^{21,high}| &\lesssim \sum_{M_{1,1}\sim M_1\leq k^{\frac{2}{3}}} \frac{kM_{1,1}}{M_1} k^{-2s} k^{-\frac{4s}{3}} \|P_k u\|^2_{L_t^\infty H_x^s} \|P_{\lesssim k} P_{\gtrsim k^{\frac{2}{3}}} u\|^2_{L_T^\infty H_x^s} \|P_{\leq k^{\frac{2}{3}}} u\|^2_{L_T^\infty H^{1/4}} \\ &\lesssim k^{-\frac{10s}{3}} \|P_{\lesssim k} u\|^6_{L_t^\infty H_x^s}\end{aligned}$$

which is acceptable for $s\geq \frac{3}{10}$.

On the other hand, in the region $|k_{12}|\not\sim|k_{13}|$, we must have $|k_{12}|\gg|k_{13}|$ and thus $M_{1,1}\sim|k_{12}|$. Moreover, (4.30) forces $|k_2|\vee|k_3|\sim|k_{12}|$. Therefore, Sobolev inequalities lead for $s<1/2$ to

$$\begin{aligned}|C_k^{21,high}| &\lesssim \sum_{k^{\frac{2}{3}}\lesssim N_{12}\lesssim k} \frac{kN_{12}^{1-2s}}{N_{12}} k^{-2s} N_{12}^{-2s} \|P_k u\|^2_{L_t^\infty H_x^s} \|P_{N_{12}} u\|^2_{L_T^\infty H_x^s} \|P_{\lesssim N_{12}} u\|^2_{L_T^\infty H_x^s} \\ &\lesssim k^{1-\frac{14s}{3}} \|P_{\lesssim k} u\|^6_{L_t^\infty H_x^s}\end{aligned}$$

which is acceptable for $s\geq\frac{3}{14}$.

Finally, we claim that $C_k^{21,low}=0$. Indeed, performing the change of variables : $(n_1,n_2,n_3)=(-k_{12},k_2+k_3,k_3)$ we first obtain

$$C_k^{21} = k|u(k)|^2 \Big[\sum_{\substack{|n_2-n_3|\vee|n_3|\vee|n_1|\vee|n_1-n_2|\leq k^{\frac{2}{3}}\\ n_2\neq 0k}} \frac{1}{n_2} \widehat{u}(-n_1)\widehat{u}(n_1-n_2)\widehat{u}(n_2-n_3)\widehat{u}(n_3)\Big]$$

But performing the change of variables $(n_1',n_2',n_3')=(-n_1,-n_2,-n_3)$ we infer that

$$C_k^{21} = -k|u(k)|^2 \Im\Big[\sum_{\substack{|n_2'-n_3'|\vee|n_3'|\vee|n_1'|\vee|n_1'-n_2'|\leq k^{\frac{2}{3}}\\ n_2'\neq 0}} \frac{1}{n_2'} \widehat{u}(n_1')\widehat{u}(n_2'-n_1')\widehat{u}(n_3'-n_2')\widehat{u}(-n_3')\Big] = -C_k^{21}$$

which ensures that $C_k^{11}=0$.

Estimates on B_k^4

By symmetry we can assume $|k_{41}|\geq|k_{42}|\geq|k_{43}|$. B_k^4 can be controlled exactly as B_k^1 and is even easier (see (4.2)) except for the treatment of the region ($k_{41}=k_1$ and $|k_{42}|\vee|k_{43}|\leq k^{\frac{2}{3}}$) which is slightly different to the treatment of the region ($k_1=k$ and $|k_{12}|\vee|k_{13}|\leq k^{\frac{2}{3}}$) for B_k^1. We thus only consider the region $k_1+k_{41}=0$ and $|k_{42}|\vee|k_{43}|\leq k^{\frac{2}{3}}$. In this region, according to (4.29) we can decompose B_k^4 as

$$\begin{aligned} B_k^4 &= \Im\Big(\sum_{\Lambda(k)}\Big[\Big(\frac{k^2}{(k_1+k_2)(k_1+k_3)}-1\Big)+1\Big] \\ &\qquad\qquad \frac{k}{(k_2+k_3)}\widehat{u}(k_1)\widehat{u}(k_2)\widehat{u}(k_3)\widehat{u}(-k_1)\widehat{u}(k_{42})\widehat{u}(k_{43})\Big) \\ &= B_k^{41}+B_k^{42} \end{aligned}$$

with

$$\Lambda(k)=\{(k_1,k_2,k_3,k_{42},k_{43})\in\mathbb{Z}^5,\ k_1+k_2+k_3=k,\ k_{42}+k_{43}=-k_2-k_3\neq 0,\ |k_2|\vee|k_3|\vee|k_{42}|\vee|k_{43}|\leq k^{\frac{2}{3}}\}\ .$$

B_k^{41} can be easily estimated as C_k^1 (actually it is is even easier) by using (4.11) and the fact that $|k_{42}+k_{43}|=|k_2+k_3|$.

Finally, we claim that $B_k^{42}=0$. We start by performing the change of variables $(n_1,n_2,n_3)=(k_2,k_2+k_3,k_{42})$ to obtain

$$B_k^{42} = -\Im\Big(\sum_{\Delta(k)}\frac{k}{n_2}|\widehat{u}(k-n_2)|^2\widehat{u}(n_1)\widehat{u}(n_2-n_1)\widehat{u}(n_3)\widehat{u}(-n_2-n_3)\Big)$$

with

$$\Delta(k)=\{(n_1,n_2,n_3)\in\mathbb{Z}^3,\ n_2\neq 0,\ |n_1|\vee|n_2-n_1|\vee|n_3|\vee|n_3-n_2|\leq k^{\frac{2}{3}}\}\ .$$

Then we separate the contributions of the regions $(n_1n_3>0)$, $(n_1n_3<0)$, $(n_1n_3=0, n_1+n_3\neq 0)$ and $(n_1=0,n_3=0)$ to obtain

$$\begin{aligned} B_k^{42} &= -\sum_{n_2\neq 0,|n_2|\lesssim k^{\frac{2}{3}}}\frac{k}{n_2}|\widehat{u}(k-n_2)|^2\Big[\\ &\qquad \Im\Big(\sum_{0<m,n<k^{\frac{2}{3}}}\widehat{u}(m)\widehat{u}(n_2-m)\widehat{u}(n)\widehat{u}(-n_2-n) \\ &\qquad\qquad +\widehat{u}(-m)\widehat{u}(n_2+m)\widehat{u}(-n)\widehat{u}(-n_2+n)\Big) \\ &\qquad +\Im\Big(\sum_{0<m,n<k^{\frac{2}{3}}}\widehat{u}(m)\widehat{u}(n_2-m)\widehat{u}(-n)\widehat{u}(-n_2+n) \\ &\qquad\qquad +\widehat{u}(-m)\widehat{u}(n_2+m)\widehat{u}(n)\widehat{u}(-n_2-n)\Big) \\ &\qquad +\Im\Big(\sum_{0<|m|<k^{\frac{2}{3}}}\widehat{u}(m)\widehat{u}(n_2-m)\widehat{u}(0)\widehat{u}(-n_2) \\ &\qquad\qquad +\widehat{u}(0)\widehat{u}(n_2)\widehat{u}(-m)\widehat{u}(-n_2+m)\Big) \\ &\qquad +\Im\Big(|\widehat{u}(0)|^2|\widehat{u}(n_2)|^2\Big)\Big] \\ &= D_k^1+D_k^2+D_k^3+D_k^4\ . \end{aligned}$$

