Metastability in reversible diffusion processes invariant under a symmetry group

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Abstract

We extend the analysis of the problem of metastability of Markovian jump processes with symmetries, initiated in a previous work, to reversible diffusion processes. Using representation theory of finite groups, we prove an Eyring-Kramers law for the low-lying spectrum of the generator of the diffusion, thereby generalizing results by A. Bovier, V. Gayrard, and M. Klein.

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1 Introduction

In this present work, we extend the analysis of the problem of metastability for Markovian jump processes with symmetries, initiated in [1], to diffusion processes. We consider the stochastic differential equation

$$dX_\varepsilon(t) = -\nabla V(X_\varepsilon(t))dt + \sqrt{2\varepsilon}dW(t), \tag{1.1}$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is a confining potential, and W_t is a d-dimensional standard Brownian motion. Recall that if V has just two quadratic local minima x^* and y^*, separated by a quadratic saddle z^*, the Eyring–Kramers law, which characterises the mean transition times between local minima of a diffusion in a potential landscape, states that the expected first-passage time τ from x^* to a small ball around y^* is given by

$$\mathbb{E}^{x^*}[\tau] = \frac{2\pi}{|\lambda_-(z^\star)|}\sqrt{\frac{|\det(\nabla^2 V(z^\star))|}{\det(\nabla^2 V(x^\star))}}\,\mathrm{e}^{[V(z^\star)-V(x^\star)]/\varepsilon}[1+\mathcal{O}(\sqrt{\varepsilon}|\log\varepsilon|^{3/2})]\,, \tag{1.2}$$

where $\nabla^2 V(x)$ denote the Hessian matrices of V at x, and $\lambda_-(z^\star)$ is the unique negative eigenvalue of $\nabla^2 V(z^\star)$.

We will consider a potential V with $N > 2$ local minima, and in this case the characterisation of metastable timescales becomes more involved. It has been known for a long time that the diffusion's generator admits N exponentially small eigenvalues, and that they are connected to metastable timescales [13, 15, 11, 10]. The generator of the diffusion process (1.1) is the linear operator L_ε of the form:

$$L_\varepsilon = -\varepsilon\,\mathrm{e}^{V(\cdot)/\varepsilon}\,\nabla\,\mathrm{e}^{-V(\cdot)/\varepsilon}\,\nabla = -\varepsilon\Delta + \langle\nabla V(\cdot),\nabla\rangle. \tag{1.3}$$

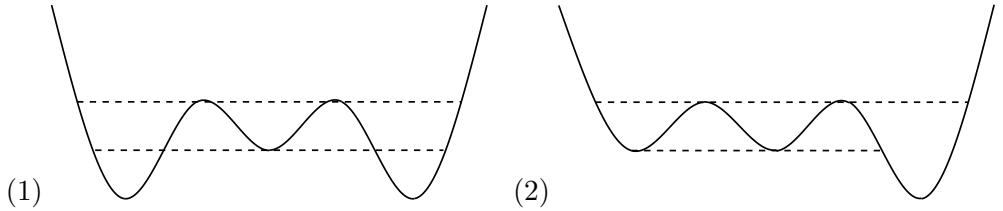


FIGURE 1. Examples of potentials : in (1) we get all eigenvalues, in (2) our theory doesn't work but [12] work.

Remark that the sign of the generator was reversed intentionally because this will make all the eigenvalues positive.

Results in [5] show that one can order the local minima $x_1, \ldots, x_N$ of V in such a way that the mean transition time from each x_k to the set $\{x_1, \ldots, x_{k-1}\}$ of its predecessors is close to the inverse of the kth small eigenvalue. But these results have a limitation, they require that all relevant saddle heights have to be different (see subsection 2.1 for a precise formulation).

In our case, this condition fails because we will suppose that the potential V is invariant under a symmetry group G. Since we can't apply some parts of the results in [5], we will use the representation theory of finite groups to avoid the condition that all relevant saddle heights have to be different.

Results obtained in this work allow us to say that the stochastic system behaves for small noise intensity ε like a Markovian jump process, which in particular proves the results announced in Corollary 4.5 and Theorem 5.1 in [2].

Our approach is similar to [12] in the sense that we make packages of local minima by group orbits which allows us to compute all N small eigenvalues of the generator (for example the potential (1) in Figure 1) instead of only part of them as in [12]. But on the other hand, our hypothesis does not allow us to deal with, for example, the potential (2) in Figure 1 while the approach in [12] can deal with this situation.

The remainder of the article is organised as follows. In subsection 2.1, we define the main setting and recall some elements of representation theory of finite groups. Subsections 2.2, 2.3 and 2.4 contain the results on eigenvalues and transition times for the processes. These results are illustrated in Subsection 2.5 on an example. In Section 3 we explain the main strategy that we will follow and recall elements of potential theory. In Section 4 we define the capacity matrix and make the connection between this matrix and the eigenvalues of the generator. The last section contains the proofs of the main results.

Notations: If $i \leqslant j$ are integers, $[\![i, j]\!]$ denotes the set $\{i, i+1, \ldots, j\}$. The cardinality of a finite set A is denoted by $|A|$. The spectrum of a matrix is denoted by Sp.

2 Results

2.1 Setting

In this subsection we will start by giving the necessary definitions and assumptions, from potential theory and representation theory of finite groups, which will allow us to state the main results. Let us start by some smoothness on the potential V.

Assumption 2.1. We suppose that $V \in \mathcal{C}^3(\mathbb{R}^d)$ and

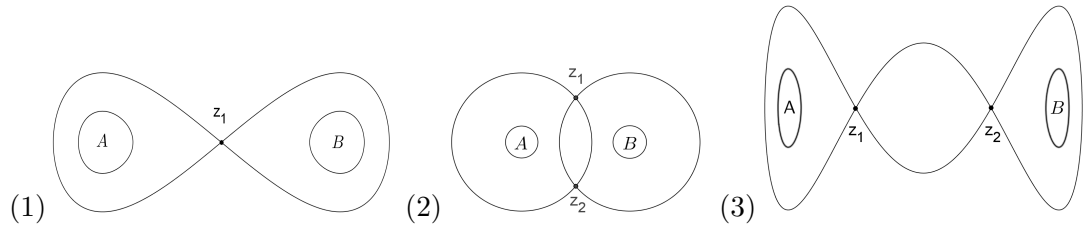


FIGURE 2. Examples of potentials and gates. (1) $\mathcal{G}^*(A,B) = \{z_1\}$. (2) $\mathcal{G}^*(A,B) = \{z_1, z_2\}$. (3) $\mathcal{G}^*(A,B) = \{z_1\}$ or $\{z_2\}$.

1. $\lim_{x\to\infty} \|\nabla V(x)\| = \infty$,
2. $\lim_{x\to\infty} \|\nabla V(x)\|^2 - 2\Delta V(x) = \infty$.

In all this paper, $\|\cdot\|$ denotes the classic scalar product on $\mathbb{R}^d$ and $\|\cdot\|_2$ denotes the norm on $L^2(\mathbb{R}^d, \mathrm{e}^{-V(x)/\varepsilon}\,dx)$.

Remark 2.2. Assumption 2.1 ensures that the resolvent of the generator L_ε is compact for ε sufficiently small. Indeed using the relation

$$\int_{B_R} \|\nabla u(x)\|^2\, \mathrm{e}^{-V(x)/\varepsilon}\,dx = \int_{B_R} \|\nabla g(x)\|^2 + \frac{1}{4\varepsilon^2} g^2 \|\nabla V(x)\|^2 + \frac{1}{2\varepsilon}\langle \nabla g^2(x), \nabla V(x)\rangle dx, \quad (2.1)$$

where B_R is the ball of radius R and centred on 0, $u \in H^1(\mathbb{R}^d, \mathrm{e}^{-V(x)/\varepsilon}\,dx)$ and $g = u\,\mathrm{e}^{-V/2\varepsilon}$ we get

$$\int_{\mathbb{R}^d} \|\nabla u(x)\|^2\, \mathrm{e}^{-V(x)/\varepsilon}\,dx \geq C \int_{\mathbb{R}^d} u(x)^2\, \mathrm{e}^{-V(x)/\varepsilon}\,dx, \quad (2.2)$$

for ε sufficiently small, which leads to the coercivity of L_ε and yields compactness using the Lax-Milgram theorem. Moreover, it implies that V has exponentially tight level in sense that for all a$\in \mathbb{R}$

$$\int_{y\,:\,V(y)\geq a} \mathrm{e}^{-V(y)/\varepsilon}\,dy \leq C\,\mathrm{e}^{-a/\varepsilon}, \quad (2.3)$$

where $C = C(a) < \infty$ is uniform in $\varepsilon \leq 1$.

The set of saddle points is intuitively the subset of the level set $\mathcal{G}(A,B) = \{z \in \mathbb{R}^d \mid V(z) = \hat{V}(A,B)\}$ that cannot be avoided by any paths ω that try to stay as low as possible. We define this set as follows:

Definition 2.3. *Let $A, B \subset \mathbb{R}^d$ be two disjoint sets.*

1. *We define $\hat{V}(A,B)$, the height of the saddle between A and B, by*

$$\hat{V}(A,B) = \inf_{\substack{\omega:[0,1]\to\mathbb{R}^d \\ \omega(0)\in A, \omega(1)\in B}} \sup_{t\in[0,1]} V(\omega(t)), \quad (2.4)$$

   where ω is a continuous path. By abuse of notation we write $\hat{V}(x,B)$ instead of $\hat{V}(\{x\},B)$.

2. *We define the communication height from $x \in \mathbb{R}^d$ to B by*

$$H(x,B) = \hat{V}(x,B) - V(x) \quad (2.5)$$

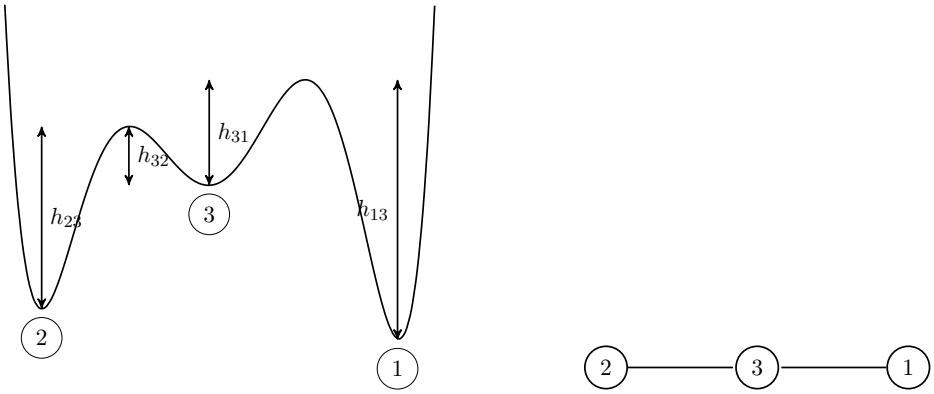


FIGURE 3. Example of a graph associated to a potential.

3. *We define $\mathcal{P}(A,B)$, the set of minimal paths from A to B, by*

$$\mathcal{P}(A,B) = \{\omega \in \mathcal{C}([0,1],\mathbb{R}^d) \mid \omega(0) \in A\ ,\ \omega(1) \in B\ ,\ \sup_{t\in[0,1]} V(\omega(t)) = \hat{V}(A,B)\}. \tag{2.6}$$

4. *A gate $\mathcal{G}^*(A,B)$ is a minimal subset of $\mathcal{G}(A,B)$ with the property that all minimal paths intersect $\mathcal{G}(A,B)$. Note that $\mathcal{G}^*(A,B)$ is in general not unique (see Figure 2). Then the set $\mathcal{S}(A,B)$ of saddle points is the union of all gates $\mathcal{G}^*(A,B)$.*

To avoid some complications, we will make the assumption that all saddle points are non-degenerate in the following sense :

Assumption 2.4.

1. The set $\mathcal{M}$, of local minima of V, is finite. For any two local minima x, y of V, any set $\mathcal{G}^*(x,y)$ consists of a finite set of isolated points $z_i^*(x,y)$.

2. The Hessian matrix of V at all local minima $x_i \in \mathcal{M}$ and all saddle points z_i^* is non-degenerate (i.e. has only non-zero eigenvalues).

Note that the definition of saddle point using paths is equivalent to the definition of saddle points from an analytical point of view, based on the Morse lemma (see [14] for Morse lemma and [3] for a proof of the equivalence).

Remark that under Assumption 2.4, all eigenvalues of a local minimum are strictly positive and all eigenvalues of a saddle point are strictly positive except one which is strictly negative.

As in [1], it will be convenient to consider the undirected graph associated to the potential V.

Definition 2.5. *Let $\mathcal{G} = (\mathcal{M}, E)$ be the undirected graph associated to the potential V where the edges are defined as follows : for all $x \in \mathcal{M}$ let D_x, the basin of attraction of x, be the set of points $y \in \mathbb{R}^d$ such that the solution of the differential equation $\frac{d}{dt}y(t) = -\nabla V(y(t))$, with initial condition $y(0) = y$, converges to x. Then*

$$E = \{(x_1,x_2) \in \mathcal{M}^2 \mid \exists \omega \in \mathcal{P}(\{x_1\},\{x_2\})\ ,\ \forall y \in \mathcal{M} \setminus \{x_1,x_2\}\ ,\ \mathrm{Im}\,\omega \cap D_y = \emptyset\} \tag{2.7}$$

In words, this means that there is a minimal path from x_1 to x_2 which does not pass through any other basin of attraction than D_{x_1} and D_{x_2}. We will say that x_1 and x_2 are neighbours, denoted $x_1 \overset{\mathcal{M}}{\sim} x_2$, if $(x_1, x_2) \in E$. Moreover, to each edge $(x_1, x_2) \in E$ we will associate the communication height $H(x_1, x_2)$ (see Figure 3 for an example).

Recall the algorithm of removing vertices in [1] subsection 2.3. If we remove a set of vertices $\{y_1, \dots, y_k\}$ and recompute the edges, then we get a new graph $\mathcal{G} = (\mathcal{M} \setminus \{y_1, \dots, y_k\}, E')$. We will say that x_1 and x_2 are neighbours with respect to $\mathcal{M}\setminus\{y_1, \dots, y_k\}$, denoted $x_1 \overset{\mathcal{M}\setminus\{y_1,\dots,y_k\}}{\sim} x_2$, if $(x_1, x_2) \in E'$. For example, in Figure 3 we have $1 \overset{\{1,2\}}{\sim} 2$. Remark that we can see this definition as follows : for all $a \in \mathcal{M}$ we have

$$x \overset{\mathcal{M}\setminus\{a\}}{\sim} y \Leftrightarrow x \overset{\mathcal{M}}{\sim} a \text{ and } a \overset{\mathcal{M}}{\sim} y. \tag{2.8}$$

Unlike in [4, 5] we do not make the assumptions on the fact that communication heights are all different since we want to make a generalisation to the symmetric case. Finally we give some elements of representation theory of finite groups. Let G be a finite group of isometries $g : \mathbb{R}^d \to \mathbb{R}^d$ such that the potential V is invariant under the group G in the following sense

$$\forall g \in G \ , \ V \circ g = V. \tag{2.9}$$

We denote by $\pi(g) \in \mathbb{R}^{|\mathcal{M}|\times|\mathcal{M}|}$ the permutation matrix

$$\pi(g)_{ab} = \begin{cases} 1 & \text{if } g(a) = b \ , \\ 0 & \text{otherwise .} \end{cases} \tag{2.10}$$

Let us recall a few definitions from basic group theory.

Definition 2.6.

1. *For $a \in \mathcal{M}$, $O_a = \{g(a) \colon g \in G\} \subset \mathcal{M}$ is called the* orbit *of a.*
2. *For $a \in \mathcal{M}$, $G_a = \{g \in G \colon g(a) = a\} \subset G$ is called the* stabiliser *of a.*
3. *For $g \in G$, $\mathcal{M}^g = \{a \in \mathcal{M} \colon g(a) = a\} \subset \mathcal{M}$ is called the* fixed-point set *of g.*

The following facts are well known:

- The orbits form a partition of $\mathcal{M}$, denoted $\mathcal{M}/G$.
- For any $a \in \mathcal{M}$, the stabiliser G_a is a subgroup of G.
- For any $a \in \mathcal{M}$, the map $\varphi : gG_a \mapsto g(a)$ provides a bijection from the set G/G_a of left cosets to the orbit O_a of a, and thus $|G|/|G_a| = |O_a|$.
- For any $g \in G$ and any $a \in \mathcal{M}$, one has $G_{g(a)} = gG_ag^{-1}$, i.e. stabilisers of a given orbit are conjugated.
- *Burnside's lemma:* $\sum_{g\in G}|\mathcal{M}^g| = |G||\mathcal{M}/G|$.

