HAL open science

Directed polymer in random environment and two points state space

Philippe Carmona

▶ To cite this version:

Philippe Carmona. Directed polymer in random environment and two points state space. 2016. hal-01345436

HAL Id: hal-01345436
https://hal.science/hal-01345436v1

Preprint submitted on 15 Jul 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Directed polymer in random environment and two points state space

Philippe Carmona

Laboratoire de Mathématiques Jean Leray UMR 6629
Université de Nantes, 2 Rue de la Houssinière
BP 92208, F-44322 Nantes Cedex 03, France
e-mail: philippe.carmona@univ-nantes.fr

Abstract:
We give an exact expression for the partition function of a continuous time DPRE on a two points state space.

MSC 2010 subject classifications: Primary 60K35.
Keywords and phrases: Directed polymer, random environment, free energy.

1. Introduction

We are aware of just three[1] models of directed polymer in a random environment for which the free energy at finite inverse temperature is well known:

1. The discrete polymer in dimension $d = 1$ with simple random walk paths and a log-gamma environment and boundary conditions: see Seppäläinen (2012).
2. The continuous time directed polymer with Poisson process paths and a Brownian environment : see O'Connell and Yor (2001).
3. The continuum random polymer with brownian paths and a white noise environment : see Amir, Corwin and Quastel (2011).

The origin of this note is to find the simplest possible model where some direct computations can be performed. Of course, this model is much less interesting than the three models above. Note also that we are even unable to generalize it to a three points state space !
Let us consider the simplest model of continuous time directed polymer in a random environment. Let $\omega = (\omega(t))_{t\geq 0}$ be the continuous time Markov chain on $\{1,2\}$ with generator:

$$Lf(1) = f(2) - f(1), \quad Lf(2) = f(1) - f(2), \tag{1.1}$$

that is the chain that spends an exponential time on 1 (resp 2) and the jumps on 2 (resp. 1). We let P_i denote the law of the Markov chain starting from i, E_i the associated expectation, and $(W, \mathscr{W})$ the path space of piecewise constant cadlag functions from $[0, +\infty[$ to $\{1,2\}$. We set $P = P_1$ and $E = E_1$.
The random environment consists of two independent standard Brownian motions $(B_i(t), t \geq 0)$ defined on another probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
For any $t > 0$ the (random) *polymer measure* μ_t is the probability defined on the path space $(W, \mathscr{W})$ by

$$\mu_t(d\omega) = \frac{1}{Z_t} e^{\beta H_t(\omega) - t\beta^2/2} \mathbb{P}(d\omega)$$

where $\beta \geq 0$ is the inverse temperature, the Hamiltonian is

$$H_t(\omega) = \int_0^t dB_{\omega(s)}(s)$$

[1] If some interested reader finds another example where an exact computation of the free energy occurs, we are more than willing to incorporate it in this list

and the *partition function* is

$$Z_t = Z_t(\beta) = E\Big[e^{\beta H_t(\omega) - t\beta^2/2}\Big],$$

It is well known, and for example established in (Carmona and Hu, 2006, Proposition 2.4) that the *point to point partition function*

$$Z_t(x,y) := E_x\Big[e^{\beta H_t(\omega) - t\beta^2/2}\,\mathbf{1}_{(\omega(t)=y)}\Big] \tag{1.2}$$

satisfy the *Discrete Stochastic Heat Equation*:

$$dZ_t(x,y) = LZ_t(x,.)(y)\,dt + \beta Z_t(x,y)dB_y(t), \tag{1.3}$$

in the Itô sense with inital conditions $Z_0(x,y) = \delta_x(y)$ (from now on we shall fix as a starting point $x = 1$). Furthermore, the free energy is well defined

$$p(\beta) := \lim_{t\to+\infty} \frac{1}{t}\log Z_t(\beta) \qquad \text{(a.s. and in } L^1(\mathbb{P})), \tag{1.4}$$

and is given by the limit (see (Carmona and Hu, 2006, Formula (15)))

$$p(\beta) = -\frac{\beta^2}{2}\lim_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}[I_s]\,ds, \tag{1.5}$$

with I_t the *overlap*

$$I_t = \mu_t^{\otimes 2}(\omega_1(t) = \omega_2(t)) = \frac{1}{Z_t^2}\sum_{y=1}^{2} Z_t(1,y)^2.$$

Theorem 1.1. *For this model of DPRE the mean ovelap converges:*

$$\lim_{t\to+\infty}\mathbb{E}[I_t] = \alpha_-(\beta) \tag{1.6}$$

with $\alpha_-(\beta) \in (0,1)$ *the smallest root of the polynomial*

$$P_\beta(X) = 3\beta^2X^2 - (5\beta^2+4)X + 2(1+\beta^2).$$

Consequently the free energy is

$$p(\beta) = -\frac{\beta^2}{2}\alpha_-(\beta).$$

Observe that as $\beta \to 0^+$, $\alpha_-(\beta) \to \frac{1}{2}$ as expected.