Since u is real-valued, it is easy to check that $D_k^1 = D_k^3 = D_k^4 = 0$. Now, to compute D_k^2 we separate the contributions of the regions $(m \neq n)$ and $(m = n)$ to obtain

$$\begin{aligned}
D_k^2 &= -\sum_{n_2\neq 0, |n_2|\lesssim k^{\frac{2}{3}}} \frac{k}{n_2}|\widehat{u}(k-n_2)|^2\Big[\\
&\qquad +\Im\Big(\sum_{mn>0,\, 0<|m|<|n|<k^{\frac{2}{3}}} \widehat{u}(m)\widehat{u}(n_2-m)\widehat{u}(-n)\widehat{u}(-n_2+n) \\
&\qquad\qquad + \sum_{mn>0,\, 0<|n|<|m|<k^{\frac{2}{3}}} \widehat{u}(m)\widehat{u}(n_2-m)\widehat{u}(-n)\widehat{u}(-n_2+n)\Big) \\
&\qquad +\Im\Big(\sum_{n\neq 0,\, |n|<k^{\frac{2}{3}}} |\widehat{u}(n)|^2|\widehat{u}(n_2-n)|^2\Big)\Big] \\
&= -\sum_{n_2\neq 0, |n_2|\lesssim k^{\frac{2}{3}}} \frac{k}{n_2}|\widehat{u}(k-n_2)|^2\Big[\\
&\qquad +\Im\Big(\sum_{mn>0,\, 0<|m|<|n|<k^{\frac{2}{3}}} \widehat{u}(m)\widehat{u}(n_2-m)\widehat{u}(-n)\widehat{u}(-n_2+n) \\
&\qquad\qquad +\widehat{u}(n)\widehat{u}(n_2-n)\widehat{u}(-m)\widehat{u}(-n_2+m)\Big) \\
&\qquad +\Im\Big(\sum_{n\neq 0,\, |n|<k^{\frac{2}{3}}} |\widehat{u}(n)|^2|\widehat{u}(n_2-n)|^2\Big)\Big] \\
&= 0\,.
\end{aligned}$$

This completes the proof of the proposition. □

Remark 4.10. For the same reasons explain in ([19], Remark 3.2) our method of proof of the smoothing effect seems to break down for $s < 1/3$. The reason is that the term A_k^1 can neither be controlled for $s < 1/3$ nor be canceled by adding a term of order 7 in the modified energy. Indeed, it is shown in [19] that for any k large enough one can find many couples of triplets $(\vec{k}_{(3)}, \vec{k}_{1(3)})$ such that $\vec{k}_{(3)} \in D^1(k)$, $\vec{k}_{1(3)} \in D^1(k_1)$ and $\Omega_5(\vec{k}_{1(3)}, k_2, k_3, -k) \lesssim 1$. Therefore, a supplementary term in the modified energy will not be useful to treat this term since we would not be able to control this term for $s < 1/3$ and the "nonlinear contribution" of the time derivative of this term would be even worst.

On the other hand, note that even if we only give an estimate of $A^{1,0}$ for $s \geq 1/3$, we could lower the Sobolev index here by adding a supplementary term in the modified energy. This is due to the fact that on the support of $A^{1,0}$ we have $\Omega_5(k_1, -k_1, k_1, k_2, k_3, k_4) = \Omega_3(k_1, k_2, k_3) \gtrsim k$.

The following corollary of Theorem 4.1 will be crucial for the local well-posedness result.

Corollary 4.1. *Assume that $s \geq 1/3$, $0 < T \leq 1$ and $u, v \in Z_T^s$ are two solutions to* (2.1) *defined in the time interval $[0, T]$. Then, for all $k \in \mathbb{N}^*$ such that $|\widehat{u}(0,k)| =$*

$|\widehat{v}(0,k)|$ *and all* $0<s'<s$ *it holds*

(4.31)
$$\sup_{t\in]0,T[} k^{1+s'-s}\Big||\widehat{u}(t,k)|^2-|\widehat{v}(t,k)|^2\Big| \lesssim \|u-v\|_{Z^{s'}_T}(\|u\|_{Z^s_T}+\|v\|_{Z^s_T})^3(1+\|u\|_{Z^s_T}+\|v\|_{Z^s_T})^4$$

where the implicit constant is independent of k.

Proof. Proceeding as in the proof of Proposition 4.9 we obtain (4.24) for u and for v. Taking the difference of these two identities and estimating the right-hand side member as in Proposition 4.9 and estimating the non quadratic terms of the modified energy as in Lemma 4.8, the triangular inequality leads for any $k\in\mathbb{N}^*$ to

$$\begin{aligned}\sup_{t\in]0,T[} k\Big||\widehat{u}(t,k)|^2-|\widehat{v}(t,k)|^2\Big| \quad &\lesssim \quad \sup_{N\geq k}\Big(\frac{k}{N}\Big)^{s-}\|P_{\leq N}(u-v)\|_{Z^s_T}\\ &(\|P_{\leq N}u\|_{Z^s_T}+\|P_{\leq N}v\|_{Z^s_T})^3(1+\|P_{\leq N}u\|_{Z^s_T}+\|P_{\leq N}v\|_{Z^s_T})^4\end{aligned}$$

This last inequality clearly yields (4.31) □

5. Estimates on the difference

We will need the following multilinear estimate of order five.

Proposition 5.1. *Assume that* $0<T\leq 1$, η_1, η_2 *are bounded functions and* u_i *are functions in* $Z^0:=X^{-\frac{11}{10},1}\cap L^\infty_T L^2$. *Assume also that* $N\gg 1$, $M\geq 1$ *and* $j\in\{1,2,3\}$. *We define*

$$G^T_{\eta,M}(u_1,u_2,u_3,u_4):=\int_{]0,T[\times\mathbb{T}}\Pi^j_{\eta,M}(u_1,u_2,u_3)u_4dxdt\,. \tag{5.1}$$

Then

$$\big|G^T_{\eta_1,M}(\Pi^j_{\eta_2,M'}(u_1,u_2,u_3),u_4,u_5,u_6)\big|\lesssim TMM'\prod_{i=1}^6\|u_i\|_{L^\infty_T L^2_x}\,. \tag{5.2}$$

Let also $N_1,N_2,N_3\geq 1$ *be dyadic integers and* $(K_1,K_2)\in(\mathbb{R}^*_+)^2$ *such that* $K_2\gg K_1$. *Then it holds*

$$\begin{aligned}&\big|G^T_{\eta_1 1_{D^1}1_{\Omega_3\sim K_1},M}\Big(\Pi^j_{\eta_2 1_{\Omega_3\geq K_2},M'}(P_{N_1}u_1,P_{N_2}u_2,P_{N_3}u_3),u_4,u_5,P_Nu_6\Big)\big|\\ &\qquad\lesssim\frac{T^{1/8}}{K_2}MM'\max(N_1,N_2,N_3)^{\frac{11}{10}}\prod_{i=1}^4\|u_6\|_{Z^0_T}\,,\end{aligned} \tag{5.3}$$

where D^1 *is defined in* (3.12). *Moreover, the implicit constant in estimate* (3.14) *only depends on the* L^∞*-norm of the function* η.