We will denote the orbits of G by $A_1, \dots, A_{n_G}$. The value of the communication height $H(a, A_j)$ is the same for all $a \in A_i$, and we will denote it $H(A_i, A_j)$. Similarly, we write V_{A_i} for the common value of all $V(a)$, $a \in A_i$. We shall make the following non-degeneracy assumption:

Assumption 2.7 (Metastable order of orbits)**.** Let $\mathcal{M}_k = A_1 \cup \cdots \cup A_k$. One can order the orbits in such a way that

$$H(A_k, \mathcal{M}_{k-1}) \leqslant \min_{i<k} H(A_i, \mathcal{M}_k \setminus A_i) - \theta \ , \qquad k = 2, \dots, n_G \tag{2.11}$$

for some $\theta > 0$. We indicate this by writing $A_1 \prec A_2 \prec \cdots \prec A_{n_G}$.

In Figure 3 we have ① ≺ ② ≺ ③. See [7, 6, section 4.3] for an algorithm determining the metastable hierarchy.

In [1] we made an assumption of absence of accidental degeneracy to simplify the expression of eigenvalues. Now we do the same for diffusion processes.

Assumption 2.8 (Absence of accidental degeneracy)**.** Whenever there are elements $x_{i_1} \overset{\mathcal{M}}{\sim} x_{i_2}, x_{j_1} \overset{\mathcal{M}}{\sim} x_{j_2} \in \mathcal{M}$ such that $\hat{V}(x_{i_1}, x_{j_1}) - V(x_{i_1}) = \hat{V}(x_{i_2}, x_{j_2}) - V(x_{i_2})$, there exist $g \in G$ such that $g(x_{i_1}) = x_{i_2}$ and $g(x_{j_1}) = x_{j_2}$.

Remark 2.9. Assumption 2.8 implies that all communication height between orbits are different and also that we can only have at most two saddles in series.

Assumptions 2.1, 2.4, 2.7 and 2.8 will be assumed to hold throughout this paper.

The map π defined by (2.10) is a morphism from G to $\mathrm{GL}(|\mathcal{M}|, \mathbb{C})$, and thus defines a *representation* of G (of dimension $\dim \pi = n$). In what follows, we will draw on some facts from representation theory of finite groups (see for instance [16]):

- A representation of G is called *irreducible* if there is no proper subspace of $\mathbb{C}^n$ which is invariant under all $\pi(g)$.
- Two representations π and π' of dimension d of G are called *equivalent* if there exists a matrix $S \in \mathrm{GL}(d, \mathbb{C})$ such that $S\pi(g)S^{-1} = \pi'(g)$ for all $g \in G$.
- Any finite group G has only finitely many inequivalent irreducible representations $\pi^{(0)}, \ldots, \pi^{(r-1)}$. Here $\pi^{(0)}$ denotes the *trivial representation*, $\pi^{(0)}(g) = 1 \ \forall g \in G$.
- Any representation π of G can be decomposed into irreducible representations:

$$\pi = \bigoplus_{p=0}^{r-1} \alpha^{(p)} \pi^{(p)} , \qquad \alpha^{(p)} \geqslant 0 , \quad \sum_{p=0}^{r-1} \alpha^{(p)} \dim(\pi^{(p)}) = \dim(\pi) = n . \tag{2.12}$$

  This means that we can find a matrix $S \in \mathrm{GL}(n, \mathbb{C})$ such that all matrices $S\pi(g)S^{-1}$ are block diagonal, with $\alpha^{(p)}$ blocks given by $\pi^{(p)}(g)$. This decomposition is unique up to equivalence and the order of factors.
- For any irreducible representation $\pi^{(p)}$ contained in π, let $\chi^{(p)}(g) = \mathrm{Tr}\, \pi^{(p)}(g)$ denote its *characters*. Then

$$P^{(p)} = \frac{\dim(\pi^{(p)})}{|G|} \sum_{g \in G} \overline{\chi^{(p)}(g)} \pi(g) \tag{2.13}$$

  is the projector on the invariant subspace of $\mathbb{C}^n$ associated with $\pi^{(p)}$. In particular,

$$\alpha^{(p)} \dim(\pi^{(p)}) = \mathrm{Tr}\, P^{(p)} = \frac{\dim(\pi^{(p)})}{|G|} \sum_{g \in G} \overline{\chi^{(p)}(g)} \chi(g) , \tag{2.14}$$

  where $\chi(g) = \mathrm{Tr}\, \pi(g)$. Note that for the representation defined by (2.10), we have $\chi(g) = |\mathcal{M}^g|$.

Example 2.10 (Irreducible representations of the permutation group S_4, [8])**.** The permutation group S_4 can be seen as the group of orientation-preserving symmetries of a cube and octahedron. It is generated by the transpositions $(1, 2)$, $(2, 3)$ and $(3, 4)$. S_4 has :

- 2 irreducible representations of dimension 1, the trivial one ($\pi^{(0)}(g) = 1$ for all $g \in G$) and the signature ($\pi^{(1)}(g) = \sigma(g) = \mathrm{sign}(g)$ for all $g \in G$),

- 1 irreducible representation $\pi^{(2)} = \theta$ of dimension 2, equivalent to

$$\theta((1,2)) = \theta((3,4)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \theta((2,3)) = \begin{pmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & -\cos((\frac{2\pi}{3})) \end{pmatrix}. \quad (2.15)$$

- 2 irreducible representations of dimension 3, $\pi^{(3)} = \varphi_1$ equivalent to

$$\varphi_1((1,2)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \ \varphi_1((2,3)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi_1((3,4)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (2.16)$$

  and $\pi^{(4)} = \varphi_2$ equivalent to $\varphi_2(g) = \sigma(g)\varphi_1(g)$ for all $g \in G$.

The associated characters are given by

	id	(ab)	(abc)	(ab)(cd)	(abcd)
χ^0	1	1	1	1	1
χ^1	1	-1	1	1	-1
χ^2	2	0	-1	2	0
χ^3	3	1	0	-1	-1
χ^4	3	-1	0	-1	1

for any a, b, c, d such that $\{a, b, c, d\} = \{1, 2, 3, 4\}$.
There are no irreducible representations of dimension larger than 3.

We use representation theory of finite groups because, each irreducible representation of G will give us a subset of the eigenvalues and eigenfunctions more easily. Indeed each irreducible representations of G will provide us a subspace invariant under L_ε which will lead to find eigenvalues and eigenfunctions. At the end, we will see that this procedure indeed gives as many eigenvalues as there are local minima, having that all exponentially small eigenvalues have been found. In the next three subsections we will denote by $B_i = B_\varepsilon(x_i)$ the ball of radius ε centred on local minima $x_i \in \mathcal{M}$ and $S_k = \bigcup_{j=1}^{k} \bigcup_{i:x_i \in A_j} B_i$ for $k \in [\![1, n_G]\!]$.

2.2 The trivial representation

Let us start by the trivial representation $\pi^{(0)}$.

Theorem 2.11. *For ε small enough, the spectrum of L_ε contains $\lambda_1^{\pi^{(0)}} = 0$ and*

$$\lambda_k^{\pi^{(0)}} = \frac{n_k |\lambda_-(s)|}{2\pi} \sqrt{\frac{\det(\nabla^2 V(a))}{|\det(\nabla^2 V(s))|}}\, \mathrm{e}^{-H(A_k, \mathcal{M}_{k-1})/\varepsilon} (1 + \mathcal{O}(\sqrt{\varepsilon} \ln \varepsilon^{3/2})) \quad (2.17)$$

where $k \in [\![2, n_G]\!]$, $\nabla^2 V(x)$ denotes the Hessian matrix of V at x, a is any element of A_k, n_k is the number of optimal saddles between a and $\mathcal{M}_{k-1}$, s any of these saddles and $\lambda_-(s)$ is the unique negative eigenvalue of $\nabla^2 V(s)$. Moreover the normalized eigenfunction

associated to $\lambda_1^{\pi^{(0)}}$ *is* $\phi_1^{\pi^{(0)}} = \frac{1}{\sqrt{|A_k|}\|h_i\|_2}$ *and the normalized eigenfunction associated to* $\lambda_k^{\pi^{(0)}}$ *for* $k \in [\![2, n_G]\!]$ *is given by*

$$\phi_1^{\pi^{(0)}}(y) = \frac{1}{\sqrt{|A_k|}\|h_i\|_2} \sum_{j \in A_k} h_j(y)(1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) + \sum_{j \in \mathcal{M}_{k-1}} \frac{h_j(y)}{\|h_j\|_2} \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}), \tag{2.18}$$

where $\delta > 0$, $h_i(y) = \mathbb{P}^y\left[\tau_{B_i} < \tau_{S_k \setminus B_i}\right]$ *and* τ_A *is the first hitting time of a set* $A \subset \mathbb{R}^d$. *Furthermore, for* $k \in [\![2, n_G]\!]$

$$\mathbb{E}^x\left[\tau_{S_{k-1}}\right] = \frac{1}{\lambda_k^{\pi^{(0)}}}\left[1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})\right] \tag{2.19}$$

holds for all $x \in S_{n_G} \setminus S_{k-1}$.

Remark 2.12. the integer n_k is the main difference with the results of [4]. Another way to define n_k is that $n_k = |\mathcal{G}^*(A_k, \mathcal{M}_{k-1})|$ or as in [1] if we take $a \in A_k$, a minimal path γ from a to $\mathcal{M}_{k-1}$ in $\mathcal{G}$ and let the unique $i, j \in \mathcal{M}$ such that i and j are in the minimal path γ and the highest saddle point of γ is between i and j, then

$$n_k = \frac{|G_a|}{|G_i \cap G_j|}. \tag{2.20}$$

2.3 Other irreducible representations of dimension 1

Theorem 2.11 only accounts for a small subset of n_G eigenvalues of the generator, associated with distributions that are uniform on each orbit A_i. All other eigenvalues of L will be associated to the rate at which non-uniform initial distributions approach the uniform one. We first determine eigenvalues associated with nontrivial representations of dimension 1, which are easier to obtain. The following lemma shows that given such a representation, only part of the orbits may be present in the image of the associated projector.

Lemma 2.13 ([1], Lemma 3.3)**.** *Let* $\pi^{(p)}$ *be an irreducible representation of dimension* 1 *of* G, *let* A_i *be an orbit of* G *and fix any* $a \in A_i$. *Denote by* $\pi_i(g)$ *the permutation induced by* $g \in G$ *on* A_i *and let* $P_i^{(p)}$ *be the associated projector, cf.* (2.13). *Then one of two following cases holds:*

- *either* $\pi^{(p)}(h) = 1$ *for all* $h \in G_a$, *and then* $\operatorname{Tr} P_i^{(p)} = 1$;
- *or* $\sum_{h \in G_a} \pi^{(p)}(h) = 0$, *and then* $\operatorname{Tr} P_i^{(p)} = 0$.

Let us call *active* (with respect to the representation $\pi^{(p)}$) the orbits A_i such that $\operatorname{Tr} P_i^{(p)} = 1$, and *inactive* the other orbits. We denote $n_{\pi^{(p)}}$ the number of active orbits with respect to the representation $\pi^{(p)}$. From the representation $\pi^{(p)}$ we can deduce a number of eigenvalues and eigenfunctions equal to the number of active orbits. We need to define one additional object before getting the result

Definition 2.14. *For any* $a \in \mathcal{M}$ *let*

$$\mathcal{C}_a = \{g(a) \mid \pi^{(p)}(g) = 1\}. \tag{2.21}$$

For representations other than the trivial, we will always have two possible cases. The first one ($l = 1$ in the Theorem) with correspond to optimal paths between $\mathcal{C}_a$ and $\mathcal{M}_k \setminus \mathcal{C}_a$

are paths between $x \in \mathcal{C}_a$ and $y \in \mathcal{M}_{k-1}$, that is to say there is no optimal path inner the orbit A_k. And the first one ($l = 2$ in the Theorem) with correspond to optimal paths between $\mathcal{C}_a$ and $\mathcal{M}_k \setminus \mathcal{C}_a$ are paths between $x \in \mathcal{C}_a$ and $y \in A_k \setminus \mathcal{C}_a$, that is to say there is optimal path inner the orbit A_k. Now we can give the result, which reads as follows

Theorem 2.15. *Let $\pi^{(p)}$ be an irreducible representation of dimension 1 of G. For ε small enough, the spectrum of L_ε contains $n_{\pi^{(p)}}$ eigenvalues of geometric multiplicity 1 given by*

$$\lambda_k^{\pi^{(p)}} = \frac{n_k^l |\lambda_-(s^l)|}{2\pi} \mathcal{D}_l \sqrt{\frac{\det(\nabla^2 V(a))}{|\det(\nabla^2 V(s^l))|}}\, \mathrm{e}^{-H_l/\varepsilon}(1 + \mathcal{O}(\sqrt{\varepsilon}\ln\varepsilon^{3/2})) \tag{2.22}$$

where

$$l = \begin{cases} 1 & \text{if } H(\mathcal{C}_a, \mathcal{M}_k \setminus \mathcal{C}_a) = H(A_k, \mathcal{M}_{k-1}), \\ 2 & \text{if } H(\mathcal{C}_a, \mathcal{M}_k \setminus \mathcal{C}_a) < H(A_k, \mathcal{M}_{k-1}), \end{cases} \quad H_l = \begin{cases} H(A_k, \mathcal{M}_{k-1}) & \text{if } l = 1, \\ H(\mathcal{C}_a, \mathcal{M}_k \setminus \mathcal{C}_a) & \text{if } l = 2, \end{cases} \tag{2.23}$$

$k \in [\![1, n_G]\!]$ such that A_k is active, $\nabla^2 V(x)$ denotes the Hessian matrix of V at x, a is an element of A_k, n_k^1(respectively n_k^2) is the number of optimal saddles between a and $\mathcal{M}_{k-1}$ (respectively $\mathcal{M}_k \setminus \mathcal{C}_a$), s^1 (respectively s^2) any of these saddles, $\lambda_-(s)$ is the unique negative eigenvalue of $\nabla^2 V(s)$, $\mathcal{D}_1 = 1$ and

$$\mathcal{D}_2 = \begin{cases} \frac{1}{\mathcal{N}_k} \sum\limits_{\substack{g \in G/G_a \\ g(a) \overset{\mathcal{M}_k}{\sim} a}} (1 - \overline{\pi(g)}) & \text{if there is an unique choice of gate } \mathcal{G}^*(\mathcal{C}_a, S_k \setminus \mathcal{C}_a), \\ \frac{1}{4\mathcal{N}_k} \sum\limits_{\substack{g \in G/G_a \\ g(a) \overset{\mathcal{M}_k}{\sim} a}} (1 - \overline{\pi(g)}) & \text{otherwise.} \end{cases} \tag{2.24}$$

Here

$$\mathcal{N}_k = |\{g \in G/G_a \mid \pi(g) \neq 1 \text{ and } a \overset{\mathcal{M}_k}{\sim} g(a)\}|, \tag{2.25}$$

moreover the associated normalized eigenfunction to $\lambda_k^{\pi^{(p)}}$ is

$$\phi_k^{\pi^{(p)}}(y) = \sum_{g \in G/G_a} \frac{\overline{\pi(g)}}{\sqrt{|A_k|}} \frac{h_{g(a)}(y)}{\|h_a\|_2}(1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) + \sum_{j \in \mathcal{M}_{k-1}} \frac{h_j(y)}{\|h_j\|_2} \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}), \tag{2.26}$$

where $\delta > 0$ and $h_i(y) = \mathbb{P}^y\left[\tau_{B_i} < \tau_{S_k \setminus B_i}\right]$.

Remark 2.16. n_k^1 and n_k^2 can be also defined as in Remark 2.12. Because of Assumption 2.7 it is easy to see that we always have $H(\mathcal{C}_a, \mathcal{M}_k \setminus \mathcal{C}_a) \leq H(A_k, \mathcal{M}_{k-1})$.

2.4 Irreducible representations of dimension larger than 1

We dealt with all irreducible representations of dimension 1 now it remains irreducible representations of dimension larger than 1. As for irreducible representations of dimension 1, let us start with a lemma shows that given such a representation, only part of the orbits may be present in the image of the associated projector.