2. Proof of Theorem 1.1

To simplify notations we let $X_i(t) = Z_t(1,i)$ and set $Z_0 = 1$ so that we have $X_1(0) = 1$, $X_2(0) = 0$, $I_0 = 1$ and (X_1, X_2) is solution of the following *simple* system of stochastic differential equations:

$$\begin{cases} dX_1(t) = (X_2 - X_1)\,dt + \beta X_1 dB_1(t) \\ dX_2(t) = -(X_2 - X_1)\,dt + \beta X_2 dB_2(t) \end{cases} \tag{2.1}$$

It is easy to check that $Z_t = X_1(t) + X_2(t)$ is a martingale

$$dZ_t = \beta(X_1 dB_1(t) + X_2 dB_2(t))$$

with quadratic variation

$$d\langle Z\rangle_t = \beta^2(X_1^2 + X_2^2)\,dt = \beta^2 N_t\,dt = \beta^2 Z_t^2 I_t\,dt,$$

where we have set

$$N_t = X_1(t)^2 + X_2(t)^2 \quad \text{so that} \quad I_t = \frac{N_t}{Z_t^2}.$$

Without having to read Carmona and Hu (2006), one can infer directly that

$$\log Z_t = \int_0^t \frac{dZ_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle Z\rangle_s}{Z_s^2} = \int_0^t \frac{dZ_s}{Z_s} - \frac{\beta^2}{2}\int_0^t I_s\, ds$$

so that

$$\frac{1}{t}\mathbb{E}[\log Z_t] = -\frac{\beta^2}{2}\frac{1}{t}\int_0^t \mathbb{E}[I_s]\, ds.$$

Let us do now some straightforward computations using Ito's formula

$$\begin{aligned}
dX_1^2(t) &= ((\beta^2-2)X_1^2 + 2X_1X_2)\,dt + 2\beta X_1^2 dB_1\\
dN_t &= (4X_1X_2 + (\beta^2-2)N_t)dt + 2\beta(X_1^2 dB_1 + X_2^2 dB_2) = (2Z_t^2 + (\beta^2-4)N_t)\,dt + 2\beta(X_1^2 dB_1 + X_2^2 dB_2)\\
d\langle N, Z\rangle_t &= 2\beta^2(X_1^3 + X_2^3)dt = \beta^2(3Z_tN_t - Z_t^3)\,dt = \beta^2 Z_t^3(3I_t - 1)\,dt.
\end{aligned}$$

In the last equation, we use the identity $2(a^3+b^3) = 3(a+b)(a^2+b^2) - (a+b)^3$. It is the only place where we really use the fact that the state space has only two points.
Let use the notation $U_t \sim V_t$ if $U_t - V_t$ is a martingale. We have:

$$\begin{aligned}
dI_t &= \frac{dN_t}{Z_t^2} - \frac{2N_t dZ_t}{Z_t^3} - 2\frac{d\langle N,Z\rangle_t}{Z_t^3} + 3N_t\frac{d\langle Z\rangle_t}{Z_t^4}\\
&\sim ((\beta^2-4)I_t + 2 - 2\beta^2(3I_t-1) + 3\beta^2 I_t^2)\,dt\\
&\sim (3\beta^2 I_t^2 - (5\beta^2+4)I_t + 2(1+\beta^2))dt = P(I_t)\,dt
\end{aligned} \tag{2.2}$$

with $a = \frac{5\beta^2+4}{6\beta^2} = \frac{5}{6} + \frac{2}{3\beta^2}$ and

$$P(X) = 3\beta^2(X^2 - 2aX + a - \frac{1}{6}) = 3\beta^2(X-\alpha_+)(X-\alpha_-)$$

with $\alpha_\pm := a \pm \sqrt{a^2 - a + \frac{1}{6}}$. Since $P(1) = 3\beta^2(1 - a - \frac{1}{6}) = -2$ we have $\alpha_- < 1 < \alpha_+$.
We now take expectations in (2.2) and get

$$\frac{d\mathbb{E}[I_t]}{dt} = \mathbb{E}[P(I_t)]. \tag{2.3}$$

Since $I_0 = 1$, and $P(1) < 0$, at least on a non empty interval $[0,\delta[$, the function $u(t) := \mathbb{E}[I_t]$ is non increasing (something we expected since $0 \le I_t \le 1$ donc $0 \le u(t) \le 1$).
Assume that there exists $t_0 > 0$ such that $u(t_0) < \alpha_-$. Since $\mathbb{E}\left[I_t^2\right] \ge \mathbb{E}[I_t]^2$ we have

$$u'(t) = \mathbb{E}[P(I_t)] \ge P(u(t)).$$

Let $T = \sup\{t < t_0 : u(t) = \alpha_-\}$. On $]T, t_0[$ we have $u'(t) \ge P(u(t)) > 0$ so u strictly increases on $]T, t_0[$, which is absurd since $u(T) = \alpha_-$ and $u(t_0) < \alpha_-$.
Therefore, we have:

$$\forall t > 0, u(t) \ge \alpha_-.$$

Observe that:

$$\begin{aligned}
u'(t) &= \mathbb{E}[P(I_t)] = \mathbb{E}[P(I_t) - P(\alpha_-)] = \mathbb{E}\left[3\beta^2(I_t^2 - \alpha_-^2) - (5\beta^2+4)(I_t - \alpha_-)\right]\\
&= 3\beta^2\mathbb{E}[(I_t+\alpha_-)(I_t-\alpha_-)] - (5\beta^2+4)\mathbb{E}[I_t - \alpha_-]\\
&\le \left(3\beta^2(1+\alpha_-) - (5\beta^2+4)\right)\mathbb{E}[I_t - \alpha_-] = -\lambda\mathbb{E}[I_t - \alpha_-].
\end{aligned}$$

We shall check that $\lambda > 0$, and this implies, by Gronwall's Lemma,

$$(u(t) - \alpha_-) \leq (u(0) - \alpha_-)e^{-\lambda t} \to 0 \quad \text{when } t \to +\infty\,.$$

It remains to prove that $\lambda = -3\beta^2(1+\alpha_-) + (5\beta^2+4) > 0$ that is $1+\alpha_- < 2a$, i.e. $\sqrt{a^2 - a + \frac{1}{6}} > 1-a$. Either $\beta \leq 2$ and then $a \geq 1$ and we are done, or $\beta > 2$, $a < 1$, and we have to show that $a^2 - a + \frac{1}{6} > (1-a)^2$ i.e. $a > 5/6$ which is true.