Proof. (5.2) follows by using twice (3.10). To prove (5.3), we first notice that $K_2\gg K_1$ and (4.7) ensure that $|\Omega_5(\vec{k}_{(5)})|\sim|\Omega_3(k_1,k_2,k_3)|\geq K_2$. Then the result follows by proceeding as in the proof of (4.13) with the help of (5.2) and by taking $R=\max(N_1,N_2,N_3)^{\frac{11}{10}}/\max(K_2,\max(N_1,N_2,N_3))$. Note that $|\Omega_3(k_1,k_2,k_3)|\gtrsim\max(N_1,N_2,N_3)$ ensures that $|\Omega_5(\vec{k}_{(5)})|\gg R$. □

5.1. **Definition of the modified energy for the difference.** Let $N_0 \geq 2^9$, N be a nonhomogeneous dyadic number and $(u,v) \in (H^s(\mathbb{T}))^2$ with $s \in \mathbb{R}$. We define the modified energy of the difference at the dyadic frequency N by

$$\mathcal{E}_N[u,v,N_0] = \begin{cases} \frac{1}{2}\|P_N(u-v)\|^2_{L^2_x} & \text{for } N \leq N_0 \\ \frac{1}{2}\|P_N(u-v)\|^2_{L^2_x} + \mathcal{E}^3_N[u,v] & \text{for } N > N_0\,, \end{cases} \tag{5.4}$$

where

$$\begin{aligned} \mathcal{E}^3_N[u,v] \;=\; & \sum_{k\in\mathbb{Z}} \sum_{\substack{\vec{k}_{(3)}\in\mathcal{A} \\ k_1+k_2+k_3=k}} \frac{k}{\Omega(\vec{k}_{(3)})}\varphi^2_N(k) \\ & \Re\Big[\Big(\hat{u}(k_1)\hat{u}(k_2)+\hat{u}(k_1)\hat{v}(k_2)+\hat{v}(k_1)\hat{v}(k_2)\Big)(\hat{u}-\hat{v})(k_3)(\hat{u}-\hat{v})(-k)\Big] \end{aligned}$$

where $\vec{k}_{(3)} = (k_1,k_2,k_3)$. The modified energy $E^{s'}[u,v,N_0]$ of the difference $u-v$ is defined by

$$E^{s'}[u,v,N_0] = \sum_{N\geq 1} N^{2s'}\mathcal{E}_N[u,v,N_0]\,.$$

The following lemma ensures that $E^{s'}[u,v,N_0]$ is well-defined as soon as $(u,v) \in (H^s(\mathbb{T}))^2$ with $s>0$. Moreover, for $N_0 > 2^9$ large enough we have $E^{s'}[u,v,N_0] \sim \|u-v\|^2_{H^{s'}}$.

Lemma 5.2. *Let $(u,v) \in (H^s(\mathbb{T}))^2$ with $s>0$.Then, for any $s' \in \mathbb{R}$ and any $N_0 \gg (\|u\|_{H^s}+\|v\|_{H^s})^{1/s}$, it holds*

$$\frac{1}{2}\|u-v\|^2_{H^{s'}} \leq E^{s'}[u,v,N_0] \leq 2\|u-v\|^2_{H^{s'}} \tag{5.5}$$

Proof. Let us recall that on D^1, it holds $|k_1| \sim |k_2| \sim |k_3| \sim |k|$. Therefore, a direct application of (3.10) leads to

$$N^{2s'}|\mathcal{E}^3_N[u,v]| \lesssim \sum_{1\leq M_{min}\lesssim N} \frac{N^{2s'}NM_{min}}{M^2_{min}N} N^{-2s'}N^{-2s}(\|P_{\sim N}u\|^2_{H^s}+\|P_{\sim N}v\|^2_{H^s})\|P_{\sim N}(u-v)\|^2_{H^{s'}}\,.$$

Summing over M_{min} and $N \geq N_0$, we obtain

$$\sum_{N\geq N_0} N^{2s'}|\mathcal{E}^3_N[u,v]| \lesssim N_0^{-2s}(\|u\|^2_{H^s}+\|v\|^2_{H^s})\|u-v\|^2_{H^{s'}}\,.$$

that clearly implies (5.5) for $N_0 \gg (\|u\|_{H^s}+\|v\|_{H^s})^{1/s}$. □

Let now (u,v) be a couple of solutions to the renormalized mKdV equation on $]0,T[$. The following proposition enables to control $E^{s'}[u,v,N_0]$ on $]0,T[$.

Proposition 5.3. *Let $0<T<1$. Let u and v be two solutions of the renormalized mKdV (2.1) belonging to $L^\infty(0,T;H^s)$ with $s \geq 1/3$ and associated with the same initial data $u_0 \in H^s(\mathbb{T})$. Then for $s/2 < s' < s - \frac{1}{10}$ and any $N_0 \gg 1$ it holds*

$$\sup_{t\in[0,T]} E^{s'}[u(t),v(t),N_0] \lesssim T^{1/8}N_0^{3/2}(1+\|u\|_{Z^s_T}+\|v\|_{Z^s_T})^8\|w\|^2_{Z^{s'}_T} \tag{5.6}$$

Proof. To simplify the notation, we denote $\mathcal{E}_N[u(t),v(t),N_0]$ simply by $\mathcal{E}_N(t)$. Note that $u(t)$ and $v(t)$ are well defined for any $t \in [0,T]$ since, by the equation, $(u,v) \in$

$(C([0,T];H^{s-3}))^2$ and that, for any $N\in 2^{\mathbb{N}}$, $E_N(0)=0$ since $u(0)=v(0)=u_0$. For $N\le N_0$, the definition of $\mathcal{E}_N(t)$ easily leads to

$$\frac{d}{dt}\mathcal{E}_N(t)=c\int_{\mathbb{T}}[P_N(u^2-v^2)P_Nu_x+P_N(P_0(v^2)-P_0(u^2))P_Nu_x+P_N(u^2-P_0(u^2))P_Nw_x]$$

which yields after applying Bernstein inequalities, integrating on $]0,t[$ and summing over $N\le N_0$,

$$\sum_{N\le N_0}\mathcal{E}_N(t)\lesssim N_0^{3/2}(\|u\|^2_{L^\infty_TL^2}+\|v\|^2_{L^\infty_TL^2})\|w\|^2_{L^\infty_TL^2}$$

Now for $N>N_0$, we first notice that the difference $w=u-v$ satisfies

$$\begin{aligned} w_t+\partial_x^3w &= -\partial_xA(u,u,w)-\partial_xA(u,v,w)-\partial_xA(v,v,w)\\ &\quad -\partial_x\Big(B(u,u,w)+(B(u,u,v)-B(v,v,v))\Big)\,, \end{aligned}\tag{5.7}$$

where A and B are defined in (3.36). Therefore, differentiating $\mathcal{E}_N$ with respect to time and integrating between 0 and t we get

$$\begin{aligned} N^{2s'}\mathcal{E}_N(t) &= N^{2s'}\mathcal{E}_N(0)-N^{2s'}\int_0^t\Re\Big(\int_{\mathbb{T}}\partial_xP_N[A(u,u,w)+A(u,v,w)+A(v,v,w)]P_Nw\Big)\\ &\quad -N^{2s'}\int_0^t\Re\Big(\int_{\mathbb{T}}\partial_xP_NB(u,u,w)P_Nw\Big)\\ &\quad -N^{2s'}\int_0^t\Re\Big(\int_{\mathbb{T}}\partial_xP_N(B(u,u,v)-B(v,v,v))P_Nw\Big)\\ &\quad +N^{2s'}\int_0^t\Re\frac{d}{dt}\mathcal{E}_N^3(\tau)d\tau\\ &= C_N(t)+D_N(t)+F_N(t)+G_N(t)\,. \end{aligned}$$