Lemma 2.17 ([1], Lemma 3.6)**.** *Let $\pi^{(p)}$ be an irreducible representation of G of dimension $d \geqslant 2$, and let A_i be an orbit of G. Denote by π_i the permutation induced by G on A_i, and let $P_i^{(p)}$ be the associated projector, cf.* (2.13)*. Then for arbitrary $a \in A_i$,*

$$\mathrm{Tr}(P_i^{(p)}) = d\alpha_i^{(p)}\ , \qquad \alpha_i^{(p)} = \frac{1}{|G_a|} \sum_{h \in G_a} \chi^{(p)}(h) \in \{0, 1, \ldots, d\}\ . \tag{2.27}$$

Here $\chi^{(p)}(h) = \operatorname{Tr} \pi^{(p)}(h)$ denotes the characters of the irreducible representation.

Let us again call *active* (with respect to the irreducible representation $\pi^{(p)}$) those orbits for which $\operatorname{Tr}(P_i^{(p)}) > 0$. We denote again $n_{\pi^{(p)}}$ the number of active orbits with respect to the representation $\pi^{(p)}$.

Definition 2.18. *For any $a \in \mathcal{M}$ let*

$$c_h(a) = \sum_{g \in G_a} \overline{\chi^{(p)}(gh)}, \tag{2.28}$$

$$\mathcal{C}_a = \{h(a) \mid c_h(a) = c_{\mathrm{id}}(a)\}. \tag{2.29}$$

Choose $d\alpha_i$ elements $a^i_{l \in [\![1, d\alpha_i]\!]}$ of A_i such that if we define

$$J(a_l^i) = \{h(a_l^i) \mid h \in G/G_{a_l^i} \text{ and } c_h(a_l^i) \neq 0\}, \tag{2.30}$$

$$\hat{c}_h(a_l^i) = \frac{c_h(a_l^i)}{(\sum\limits_{k \in G/G_{a_l^i}} (c_k(a_l^i))^2)^{1/2}}, \tag{2.31}$$

then the l vectors $\hat{c}^{l,i}$ are linearly independent.

Remark 2.19. This new definition of $\mathcal{C}_a$ in (2.29) is a generalisation of (2.21). Indeed for an irreducible representation of dimension 1 of G we have

$$c_h(a) = c_{\mathrm{id}}(a) \Leftrightarrow \sum_{g \in G_a} \overline{\chi^{(p)}(gh)} = \sum_{g \in G_a} \overline{\chi^{(p)}(g)} \Leftrightarrow \sum_{g \in G_a} \overline{\pi(g)\pi(h)} = \sum_{g \in G_a} \overline{\pi(g)} \Leftrightarrow \pi(h) = 1$$

With this definition we can give the remaining eigenvalues of L_ε

Theorem 2.20. *Let $\pi^{(p)}$ be an irreducible representation of dimension larger than 1 of G. For ε small enough, the spectrum of L_ε contains $n_{\pi^{(p)}}$ eigenvalues of geometric multiplicity $d\alpha_k$ given by*

$$\lambda_k^{\pi^{(p)}} = \frac{n_k^l |\lambda_-(s^l)|}{2\pi} \mathcal{D}_l \sqrt{\frac{\det(\nabla^2 V(a))}{|\det(\nabla^2 V(s^l))|}}\, \mathrm{e}^{-H_l/\varepsilon} (1 + \mathcal{O}(\sqrt{\varepsilon} \ln \varepsilon^{3/2})) \tag{2.32}$$

where

$$l = \begin{cases} 1 & \text{if } H(\mathcal{C}_a, \mathcal{M}_k \setminus \mathcal{C}_a) = H(A_k, \mathcal{M}_{k-1}), \\ 2 & \text{if } H(\mathcal{C}_a, \mathcal{M}_k \setminus \mathcal{C}_a) < H(A_k, \mathcal{M}_{k-1}), \end{cases} \quad H_l = \begin{cases} H(A_k, \mathcal{M}_{k-1}) & \text{if } l = 1, \\ H(\mathcal{C}_a, \mathcal{M}_k \setminus \mathcal{C}_a) & \text{if } l = 2, \end{cases} \tag{2.33}$$

$k \in [\![1, n_G]\!]$ such that A_k is active, $\nabla^2 V(x)$ denotes the Hessian matrix of V at x, a is an element of A_k, n_k^1(respectively n_k^2) is the number of optimal saddles between a and $\mathcal{M}_{k-1}$ (respectively $\mathcal{M}_k \setminus \mathcal{C}_a$), s^1 (respectively s^2) any of these saddle, $\lambda_-(s)$ is the unique negative eigenvalue of $\nabla^2 V(s)$, $\mathcal{D}_1 = 1$ and

$$\mathcal{D}_2 = \begin{cases} \frac{1}{\mathcal{N}_k} \sum\limits_{\substack{q \in G/G_a \\ q(a) \overset{\mathcal{M}_k}{\sim} a}} \left(1 - \frac{c_q(a)}{c_{\mathrm{id}}(a)}\right) & \text{if } \mathcal{G}^*(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) \text{ is unique,} \\ \frac{1}{4\mathcal{N}_k} \sum\limits_{\substack{q \in G/G_a \\ q(a) \overset{\mathcal{M}_k}{\sim} a}} \left(1 - \frac{c_q(a)}{c_{\mathrm{id}}(a)}\right) & \text{otherwise.} \end{cases} \tag{2.34}$$

Here

$$\mathcal{N}_k = |\{q \in G/G_a \mid c_q(a) \neq c_{\mathrm{id}}(a) \text{ and } a \overset{\mathcal{M}_k}{\sim} q(a)\}|, \tag{2.35}$$

moreover the associated normalized eigenfunctions to $\lambda_k^{\pi^{(p)}}$ *are*

$$\phi_{k,l}^{\pi^{(p)}}(y) = \sum_{g \in G/G_{a_l^k}} \hat{c}_g(a_l^k) \frac{h_{g(a_l^k)}(y)}{\|h_{a_l^k}\|_2}(1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) + \sum_{j \in \mathcal{M}_k \setminus J(a_l^k)} \frac{h_j(y)}{\|h_j\|_2}\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}), \tag{2.36}$$

where $\delta > 0$ *and* $h_i(y) = \mathbb{P}^y\left[\tau_{B_i} < \tau_{S_k \setminus B_i}\right]$.

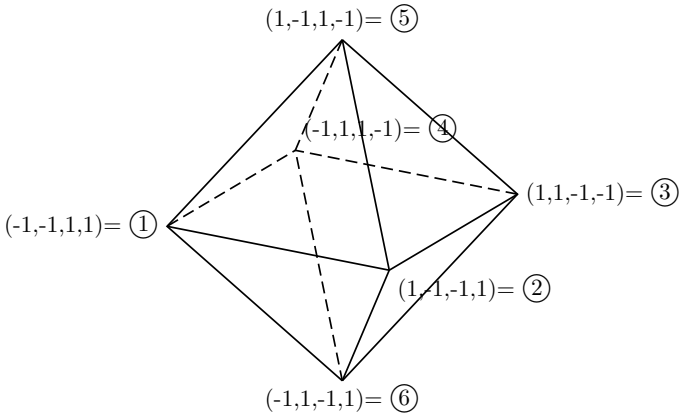
Remark 2.21. The new definition of $\mathcal{N}_k$ in (2.35) is also a generalisation of (2.25).

2.5 Example

To illustrate these results, as in [1], we will consider the following potential

$$V(x) = \sum_{i=1}^{d} \frac{1}{4}x_i^4 - \frac{1}{2}x_i^2 \ , \tag{2.37}$$

restricted to the space $\{x \in \mathbb{R}^d \colon \sum x_i = 0\}$. We have seen in [2] that the set of local minima consists of elements with two coordinates equal to 1 and two coordinates equal to -1, and the set of saddle points consists of elements with two coordinates equal to 0, one coordinate equal to 1 and one coordinate equal to -1. The graph associated to this potential is



We can take $G = S_4 \times \mathbb{Z}/2\mathbb{Z} = \langle (ab), (bc), (cd) \rangle \times \langle f \rangle$ which can be seen as the group of symmetries of an octahedron where

- a represents the line passing through the two barycentres of $\{1, 4, 5\}$ and of $\{2, 3, 6\}$,
- b represents the line passing through the two barycentres of $\{3, 4, 5\}$ and of $\{1, 2, 6\}$,
- c represents the line passing through the two barycentres of $\{2, 3, 5\}$ and of $\{1, 4, 6\}$,
- d represents the line passing through the two barycentres of $\{1, 2, 5\}$ and of $\{3, 4, 6\}$,
- f is a point reflection in the barycentre of the octahedron.

We use the notation (ab) for permutation of the lines a and b, and similarly for (bc) and (cd). We have

$$\begin{aligned} \forall x \in \mathcal{M}\ ,\ V(x) &= -1, \\ \forall s \in \mathcal{S}\ ,\ V(s) &= -\frac{1}{2}. \end{aligned} \tag{2.38}$$

We have only one orbit $A = \mathcal{M} = \{1, 2, 3, 4, 5, 6\}$. The elements of G can be seen as permutations of local minima and are given by this table :

(ab)	$(26)(45)(13)$
$(ab)f$	$(25)(46)$
(bc)	$(35)(16)(24)$
$(bc)f$	$(15)(36)$
(cd)	$(25)(46)(13)$
$(cd)f$	$(26)(45)$

Then the stabiliser of 1 is

$$G_1 = \{\mathrm{id}, (ab)f, (cd)f, (ab)(cd), (ac)(bd)f, (ac)(bd)f, (acbd), (adbc)\}. \tag{2.39}$$

Using example 2.10, the irreducible representations of G are

$$\forall i \in [\![0, 4]\!]\ ,\ \forall g \in S_4\ ,\ \pi^{i\pm}(gf^j) = (\pm 1)^j \pi^{(i)}(g) \tag{2.40}$$

By applying Theorem 2.11 we get the eigenvalue $\lambda_0 = 0$. The orbit A is inactive for all other irreducible representations of dimension 1. The orbit A is active only for the following two representations :

- for π^{2+} , Theorem 2.20 gives $l = 2$, $H_2 = \frac{1}{2}$, $n_2^2 = 4$, $\mathcal{N}_2 = 4$, $\mathcal{D}_2 = \frac{3}{2}$ and the eigenvalue $\lambda_2 = \frac{6\sqrt{2}}{\pi}\,\mathrm{e}^{-\frac{1}{2\varepsilon}}(1+\mathcal{O}(\sqrt{\varepsilon}\ln\varepsilon^{3/2}))$ of geometric multiplicity 2 with associated eigenfunctions :

$$\begin{aligned} \phi_1^{\pi^{2+}}(y) &= \left(\frac{\sqrt{3}}{3}\sum_{i\in\{1,3\}}\frac{h_i(y)}{\|h_i\|_2} - \frac{\sqrt{3}}{6}\sum_{j\in\{2,4,5,6\}}\frac{h_j(y)}{\|h_j\|_2}\right)(1+\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) \\ \phi_2^{\pi^{2+}}(y) &= \left(\frac{\sqrt{3}}{3}\sum_{i\in\{2,4\}}\frac{h_i(y)}{\|h_i\|_2} - \frac{\sqrt{3}}{6}\sum_{j\in\{1,3,5,6\}}\frac{h_j(y)}{\|h_j\|_2}\right)(1+\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) \end{aligned} \tag{2.41}$$

- for π^{4-} , Theorem 2.20 gives $l = 2$, $H_2 = \frac{1}{2}$, $n_1^2 = 4$, $\mathcal{N}_1 = 4$, $\mathcal{D}_2 = 1$ and the eigenvalue $\lambda_1 = \frac{4\sqrt{2}}{\pi}\,\mathrm{e}^{-\frac{1}{2\varepsilon}}(1+\mathcal{O}(\sqrt{\varepsilon}\ln\varepsilon^{3/2}))$ of geometric multiplicity 3 with associated eigenfunctions :

$$\begin{aligned} \phi_1^{\pi^{3+}}(y) &= \frac{\sqrt{2}}{2}\left(\frac{h_1(y)}{\|h_1\|_2} - \frac{h_3(y)}{\|h_3\|_2}\right)(1+\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) + \sum_{i\in\{2,4,5,6\}}\frac{h_i(y)}{\|h_i\|_2}\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}) \\ \phi_2^{\pi^{3+}}(y) &= \frac{\sqrt{2}}{2}\left(\frac{h_2(y)}{\|h_2\|_2} - \frac{h_4(y)}{\|h_4\|_2}\right)(1+\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) + \sum_{i\in\{1,3,5,6\}}\frac{h_i(y)}{\|h_i\|_2}\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}) \\ \phi_5^{\pi^{3+}}(y) &= \frac{\sqrt{2}}{2}\left(\frac{h_5(y)}{\|h_5\|_2} - \frac{h_6(y)}{\|h_6\|_2}\right)(1+\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) + \sum_{i\in\{1,2,3,4\}}\frac{h_i(y)}{\|h_i\|_2}\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}) \end{aligned} \tag{2.42}$$

3 Strategy

3.1 Boundary value problem

Let us explain the main strategy that we will follow to get the main results. Consider balls $B_i = B_\varepsilon(x_i)$ of radius ε centred on local minima for $i \in [\![1, |\mathcal{M}|]\!]$. Let $\overline{\lambda}_k$ be the principal eigenvalue of the Dirichlet problem

$$\begin{aligned} (L_\varepsilon - \lambda) f(x) = 0, \quad & x \in \mathbb{R}^d \setminus \partial S_k, \\ f(x) = 0, \quad & x \in \partial S_k, \end{aligned} \tag{3.1}$$

where $S_k = \bigcup_{j=1}^{k} \bigcup_{i: x_i \in A_j} B_i$ and $k \in [\![1, n_G]\!]$. Note that the definition of S_k is slightly different from that in [4], since because of the symmetry we need to work with unions by orbits instead of elements.

Consider, for $\lambda < \overline{\lambda}_k$, the solution of the Dirichlet problem

$$\begin{aligned} (L_\varepsilon - \lambda) f^\lambda(x) = 0, \qquad & x \in \mathbb{R}^d \setminus \partial S_k, \\ f^\lambda(x) = \phi(x), \quad & x \in \partial S_k. \end{aligned} \tag{3.2}$$

The idea is to construct an eigenfunction of the operator on all $\mathbb{R}^d$ as a solution of the problem (3.2) for a well chosen function ϕ. In fact we can see that, if λ an eigenvalue of L_ε and if we choose ϕ as the eigenfunction associated to the eigenvalue λ, then $f^\lambda = \phi$ on $\mathbb{R}^d$. To see this, remark that if we have the equality on ∂S_k, then for all $x \in \mathbb{R}^d \setminus S_k$ we have

$$(L_\varepsilon - \lambda)(f^\lambda - \phi)(x) = 0. \tag{3.3}$$

But if λ is not in the spectrum of (3.1), then $(L_\varepsilon - \lambda)$ is invertible so we obtain $f^\lambda = \phi$ on $\mathbb{R}^d \setminus S_k$. The same argument works on the interior of S_k so we have the equality on all $\mathbb{R}^d$.

Thus $\lambda < \overline{\lambda}_k$ is an eigenvalue of L_ε if and only if we can find a function on ∂S_k such that the solution of the Dirichlet problem (3.2) is an eigenfunction of L_ε associated to the eigenvalue λ. So any eigenfunction associated to an eigenvalue $\lambda < \overline{\lambda}_k$ can be represented as a solution of the Dirichlet problem (3.2).

The problem of finding eigenvalues boils down to finding for what value λ for a well chosen function ϕ on the boundary ∂B_i we have $(L_\varepsilon - \lambda) f^\lambda = 0$ on all $\mathbb{R}^d$. One can interpret $(L_\varepsilon - \lambda) f^\lambda$ as a measure concentrated on ∂S_k. That means, for any test function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \int_{\mathbb{R}^d} \mathrm{e}^{-V(y)/\varepsilon} g(y)(L_\varepsilon - \lambda) f^\lambda(y) dy &= \int_{\mathbb{R}^d} \mathrm{e}^{-V(y)/\varepsilon} f^\lambda(y)(L_\varepsilon - \lambda) g(y) dy \\ &= \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon} f^\lambda(y)(L_\varepsilon - \lambda) g(y) dy + \int_{S_k} \mathrm{e}^{-V(y)/\varepsilon} f^\lambda(y)(L_\varepsilon - \lambda) g(y) dy \\ &= \varepsilon \int_{\partial S_k} \mathrm{e}^{-V(y)/\varepsilon} (g(y)\partial_{n(y)} f^\lambda(y) - f^\lambda(y)\partial_{n(y)} g(y)) d\sigma_{S_k}(y) \\ &\quad + \varepsilon \int_{\partial S_k} \mathrm{e}^{-V(y)/\varepsilon} (g(y)\partial_{-n(y)} f^\lambda(y) - f^\lambda(y)\partial_{-n(y)} g(y)) d\sigma_{S_k}(y) \\ &= \varepsilon \int_{\partial S_k} \mathrm{e}^{-V(y)/\varepsilon} (g(y)\partial_{n(y)} f^\lambda(y) + g(y)\partial_{-n(y)} f^\lambda(y)) d\sigma_{S_k}(y), \end{aligned} \tag{3.4}$$

where $\partial_{n(y)}$ denotes the derivative in the direction of the exterior normal vector to the surface ∂S_k.Requesting that the measure $(L_\varepsilon - \lambda)f^\lambda$ vanishes is equivalent to requesting that the measure

$$\varepsilon \, \mathrm{e}^{-V(y)/\varepsilon}(\partial_{n(y)} f^\lambda(y) + \partial_{-n(y)} f^\lambda(y)) d\sigma_{S_k}(y), \tag{3.5}$$

vanishes. Remark that the problem would be easier if the function ϕ were constant on the boundary ∂B_i. Although this is not true, we will see that either the function ϕ does not change sign around a minimum, and therefore we will be able to use Harnack and Hölder inequalities. Or the function ϕ changes sign and in this case it is exponentially small.

