

# A test of Gaussianity based on the Euler characteristic of excursion sets - Supplementary material

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## Introduction

The only purpose of this note is to provide the technical proofs of the results that are presented in our paper “A test of Gaussianity based on the Euler characteristic of excursion sets”. In order to make the present note selfcontained, the setting and the main environment of the referred-to paper has been reproduced here.

**Outline of the note.** Section 1 contains the general setting, in particular the definition of the Euler characteristic (EC) of excursion sets. In Section 2, we focus on the Gaussian hypothesis. Assuming that the field under study is a stationary Gaussian field, we give explicit formulas for the first two moments of the excursions EC (the first moment formula is well known, the second moment formula is new, see Proposition 1 and Proposition 2). We also recall the Central Limit Theorem satisfied by the excursions EC when the domain tends to  $\mathbb{R}^d$  (see [4]) and give an extension for the joint excursions EC concerned with different levels and disjoint domains, see Proposition 4. Section 3 is concerned with the study of an alternative distribution for the observed random field, namely  $\chi^2$ . Section 4 is devoted to the explicit computation of the variance of the excursion EC for an univariate process, i.e. in the one dimensional case.

## 1 Setting

All over the paper, we consider a real valued random field  $X$  defined on  $\mathbb{R}^d$  that satisfies the following assumption.

**Assumption (A):** The random field  $X$  is stationary, isotropic,  $\mathbb{E}(X(0)) = 0$ ,  $\text{Var}(X(0)) = 1$  and almost all realisations belong to  $C^3(\mathbb{R}^d)$ . Let  $\nabla^2 X(t)$  stand for the  $\frac{1}{2}d(d+1)$  random vector that contains the upper coefficients of the symmetric Hessian matrix  $X''(t)$  and  $\mathbf{X}(t)$  for the  $d + \frac{1}{2}d(d+1) + 1$  random vector  $(X'(t), \nabla^2 X(t), X(t))$ . For any fixed  $t$  in  $\mathbb{R}^d$ ,

the covariance matrix of the random vector  $(\mathbf{X}(0), \mathbf{X}(t))$  has full rank.

We denote by  $r$  the covariance function of  $X$ ,  $r(t) = \text{Cov}(X(0), X(t))$ ,  $t \in \mathbb{R}^d$ , which belongs to  $C^6(\mathbb{R}^d)$ . The isotropy assumption implies that the Hessian matrix  $r''(0)$  is equal to  $-\lambda I_d$  for some  $\lambda > 0$ , usually named as *second spectral moment*. At last,  $r$  is such that,

$$\psi(t) \rightarrow 0 \text{ when } \|t\| \rightarrow +\infty \text{ and } \psi \in L^1(\mathbb{R}^d),$$

where  $\psi(t) = \max \left( \left| \frac{\partial^{\mathbf{k}} r}{\partial t^{\mathbf{k}}} (t) \right| ; \mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d, k_1 + \dots + k_d \leq 4 \right)$ .

### Notations.

• for any  $u \in \mathbb{R}$  and any compact  $T \subset \mathbb{R}^d$ , we call “excursion set of  $X$  above the level  $u$  within the domain  $T$ ” the following set  $\{t \in T : X(t) \geq u\}$ ,

- $p_Z(\cdot)$  denotes the probability density function of any random vector  $Z$  (assuming it exists),
- $|\cdot|$  denotes without any ambiguity, either the absolute value, or the  $d$ -dimensional Lebesgue measure.

**Euler characteristic.**

The Euler characteristic of a compact domain  $K$  in  $\mathbb{R}^d$  can be heuristically defined in the case  $d = 1$  as the number of connected components of  $K$ , or in the case  $d = 2$  as the number of connected components minus the number of holes in  $K$ . In the case where  $K$  is an excursion set  $\{t \in T : X(t) \geq u\}$ , with  $T$  a rectangle in  $\mathbb{R}^d$  and  $u$  a real number, there exists a rather tractable formula that uses the theory of Morse functions, see [1] Chapter 9 for instance. It states that the Euler characteristic of  $\{t \in T : X(t) \geq u\}$  is equal to a sum of two terms. The first one only depends on the restriction of  $X$  to the interior of  $T$ , it is given by the quantity  $\varphi(X, T, u)$  defined in Equation (1) below. The second one exclusively depends on the behaviour of  $X$  on the  $l$ -dimensional faces of  $T$ , with  $0 \leq l < d$ . From now on, we focus on the term  $\varphi(X, T, u)$ , named as “modified Euler characteristic” in [4], and we still call it Euler characteristic (EC). It is defined by the following,

$$\begin{aligned} \varphi(X, T, u) &= \sum_{k=0}^d (-1)^k \mu_k(T, u), \quad \text{where} & (1) \\ \mu_k(T, u) &= \#\{t \in \overset{\circ}{T} : X(t) \geq u, X'(t) = 0, \text{index}(X''(t)) = d - k\}. \end{aligned}$$