As in (3.37) we notice that, since u and v are real-valued,

$$\int_{\mathbb{T}}\partial_xP_NB(u,u,w)P_Nw=ik\sum_{k\in\mathbb{Z}}|\hat{u}(k)|^2|\varphi_N(k)\hat{w}(k)|^2\in i\mathbb{R}\,.$$

and thus $D_N(t)=0$. On the other hand, the smoothing effect (4.31) leads to

$$\begin{aligned} |F_N(t)| &\lesssim N^{2s'}\Big|\int_0^t k\sum_{k\in\mathbb{Z}}(|\hat{u}(\tau,k)|^2-|\hat{v}(\tau,k)|^2)\varphi_N(k)^2\hat{v}(\tau,k)\hat{w}(\tau,-k)\Big|\\ &\lesssim \sup_{\tau\in[0,T]}\sup_{\substack{k\in\mathbb{Z}\\|k|\sim N}}\Big(|k|^{1+(s'-s)}\Big||\hat{u}(t,k)|^2-|\hat{v}(\tau,k)|^2\Big|\Big)\int_0^t\|v_N(\theta)\|_{H^s}\|w_N(\theta\|_{H^{s'}}\,d\theta\\ &\lesssim \delta_N\,T(1+\|u\|^7_{Z^s}+\|v\|^7_{Z^s})\|w\|_{Z^{s'}_T}\|v\|_{L^\infty_TH^s}\|w\|_{L^\infty_TH^{s'}} \end{aligned}$$

with $\|(\delta_{2^j})_j\|_{l^1(\mathbb{N})}\lesssim 1$. It thus remains to control $C_N(t)$. We notice that C_N can be decomposed as

$$\begin{aligned} C_N &= N^{2s'}\sum_{k\in\mathbb{Z}}(\sum_{k_{(3)}\in D^1(k)}+\sum_{k_{(3)}\in D^2(k)})k\varphi_N^2(k)\\ &\quad \Im\Big[\Big(\hat{u}(k_1)\hat{u}(k_2)+\hat{u}(k_1)\hat{v}(k_2)+\hat{v}(k_1)\hat{v}(k_2)\Big)\hat{w}(k_3)\hat{w}(-k)\Big]\\ &= C_N^{low}+C_N^{high}\,. \end{aligned}$$

Estimate on C_N^{high}. We notice that C_N^{high} is composed of three terms of the form

$$N^{2s'}\sum_{k\in\mathbb{Z}}\sum_{N_1,N_2,N_3}\sum_{k_{(3)}\in D^2(k)} k\varphi_N^2(k)\varphi_{N_1}(k_1)\varphi_{N_2}(k_2)\varphi_{N_3}(k_3)\Im\Big[\Big(\hat{z}_1(k_1)\hat{z}_2(k_2)\hat{w}(k_3)\hat{w}(-k)\Big]$$

with $z_i\in Z^s_{T,}$. We separate different contributions of the sum over N_1, N_2, N_3. By symmetry we can assume that $N_1\geq N_2$.
1. $N_1\geq N_3$**.** Then, (3.15) leads to

$$\begin{aligned}|C_N^{high}(t)| &\lesssim \sum_{N_1\gtrsim N}\sum_{1\leq M\lesssim N_2\vee N_3\lesssim N_1} T^{1/8}\frac{N^{2s'}N}{N^{\frac{9}{10}}}N_1^{-s}N^{-s'}N_2^{-s}N_3^{-s'}\\ &\quad \|P_{N_1}z_1\|_{Z^s}\|P_{N_2}z_2\|_{Z^s}\|P_{N_3}w\|_{Z^{s'}}\|P_Nw\|_{Z^{s'}}\\ &\lesssim T^{1/8}N^{s'-s+\frac{1}{10}}\|z_1\|_{Z^s}\|z_2\|_{Z^s}\|w\|^2_{Z^{s'}}\end{aligned}$$

which is acceptable for $s>s'+\frac{1}{10}$.
2. $N_1<N_3$**.** By symmetry, we assume that $N_1\geq N_2$.
2.1 $N_1\geq N^{1/2}$ Then (3.15) yields

$$\begin{aligned}|C_N^{high}| &\lesssim \sum_{\substack{N_3\geq N_1\geq N_2\\ N_3\gtrsim N,\ N_1\geq N^{1/2}}}\sum_{1\leq M\lesssim N_1} T^{1/8}\frac{N^{2s'}N}{N^{\frac{9}{10}}}N_3^{-s'}N^{-s'}N_1^{-s}N_2^{-s}\\ &\quad \|P_{N_1}z_1\|_{Z^s}\|P_{N_2}z_2\|_{Z^s}\|P_{N_3}w\|_{Z^{s'}}\|P_Nw\|_{Z^{s'}}\\ &\lesssim T^{1/8}N^{s/2-\frac{1}{10}}\|z_1\|_{Z^s}\|z_2\|_{Z^s}\|w\|^2_{Z^{s'}}\end{aligned}$$

which is acceptable for $s>1/5$.
2.2 $N_1<N^{1/2}$ Then $N_3\sim N$ and $M_{min}=M_3\lesssim N^{1/2}$. Then, according to (3.7) we have

$$C_N^{high}=\sum_{\substack{N_3\geq N_1\geq N_2\\ N_3\sim N,\ N_1<N^{1/2}}}\sum_{1\leq M\lesssim N^{1/2}} MN^{2s'}\int_{\mathbb{T}}\Pi^3_{\eta_3,M}(P_{N_1}z_1,P_{N_2}z_2,P_{N_3}w)P_Nw\,dx,$$

where η_3 is a function of (ξ_1,ξ_2,ξ_3) whose $L^\infty-$norm is uniformly bounded in N and M. Therefore, by (3.15), we get

$$\begin{aligned}|C_N^{high}| &\lesssim \sum_{\substack{N_3\geq N_1\geq N_2\\ N_3\sim N,\ N_1<N^{1/2}}}\sum_{1\leq M\lesssim N^{1/2}} T^{1/8}\frac{N^{2s'}M}{N^{\frac{9}{10}}}N^{-2s'}N_1^{-s}N_2^{-s}\\ &\quad \|P_{N_1}z_1\|_{Z^s}\|P_{N_2}z_2\|_{Z^s}\|P_{N_3}w\|_{Z^{s'}}\|P_Nw\|_{Z^{s'}}\\ &\lesssim T^{1/8}N^{-\frac{2}{5}}\|z_1\|_{Z^s}\|z_2\|_{Z^s}\|w\|^2_{Z^{s'}}\end{aligned}$$

which is acceptable.
Estimate on $C_N^{low}+G_N$. We have

$$
\begin{aligned}
G_N(t) &= N^{2s'}\frac{d}{dt}\mathcal{E}_N^3(t)\\
&= -N^{2s'}\sum_{k\in\mathbb{Z}}\sum_{k_{(3)}\in D^1(k)} k\varphi_N^2(k)\Im\Big[\Big(\hat u(k_1)\hat u(k_2)+\hat u(k_1)\hat v(k_2)+\hat v(k_1)\hat v(k_2)\Big)\hat w(k_3)\hat w(-k)\Big]\\
&\quad +N^{2s'}\sum_{k\in\mathbb{Z}}\sum_{k_{(3)}\in D^1(k)}\sum_{i=1}^{3}\frac{k\varphi_N^2(k)k_1}{\Omega^3(\vec k_{(3)})}\Im\Big[\hat z_{2,i}(k_2)\hat w(k_3)\hat w(-k)\\
&\quad \Big(3|\widehat z_{1,i}(k_1)|^2\widehat z_{1,i}(k_1)+\sum_{\vec k_{1(3)}\in D(k_1)}\prod_{q=1}^{3}\widehat z_{1,i}(k_{1,q})\Big)\Big]\\
&\quad +N^{2s'}\sum_{k\in\mathbb{Z}}\sum_{k_{(3)}\in D^1(k)}\sum_{i=1}^{3}\frac{k\varphi_N^2(k)k_3}{\Omega^3(\vec k_{(3)})}\Im\Big[\hat w(-k)\Big(\sum_{i=1}^{3}\widehat z_{1,i}(k_1)\widehat z_{2,i}(k_2)\Big)\\
&\quad \Big((|\widehat u(k_3)|^2+|\widehat v(k_3)|^2)\widehat w(k_3)+\widehat w(-k_3)\widehat u(k_3)\widehat v(k_3)\\
&\qquad +\sum_{\vec k_{3(3)}\in D(k_3)}\sum_{i=1}^{3}[\widehat u(k_{3,1}\widehat u(k_{3,2})+\widehat u(k_{3,1})\widehat v(k_{3,2})+\widehat v(k_{3,1})\widehat v(k_{3,2})]\widehat w(k_{3,3})\Big)\Big]\\
&\quad +N^{2s'}\sum_{k\in\mathbb{Z}}\sum_{k_{(3)}\in D^1(k)}\sum_{i=1}^{3}\frac{k^2\varphi_N^2(k)}{\Omega^3(\vec k_{(3)})}\Im\Big[\hat w(k_3)\Big(\sum_{i=1}^{3}\widehat z_{1,i}(k_1)\widehat z_{2,i}(k_2)\Big)\\
&\quad \Big((|\widehat u(k)|^2+|\widehat v(k)|^2)\widehat w(-k)+\widehat w(k)\widehat u(-k)\widehat v(-k)\\
&\qquad +\sum_{\vec k_{4(3)}\in D(-k)}\sum_{i=1}^{3}[\widehat u(k_{4,1}\widehat u(k_{4,2})+\widehat u(k_{4,1})\widehat v(k_{4,2})+\widehat v(k_{4,1})\widehat v(k_{4,2})]\widehat w(k_{4,3})\Big)\Big]\\
&= -C_N^{low}+A_1+A_2+A_3\,,
\end{aligned}
$$