Now it will be easier than in [4]. Indeed, we will easily find the eigenfunctions because the values around the minima can be deduced from the capacity matrix (defined in section 4) which is invariant under the symmetry group G. So using the irreducible representation of G (see [1]), we already have a good candidate for the eigenfunctions, so all we have to do is to prove that indeed they are the eigenfunctions and at the same time get the eigenvalues.

3.2 Potential theory

Let us recall some important facts from potential theory which come from [5].

Proposition 3.1. *By construction the operator L_ε given in* (1.3) *is symmetric on $L^2(\mathbb{R}^d, e^{-V(x)/\varepsilon}dx)$, i.e.*

$$\langle L_\varepsilon f, g\rangle := \varepsilon \int_{\mathbb{R}^d} \nabla f(x) \nabla g(x) \, \mathrm{e}^{-V(x)/\varepsilon} \, dx = \langle f, L_\varepsilon g\rangle, \tag{3.6}$$

for any function $f, g \in H^1(\mathbb{R}^d, e^{-V(x)/\varepsilon}dx)$ in weak sense.

We want to find eigenvalues and eigenfunctions of the operator L_ε, so we introduce a useful tool, the Green's function.

Definition 3.2 (Green's function)**.** *Consider for $\lambda \in \mathbb{C}$ the Dirichlet problem:*

$$\begin{aligned} (L_\varepsilon - \lambda) f(x) &= g(x), \quad x \in \Omega, \\ f(x) &= 0, \qquad x \in \Omega^c, \end{aligned} \tag{3.7}$$

where $\Omega \subset \mathbb{R}^d$. We call Green function $G^\lambda_\Omega(x, y)$ associated to the inverse of the operator $L_\varepsilon - \lambda$, the function which satisfies :

$$(L_\varepsilon - \lambda) G^\lambda_\Omega(x, y) = \delta(x - y). \tag{3.8}$$

Proposition 3.3. *The solution f, of the Dirichlet problem* (3.7)*, can be written*

$$f(x) = \int_\Omega G^\lambda_\Omega(x, y) g(y) dy. \tag{3.9}$$

Moreover the Green function is symmetric with respect to the measure $\mathrm{e}^{-V(x)/\varepsilon}$, i.e.

$$G^\lambda_\Omega(x, y) = \mathrm{e}^{-V(y)/\varepsilon} \, G^\lambda_\Omega(y, x) \, \mathrm{e}^{V(x)/\varepsilon} \tag{3.10}$$

A tool often used in differential geometry is given by the Green identities, which here take the form

Proposition 3.4. *First Green identity*

$$\int_\Omega e^{-V(x)/\varepsilon}(\varepsilon\nabla\phi(x)\cdot\nabla\psi(x)-\psi(x)(L_\varepsilon\phi)(x))dx=\varepsilon\int_{\partial\Omega}e^{-V(x)/\varepsilon}\psi(x)\partial_{n(x)}\phi(x)d\sigma_\Omega(x), \tag{3.11}$$

and Second Green identity

$$\int_\Omega e^{-V(x)/\varepsilon}(\phi(x)(L_\varepsilon-\lambda)\psi(x)-\psi(x)(L_\varepsilon-\lambda)\phi(x))dx= \varepsilon\int_{\partial\Omega}e^{-V(x)/\varepsilon}(\psi(x)\partial_{n(x)}\phi(x)-\phi(x)\partial_{n(x)}\psi(x))d\sigma_\Omega(x), \tag{3.12}$$

where $\phi,\psi\in C^2(\Omega)$.

We introduce another useful tool, the Poisson kernel.

Definition-Proposition 3.5 (Poisson kernel)**.** *Consider for* $\lambda\in\mathbb{C}$ *the boundary value problem:*

$$\begin{aligned}(L_\varepsilon-\lambda)f(x)&=0, \qquad x\in\Omega,\\ f(x)&=\phi(x), \quad x\in\Omega^c.\end{aligned} \tag{3.13}$$

We denote by H^λ_Ω*, the Poisson kernel, the solution operator associated to* (3.13) *which can be represented in the form*

$$f(x)=(H^\lambda_\Omega\phi)(x)=-\varepsilon\int_{\partial\Omega}e^{-[V(y)-V(x)]/\varepsilon}\phi(y)\partial_{n(y)}G^\lambda_\Omega(y,x)d\sigma_\Omega(y), \tag{3.14}$$

where $d\sigma_\Omega(y)$ *denotes the surface measure on* $\partial\Omega$ *and* $\partial_{n(y)}$ *denotes the derivative in the direction of the exterior normal vector to the surface* $\partial\Omega$ *at* y *acting on the first argument of the function* $G^\lambda_\Omega(y,x)$.

The last object we need to introduce to get the results are the following

Definition 3.6. *Let* $A,B\subset\mathbb{R}^d$ *be two disjoints sets.*

1. *The equilibrium potential* $h^\lambda_{A,B}$ *is defined as the solution of the Dirichlet problem:*

$$\begin{aligned}(L_\varepsilon-\lambda)h^\lambda_{A,B}(x)&=0, \quad x\in(A\cup B)^c,\\ h^\lambda_{A,B}(x)&=1, \quad x\in A,\\ h^\lambda_{A,B}(x)&=0, \quad x\in B.\end{aligned} \tag{3.15}$$

2. *The equilibrium measure* $e^\lambda_{A,B}$ *is defined as the unique measure on* ∂A *such that :*

$$h^\lambda_{A,B}(x)=\int_{\partial A}G^\lambda_{A^c}(x,y)e^\lambda_{A,B}(dy), \tag{3.16}$$

   where $G^\lambda_{A^c}$ *is the Green function (recall in Definition 3.2)*

3. *The capacity, for* $A,B\subset\mathbb{R}^d$ *and* $\lambda\in\mathbb{R}$*, is defined by :*

$$\mathrm{cap}^\lambda_A(B)=\int_{\partial A}e^{-V(y)/\varepsilon}e^\lambda_{A,B}(dy). \tag{3.17}$$

Remark 3.7. If $\lambda = 0$ the equilibrium potential has the following probabilistic interpretation:

$$h_{A,B}(x) \equiv h^0_{A,B}(x) = \mathbb{P}_x\left[\tau_A < \tau_B\right]. \tag{3.18}$$

Remark 3.8. The relation (3.14) comes from the two Green identities 3.11 and 3.12. The relation (3.16) comes from the representation of the Poisson kernel (3.14) with $\Omega = A^c$.

Remark that we can express the capacity in another way. Indeed, if we consider L_ε as a map from $H^n(\mathbb{R}^d)$ to $H^{n-2}(\mathbb{R}^d)$ we can write the relation (3.16) as :

$$\begin{aligned}(L_\varepsilon - \lambda)h^\lambda_{A,B}(x) &= (L_\varepsilon - \lambda)\int_{\partial A} G^\lambda_{A^c}(x,y)e^\lambda_{A,B}(dy) = \int_{\partial A}(L_\varepsilon - \lambda)G^\lambda_{A^c}(x,y)e^\lambda_{A,B}(dy)\\ &= \int_{\partial A}\delta(x-y)e^\lambda_{A,B}(dy) = e^\lambda_{A,B}(dx)\end{aligned} \tag{3.19}$$

Using the second Green identity (3.12) and the representation of Poisson kernel (3.14) we have :

$$(L_\varepsilon - \lambda)h^\lambda_{A,B}(x) = \varepsilon\partial_{n(x)}h^\lambda_{A,B}(x)d\sigma_{A\cup B}(x) - \lambda\mathbb{1}_A dx \tag{3.20}$$

Using the second Green identity (3.12) we deduce that:

$$\mathrm{cap}^\lambda_A(B) = \varepsilon\int_{(A\cup B)^c} e^{-F(x)/\varepsilon}\left[\parallel \nabla h^\lambda_{A,B}(x)\parallel_2^2 - \frac{\lambda}{\varepsilon}(h^\lambda_{A,B}(x))^2\right]dx \equiv \Phi^\lambda_{(A\cup B)^c}(h^\lambda_{A,B}), \tag{3.21}$$

where $\Phi^\lambda_{(A\cup B)^c}$ is called the Dirichlet form associated to the operator $L_\varepsilon - \lambda$.

A fundamental consequence of the previous relation is the variational representation of the capacity if $\lambda \in \mathbb{R}_-$:

$$\mathrm{cap}^\lambda_A(B) = \inf_{h\in\mathcal{H}_{A,B}}\Phi^\lambda_{(A\cup B)^c}(h), \tag{3.22}$$

where $\mathcal{H}_{A,B}$ denotes the following set of functions

$$\mathcal{H}_{A,B} = \{h \in W^{1,2}(\mathbb{R}^d) \mid h(x) = 0 \text{ for } x \in B\ ,\ h(x) = 1 \text{ for } x \in A\}. \tag{3.23}$$

Moreover, for all λ and all $x \in (A\cup B)^c$, we have :

$$h_{A,B}(x) = \mathbb{E}_x e^{\lambda\tau_A}\mathbb{1}_{\tau_A<\tau_B}, \tag{3.24}$$

That implies

$$\frac{d}{d\lambda}h^{\lambda=0}_{A,B}(x) = \mathbb{E}_x\tau_A\mathbb{1}_{\tau_A<\tau_B}. \tag{3.25}$$

From this derivative we deduce that the function

$$w_{A,B}(x) = \begin{cases}\mathbb{E}_x\tau_A\mathbb{1}_{\tau_A<\tau_B}, & x \in (A\cup B)^c,\\ 0, & x \in (A\cup B),\end{cases} \tag{3.26}$$

is solution of the Dirichlet problem

$$\begin{aligned}L_\varepsilon w_{A,B}(x) = h_{A,B}(x), &\quad x \in (A\cup B)^c,\\ w_{A,B}(x) = 0, &\quad x \in (A\cup B).\end{aligned} \tag{3.27}$$

Remark that in the particular case where $B = \emptyset$ we obtain the Dirichlet problem with the representation

$$\begin{aligned}L_\varepsilon w_A(x) = 1, &\quad x \in A^c,\\ w_A(x) = 0, &\quad x \in A,\end{aligned} \tag{3.28}$$

4 Capacity matrix

4.1 Definition and generalities

Before defining the capacity matrix we have to establish some estimates on the behaviour of eigenfunctions near local minima of the potential V. For the analysis of harmonic functions that are not necessarily positive, we need an estimate for sub-harmonic functions that allows us to compare the oscillation to the L^2 norm (a local maximum principle).

Lemma 4.1 ([4], Lemma 4.1)**.** *Let ϕ be a strong solution of $(L_\varepsilon - \lambda)\phi = 0$ on a ball $B_{c\sqrt{\varepsilon}}(x)$. Then there exists a constant C independent of ε such that*

$$\sup_{y \in B_{c\sqrt{\varepsilon}}(x)} \phi(y) - \inf_{y \in B_{c\sqrt{\varepsilon}}(x)} \phi(y) \leq C\varepsilon^{-d/4} \left(\int_{B_{2c\sqrt{\varepsilon}}(x)} \phi(y)^2 dy \right)^{1/2} \tag{4.1}$$

Proof: See [9], Theorem 9.20 . □

Our goal is to show that, in the balls of radius $\sqrt{\varepsilon}$ centered on the local minima of the potential V, the eigenfunctions associated to the exponentially small eigenvalues of L_ε are either of constant sign or exponentially small. This is suggested by the following result we have taken in [10](Chapter 8, Proposition 2.2):

Proposition 4.2 ([4], Proposition 4.2)**.** *Let ϕ be a normalized eigenfunction of L_ε associated to one of the $|\mathcal{M}|$ eigenvalues. Let $\gamma < \hat{\gamma} = \min_{x,y \in \mathcal{M}} \left(\hat{V}(x,y) - V(y) \right)$. Let D_i, basin of attraction of x_i, be the set of points $y \in \mathbb{R}^d$ such that the solution of the differential equation $\frac{d}{dt} y(t) = -\nabla V(y(t))$, with initial condition $y(0) = y$, converges to $x_i \in \mathcal{M}$. Then there exist a constant $C < \infty$ dependent of γ and constants c_i such that*

$$\| \phi - \sum_i c_i \mathbb{1}_{D_i} \|_2 \leq C \, \mathrm{e}^{-\gamma/\varepsilon} \tag{4.2}$$

Proof: See [10]. □

This estimate does not allow to conclude that the normalized eigenfunction ϕ does not change sign in the neighbourhood of the local minima. We will show, for a given local minimum, that ϕ does not change sign if the contribution of ϕ is important. For $D \subset \mathbb{R}^d$ define

$$\| f \|_{2,D} = \left(\int_D \mathrm{e}^{-V(y)/\varepsilon} \, f(y)^2 dy \right)^{1/2}, \tag{4.3}$$

and for a given eigenfunction ϕ define

$$J = \{ j \in \mathcal{M} \mid \| \phi \|_{2,D_j} \geq \mathrm{e}^{-\gamma/2\varepsilon} \}. \tag{4.4}$$

Lemma 4.3 ([4], Lemma 4.3)**.** *Let ϕ be an eigenfunction like in Proposition 4.2. If $j \in J$ then there exist a constant c_j, a positive constant $C < \infty$ independent of ε and a constant α such that*

$$\sup_{x \in B_{\sqrt{\varepsilon}}(x_j)} |\phi(x) - c_j| \leq C\varepsilon^{\alpha/2} |c_j|. \tag{4.5}$$

Proof: Same proof as in [4] with the change $|c_j|$ instead of c_j because in the symmetric case eigenfunctions are not always positive. □

Remark 4.4. The Corollary 9.4 in [9] we used for the proof of the Lemma 4.3, is a consequence of Theorem 8.22 of [9], and by analysing the proof we see that $\alpha \in]0,1[$.

Lemma 4.5 ([4], Lemma 4.4)**.** *Let ϕ be an eigenfunction like in Proposition 4.2. If $j \notin J$ then there exists a positive constant $C < \infty$ independent of ε such that*

$$\sup_{x \in B_{\sqrt{\varepsilon}}(x_j)} |\phi(x)| \leq C\varepsilon^{-d/4}\, \mathrm{e}^{-\gamma/2\varepsilon}\, \mathrm{e}^{V(x_j)/2\varepsilon}\,. \tag{4.6}$$

Proof: See [4]. □

The analysis of the eigenvalues, in subsection 3.1, shows that the necessary and sufficient condition for such λ to exist is the existence of a nontrivial function ϕ^λ on ∂S_k such that the surface measure

$$\mathrm{e}^{-V(y)/\varepsilon}(L_\varepsilon - \lambda) f^\lambda(y) dy = \mathrm{e}^{-V(y)/\varepsilon}(\partial_{n(y)} + \partial_{-n(y)}) f^\lambda(y) d\sigma_{S_k}(y) \tag{4.7}$$

vanishes. An obvious necessary condition for this to be verified is

$$\int_{\partial B_i} \mathrm{e}^{-V(y)/\varepsilon}(\partial_{n(y)} + \partial_{-n(y)}) f^\lambda(y) d\sigma_{S_k}(y) = 0 \tag{4.8}$$

for all $B_i \subset S_k$.