**Special case 1: dimension one.** When  $d = 1$ , writing  $[0, T]$  instead of  $T$  for a while, Equation (1) becomes

$$\begin{aligned} \varphi(X, [0, T], u) &= \#\{\text{local maxima of } X \text{ above } u \text{ in } (0, T)\} & (2) \\ &\quad - \#\{\text{local minima of } X \text{ above } u \text{ in } (0, T)\}. \end{aligned}$$

Morse’s theorem says that this quantity is linked with the number of up-crossings,

$$U(X, [0, T], u) = \#\{t \in [0, T] : X(t) = u, X'(t) \geq 0\},$$

by the following

$$\varphi(X, [0, T], u) + \mathbf{1}_{\{X(0) > u, X'(0) < 0\}} + \mathbf{1}_{\{X(T) > u, X'(T) > 0\}} = U(X, [0, T], u) + \mathbf{1}_{\{X(0) > u\}}. \quad (3)$$

Taking expectation in both expressions and using stationarity yield the next formula that we will use in Sections 3 and 4,

$$\mathbb{E}[\varphi(X, [0, T], u)] = \mathbb{E}[U(X, [0, T], u)].$$

**Special case 2: dimension two.** When  $d = 2$ , Equation (1) can be expressed in a more simple way. Since  $\mu_0(T, u)$  denotes the number of local maxima above  $u$ ,  $\mu_2(T, u)$  denotes the number of local minima above  $u$  and  $\mu_1(T, u)$  the number of local saddle points above  $u$ , we get

$$\begin{aligned} \varphi(X, T, u) &= \#\{\text{local extrema of } X \text{ above } u \text{ in } \overset{\circ}{T}\} & (4) \\ &\quad - \#\{\text{local saddle points of } X \text{ above } u \text{ in } \overset{\circ}{T}\}. \end{aligned}$$

## 2 Under Gaussian hypothesis

In this section, we assume that  $X$  is Gaussian and satisfies Assumption described in Section 1.

## 2.1 First two moments of the Euler characteristic of an excursion set

Let  $T$  be a cube in  $\mathbb{R}^d$ . This section is devoted to explicit formulas for the first two moments of  $\varphi(X, T, u)$ . They are based on the decomposition (1) and on Rice formulas for the factorial moments of  $\mu_k(T, u)$  (see for instance [1] Chapter 11 or [3] Chapter 6).

In particular, using the stationarity of  $X$ , the expectation can be computed as follows

$$\begin{aligned}\mathbb{E}[\varphi(X, T, u)] &= (-1)^d \int_T \mathbb{E}[\mathbf{1}_{[u, \infty)}(X(t)) \det(X''(t)) | X'(t) = 0] p_{X'(t)}(0) dt \\ &= (-1)^d |T| (2\pi\lambda)^{-d/2} \mathbb{E}[\mathbf{1}_{[u, \infty)}(X(0)) \det(X''(0))].\end{aligned}$$

Moreover, it is proved in [1] Lemma 11.7.1, through a regression and thanks to Wick's formula, that

$$\mathbb{E}[\mathbf{1}_{[u, \infty)}(X(0)) \det(X''(0))] = (-1)^d (2\pi)^{-1/2} \lambda^d H_{d-1}(u) e^{-u^2/2},$$

where  $H_k$  stands for the Hermite polynomial of order  $k$ . Hence, the next formula holds

$$\mathbb{E}[\varphi(X, T, u)] = |T| (2\pi)^{-(d+1)/2} \lambda^{d/2} H_{d-1}(u) e^{-u^2/2}. \quad (5)$$

In what follows, we will be particularly interested in the next function

$$C(u) = (2\pi)^{-(d+1)/2} \lambda^{d/2} H_{d-1}(u) e^{-u^2/2}, \quad (6)$$

that yields  $\mathbb{E}[\varphi(X, T, u)] = |T| C(u)$ . Equation (6) shows that  $C(u)$  implicitly depends on  $X$  through its dimension parameter  $d$  and its second spectral moment  $\lambda$ . Whenever necessary in the next sections, we will emphasize this dependence by writing  $C(u) = C(u, \lambda)$ .

For the second moment, a so nice formula as (5) seems to be out of reach. Nevertheless, in the next proposition, we provide the second moment as an integral that can be numerically evaluated (see also [10] for another formula, which is valid under restrictive assumptions on  $X$ ). We will use the following functions, defined for  $u \in \mathbb{R}$  and  $t \in \mathbb{R}^d$ ,

$$\left. \begin{aligned}g(u) &= \mathbb{E}[\mathbf{1}_{[u, \infty)}(X(0)) | \det(X''(0))] \\ D(t) &= (2\pi)^{2d} \det(\lambda^2 I_d - r''(t)^2) \\ G(u, t) &= \mathbb{E}[\mathbf{1}_{[u, \infty)}(X(0)) \mathbf{1}_{[u, \infty)}(X(t)) \det(X''(0)) \det(X''(t)) | X'(0) = X'(t) = 0]\end{aligned}\right\} \quad (7)$$