where we set $(z_{1,1},z_{2,1})=(u,u)$, $(z_{1,2},z_{2,2})=(u,v)$ and $(z_{1,3},z_{2,3})=(v,v)$. Hence,

$$
C_N^{low}+G_N=\sum_{j=1}^{3}A_j\,.
$$

For any sextuples $\vec N=(N_{1,1},N_{1,2},N_{1,3},N_2,N_3,N)\in(2^{\mathbb{N}})^6$ and any $\vec z=(z_{1,1},z_{1,2},z_{1,3},z_2,z_3,z_4)\in(Z^s)^6$, we set

$$
\begin{aligned}
R_{\vec N,\vec z} &= N^{2s'}\sum_{k\in\mathbb{Z}}\sum_{k_{(3)}\in D^1(k)}\frac{k\varphi_N(k)k_1}{\Omega^3(\vec k_{(3)})}\Big|P_{N_2}\widehat z_2(k_2)P_{N_3}\widehat z_3(k_3)P_N\widehat z_4(-k)\\
&\qquad P_{N_{1,i}}\widehat z_{1,1}(k_1)P_{N_{1,2}}\widehat z_{1,2}(-k_1)P_{N_{1,3}}\widehat z_{1,3}(k_1)\Big|
\end{aligned}
$$

and

$$S_{\vec{N},\vec{z}} = N^{2s'} \sum_{k\in\mathbb{Z}} \sum_{k_{(3)}\in D^1(k)} \frac{k\varphi_N(k)k_1}{\Omega^3(\vec{k}_{(3)})} \Big| P_{N_2}\widehat{z}_2(k_2)P_{N_3}\widehat{z}_3(k_3)P_N\widehat{z}_4(-k) \sum_{\vec{k_1}_{(3)}\in D(k_1)} P_{N_{1,i}}\widehat{z}_{1,1}(k_{1,1})P_{N_{1,2}}\widehat{z}_{1,2}(k_{1,2})P_{N_{1,3}}\widehat{z}_{1,3}(k_{1,3})\Big|$$

We observe that to get the desired estimates on the A_j it suffices to prove that for any sextuples $\vec{N} = (N_{1,1}, N_{1,2}, N_{1,3}, N_2, N_3, N) \in (2^{\mathbb{N}})^6$ and any $\vec{z} = (z_{1,1}, z_{1,2}, z_{1,3}, z_2, z_3, z_4) \in (Z^s)^6$,

$$R_{\vec{N},\vec{z}} + S_{\vec{N},\vec{z}} \lesssim T^{1/8}\widetilde{N}_{max}^{-2(s-s')+}\|P_N z_4\|_{Z^s}\prod_{i=2}^{3}\|P_{N_i}z_i\|_{Z^s}\prod_{j=1}^{3}\|P_{N_{1,j}}z_{1,j}\|_{Z^s} \tag{5.8}$$

where $\widetilde{N}_{max} = \max(N_{1,1}, N_{1,2}, N_{1,3}, N_2, N_3, N)$.
Indeed, the modulus of the $A_j$are controlled by sums of terms of this form where w appears two times in the components of $\vec{z}$ and all the other components are u or v. Therefore (5.8) leads to

$$R_{\vec{N},\vec{z}} + S_{\vec{N},\vec{z}} \lesssim T^{1/8}\widetilde{N}_{max}^{0-}\|P_{\lesssim\widetilde{N}_{max}}w\|_{Z^{s'}}^2\Big(\|P_{\lesssim\widetilde{N}_{max}}u\|_{Z^s} + \|P_{\lesssim\widetilde{N}_{max}}v\|_{Z^s}\Big)^4 \tag{5.9}$$

and (5.6) then follows by summing over $(N_{1,1}, N_{1,2}, N_{1,3}, N_2, N_3, N)$ thanks to the factor $\widetilde{N}_{max}^{0-}$.

To simplify the notation, we denote $P_{N_i}z_i$, $P_{N_{1,j}}z_{1,j}$ and $P_N z_4$ by respectively z_i and $z_{1,j}$ and z_4 in the sequel.

Estimate on $R_{\vec{N},\vec{z}}$: We recall that on D^1 we must have $|k_1| \sim |k_2| \sim |k_3| \sim |k|$ and thus $R_{\vec{N},\vec{z}}$ vanishes except if $N_{1,1} \sim N_{1,2} \sim N_{1,3} \sim N_2 \sim N_3 \sim N$. In particular, $N_{max} \sim N$. (3.10) then leads to

$$\begin{aligned} R_{\vec{N},\vec{z}} &\lesssim \sum_{M\geq 1} T\frac{N^{2s'}N^2M}{M^2N}N^{-6s}\prod_{i=2}^{4}\|z_i\|_{L^\infty_T H^s}\prod_{j=1}^{3}\|z_{1,j}\|_{L^\infty_T H^s} \\ &\lesssim T\widetilde{N}_{max}^{-2(s-s')}\widetilde{N}_{max}^{1-4s}\prod_{i=2}^{4}\|z_i\|_{L^\infty_T H^s}\prod_{j=1}^{3}\|z_{1,j}\|_{L^\infty_T H^s} \end{aligned}$$

which is acceptable since for $s \geq 1/3$ we have $1 - 4s < 0$.

Estimate on $S_{\vec{N},\vec{z}}$: We set $k_{1(3)} = (k_{1,1}, k_{1,2}, k_{1,3})$. By symmetry, we may assume that $N_{1,1} \geq N_{1,2} \geq N_{1,3}$. On $\mathcal{A}$ we must have $N_{1,1} \sim \widetilde{N}_{max}$. We separate different contributions.