Because of Lemmas 4.3 and 4.5, by defining $c_i = \inf_{y \in B_i} \phi^\lambda(y)$ one has the alternative :

1. Either
$$\sup_{y \in B_i} \left| \frac{\phi^\lambda(y)}{c_i} - 1 \right| \leq C\varepsilon^{\alpha/2}. \tag{4.9}$$
2. Or
$$\sup_{y \in B_i} |\phi^\lambda(y)| \leq C\varepsilon^{-d/4}\, \mathrm{e}^{-\gamma/2\varepsilon}\, \mathrm{e}^{V(x_i)/2\varepsilon}\,. \tag{4.10}$$

We will now consider all possible cases. Let $J \subset \mathcal{M}$ be the set of indices where (4.9) holds and J^c the set of indices where (4.10) holds. With this partition let

$$f^\lambda = \sum_{j \in J} c_j (h^\lambda_{B_j, S_k \setminus B_j} + \psi^\lambda_j) + \sum_{j \in J^c} \psi^\lambda_j, \tag{4.11}$$

where $h^\lambda_{B_j, S_k \setminus B_j}$ are the equilibrium potentials defined in (3.15) and the functions ψ^λ_j satisfy for $j \in J$ the problems

$$\begin{aligned} (L_\varepsilon - \lambda)\psi^\lambda_j(y) &= 0, && y \in \mathbb{R}^d \setminus S_k, \\ \psi^\lambda_j(y) &= \frac{\phi^\lambda(y)}{c_j} - 1, && y \in \partial B_j, \\ \psi^\lambda_j(y) &= 0, && y \in \partial S_k \setminus B_j, \end{aligned} \tag{4.12}$$

and for $j \in J^c$

$$\begin{aligned} (L_\varepsilon - \lambda)\psi^\lambda_j(y) &= 0, && y \in \mathbb{R}^d \setminus S_k, \\ \psi^\lambda_j(y) &= \phi^\lambda(y), && y \in \partial B_j, \\ \psi^\lambda_j(y) &= 0, && y \in \partial S_k \setminus B_j, \end{aligned} \tag{4.13}$$

In the sequel we will denote $h_i = h_{B_i, S_k \setminus B_i}$ for all $i \in \mathcal{M}$. With the decomposition (4.11) we can decompose the necessary condition (4.8) as follows

$$\begin{aligned}
0 &= \int_{\partial B_i} \mathrm{e}^{-V(y)/\varepsilon} (\partial_{n(y)} + \partial_{-n(y)}) f^\lambda(y) d\sigma_{S_k}(y) \\
&= \int_{\partial B_i} \mathrm{e}^{-V(y)/\varepsilon}\, h_i(y) (\partial_{n(y)} + \partial_{-n(y)}) f^\lambda(y) d\sigma_{S_k}(y) \\
&= \int_{\partial S_k} \mathrm{e}^{-V(y)/\varepsilon}\, \partial_{n(y)} h_i(y) f^\lambda(y) d\sigma_{S_k}(y) - \frac{\lambda}{\varepsilon} \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_i(y) f^\lambda(y) dy \\
&= \sum_{j \in J} c_j \left[\int_{\partial B_j} \mathrm{e}^{-V(y)/\varepsilon}\, \partial_{n(y)} h_i(y) (1 + \psi_j^\lambda(y)) d\sigma_{S_k}(y) - \frac{\lambda}{\varepsilon} \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_i(y) (h_j^\lambda(y) + \psi_j^\lambda(y)) dy \right] \\
&\quad + \sum_{j \in J^c} \left[\int_{\partial B_j} \mathrm{e}^{-V(y)/\varepsilon}\, \partial_{n(y)} h_i(y) \psi_j^\lambda(y) d\sigma_{S_k}(y) - \frac{\lambda}{\varepsilon} \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_i(y) \psi_j^\lambda(y) dy \right]
\end{aligned} \tag{4.14}$$

This decomposition of the necessary condition then motivates the following definition

Definition 4.6. *Define the capacity matrix $\mathcal{C}$ as the matrix of size $|\mathcal{M}| \times |\mathcal{M}|$ with elements*

$$\mathcal{C}_{ij} = \varepsilon \int_{\partial B_j} \mathrm{e}^{-V(y)/\varepsilon}\, h_j(y) \partial_{n(y)} h_i(y) d\sigma_{B_j}(y), \tag{4.15}$$

and its normalized version $\mathcal{K}$ with elements :

$$\mathcal{K}_{ij} = \frac{\mathcal{C}_{ij}}{\|h_i\|_2 \|h_j\|_2}. \tag{4.16}$$

Define further the auxiliary matrices

$$A_{ij} = \frac{\varepsilon}{\|h_i\|_2 \|h_j\|_2} \int_{\partial B_j} \mathrm{e}^{-V(y)/\varepsilon}\, \partial_{n(y)} h_i(y) \psi_j^\lambda(y) d\sigma_{\partial S_k}(y), \tag{4.17}$$

$$\begin{aligned}
B_{ij} &= \frac{1 - \delta_{ij}}{\|h_i\|_2 \|h_j\|_2} \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_i(y) (h_j^\lambda(y) + \psi_j^\lambda(y)) dy, \quad j \in J, \\
B_{ij} &= \frac{1 - \delta_{ij}}{\|h_i\|_2 \|h_j\|_2} \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_i(y) \psi_j^\lambda(y) dy, \qquad\qquad j \in J^c,
\end{aligned} \tag{4.18}$$

$$D_{ij} = \frac{\delta_{ij}}{\|h_j\|_2^2} \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_i(y) (h_j^\lambda(y) - h_j(y) + \psi_j^\lambda(y)) dy, \quad j \in J \tag{4.19}$$

Remark 4.7. Using (3.17), (3.19) and (3.20) then the diagonal elements of the capacity matrix $\mathcal{C}$ in (4.15) can be see as

$$\forall i \in \mathcal{M}\ ,\ \mathcal{C}_{ii} = \mathrm{cap}(B_i, S_k \setminus B_i). \tag{4.20}$$

With the notation of the Definition 4.6 and the necessary condition (4.14) on λ we get

Proposition 4.8. *If for all $i \in J$ we have*

$$\sum_{j \in J} \hat{c}_j \left(\mathcal{K}_{ij} - \lambda \delta_{ij} + A_{ij} - \lambda (D_{ij} + B_{ij}) \right) + \sum_{j \in J^c} \|h_j\|_2 \left(A_{ij} - \lambda B_{ij} \right) = 0, \tag{4.21}$$

where $\hat{c}_j = \|h_j\|_2 c_j$, then λ can be an eigenvalue of L_ε.

PROOF: We write (4.14) using the notations of definition 4.6, simplifying $\frac{\|h_i\|_2}{\varepsilon}$ and replacing $\|h_j\|_2 c_j$ by $\hat{c}_j$, we obtain the result. □

To continue our analysis we will need various estimates and properties of these matrices, starting with almost orthogonality on the functions h_i.

Lemma 4.9 (See [4], Lemma 4.5)**.** *We have*

1. *for all $i \neq j$*

$$\begin{aligned}\int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_i(y) h_j(y) dy &\leq C\varepsilon^{-1/2} \min\left(\mathrm{e}^{-\hat{V}(x_i, S_k \setminus B_i)/\varepsilon}, \mathrm{e}^{-\hat{V}(x_j, S_k \setminus B_j)/\varepsilon}\right) \\ &\leq C\varepsilon^{-d-\frac{1}{2}} \sqrt{\mathcal{C}_{ii}\mathcal{C}_{jj}},\end{aligned} \tag{4.22}$$

2. *for all j*

$$\int_{\mathbb{R}^d} \mathrm{e}^{-V(y)/\varepsilon}\, h_j(y)^2 dy = \mathrm{e}^{-V(x_j)/\varepsilon}\, \frac{(2\pi\varepsilon)^{d/2}}{\sqrt{\det(\nabla^2 V(x_j))}} (1 + \mathcal{O}(\sqrt{\varepsilon}|\ln\varepsilon|^{3/2})). \tag{4.23}$$

PROOF: See [4]. □

Then the auxiliary matrices (A_{ij}, B_{ij}, D_{ij})

Lemma 4.10 (See [4], Lemma 4.6)**.** *We have :*

1. *for all $j \in J$ and all i*

$$|A_{ij}| \leq C\varepsilon^{\alpha/2} |\mathcal{K}_{ij}|, \tag{4.24}$$

2. *for all $j \in J$*

$$|D_{jj}| \leq C\varepsilon^{\alpha/2}, \tag{4.25}$$

3. *for all $j \in J$ and all $i \neq j$*

$$|B_{ij}| \leq C\varepsilon^{-d-\frac{1}{2}} \sqrt{\mathcal{K}_{ii}\mathcal{K}_{jj}}, \tag{4.26}$$

4. *for all $j \in J^c$ and all $i \neq j$*

$$\|h_j\|_2 |B_{ij}| \leq C\varepsilon^{-d-\frac{1}{2}}\, \mathrm{e}^{-\gamma/2\varepsilon} \sqrt{\mathcal{K}_{ii}\mathcal{K}_{jj}} \tag{4.27}$$

PROOF: See [4]. □

Finally the two capacity matrices. The first two points comes also from [4] and the last point is also true in the asymmetric case but it will be very useful in our case.

Proposition 4.11. *The matrices $\mathcal{C}$ and $\mathcal{K}$ are symmetric and satisfy :*

1. *for all i, j*

$$\mathcal{C}_{ij} = \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, \varepsilon \langle \nabla h_i(y), \nabla h_j(y) \rangle dy, \tag{4.28}$$

2. *for all i, j*

$$|\mathcal{K}_{ij}| \leq \sqrt{\mathcal{K}_{ii}\mathcal{K}_{jj}}, \tag{4.29}$$

3. for all i

$$\sum_j \mathcal{C}_{ij} = 0. \tag{4.30}$$

PROOF: 1. As $h_j(y) = 0$ for all $y \in S_k \setminus B_j$ then

$$\mathcal{C}_{ij} = \varepsilon \int_{\partial S_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_j(y) \partial_{n(y)} h_i(y) d\sigma_{S_k}(y).$$

To use the first Green formula, as $n(y)$ is the exterior normal vector, one must move on to complementary to get the interior normal vector

$$\begin{aligned}\mathcal{C}_{ij} &= \varepsilon \int_{\partial S_k^c} \mathrm{e}^{-V(y)/\varepsilon}\, h_j(y) \partial_{-n(y)} h_i(y) d\sigma_{S_k^c}(y) \\ &= \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon} (\varepsilon \langle \nabla h_i(y), \nabla h_j(y) \rangle - h_j(y) L_\varepsilon h_i(y)) dy.\end{aligned}$$

As $L_\varepsilon h_i(y) = 0$ for all $y \in \mathbb{R}^d \setminus S_k$, we get the result.

2. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\mathcal{C}_{ij}| &= \left| \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, \varepsilon \langle \nabla h_i(y), \nabla h_j(y) \rangle dy \right| \leq \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, \varepsilon \left| \langle \nabla h_i(y), \nabla h_j(y) \rangle \right| dy \\ &\leq \left(\int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, \varepsilon \langle \nabla h_i(y), \nabla h_i(y) \rangle dy \right)^{1/2} \left(\int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, \varepsilon \langle \nabla h_j(y), \nabla h_j(y) \rangle dy \right)^{1/2} \\ &= \sqrt{\mathcal{C}_{ii} \mathcal{C}_{jj}}.\end{aligned}$$

Hence

$$|\mathcal{K}_{ij}| = \frac{|\mathcal{C}_{ij}|}{\|h_i\| \|h_j\|} \leq \frac{\sqrt{\mathcal{C}_{ii} \mathcal{C}_{jj}}}{\sqrt{\|h_i\|^2 \|h_j\|^2}} = \sqrt{\mathcal{K}_{ii} \mathcal{K}_{jj}}$$

3. Using the previous point, the linearity of the integral and of the gradient, we have

$$\sum_j \mathcal{C}_{ij} = \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, \varepsilon \langle \nabla h_i(y), \nabla \big(\sum_j h_j \big)(y) \rangle dy.$$

Due to the probabilistic interpretation of the equilibrium potentials (3.18), we know that

$$h_j(y) = \mathbb{P}^y[\tau_{B_j} < \tau_{S_k \setminus B_j}].$$

So, for all $y \in \mathbb{R}^d$

$$\sum_j h_j(y) = 1,$$

hence

$$\nabla \big(\sum_j h_j \big)(y) = 0.$$

Then

$$\sum_j \mathcal{C}_{ij} = \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, \varepsilon \langle \nabla h_i(y), 0 \rangle dy = 0.$$

□

By grouping all of these results we obtain

Theorem 4.12 (See [4], Theorem 4.7)**.** *Let S_k be defined as in Assumption 2.7 and let $\overline{\lambda}_k$ be the principal eigenvalue of the generator L_ε. Then a $\lambda < \overline{\lambda}_k$ can be an eigenvalue of the generator L_ε if there exist a non-empty set J and constants $\hat{c}_j$ such that $\sum_{j\in J} \hat{c}_j^2 = 1$ and for all $i \in J$ the relation* (4.21) *holds.*

4.2 Elements of the symmetric case

It is expected that the eigenvalues of the generator L_ε are close to the eigenvalues of the matrix $\mathcal{K}$. Since we are in the case where there may be symmetries, we cannot prove it as in [4]. For now, we will give some properties on the equilibrium potential h_j as well as the capacity matrices, due to the symmetries in the general case of several orbits.

Lemma 4.13. *For all $g \in G$ we have*

$$\forall x \in \mathbb{R}^d \ , \ \nabla V(g(x)) = g(\nabla V(x)). \tag{4.31}$$

Proof: Let $x \in \mathbb{R}^d$. Using the chain rule formula we get

$$\nabla V \circ g(x) = g^{-1}(\nabla V(g(x))).$$

Let $y = g(x)$, then using (2.9)

$$g(\nabla V(x)) = g(\nabla V(g^{-1}(y))) = \nabla V \circ g^{-1}(y) = \nabla V(g(x)).$$

□

Proposition 4.14. *Let X_t^u be the solution of* (1.1) *with initial condition $X_0^t = u$. Then $X_t^{g(x_0)}$ et $g(X_t^{x_0})$ have the same law.*

Proof: $X_t^{g(x_0)}$ and $X_t^{x_0}$ satisfy

$$X_t^{g(x_0)} = g(x_0) - \int_0^t \nabla V(X_u^{g(x_0)})du + \int_0^t \sqrt{2\varepsilon}dW(u),$$
$$X_t^{x_0} = x_0 - \int_0^t \nabla V(X_u^{x_0})du + \int_0^t \sqrt{2\varepsilon}dW(u).$$

Since g is a linear isometry

$$g(X_t^{x_0}) = g(x_0) - \int_0^t g(\nabla V(X_u^{x_0}))du + \int_0^t \sqrt{2\varepsilon}dW(u).$$

We know that Brownian motion is invariant under isometries, using Lemma 4.13, we get

$$g(X_t^{x_0}) = g(x_0) - \int_0^t \nabla V(g(X_u^{x_0}))du + \int_0^t \sqrt{2\varepsilon}dW(u).$$

□

Corollary 4.15. *Let $g \in G$ and $i \in \mathcal{M}$. Let $j = g(i)$, then*

$$\forall y \in \mathbb{R}^d \ , \ h_i(y) = h_j(g(y)). \tag{4.32}$$

Moreover

$$\|h_i\|_2 = \|h_j\|_2 \tag{4.33}$$

PROOF: By Proposition 4.14, $X_t^{g(x_0)}$ and $g(X_t^{x_0})$ have the same law so we deduce

$$\mathbb{P}^{x_0}\left[\tau_A < \tau_B\right] = \mathbb{P}^{g(x_0)}\left[\tau_{g(A)} < \tau_{g(B)}\right], \tag{4.34}$$

for all subsets $A, B \subset \mathbb{R}^d$, all $x_0 \in \mathbb{R}^d$ and all $g \in G$. Let $g \in G$, $i \in \mathcal{M}$ and $j = g(i)$. Using (3.18) and (4.34), we deduce

$$h_i(y) = \mathbb{P}^{g(y)}[\tau_{g(B_i)} < \tau_{g(S_k \setminus B_i)}].$$

Since $g(B_i) = B_j$ and $g(S_k \setminus B_i) = S_k \setminus B_j$, then we get (4.32). By a change of variable, we have

$$\|h_j\|_2^2 = \int_{\mathbb{R}^d} |\det J_g(z)|\, \mathrm{e}^{-V(g(z))/\varepsilon}\, h_j(g(z))^2 dz.$$

All the eigenvalues of a linear isometry have module 1, thus $|\det J_g(z)| = 1$. Using (2.9) and (4.32) we get (4.33). □

Proposition 4.16. *For all $g \in G$ and all i, j we have*

$$\mathcal{C}_{ij} = \mathcal{C}_{g(i)g(j)}, \tag{4.35}$$

$$\mathcal{K}_{ij} = \mathcal{K}_{g(i)g(j)}. \tag{4.36}$$

PROOF: Let i, j and $g \in G$. Using (4.28) and Corollary 4.15, we have

$$\mathcal{C}_{g(i)g(j)} = \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon}\, \varepsilon \langle \nabla(h_i \circ g^{-1})(y), \nabla(h_j \circ g^{-1})(y) \rangle dy.$$

By a change of variable, we get

$$\mathcal{C}_{g(i)g(j)} = \int_{\mathbb{R}^d \setminus S_k} |\det J_g(z)|\, \mathrm{e}^{-V(g(z))/\varepsilon}\, \varepsilon \langle \nabla(h_i \circ g^{-1})(g(z)), \nabla(h_j \circ g^{-1})(g(z)) \rangle dz.$$

All the eigenvalues of a linear isometry have module 1, thus $|\det J_g(z)| = 1$. Using the chain rule formula we have

$$\langle \nabla(h_i \circ g^{-1})(g(z)), \nabla(h_j \circ g^{-1})(g(z)) \rangle = \sum_{\alpha,\beta=1}^{d} \frac{\partial h_i}{\partial x_\alpha}(z) \frac{\partial h_j}{\partial x_\beta}(z) \sum_{k=1}^{d} \frac{\partial g_\alpha^{-1}}{\partial x_k}(z) \frac{\partial g_\beta^{-1}}{\partial x_k}(z).$$

As g is a linear isometry, g^{-1} is too, so

$$\langle \nabla(h_i \circ g^{-1})(g(z)), \nabla(h_j \circ g^{-1})(g(z)) \rangle = \sum_{\alpha,\beta=1}^{d} \frac{\partial h_i}{\partial x_\alpha}(z) \frac{\partial h_j}{\partial x_\alpha}(z) \delta_{\alpha\beta} = \langle \nabla h_i(z), \nabla h_j(z) \rangle.$$

Using (2.9), we deduce (4.35). Using Corollary 4.15 and (4.35) we get (4.36). □

5 Proofs

5.1 Irreducible representations of dimension 1

Let π be an irreducible representation of dimension 1 of G, $k \in [\![1, n_G]\!]$ such that A_k is active with respect to the representation π and $a \in A_k$. Let us consider the space

$$E_\pi = \{\phi : \mathbb{R}^d \to \mathbb{R} \mid \forall g \in G,\ \phi \circ g = \overline{\pi(g)}\phi\}. \tag{5.1}$$

Then we have

Proposition 5.1. *The space E_π is stable under L_ε.*

PROOF: Let $\phi \in E_\pi$, $g \in G$ and $x \in \mathbb{R}^d$. Since $\phi \in E_\pi$ and L_ε is a linear operator then

$$\overline{\pi(g)} L_\varepsilon \phi(x) = L_\varepsilon(\phi \circ g)(x) = -\varepsilon \Delta(\phi \circ g)(x) + \langle \nabla V(x), \nabla(\phi \circ g)(x) \rangle \tag{5.2}$$

Using the fact that g is an isometry and the chain rule formula we obtain

$$\begin{aligned} \langle \nabla V(x), \nabla(\phi \circ g)(x) \rangle &= \langle g^{-1}(\nabla V(g(x))), g^{-1}(\nabla \phi(g(x))) \rangle \\ &= \langle \nabla V(g(x)), \nabla \phi(g(x)) \rangle \end{aligned} \tag{5.3}$$

Using again the fact that g is an isometry and the chain rule formula, for all $i \in [\![1, d]\!]$ we obtain

$$\frac{\partial \phi \circ g}{\partial x_i^2}(x) = (J_g^{-1} H_\phi(g(x)) J_g)_{ii} \tag{5.4}$$

where $H_\phi(g(x))$ is the Hessian matrix of ϕ in $g(x)$ and J_g is the Jacobian matrix of g (remark that $g(x) = J_g x$). So

$$\Delta(\phi \circ g)(x) = \mathrm{Tr}(J_g^{-1} H_\phi(g(x)) J_g) = \mathrm{Tr}(H_\phi(g(x))) = \Delta\phi(g((x)) \tag{5.5}$$

Combining (5.3) and (5.5) we get the result. □

We see, in Theorem 4.12, that we need to find a set J and constants $\hat{c}_j$ such that for all $i \in J$ the relation (4.21) holds. So let $J = A_k$ and

$$\forall g \in G/_{G_a} \ , \ \hat{c}_{g(a)} = \Big(\sum_{h \in G/_{G_a}} \overline{\pi(h)}^2 \Big)^{-1/2} \overline{\pi(g)}. \tag{5.6}$$

Remark that we voluntarily leaves J instead of A_k in this subsection because in the next subsection the subset J is not equal to A_k in general.

Recall Definition 2.14, as in the proof of the fact that stabilisers of a given orbit are conjugated, it is easy to remark that

$$\forall k \in G \ , \ \mathcal{C}_{k(a)} = k\mathcal{C}_a. \tag{5.7}$$

Since E_π is stable, we focus on finding eigenfunctions of L_ε in E_π and their eigenvalues. Let us start by a technical lemma.

Lemma 5.2. *If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) = H(A_k, S_{k-1})$ then for all $j \in A_k \setminus \mathcal{C}_a$*

$$\mathcal{K}_{aj} = \frac{\mathrm{cap}(A_k, S_{k-1})}{\|h_a\|_2^2} \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}) \tag{5.8}$$

If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) < H(A_k, S_{k-1})$ then for all $j \in A_k \setminus \mathcal{C}_a$

$$\mathcal{K}_{aj} = -\frac{\mathrm{cap}(\mathcal{C}_a, S_k \setminus \mathcal{C}_a)}{\mathcal{N}_k^\pi |\mathcal{C}_a| \|h_a\|_2^2} (\mathbb{1}_{j \overset{\mathcal{M}_k}{\sim} a} + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) \tag{5.9}$$

where $\mathcal{N}_k^\pi = |\{g \in G/_{G_a} | \pi(g) \neq 1$ and $a \overset{\mathcal{M}_k}{\sim} g(a)\}|$.

We can deduce an estimate of the sum of elements of $\mathcal{K}$ in J.

Proposition 5.3. *For all $i \in J$ we have*

$$\sum_{j\in J} \mathcal{K}_{ij} = \frac{1}{|A_k|\|h_i\|_2^2}\operatorname{cap}(A_k, S_{k-1}) \tag{5.10}$$

PROOF: First remark that $\sum_{j\in J} h_j = h_{A_k,\mathcal{M}_{k-1}}$. Let $i \in J$ then

$$\sum_{j\in J} \mathcal{K}_{ij} = \frac{\varepsilon}{\|h_i\|_2^2}\int_{\partial A_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_{A_k,\mathcal{M}_{k-1}}(y)\partial_{n(y)}h_i(y)d\sigma_{A_k}(y). \tag{5.11}$$

Moreover, let $i' \in J$ then there exist $g \in G$ such that $g(i') = i$, and using Proposition 4.16 we obtain

$$\sum_{j\in J} \mathcal{K}_{i'j} = \sum_{j\in J} \mathcal{K}_{g(i')g(j)} = \sum_{j\in J} \mathcal{K}_{ig(j)} = \sum_{j\in J} \mathcal{K}_{ij}. \tag{5.12}$$

Hence, using Corollary 4.15 we get

$$\begin{aligned}\sum_{j\in J} \mathcal{K}_{ij} &= \frac{1}{|A_k|}\sum_{i',j\in J} \mathcal{K}_{i'j} = \frac{1}{|A_k|}\sum_{i'\in J}\frac{\varepsilon}{\|h_{i'}\|_2^2}\int_{\partial A_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_{A_k,\mathcal{M}_{k-1}}(y)\partial_{n(y)}h_{i'}(y)d\sigma_{A_k}(y)\\ &= \frac{\varepsilon}{|A_k|\|h_{i'}\|_2^2}\int_{\partial A_k} \mathrm{e}^{-V(y)/\varepsilon}\, h_{A_k,\mathcal{M}_{k-1}}(y)\partial_{n(y)}h_{A_k,\mathcal{M}_{k-1}}(y)d\sigma_{A_k}(y)\end{aligned} \tag{5.13}$$

□

Then, we can give an estimate of the sum of elements of $\hat{c}\mathcal{K}$ in J, which is the important point of the asymmetric case.

Proposition 5.4. *If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) = H(A_k, S_{k-1})$ then for all $i \in J$*

$$\sum_{j\in J} \hat{c}_j\mathcal{K}_{ij} = \frac{\hat{c}_i \operatorname{cap}(A_k, S_{k-1})}{|A_k|\|h_a\|_2^2}(1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) \tag{5.14}$$

If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) < H(A_k, S_{k-1})$ then for all $i \in J$

$$\sum_{j\in J} \hat{c}_j\mathcal{K}_{ij} = \frac{\hat{c}_i \operatorname{cap}(\mathcal{C}_a, S_k \setminus \mathcal{C}_a)}{\mathcal{N}_k^\pi|\mathcal{C}_a|\|h_a\|_2^2}\sum_{\substack{g\in G/G_a\\ g(a)\overset{\mathcal{M}_k}{\sim}a}} (1 - \overline{\pi(g)})(1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) \tag{5.15}$$

where $\mathcal{N}_k^\pi = |\{g \in G/_{G_a} | \pi(g) \neq 1$ and $a \overset{\mathcal{M}_k}{\sim} g(a)\}|$.

PROOF: Let $i \in J$ and $k \in G$ such that $i = k(a)$. Using Proposition 4.16 we can write the sum as

$$\sum_{j\in J} \hat{c}_j\mathcal{K}_{ij} = \overline{\pi(k)}\sum_{j\in J} \hat{c}_j\mathcal{K}_{aj}.$$

If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) = H(A_k, S_{k-1})$, using Proposition 5.3 and (5.8), we have

$$\begin{aligned}\sum_{j\in J} \hat{c}_j\mathcal{K}_{ij} &= \overline{\pi(k)}\hat{c}_a\sum_{j\in A_k} \mathcal{K}_{aj} + \overline{\pi(k)}\sum_{j\in A_k\setminus\mathcal{C}_a} (\hat{c}_j - \hat{c}_a)\mathcal{K}_{aj}\\ &= \frac{\hat{c}_i \operatorname{cap}(A_k, S_{k-1})}{|A_k|\|h_a\|_2^2} + \overline{\pi(k)}\sum_{j\in A_k\setminus\mathcal{C}_a} (\hat{c}_j - \hat{c}_a)\frac{\operatorname{cap}(A_k, S_{k-1})}{\|h_a\|_2^2}\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})\\ &= \frac{\hat{c}_i \operatorname{cap}(A_k, S_{k-1})}{|A_k|\|h_a\|_2^2}(1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}))\end{aligned}$$

If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) < H(A_k, S_{k-1})$, using the same argument as in Proposition 5.3 and (5.9), we have

$$\begin{aligned}
\sum_{j\in J} \hat{c}_j \mathcal{K}_{ij} &= \overline{\pi(k)}\hat{c}_a \sum_{j\in\mathcal{C}_a} \mathcal{K}_{aj} + \overline{\pi(k)} \sum_{j\in J\setminus\mathcal{C}_a} \hat{c}_j \mathcal{K}_{aj} \\
&= \frac{\overline{\pi(k)}\hat{c}_a \operatorname{cap}(\mathcal{C}_a, S_k\setminus\mathcal{C}_a)}{|\mathcal{C}_a|\|h_a\|_2^2} - \overline{\pi(k)} \sum_{j\in J\setminus\mathcal{C}_a} \frac{\hat{c}_j \operatorname{cap}(\mathcal{C}_a, S_k\setminus\mathcal{C}_a)}{\mathcal{N}_k^\pi |\mathcal{C}_a|\|h_a\|_2^2} (\mathbb{1}_{j \overset{\mathcal{M}_k}{\sim} a} + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) \\
&= \frac{\overline{\pi(k)} \operatorname{cap}(\mathcal{C}_a, S_k\setminus\mathcal{C}_a)}{\mathcal{N}_k^\pi |\mathcal{C}_a|\|h_a\|_2^2} (\mathcal{N}_k^\pi \hat{c}_a - \sum_{j\in J\setminus\mathcal{C}_a} \hat{c}_j (\mathbb{1}_{j \overset{\mathcal{M}_k}{\sim} a} + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}))) \\
&= \frac{\overline{\pi(k)} \operatorname{cap}(\mathcal{C}_a, S_k\setminus\mathcal{C}_a)}{\mathcal{N}_k^\pi (\sum_{h\in G/G_a} \overline{\pi(h)}^2)^{1/2} |\mathcal{C}_a|\|h_a\|_2^2} (\mathcal{N}_k^\pi - \sum_{\substack{g\in G/G_a \\ \pi(g)\neq 1}} \overline{\pi(g)} (\mathbb{1}_{g(a) \overset{\mathcal{M}_k}{\sim} a} + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}))) \\
&= \frac{\hat{c}_i \operatorname{cap}(\mathcal{C}_a, S_k\setminus\mathcal{C}_a)}{\mathcal{N}_k^\pi |\mathcal{C}_a|\|h_a\|_2^2} \sum_{\substack{g\in G/G_a \\ g(a) \overset{\mathcal{M}_k}{\sim} a}} (1 - \overline{\pi(g)})(1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}))
\end{aligned}$$

□

Now, it remains to control all the error terms. We need to get an estimate of the sums over the elements of the matrix A to get the result. Let us start by a technical lemma

Lemma 5.5. *For all $i \in J$ and all $g \in G$ we have*

$$\psi^\lambda_{g(i)} \circ g = \psi^\lambda_i, \tag{5.16}$$

where ψ^λ is the solution of (4.12).

Proof: Let $x \in \partial_{B_i}$. Then $g(x) \in \partial_{B_{g(i)}}$ so

$$\psi^\lambda_{g(i)}(g(x)) = \frac{\phi(g(x))}{c_{g(i)}} - 1 = \frac{\phi(x)}{c_i} - 1 = \psi^\lambda_i(x)$$

Let $x \in \partial_{B_j}$ with $j \neq i$. Then $g(x) \in \partial_{B_{g(j)}}$ and $g(j) \neq g(i)$ so

$$\psi^\lambda_{g(i)}(g(x)) = 0 = \psi^\lambda_i(x)$$

Let $x \in \mathbb{R}^d \setminus S_k$. Using the same argument as in Proposition 5.1 we get

$$(L_\varepsilon - \lambda)(\psi^\lambda_{g(i)} \circ g)(x) = (L_\varepsilon - \lambda)(\psi^\lambda_{g(i)})(g(x)) = 0.$$

So $\psi^\lambda_{g(i)} \circ g$ is a solution of a Dirichlet problem whose only solution is ψ^λ_i. □

Then we get an estimate of the sum over the elements of the matrix A.

Proposition 5.6. *For all $i, i' \in J$ we have*

$$\sum_{j\in J} A_{ij} = \sum_{j\in J} A_{i'j} \tag{5.17}$$

For all $i, i' \in \mathcal{C}_a$ we have

$$\sum_{j\in \mathcal{C}_a} A_{ij} = \sum_{j\in \mathcal{C}_a} A_{i'j} \tag{5.18}$$

PROOF: As in the first point of Proposition 4.11 (using the first Green identity (3.11)) we have

$$A_{ij} = \frac{\varepsilon}{\|h_i\|_2 \|h_j\|_2} \int_{\mathbb{R}^d \setminus S_k} \mathrm{e}^{-V(y)/\varepsilon} \langle \nabla h_i(y), \nabla \psi_j^\lambda(y) \rangle dy, \tag{5.19}$$

for all $i, j \in \mathcal{M}$. Then as in Proposition 4.16, using Corollary 4.15 and Lemma 5.5, we have

$$A_{g(i)g(j)} = A_{ij}, \tag{5.20}$$

for all $i, j \in \mathcal{M}$ and all $g \in G$. Hence we deduce the result. □

So we get an estimate of the sum over the elements $\hat{c}A$ in J.