1. $M_{1,med} \geq 2^{-9}N$**.**

1.1. $\Omega_3(k_{1(3)}) \gg \Omega_3(k_{(3)})$**.** Noticing that $\Omega_3(k_{(3)}) \sim M_{min}M_{med}N$ and $\Omega_3(k_{1(3)}) \gtrsim 2^{-9}M_{1,min}NN_{1,1}$ in this region, (5.3) leads to

$$\begin{aligned} S_{\vec{N},\vec{z}} &\lesssim \sum_{1\leq M_{1,min}\lesssim N_{1,2}} \sum_{M_{med}\gtrsim M_{min}\geq 1} T^{1/8}\frac{N^{2s'}N^2M_{min}}{M_{min}M_{med}N^2}\widetilde{N}_{max}^{\frac{1}{10}}\widetilde{N}_{max}^{-2s}N^{-2s}\prod_{i=2}^{4}\|z_i\|_{Z^s}\prod_{j=1}^{3}\|z_{1,j}\|_{Z^s} \\ &\lesssim T^{1/8}N_{max}^{-2(s-s')}\widetilde{N}_{max}^{-2s'+\frac{1}{10}+}N^{2(s'-s)}\prod_{i=2}^{4}\|z_i\|_{Z^s}\prod_{j=1}^{3}\|z_{1,j}\|_{Z^s} \end{aligned}$$

which is acceptable since for $s' > s/2$ and $s \geq 1/3$ force $s' > 1/6$.
1.2 $|\Omega_3(k_{1(3)})| \lesssim |\Omega_3(k_{(3)})|$**.**
1.2.1 $M_{med} \geq N_{1,1}^{\frac{2}{3}}$**.** Then (3.10) and Sobolev embedding theorems lead, for $s < 1/2$, to

$$\begin{aligned}
S_{\vec{N},\vec{z}} &\lesssim T \sum_{1\leq M_{min}\lesssim N} \sum_{M_{med}\geq N_{1,1}^{\frac{2}{3}}} \frac{N^{2s'}N^2 M_{min}}{M_{min}M_{med}N} N^{-3s} \prod_{i=2}^{4} \|z_i\|_{L^\infty_T H^s} \|A(z_{1,1},z_{1,2},z_{1,3})\|_{L^\infty_T L^2} \\
&\lesssim TN^{2s'+1+} N_{1,1}^{-\frac{2}{3}} N^{-3s} \prod_{i=2}^{4} \|z_i\|_{L^\infty_T H^s} \|z_{1,1}\|_{L^\infty_T L^2} \|z_{1,2}\|_{L^\infty_T L^\infty} \|z_{1,3}\|_{L^\infty_T L^\infty} \\
&\lesssim TN^{2s'+1+} N_{1,1}^{-\frac{2}{3}} N^{-3s} N_{1,1}^{-3s+1} \prod_{i=2}^{4} \|z_i\|_{L^\infty_T H^s} \prod_{j=1}^{3} \|z_{1,j}\|_{L^\infty_T H^s} \\
&\lesssim T\widetilde{N}_{max}^{-2(s-s')} \widetilde{N}_{max}^{\frac{1}{3}-s-s/2'} N^{-3s-s'/2+1+} \prod_{i=2}^{4} \|z_i\|_{L^\infty_T H^s} \prod_{j=1}^{3} \|z_{1,j}\|_{L^\infty_T H^s}
\end{aligned}$$

which is acceptable for $s \geq 1/3$ and $s' > s/2 > 0$.
1.2.2 $M_{med} \leq N_{1,1}^{\frac{2}{3}}$**.** Then $|\Omega_3(k_{1(3)})| \lesssim |\Omega_3(k_{(3)})|$ forces

$$M_{1,min} \lesssim \frac{M_{min}M_{med}N}{M_{1,med}N_{1,1}} \lesssim \frac{M_{min}}{N_{1,1}^{\frac{1}{3}}} \ .$$

1.2.2.1 $N_{1,1} \sim N_{1,2} \gg N_{1,3}$**.** Then (5.2) leads to

$$\begin{aligned}
S_{\vec{N},\vec{z}} &\lesssim T \sum_{1\leq M_{min}\leq N} \frac{N^{2s'}N^2 M_{min}^2}{M_{min}^2 N N_{1,1}^{\frac{1}{3}}} N_{1,1}^{-2s} N^{-3s} \prod_{i=2}^{4} \|z_i\|_{L^\infty_T H^s} \prod_{j=1}^{3} \|z_{1,j}\|_{L^\infty_T H^s} \\
&\lesssim T\widetilde{N}_{max}^{-2(s-s')} \widetilde{N}_{max}^{0-} N^{-3s+\frac{2}{3}+} \prod_{i=2}^{4} \|z_i\|_{L^\infty_T H^s} \prod_{j=1}^{3} \|z_{1,j}\|_{L^\infty_T H^s}
\end{aligned}$$

which is acceptable for $s > 2/9$.
1.2.2.2 $N_{1,1} \gg N_{1,2} \geq N_{1,3}$**.** Then $M_{1,med} \sim N_{1,1}$ and thus $M_{1,min} \lesssim \frac{M_{min}N}{N_{1,1}^{\frac{4}{3}}}$.
Therefore (5.2) leads to

$$\begin{aligned}
S_{\vec{N},\vec{z}} &\lesssim T \sum_{1\leq M_{min}\leq N} \frac{N^{2s'}N^2 M_{min}^2}{M_{min}^2 N N_{1,1}^{\frac{4}{3}}} N_{1,1}^{-s} N^{-3s} \prod_{i=2}^{4} \|z_i\|_{L^\infty_T H^s} \prod_{j=1}^{3} \|z_{1,j}\|_{L^\infty_T H^s} \\
&\lesssim T\widetilde{N}_{max}^{-2(s-s')} \widetilde{N}_{max}^{s-2s'-\frac{1}{3}} N^{-3s+2s'+0+} \prod_{i=2}^{4} \|z_i\|_{L^\infty_T H^s} \prod_{j=1}^{3} \|z_{1,j}\|_{L^\infty_T H^s}
\end{aligned}$$

which is acceptable for $0 < s' < s < 2s'$.
2. $M_{1,med} < 2^{-9}N$. Then $N_{1,1} \sim N_{1,2} \sim N_{1,3} \sim N_2 \sim N_3 \sim N$.

2.1 $M_{1,min} \leq 2^9 M_{med}$. Then (5.2) leads to

$$\begin{aligned} S_{\vec{N},\vec{z}} &\lesssim T \sum_{1\leq M_{min}\leq M_{med}\lesssim N} \frac{N^{2s'}N^2 M_{min}M_{med}}{M_{min}M_{med}N} N^{-6s} \prod_{i=2}^{4} \|z_i\|_{L^\infty_T H^s} \prod_{j=1}^{3} \|z_{1,j}\|_{L^\infty_T H^s} \\ &\lesssim T\widetilde{N}_{max}^{-2(s-s')} \widetilde{N}_{max}^{-4s+1+} \prod_{i=2}^{4} \|z_i\|_{L^\infty_T H^s} \prod_{j=1}^{3} \|z_{1,j}\|_{L^\infty_T H^s} \ . \end{aligned}$$

which acceptable for $s > 1/4$.
2.2 $M_{1,min} > 2^9 M_{med}$. Then

$$\Omega_3(k_{1(3)}) = M_{1,min}M_{1,med}M_{1,max} \gtrsim 2^{18} M_{min}M_{med}N \gg \Omega_3(k_{(3)})$$

and thus (5.3) leads to

$$\begin{aligned} S_{\vec{N},\vec{z}} &\lesssim T^{1/8} \sum_{1\leq M_{min}\leq M_{med}\lesssim N} \sum_{1\leq M_{1,med}\lesssim N} \frac{N^{2s'}N^2 M_{min}}{M_{min}M_{med}N\, M_{1,med}} N^{-6s} N^{\frac{1}{10}} \\ &\qquad \prod_{i=2}^{4} \|z_i\|_{Z^s} \prod_{j=1}^{3} \|z_{1,j}\|_{Z^s} \\ &\lesssim T^{1/8} \widetilde{N}_{max}^{-2(s-s')} \widetilde{N}_{max}^{-4s+\frac{11}{10}} \prod_{i=2}^{4} \|z_i\|_{Z^s} \prod_{j=1}^{3} \|z_{1,j}\|_{Z^s} \end{aligned}$$

which is acceptable for $s > \frac{11}{40}$. □

Lemma 5.4. *Assume that* $0 < T \leq 1$, $s \geq 1/3$ *and* $(u,v) \in (L^\infty(0;H^s(\mathbb{T})))^2$ *are two solution to* (2.1) *associated to the initial data* $(u_0,v_0) \in H^s(\mathbb{T})^2$. *Then, setting* $s' = \frac{1}{3} - \frac{1}{8} = \frac{5}{24}$, *it holds*

$$\|u-v\|_{Z^{s'}_T} \lesssim \|u_0 - v_0\|_{H^{s'}} + (1 + \|u\|^2_{L^\infty_T H^s} + \|v\|^2_{L^\infty_T H^s})^3 \|u-v\|_{L^\infty_T H^{s'}} \tag{5.10}$$