Corollary 5.7. *If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) = H(A_k, S_{k-1})$ then for all $i \in J$ we have*

$$\left| \sum_{j \in J} \hat{c}_j A_{ij} \right| \leq \frac{C\varepsilon^{\alpha/2} \operatorname{cap}(A_k, S_{k-1})}{\|h_a\|_2^2} \tag{5.21}$$

If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) < H(A_k, S_{k-1})$ then for all $i \in J$ we have

$$\left| \sum_{j \in J} \hat{c}_j A_{ij} \right| \leq \frac{C\varepsilon^{\alpha/2} \operatorname{cap}(\mathcal{C}_a, S_k \setminus \mathcal{C}_a)}{\|h_a\|_2^2}. \tag{5.22}$$

PROOF: As in the proof of Proposition 5.3, using Corollary 4.15 and Proposition 5.6 we get

$$\begin{aligned} \sum_{j \in J} A_{ij} &= \frac{1}{|A_k|} \sum_{i,j \in J} A_{ij} = \frac{1}{|A_k|} \sum_{i \in J} \frac{\varepsilon}{\|h_i\|_2^2} \int_{\partial A_k} \mathrm{e}^{-V(y)/\varepsilon} \, \partial_{n(y)} h_i(y) \sum_{j \in J} \psi_j^\lambda(y) d\sigma_{S_k}(y) \\ &= \frac{\varepsilon}{|A_k| \|h_a\|_2^2} \int_{\partial A_k} \mathrm{e}^{-V(y)/\varepsilon} \, \partial_{n(y)} h_{A_k, \mathcal{M}_{k-1}}(y) \sum_{j \in J} \psi_j^\lambda(y) d\sigma_{S_k}(y), \end{aligned} \tag{5.23}$$

for all $i \in J$. By definition of J, for all $x \in \partial A_k$, we have $|\sum_{j \in J} \psi_j^\lambda(x)| \leq C\varepsilon^{\alpha/2}$ and moreover $\partial_{n(x)} h_{A_k, \mathcal{M}_{k-1}}(x) \geq 0$. So we get

$$\begin{aligned} \left| \sum_{j \in J} A_{ij} \right| &= \frac{\varepsilon}{|A_k| \|h_a\|_2^2} \left| \int_{\partial A_k} \mathrm{e}^{-V(y)/\varepsilon} \, \partial_{n(y)} h_{A_k, \mathcal{M}_{k-1}}(y) \sum_{j \in J} \psi_j^\lambda(y) d\sigma_{S_k}(y) \right| \\ &\leq \frac{C\varepsilon^{\alpha/2} \operatorname{cap}(A_k, S_{k-1})}{|A_k| \|h_a\|_2^2}. \end{aligned} \tag{5.24}$$

By the same argument

$$\begin{aligned} \left| \sum_{j \in \mathcal{C}_a} A_{ij} \right| &= \frac{\varepsilon}{|\mathcal{C}_a| \|h_a\|_2^2} \left| \int_{\partial \mathcal{C}_a} \mathrm{e}^{-V(y)/\varepsilon} \, \partial_{n(y)} h_{\mathcal{C}_a, \mathcal{M}_{k-1}}(y) \sum_{j \in \mathcal{C}_a} \psi_j^\lambda(y) d\sigma_{S_k}(y) \right| \\ &\leq \frac{C\varepsilon^{\alpha/2} \operatorname{cap}(\mathcal{C}_a, S_k \setminus \mathcal{C}_a)}{|\mathcal{C}_a| \|h_a\|_2^2}, \end{aligned} \tag{5.25}$$

for all $i \in \mathcal{C}_a$. Using (5.20) we can write

$$\sum_{j \in J} \hat{c}_j A_{ij} = \overline{\pi(k)} \sum_{j \in J} \hat{c}_j A_{aj} \tag{5.26}$$

where $k \in G$ such that $k(i) = a$. If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) = H(A_k, S_{k-1})$, using (5.24), Lemma 4.10 and (5.8) we have

$$\left|\sum_{j\in J} \hat{c}_j A_{ij}\right| \leq \left|\hat{c}_a \sum_{j\in J} A_{aj}\right| + \left|\sum_{j\in J} (\hat{c}_j - \hat{c}_a) A_{aj}\right| \leq \frac{C\varepsilon^{\alpha/2}\,\mathrm{cap}(A_k, S_{k-1})}{\|h_a\|_2^2}. \tag{5.27}$$

If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) < H(A_k, S_{k-1})$, using (5.25), Lemma 4.10 and (5.9) we have

$$\left|\sum_{j\in J} \hat{c}_j A_{ij}\right| \leq \left|\hat{c}_a \sum_{j\in \mathcal{C}_a} A_{aj}\right| + \left|\sum_{j\in J\setminus\mathcal{C}_a} \hat{c}_j A_{aj}\right| \leq \frac{C\varepsilon^{\alpha/2}\,\mathrm{cap}(\mathcal{C}_a, S_k \setminus \mathcal{C}_a)}{\|h_a\|_2^2}. \tag{5.28}$$

□

And an estimate of the sum over the elements $\hat{c}A$ in J^c.

Proposition 5.8. *If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) = H(A_k, S_{k-1})$ then for all $i \in J$ we have*

$$\left|\sum_{j\in J^c} \|h_j\|_2 A_{ij}\right| \leq \frac{C\,\mathrm{e}^{-\gamma/2\varepsilon}\,\mathrm{cap}(A_k, S_{k-1})}{\|h_a\|_2^2}. \tag{5.29}$$

If $H(\mathcal{C}_a, S_k \setminus \mathcal{C}_a) < H(A_k, S_{k-1})$ then for all $i \in J$ we have

$$\left|\sum_{j\in J^c} \|h_j\|_2 A_{ij}\right| \leq \frac{C\,\mathrm{e}^{-\gamma/2\varepsilon}\,\mathrm{cap}(\mathcal{C}_a, S_k \setminus \mathcal{C}_a)}{\|h_a\|_2^2}. \tag{5.30}$$

Proof: Let $i \in J$, using the fourth point of the Lemma 4.10 we obtain

$$\left|\sum_{j\in J^c} \|h_j\|_2 A_{ij}\right| \leq \frac{\varepsilon}{|A_k|} \sum_{l=1}^{k-1} \frac{C\,\mathrm{e}^{-\gamma/2\varepsilon}}{\|h_i\|_2 \|h_{j_l}\|_2} \left|\int_{\partial A_l} \mathrm{e}^{-V(y)/\varepsilon}\, \partial_{n(y)} h_{A_k, S_{k-1}}(y) d\sigma_{S_k}(y)\right|, \tag{5.31}$$

where j_l is an element of A_l. Now let split the sum over l which satisfies $V(j_l) < V(i)$ or not. Remark that $V(j_1) \leq V(i)$ always hold. If $V(j_l) \leq V(i)$ then by (4.22) we obtain

$$\frac{\|h_i\|_2}{\|h_{j_l}\|_2} \leq 1, \tag{5.32}$$

Since $\partial_{n(y)} h_{A_k, S_{k-1}}(y) \leq 0$ for all $y \in S_{k-1}$, by using (4.30) we get

$$\begin{aligned}
&\frac{\varepsilon}{|A_k|} \sum_{\substack{l=1 \\ V(j_1)\leq V(i)}}^{k-1} \frac{C\,\mathrm{e}^{-\gamma/2\varepsilon}}{\|h_i\|_2 \|h_{j_l}\|_2} \left|\int_{\partial A_l} \mathrm{e}^{-V(y)/\varepsilon}\, \partial_{n(y)} h_{A_k, S_{k-1}}(y) d\sigma_{S_k}(y)\right| \\
&\leq \frac{C\,\mathrm{e}^{-\gamma/2\varepsilon}\,\varepsilon}{|A_k| \|h_i\|_2^2} \left(\sum_{\substack{l=2 \\ V(j_1)>V(i)}}^{k-1} \int_{\partial A_l} \mathrm{e}^{-V(y)/\varepsilon}\, \partial_{n(y)} h_{A_k, S_{k-1}}(y) d\sigma_{S_k}(y) + \mathrm{cap}(A_k, S_{k-1}) \right)
\end{aligned} \tag{5.33}$$

Hence

$$\left|\sum_{j\in J^c}\|h_j\|_2 A_{ij}\right| \leq$$

$$\frac{C\,\mathrm{e}^{-\gamma/2\varepsilon}\,\varepsilon}{|A_k|\|h_i\|_2^2}\left(\sum_{\substack{l=2\\ V(j_1)>V(i)}}^{k-1}\left|\int_{\partial A_l}\mathrm{e}^{-V(y)/\varepsilon}\,\partial_{n(y)}h_{A_k,S_{k-1}}(y)d\sigma_{S_k}(y)\right|\left(\frac{\|h_i\|_2^2}{\|h_{j_l}\|_2}-1\right)+\mathrm{cap}(A_k,S_{k-1})\right) \tag{5.34}$$

Using Corolary 4.8 of [5] and the fact that limit preserve inequalities we get

$$\varepsilon\left|\int_{\partial A_l}\mathrm{e}^{-V(y)/\varepsilon}\,\partial_{n(y)}h_{A_k,S_{k-1}}(y)d\sigma_{S_k}(y)\right| \leq \frac{C}{\sqrt{\varepsilon}}\,\mathrm{e}^{-\hat{V}(A_k,A_l)/\varepsilon} \tag{5.35}$$

Using estimate of the capacity we get

$$\varepsilon\left|\int_{\partial A_l}\mathrm{e}^{-V(y)/\varepsilon}\,\partial_{n(y)}h_{A_k,S_{k-1}}(y)d\sigma_{S_k}(y)\right| \leq C\,\mathrm{cap}(A_k,S_{k-1})\,\mathrm{e}^{-\delta/\varepsilon}, \tag{5.36}$$

where $\delta < \hat{V}(A_k,A_l)-\hat{V}(A_k,S_{k-1})$. By the Assumption 2.7 we have

$$V(i)-V(j_l) < \hat{V}(A_k,A_l)-\hat{V}(A_k,S_{k-1}), \tag{5.37}$$

so we can chose δ between this two values. Finally using (4.22) we get

$$\varepsilon\left|\int_{\partial A_l}\mathrm{e}^{-V(y)/\varepsilon}\,\partial_{n(y)}h_{A_k,S_{k-1}}(y)d\sigma_{S_k}(y)\right|\left(\frac{\|h_i\|_2^2}{\|h_{j_l}\|_2}-1\right) \leq C\,\mathrm{cap}(A_k,S_{k-1})\,\mathrm{e}^{-\delta'/\varepsilon}, \tag{5.38}$$

where $\delta' = \delta - V(i)+V(j_l) > 0$, which lead to (5.29). If $H(\mathcal{C}_a, S_k\setminus\mathcal{C}_a) < H(A_k,S_{k-1})$, we get (5.30) by using

$$\mathrm{cap}(A_k,S_{k-1}) < C\,\mathrm{cap}(\mathcal{C}_a,S_k\setminus\mathcal{C}_a). \tag{5.39}$$

□

Combining these results we finally get

Theorem 5.9. *Let $(H1)$ be the condition $H(\mathcal{C}_a,S_k\setminus\mathcal{C}_a)=H(A_k,S_{k-1})$ and $(H2)$ be the condition $H(\mathcal{C}_a,S_k\setminus\mathcal{C}_a)<H(A_k,S_{k-1})$. For ε small enough, the spectrum of L_ε contain n_π (the number of active orbits with respect to the representation π) eigenvalues of geometric multiplicity 1 given by*

$$\lambda_k^\pi = \begin{cases} \frac{\mathrm{cap}(A_k,S_{k-1})}{|A_k|\|h_a\|_2^2}(1+\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon},\varepsilon^{\alpha/2})) & \text{if } (H1),\\ \frac{\mathrm{cap}(\mathcal{C}_a,S_k\setminus\mathcal{C}_a)}{n|\mathcal{C}_a|\|h_a\|_2^2}\sum\limits_{\substack{g\in G/G_a\\ g(a)\overset{\mathcal{M}_k}{\sim}a}}(1-\overline{\pi(g)})(1+\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon},\varepsilon^{\alpha/2})) & \text{if } (H2). \end{cases} \tag{5.40}$$

Moreover the associated normalized eigenfunction to λ_k^π is

$$\phi_k^\pi(y) = \sum_{g\in G/G_a}\hat{c}_{g(a)}\frac{h_{g(a)}(y)}{\|h_i\|_2}(1+\mathcal{O}(\varepsilon^{\alpha/2}))+\sum_{j\in J^c}\frac{h_j(y)}{\|h_j\|_2}\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}) \tag{5.41}$$

PROOF: Collecting Lemma 4.10, Proposition 5.3, Corollary 5.7 and Proposition 5.8 we have

- $\displaystyle\sum_{j\in J}\hat{c}_j(\mathcal{K}_{ij}-\lambda\delta_{ij}) = \begin{cases} \left(\frac{\hat{c}_i\,\mathrm{cap}(A_k,S_{k-1})}{|A_k|\|h_i\|_2^2}-\hat{c}_i\lambda\right)(1+\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) & \text{if } (H1),\\ \hat{c}_i\left(\frac{\mathrm{cap}(\mathcal{C}_a,S_k\setminus\mathcal{C}_a)}{n|\mathcal{C}_a|\|h_a\|_2^2}\sum\limits_{\substack{g\in G/G_a\\ g(a)\overset{\mathcal{M}_k}{\sim}a}}(1-\overline{\pi(g)})-\lambda\right)(1+\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) & \text{if } (H2),\end{cases}$
- $\displaystyle\sum_{j\in J}\hat{c}_jA_{ij} = \begin{cases}\frac{\mathrm{cap}(A_k,S_{k-1})}{\|h_a\|_2^2}\mathcal{O}(\varepsilon^{\alpha/2}) & \text{if } (H1),\\ \frac{\mathrm{cap}(\mathcal{C}_a,S_k\setminus\mathcal{C}_a)}{\|h_a\|_2^2}\mathcal{O}(\varepsilon^{\alpha/2}) & \text{if } (H2),\end{cases}$,
- $\displaystyle\sum_{j\in J}\hat{c}_j\lambda(D_{ij}+B_{ij}) = \lambda\mathcal{O}(\varepsilon^{\alpha/2},\mathrm{e}^{-\delta/\varepsilon})$,
- $\displaystyle\sum_{j\in J^c}\|h_j\|_2A_{ij} = \begin{cases}\frac{\mathrm{cap}(A_k,S_{k-1})}{\|h_a\|_2^2}\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}) & \text{if } (H1),\\ \frac{\mathrm{cap}(\mathcal{C}_a,S_k\setminus\mathcal{C}_a)}{\|h_a\|_2^2}\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}) & \text{if } (H2),\end{cases}$,
- $\displaystyle\sum_{j\in J^c}\|h_j\|_2\lambda B_{ij} = \lambda\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})$.

Put all theses estimates in (4.21) we get (5.40). To get (5.41) recall that ϕ_k^π satisfies (4.11) so let

$$g = \phi_k^\pi - \sum_{j\in J}\hat{c}_j\frac{h_j}{\|h_j\|_2} = \sum_{j\in J}c_j\psi_j^\lambda + \sum_{j\in J^c}\psi_j^\lambda. \tag{5.42}$$

Then the function g is the solution of the Dirichlet problem

$$\begin{aligned}(L_\varepsilon-\lambda)f(y) &= 0, && y\in\mathbb{R}^d\setminus S_k,\\ f(y) &= \phi_k^\pi(y)-c_j, && y\in\partial B_j \text{ and } j\in J,\\ f(y) &= \phi_k^\pi(y), && y\in\partial B_j \text{ and } j\in J^c.\end{aligned} \tag{5.43}$$

Using Poisson kernel of the functions g and h_j, the fact that $\partial_n G_{\Omega\setminus S_k}<0$, (4.9) and (4.10) we get

$$|g(y)| \le \sum_{j\in J}C\varepsilon^{\alpha/2}\hat{c}_j\frac{h_j(y)}{\|h_j\|_2} + \sum_{j\in J^c}C\varepsilon^{-d/4}\,\mathrm{e}^{-(\gamma-V(j))/2\varepsilon}\|h_j\|_2\frac{h_j(y)}{\|h_j\|_2}. \tag{5.44}$$

Using (4.23) we get

$$|g(y)| \le \sum_{j\in J}C\varepsilon^{\alpha/2}\hat{c}_j\frac{h_j(y)}{\|h_j\|_2} + \sum_{j\in J^c}C\,\mathrm{e}^{-\gamma/2\varepsilon}\,\frac{h_j(y)}{\|h_j\|_2}, \tag{5.45}$$

which give (5.41) □

Then using the same technique as in [4], we can improve the error estimate of Theorem 5.9, which lead to replacing $\varepsilon^{\alpha/2}$ by $\mathrm{e}^{-\delta/\varepsilon}$.

5.2 Irreducible representations of dimension larger than 1

Let π be an irreducible representation of G of dimension larger than 1, $k \in [\![1, n_G]\!]$ such that A_k is active with respect to the representation π. In this subsection $\chi(h) = \operatorname{Tr} \pi(h)$ denotes the characters of the irreducible representation π. The proofs in this section are essentially the same as in Section 5.1 and they will be left to the reader. The eigenspace analogue to (5.1) is the following :

$$E_\pi = \{\phi \in H^1(\mathbb{R}^d) \mid \forall h \in G,\ \forall a \in A_k\ ,\ \phi \circ h = \frac{\sum_{g \in G_a} \overline{\chi(gh)}}{\sum_{g \in G_a} \overline{\chi(g)}} \phi\}. \tag{5.46}$$

Proposition 5.10. *The space E_π is stable under L_ε.*

Proof: Same argument as in the proof of Proposition 5.1. □

Since E_π is stable, we focus on finding eigenfunctions of L_ε in E_π and their eigenvalues. In [1] we have seen that $\dim E_\pi = d\alpha$ where $\alpha = \frac{1}{|G_a|} \sum_{g \in G_a} \chi(g) \in [\![0, d]\!]$. So we need to choose $d\alpha$ elements of A_k such that if we define

$$J_l = \{h(a_l) \mid h \in G/_{G_{a_l}} \text{ and } \sum_{g \in G_{a_l}} \overline{\chi(gh)} \neq 0\}, \tag{5.47}$$

$$\hat{c}^l_{h(a_l)} = \frac{\sum_{g \in G_{a_l}} \overline{\chi(gh)}}{(\sum_{k \in G/_{G_{a_l}}} (\sum_{g \in G_{a_l}} \overline{\chi(gh)})^2)^{1/2}}, \tag{5.48}$$

for $l \in [\![1, d\alpha]\!]$, the l vectors $\hat{c}^l$ are linearly independent.

Recalling (2.29), remark that

$$\forall p \in G\ ,\ \mathcal{C}_{p(a)} = p\mathcal{C}_a. \tag{5.49}$$

Lemma 5.11. *For all $l \in [\![1, d\alpha]\!]$, if*

- *$H(\mathcal{C}_{a_l}, S_k \setminus \mathcal{C}_{a_l}) = H(A_k, S_{k-1})$ then for all $j \in A_k \setminus \mathcal{C}_{a_l}$*

$$\mathcal{K}_{a_l j} = \frac{\operatorname{cap}(A_k, S_{k-1})}{\|h_{a_l}\|_2^2} \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}) \tag{5.50}$$

- *$H(\mathcal{C}_{a_l}, S_k \setminus \mathcal{C}_{a_l}) < H(A_k, S_{k-1})$ then for all $j \in A_k \setminus \mathcal{C}_{a_l}$*

$$\mathcal{K}_{a_l j} = -\frac{\operatorname{cap}(\mathcal{C}_{a_l}, S_k \setminus \mathcal{C}_{a_l})}{\mathcal{N}_k^\pi |\mathcal{C}_{a_l}| \|h_{a_l}\|_2^2} (\mathbb{1}_{j \overset{\mathcal{M}_k}{\sim} a_l} + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) \tag{5.51}$$

where $\mathcal{N}_k^\pi = |\{p \in G/_{G_{a_l}} | \sum_{g \in G_{a_l}} \overline{\chi(gp)} \neq \sum_{g \in G_{a_l}} \overline{\chi(g)}$ and $a_1 \overset{\mathcal{M}_k}{\sim} p(a_1)\}|$.

Proof: Same proof as the proof of Lemma 5.2. □

Proposition 5.12. *If $H(\mathcal{C}_{a_1}, S_k \setminus \mathcal{C}_{a_1}) = H(A_k, S_{k-1})$ then for all $l \in [\![1, d\alpha]\!]$ and for all $i \in J_l$*

$$\sum_{j \in J_l} \hat{c}^l_j \mathcal{K}_{ij} = \frac{\hat{c}^l_i \operatorname{cap}(A_k, S_{k-1})}{|A_k| \|h_{a_l}\|_2^2} (1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) \tag{5.52}$$

If $H(\mathcal{C}_{a_1}, S_k \setminus \mathcal{C}_{a_1}) < H(A_k, S_{k-1})$ then for all $l \in [\![1, d\alpha]\!]$ and for all $i \in J_l$

$$\sum_{j\in J} \hat{c}_j^l \mathcal{K}_{ij} = \frac{\hat{c}_i^l \operatorname{cap}(\mathcal{C}_{a_1}, S_k \setminus \mathcal{C}_{a_1})}{\mathcal{N}_k^\pi |\mathcal{C}_{a_1}| \|h_{a_1}\|_2^2} \sum_{p\in G/G_{a_1}} \left(1 - \frac{\sum_{g\in G_{a_1}} \overline{\chi(gp)}}{\sum_{g\in G_{a_1}} \overline{\chi(g)}}\right)(1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon})) \tag{5.53}$$

where $\mathcal{N}_k^\pi = |\{p \in G/_{G_{a_l}} | \sum_{g\in G_{a_l}} \overline{\chi(gp)} \neq \sum_{g\in G_{a_l}} \overline{\chi(g)}$ and $a_1 \overset{\mathcal{M}_k}{\sim} p(a_1)\}|$.

Proof: Same arguments as in the proof of Proposition 5.4. □

Corollary 5.13. *For all $l \in [\![1, d\alpha]\!]$, if*

- *$H(\mathcal{C}_{a_l}, S_k \setminus \mathcal{C}_{a_l}) = H(A_k, S_{k-1})$ then for all $i \in J_l$ we have*

$$\left|\sum_{j\in J_l} \hat{c}_j^l A_{ij}\right| \leq \frac{C\varepsilon^{\alpha/2} \operatorname{cap}(A_k, S_{k-1})}{\|h_{a_l}\|_2^2} \tag{5.54}$$

- *$H(\mathcal{C}_{a_l}, S_k \setminus \mathcal{C}_{a_l}) < H(A_k, S_{k-1})$ then for all $i \in J_l$ we have*

$$\left|\sum_{j\in J_l} \hat{c}_j^l A_{ij}\right| \leq \frac{C\varepsilon^{\alpha/2} \operatorname{cap}(\mathcal{C}_{a_l}, S_k \setminus \mathcal{C}_{a_l})}{\|h_{a_l}\|_2^2}. \tag{5.55}$$

Proof: Same arguments as in the proof of Corollary 5.4. □

Proposition 5.14. *For all $l \in [\![1, d\alpha]\!]$, if*

- *$H(\mathcal{C}_{a_l}, S_k \setminus \mathcal{C}_{a_l}) = H(A_k, S_{k-1})$ then for all $i \in J_l$ we have*

$$\left|\sum_{j\in (J_l)^c} \|h_j\|_2 A_{ij}\right| \leq \frac{C\,\mathrm{e}^{-\gamma/2\varepsilon} \operatorname{cap}(A_k, S_{k-1})}{\|h_{a_l}\|_2^2}. \tag{5.56}$$

- *$H(\mathcal{C}_{a_l}, S_k \setminus \mathcal{C}_{a_l}) < H(A_k, S_{k-1})$ then for all $i \in J_l$ we have*

$$\left|\sum_{j\in (J_l)^c} \|h_j\|_2 A_{ij}\right| \leq \frac{C\,\mathrm{e}^{-\gamma/2\varepsilon} \operatorname{cap}(\mathcal{C}_{a_l}, S_k \setminus \mathcal{C}_{a_l})}{\|h_{a_l}\|_2^2}. \tag{5.57}$$

Proof: Same arguments as in the proof of Proposition 5.8. □

Theorem 5.15. *Let $(H1)$ be the condition $H(\mathcal{C}_{a_1}, S_k \setminus \mathcal{C}_{a_1}) = H(A_k, S_{k-1})$ and $(H2)$ be the condition $H(\mathcal{C}_{a_1}, S_k \setminus \mathcal{C}_{a_1}) < H(A_k, S_{k-1})$. For ε small enough, the spectrum of L_ε contain an eigenvalues of geometric multiplicity $d\alpha$ given by*

$$\lambda_k^{\pi^{(p)}} = \begin{cases} \frac{\operatorname{cap}(A_k, S_{k-1})}{|A_k| \|h_{a_1}\|_2^2}(1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}, \varepsilon^{\alpha/2})) & \text{if } (H1), \\ \frac{\operatorname{cap}(\mathcal{C}_{a_1}, S_k \setminus \mathcal{C}_{a_1})}{\mathcal{N}_k |\mathcal{C}_{a_1}| \|h_{a_1}\|_2^2} \sum\limits_{\substack{p\in G/_{G_{a_1}} \\ p(a_1) \overset{\mathcal{M}_k}{\sim} a_1}} \left(1 - \frac{\sum\limits_{g\in G_{a_1}} \overline{\chi^{(p)}(gp)}}{\sum\limits_{g\in G_{a_1}} \overline{\chi^{(p)}(g)}}\right)(1 + \mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}, \varepsilon^{\alpha/2})) & \text{if } (H2), \end{cases} \tag{5.58}$$

where $k \in [\![1, n_G]\!]$ such that A_k is active and

$$\mathcal{N}_k = \{p \in G/_{G_{a_1}} \mid \sum_{g \in G_{a_1}} \overline{\chi^{(p)}(gp)} \neq \sum_{g \in G_{a_1}} \overline{\chi^{(p)}(g)} \text{ and } a_1 \overset{\mathcal{M}_k}{\sim} p(a_1)\}. \tag{5.59}$$

Moreover the associated normalized eigenfunctions to $\lambda_k^{\pi^{(p)}}$ are

$$\phi_{k,l}^{\pi^{(p)}}(y) = \sum_{g \in G/G_{a_l}} \hat{c}^l_{g(a_l)} \frac{h_{g(a_l)}(y)}{\|h_{a_l}\|_2}(1 + \mathcal{O}(\varepsilon^{\alpha/2})) + \sum_{j \in S_k \setminus J_l^k} \frac{h_j(y)}{\|h_j\|_2}\mathcal{O}(\mathrm{e}^{-\delta/\varepsilon}), \tag{5.60}$$

where $\delta > 0$ and $h_i = h_{B_i, S_k \setminus B_i}$ as in (3.15).

Proof: Same arguments as in the proof of Theorem 5.9. □

Then using the same technique as in [4], we can improve the error estimate of Theorem 5.15, which leads to replacing $\varepsilon^{\alpha/2}$ by $\mathrm{e}^{-\delta/\varepsilon}$.

5.3 Capacity estimates

In the previous subsections we have obtained the expressions of all eigenvalues and eigenfunctions. To conclude it remains to compute the capacities $\mathrm{cap}(A_k, S_{k-1})$ and $\mathrm{cap}(\mathcal{C}_a, S_k \setminus \mathcal{C}_a)$ of Theorems 5.9 and 5.15. Let us start by the first capacity.

Theorem 5.16. *We have*

$$\mathrm{cap}(A_k, S_{k-1}) = \frac{n_k |A_k| (2\pi\varepsilon)^{d/2}}{2\pi} \frac{|\lambda_-^*(s)|}{\sqrt{|\det(\nabla^2 V(s))|}} \mathrm{e}^{-H(A_k, S_{k-1})/\varepsilon}(1 + \mathcal{O}(\sqrt{\varepsilon} \ln \varepsilon^{3/2})), \tag{5.61}$$

where $k \in [\![2, n_G]\!]$, $\nabla^2 V(x)$ denotes the Hessian matrix of V at x, n_k is the number of optimal saddles between any element a of A_k and M_{k-1}, s any of these saddles and $\lambda_-(s)$ is the unique negative eigenvalue of $\nabla^2 V(s)$.

Proof: By Assumption 2.8 the gate $\mathcal{G}^*(A_k, S_{k-1}) = \{s_1, \ldots, s_l\}$ is uniquely defined so, using Theorem 3.1 in [5] we get

$$\mathrm{cap}(A_k, S_{k-1}) = \mathrm{e}^{-H(A_k, S_{k-1})/\varepsilon} \frac{(2\pi\varepsilon)^{d/2}}{2\pi} \sum_{i=1}^{l} \frac{|\lambda_-(s_l)|}{\sqrt{|\det(\nabla^2 V(s_l))|}}(1 + \mathcal{O}(\sqrt{\varepsilon} \ln \varepsilon^{3/2})). \tag{5.62}$$

By Assumption 2.8 we have

$$\forall i, i' \in [\![1, l]\!] \ , \ \exists g \in G \ , \ g(s_i) = s_{i'}. \tag{5.63}$$

Since G is a group of isometries we get

$$\forall i, i' \in [\![1, l]\!] \ , \mathrm{Sp}\, \nabla^2 V(s_i) = \mathrm{Sp}\, \nabla^2 V(s_{i'}). \tag{5.64}$$

And by Assumption 2.8 we have $l = |A_k||\mathcal{G}^*(\{a\}, S_{k-1})|$ for any $a \in A_k$ which leads to claimed result. □

To get an estimate of the second capacity in the case where the gate is not uniquely defined, we need an estimate of capacity when we have two saddle point in series

Theorem 5.17. *Let us consider the case of two saddles points in series (see picture* 3 *in Figure 2), then we have*

$$\mathrm{cap}(A,B) = \mathrm{e}^{-H(A,B)/\varepsilon}\,\frac{(2\pi\varepsilon)^{d/2}}{2\pi}\frac{1}{4}\sum_{i=1}^{2}\frac{|\lambda_-(z_l)|}{\sqrt{|\det(\nabla^2 V(z_l))|}}(1+\mathcal{O}(\sqrt{\varepsilon}\ln\varepsilon^{3/2})). \tag{5.65}$$

Proof: The proof is the same as the proof of Theorem 3.1 in [5]. The key point is to take a functions $h \in \mathcal{H}_{A,B}$ with boundary conditions 1 in D_{x_1}, $\frac{1}{2}$ in D_{x_2} and 0 in D_{x_3} where x_1 is the local minimum A, x_3 is the local minimum B and x_2 is the last local minimum. □

Remark 5.18. The generalisation to multiple parallel configurations of two saddle points in series can be easily obtained and it will be left to the reader.

The last required capacity estimate is the following :

Theorem 5.19. *We have*

$$\mathrm{cap}(\mathcal{C}_a, S_k\setminus\mathcal{C}_a) = \frac{n_k|\mathcal{C}_a|\mathcal{D}}{2\pi}\frac{|\lambda_-(s)|}{\sqrt{|\det(\nabla^2 V(s))|}}\,\mathrm{e}^{-H(\mathcal{C}_a,\mathcal{M}_k\setminus\mathcal{C}_a)/\varepsilon}(1+\mathcal{O}(\sqrt{\varepsilon}\ln\varepsilon^{3/2})) \tag{5.66}$$

where $k \in [\![1, n_G]\!]$ such that A_k is active, $\nabla^2 V(x)$ denotes the Hessian matrix of V at x, a is an element of A_k, n_k is the number of optimal saddles between a and $\mathcal{M}_k\setminus\mathcal{C}_a$, s any of these saddle, $\lambda_-(s)$ is the unique negative eigenvalue of $\nabla^2 V(s)$ and

$$\mathcal{D} = \begin{cases} 1 & \text{if } \mathcal{G}^*(\mathcal{C}_a, S_k\setminus\mathcal{C}_a) \text{ is unique,} \\ \frac{1}{4} & \text{otherwise.} \end{cases} \tag{5.67}$$

Proof: The proof of the case when the gate $\mathcal{G}^*(\mathcal{C}_a, S_k\setminus\mathcal{C}_a) = \{s_1,\dots,s_l\}$ is uniquely defined is the same as in Theorem 5.16. In the case when the gate $\mathcal{G}^*(\mathcal{C}_a, S_k\setminus\mathcal{C}_a)$ is not uniquely defined, Assumption 2.8 ensures that all minimal paths between $x \in \mathcal{C}_a$ and $S_k\setminus\mathcal{C}_a$ only cross two saddle points at the same height, so we are in the case of multiple parallel configurations of two saddle points in series. Using Theorem 5.17 and the same argument as in the proof of Theorem 5.16 we deduce the formula in the case where the gate is not uniquely defined. □

5.4 Exit times

It remains to prove relation (2.19) on exit times. Let us recall a Proposition from [5]

Proposition 5.20 (Proposition 6.1, [5])**.** *Let $x \in \mathcal{M}\setminus S_{k-1}$. Then there exists $\alpha > 0$ such that*

$$\mathbb{E}^x\big[\tau_{S_{k-1}}\big] = \frac{\int_{S_{k-1}^c}\mathrm{e}^{-V(y)/\varepsilon}\,h_{B_\varepsilon(x),S_{k-1}}(y)dy}{\mathrm{cap}_(\,B_\varepsilon(x), S_{k-1})}(1+\mathcal{O}(\varepsilon^{\alpha/2})). \tag{5.68}$$

Proof: Same proof as in [5]. □

Then, using this Proposition, Theorem 5.16, an adaptation of (4.23) and the same technique as in [4] to improve the error estimate which leads to replacing $\varepsilon^{\alpha/2}$ by $\mathrm{e}^{-\delta/\varepsilon}$ we obtain the relation (2.19).

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