Proof. We proceed as in Lemma 3.11 so that we are reduced to estimate $\|u - v\|_{X_T^{s'-\frac{11}{10},1}}$. Setting $w = u - v$, the Duhamel formula associated to (1.1), the standard linear estimates in Bourgain's spaces and the fractional Leibniz rule (*c.f.* Theorem A.12 in [10]) lead to

(5.11)

$$\begin{aligned} \|w\|_{X_T^{s'-\frac{11}{10},1}} &\lesssim \|u_0 - v_0\|_{H^{s'}} + \|\partial_x(w(u^2+uv+v^2))\|_{X_T^{s'-\frac{11}{10},0}} + \|P_0(u^2)w_x\|_{X_T^{s'-\frac{11}{10},0}} \\ &\quad + \|P_0(u^2-v^2)v_x\|_{X_T^{s'-\frac{11}{10},0}} \\ &\lesssim \|u_0 - v_0\|_{H^{s'}} + \|J_x^{s'-\frac{1}{10}}(w(u^2+uv+v^2))\|_{L^2_T L^2_x} + \|u\|^2_{L^\infty_T L^2_x} \|w\|_{L^2_T H_x^{s'-\frac{1}{10}}} \\ &\quad + (\|u\|_{L^\infty_T L^2_x} + \|v\|_{L^\infty_T L^2_x}) \|v\|_{L^2_T H_x^{s'-\frac{1}{10}}} \|w\|_{L^\infty_T L^2_x} \ . \end{aligned}$$

Then, we notice that

$$\begin{aligned} \|J_x^{s'-\frac{1}{10}}(w(u^2+uv+v^2))\|_{L^2_T L^2_x} &\lesssim \|J_x^{s'}(w(u^2+uv+v^2))\|_{L^2_T L_x^{\frac{5}{3}}} \\ &\lesssim (\|u\|^2_{L^4_T L^{20}_x} + \|v\|^2_{L^4_T L^{20}_x}) \|J_x^{s'} w\|_{L^\infty_T L^2_x} \\ &\quad + (\|u\|_{L^4_T L^{20}_x} + \|v\|_{L^4_T L^{20}_x}) \|w\|_{L^\infty_T L_x^{\frac{24}{7}}} (\|J_x^{\frac{5}{24}} u\|_{L^4_T L_x^{\frac{120}{31}}} + \|J_x^{\frac{5}{24}} v\|_{L^4_T L_x^{\frac{120}{31}}}) \end{aligned}$$

which leads to (5.10) thanks to (3.25)-(3.26) and Sobolev inequalities since $H^{\frac{5}{24}}(\mathbb{T}) \hookrightarrow L^{\frac{24}{7}}(\mathbb{T})$ and for $s \geq 1/3$, it holds $s' = s - 1/8 \geq \frac{5}{24}$ and $\frac{120}{31} < 4$. □

6. Proof of Theorem 2.4

6.1. **Unconditional uniqueness for the renormalized mKdV equation.** Let us start by proving the unconditional uniqueness of (2.1). Let $T > 0$ and $(v_1, v_2) \in L^\infty(0,T;H^{1/3})^2$ be a couple of functions that satisfies (2.1) in the distributional sense with $v_1(0) = v_2(0) = u_0 \in H^s(\mathbb{T})$. We first notice that Lemma 3.11 ensures that $(u,v) \in Z^{1/3}_{\tilde{T}}$ with $\tilde{T} = \min(1,T)$ and, from Proposition 3.12, we infer that

$$\|v_1\|_{Z^{1/3}_{\tilde{T}}} + \|v_2\|_{Z^{1/3}_{\tilde{T}}} \lesssim \|u_0\|_{H^{1/3}} + \tilde{T}^{1/8}\Big(1 + \|v_1\|^3_{L^\infty_T H^{1/3}} + \|v_2\|^3_{L^\infty_T H^{1/3}}\Big)^4 .$$

Hence, taking $\tilde{T} \leq \min(1, T, (1+\|u_0\|_{H^{1/3}})^{-16})$, we get

$$\|v_1\|_{Z^{1/3}_{\tilde{T}}} + \|v_2\|_{Z^{1/3}_{\tilde{T}}} \lesssim \|u_0\|_{H^{1/3}} .$$

Then, noticing that $\frac{1}{6} < \frac{5}{24} < \frac{1}{3} - \frac{1}{9}$, (5.5)-(5.6) and (5.10) lead to

$$\|v_1 - v_2\|^2_{L^\infty_T H^{\frac{5}{24}}} \lesssim T^{1/8} N_0^{3/2} (1 + \|u_0\|_{H^{1/3}})^{25} \|v_1 - v_2\|^2_{L^\infty_T H^{\frac{5}{24}}}$$

with $N_0 \gg \|u_0\|^{-3}_{H^{1/3}}$. This forces

$$\|v_1 - v_2\|_{L^\infty_{T'} H^{\frac{5}{24}}} = 0$$

with $T' \sim \min(\tilde{T}, (1+\|u_0\|_{H^{1/3}})^{-300})$. Hence $v_1 = v_2$ *a.e.* on $[0,T']$. Therefore there exists $t_1 \in [T'/2, T']$ such that $v_1(t_1) = v_2(t_1)$ and $\|v_1(t_1)\|_{H^{1/3}} \leq \|v_1\|_{L^\infty_T H^{1/3}}$. Using this bound we can repeat this argument a finite number of times to extend the uniqueness result on $[0,T]$.

6.2. **Local well-posedness of the renormalized mKdV equation.** We fix $1/3 \leq s < 1/2$. It is known from the classical well-posedness theory that such an initial data $u_0 \in H^\infty(\mathbb{T})$ gives rise to a global solution $u \in C(\mathbb{R}; H^\infty(\mathbb{T}))$ to the Cauchy problem (1.1). Then combining Lemma 3.11 and Proposition 3.12 we infer that u verifies

$$\|u\|^2_{L^\infty_T H^s} \lesssim \|u_0\|^2_{H^s} + T^{1/8}\Big(\|u\|_{L^\infty_T H^s} + \|u\|^3_{L^\infty_T H^s}\Big)^4 \tag{6.1}$$

for any $0 < T < 1$. Taking $T = T(\|u_0\|_{H^s}) \sim \min(1, (1+\|u_0\|_{H^s})^{-10})$, the continuity of $T \mapsto \|u\|_{L^\infty_T H^s}$ ensures that

$$\|u\|_{L^\infty_T H^s} \lesssim \|u_0\|_{H^s}$$

and Lemma 3.11 then leads to

$$\|u\|_{Z^s_T} \lesssim \|u_0\|_{H^s}(1 + \|u_0\|^2_{H^s}) . \tag{6.2}$$

Moreover, we infer from Theorem 4.1 that for any $K \in 2^{\mathbb{N}}$ it holds

$$\begin{aligned} \|P_{\geq K} u(t)\|^2_{L^\infty_T H^s} &\leq \sum_{k \geq K} \sup_{t \in [0]} |k|^{2s} |\widehat{u}(t,k)|^2 \\ &\lesssim \sum_{k \geq K} \Big(|k|^{2s} |\widehat{u_0}(k)|^2 + k^{2s-1} \|u\|^4_{Z^s} (1 + \|u\|_{Z^s})^4 \Big) \\ &\leq \|P_{\geq K} u_0\|^2_{H^s} + K^{2s-1}(1 + \|u_0\|_{H^s})^{24} . \end{aligned} \tag{6.3}$$

Now let $u_0 \in H^s(\mathbb{T})$. We set $u_{0,n} = P_{\leq n} u_0$ and we denote by $u_n \in C(\mathbb{R}; H^\infty(\mathbb{T}))$ the solutions to (2.1) emanating from $u_{0,n}$. In view of (6.2) we infer that for any $n \in \mathbb{N}^*$,

$$\|u_n\|_{Z_T^s} \lesssim \|u_0\|_{H^s}(1 + \|u_0\|_{H^s}^2) \ .$$

and (6.3) ensures that

$$\lim_{K \to +\infty} \sup_{n \in \mathbb{N}} \|P_{\geq K} u_n(t)\|_{L_T^\infty H^s} = 0 \ . \tag{6.4}$$

This proves that the sequence $\{u_n\}$ is bounded in $L^\infty(0,T; H^s(\mathbb{T}))$ and thus u_n^3 is bounded in $L^\infty(0,T; L^2(\mathbb{T}))$. Moreover, in view of the equation (2.1), the sequence $\{\partial_t u_n\}$ is bounded in $L^\infty(0,T; H^{-3}(\mathbb{T}))$. By Aubin-Lions compactness theorem, we infer that for any $T > 0$, $\{u_n\}$ is relatively compact in $L^2(]0,T[\times\mathbb{T})$.Therefore, using a diagonal extraction argument, we obtain the existence of an increasing sequence $\{n_k\} \subset \mathbb{N}$ and $u \in L^\infty(]0,T[; H^s(\mathbb{T}))$ such that

$$u_{n_k} \rightharpoonup u \text{ weak star in } L^\infty(]0,T[; H^s(\mathbb{T})) \tag{6.5}$$

$$u_{n_k} \to u \text{ in } L^2(0,T; L^2(\mathbb{T})) \cap L^3(0,T; L^3(\mathbb{T})) \tag{6.6}$$

$$u_{n_k} \to u \text{ a.e. in }]0,T[\times\mathbb{T} \tag{6.7}$$

$$u_{n_k}^3 \to u^3 \text{ in } L^1(0,T; L^1(\mathbb{T})) \tag{6.8}$$

These convergences results enable us to pass to the limit on the equation and to obtain that the limit function u satisfies (2.2) with $F(u) = u^3 - 3P_0(u^2)$. Therefore the unconditional uniqueness result ensures that u is the only accumulation point of $\{u_n\}$ and thus $\{u_n\}$ converges to u in the sense (6.5)-(6.8). Now, using the bounds on $\{u_n\}$ and $\{\partial_t u_n\}$, it is clear that for any $\phi \in C^\infty(\mathbb{T})$ and any $T > 0$, the sequence $\{t \mapsto (u_n, \phi)_{H^s}\}$ is uniformly equi-continuous on $[0,T]$. By Ascoli's theorem it follows that

$$(u_n, \phi)_{H^s} \to (u, \phi) \text{ in } C([0,T]) \ .$$

In particular, for any fixed $N \geq 1$, it holds

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \|P_{\leq N}(u_n - u)(t)\|_{H^s} = 0 \ .$$

This last limit combined with (6.4) ensures that

$$u_n \to u \text{ in } C([0,T]; H^s(\mathbb{T})) \ .$$

and thus $u \in C([0,T]; H^s(\mathbb{T}))$.
Finally, to prove the continuity with respect to initial data, we take a sequence $(u_0^m) \subset B_{H^s}(0, 2\|u_0\|_{H^s})$ that converges to u_0 in $H^s(\mathbb{T})$. Denoting by u^m the associated solutions to (2.1) that we have constructed above, we obtain in exactly the same way as above that for $T \sim \min(1, (1 + \|u_0\|_{H^s})^{-10})$ it holds

$$\|u_m\|_{Z_T^s} \lesssim \|u_0\|_{H^s}(1 + \|u_0\|_{H^s}^2), \qquad \lim_{K \to +\infty} \sup_{m \in \mathbb{N}} \|P_{\geq K} u_n(t)\|_{L_T^\infty H^s} = 0 \ .$$

and

$$(u_m, \phi)_{H^s} \to (u, \phi) \text{ in } C([0,T]) \ .$$

This ensures that $u_m \to u$ in $C([0,T]; H^s(\mathbb{T}))$ and completes the proof of the unconditional well-posedness of (2.1).

6.3. **Back to the mKdV equation.** For $s \geq 0$ we define the application

$$\Psi \ : \ \begin{array}{ccc} L^\infty_T H^s_x & \longrightarrow & L^\infty_T H^s_x \\ u = u(t,x) & \longmapsto & \Psi(u) = \Psi(u)(t,x) = u(t, x + \int_0^t P_0(u^2(\tau))\, d\tau) \end{array}$$

It is easy to check that Ψ is a bijection from $L^\infty_T H^s_x$ into itself and also from $C([0,T]; H^s(\mathbb{T})$ into itself with inverse bijection defined by

$$\Psi^{-1}(u) = u\Big(t, x - \int_0^t P_0(u^2(\tau))\, d\tau\Big)\,.$$

Moreover, for $s \geq 1/3$, it is not too hard to check that $u \in L^\infty_T H^s_x$ is a solution of (2.2) with $F(u) = u^3$ if and only if $\Psi(u) \in L^\infty_T H^s_x$ is a solution to (2.2) with $F(u) = u^3 - 3P_0(u^2)$. Finally, we claim that Ψ and Ψ^{-1} are continuous from $C([0,T]; H^s(\mathbb{T})$ into itself. Indeed, let $(v_n)_{n\geq 1} \subset C([0,T]; H^s(\mathbb{T}))$ that converges to v in $C([0,T]; H^s(\mathbb{T}))$. Then denoting $\int_0^t P_0(v_n^2)(s)\, ds$ by $\alpha_n(t)$ and $\int_0^t P_0(v^2)(s)\, ds$ by $\alpha(t)$, it is easy to check that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} (\alpha_n(t) - \alpha(t)) = 0 \tag{6.9}$$

and

$$\begin{aligned} \sup_{t\in[0,1]} \|\Psi(v_n)(t) - \Psi(v)(t)\|_{H^s} \ \leq \ & \sup_{t\in[0,1]} \Big\| v_n(t, \cdot + \alpha_n(t)) - v(t, \cdot + \alpha_n(t)) \Big\|_{H^s} \\ & + \sup_{t\in[0,1]} \Big\| v(t, \cdot + \alpha_n(t)) - v(t, \cdot + \alpha(t)) \Big\|_{H^s} \end{aligned}$$

It is clear that the first term of the right-hand side of the above estimate converges to 0. Now, the second term can be rewritten as

$$I_n = \sup_{t\in[0,1]} \Big(\sum_{k\in\mathbb{Z}} \Big| k^s e^{ik(\alpha_n(t) - \alpha(t))} \widehat{v}(t,k) \Big|^2 \Big)^{1/2}$$

Since $v \in C([0,T]; H^s(\mathbb{T}))$, $\{v(t),\, t \in [0,T]\}$ is a compact set of $H^s(\mathbb{T})$ and thus

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \sum_{|k|\geq N} |k|^{2s} |\widehat{v}_(t,k)|^2 = 0\,,$$

which combined with (6.9) ensures that $\lim_{n\to\infty} \ I_n = 0$ and completes the proof of the desired continuity result.

These properties of Ψ combined with the unconditional local well-posedness of the renormalized mKdV equation in $H^s(\mathbb{T})$, clearly leads to Theorem 2.4.

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