**Proposition 1.** *Assume that  $X$  is Gaussian and satisfies Assumption **(A)**.*

*Then, for any  $u \in \mathbb{R}$ , the map  $t \mapsto G(u, t) D(t)^{-1/2}$  is integrable on any compact set in  $\mathbb{R}^d$  and*

$$\mathbb{E}[\varphi(X, T, u)^2] = \int_{\mathbb{R}^d} |T \cap (T - t)| G(u, t) D(t)^{-1/2} dt + |T| (2\pi\lambda)^{-d/2} g(u).$$

**Proof.** Integrability comes from [4] Proposition 1.1 since  $X \in C^3$ . In order to compute the expectation of  $\varphi(X, T, u)^2$ , let us start with (1). It yields

$$\varphi(X, T, u)^2 = \sum_{0 \leq k \leq d} \mu_k(T, u) + \sum_{0 \leq k \leq d} \mu_k(T, u) (\mu_k(T, u) - 1) + \sum_{0 \leq k \neq l \leq d} (-1)^{k+l} \mu_k(T, u) \mu_l(T, u).$$

The expectation of the first term is equal to

$$\begin{aligned}\mathbb{E}\left[\sum_{0 \leq k \leq d} \mu_k(T, u)\right] &= \mathbb{E}[\#\{t \in T : X(t) \geq u, X'(t) = 0\}] \\ &= \int_T \mathbb{E}[\mathbf{1}_{[u, \infty)}(X(t)) | \det X''(t) | X'(t) = 0] p_{X'(t)}(0) dt \\ &= |T| g(u) p_{X'(0)}(0),\end{aligned}$$

where we have used Rice Formula to get the second line and stationarity as well as the independence between  $X'(0)$  and  $(X(0), X''(0))$  to get the third one.

For the second and third terms, we introduce, for  $k, l = 1, \dots, d$  and  $s, t \in \mathbb{R}^d$ ,

$$F_u^{k,l}(s, t) = \mathbb{E}[\mathbf{1}_{[u, \infty)}(X(s)) | \det X''(s)| \mathbf{1}_{D_k}(X''(s)) \\ \times \mathbf{1}_{[u, \infty)}(X(t)) | \det X''(t)| \mathbf{1}_{D_l}(X''(t)) | X'(s) = X'(t) = 0],$$

where  $D_k$  denotes the set of symmetric matrices having index equal to  $d - k$ . Hence, Rice Formula for the second factorial moment allows us to obtain

$$\mathbb{E}\left[\sum_{0 \leq k \leq d} \mu_k(T, u)(\mu_k(T, u) - 1)\right] = \int_{T \times T} \left(\sum_{0 \leq k \leq d} F_u^{k,k}(s, t)\right) p_{X'(s), X'(t)}(0, 0) ds dt.$$

Given that  $D_k \cap D_l = \emptyset$  for  $k \neq l$ , one can adapt the proof of the second moment Rice formula to get

$$\mathbb{E}\left[\sum_{0 \leq k \neq l \leq d} (-1)^{k+l} \mu_k(T, u) \mu_l(T, u)\right] \\ = \int_{T \times T} \left(\sum_{0 \leq k \neq l \leq d} (-1)^{k+l} F_u^{k,l}(s, t)\right) p_{X'(s), X'(t)}(0, 0) ds dt.$$

Let us remark that

$$\sum_{0 \leq k \leq d} F_u^{k,k}(s, t) + \sum_{0 \leq k \neq l \leq d} (-1)^{k+l} F_u^{k,l}(s, t) = G(u, t - s).$$

It yields

$$\mathbb{E}[\varphi(X, T, u)^2] = \int_{\mathbb{R}^d} |T \cap (T - t)| G(u, t) p_{X'(0), X'(t)}(0, 0) dt + |T| g(u) p_{X'(0)}(0).$$

It remains to compute the probability density function of  $(X'(0), X'(t))$ . The covariance matrix of this vector is equal to  $\begin{pmatrix} \lambda I_d & -r''(t) \\ -r''(t) & \lambda I_d \end{pmatrix}$  and so  $p_{X'(0), X'(t)}(0, 0) = D(t)^{-1/2}$ . The proposition is proved.  $\square$

**Comment concerning the one dimensional case.** Let us remark that in the one dimensional case, two important results of the present section, namely Formula (5) that gives the expected value of  $\varphi(X, T, u)$  and the fact that  $\varphi(X, T, u)$  has a finite second moment, remain valid under weaker assumptions. Actually, on the one hand, the Rice formula gives the mean number of upcrossings under the only assumption of a finite second spectral moment, and (3) allows us to conclude for  $\mathbb{E}[\varphi(X, T, u)]$ . On the other hand, it is well known (see for instance [7]) that a sufficient condition for  $U(X, u, T)$  to have a finite second moment is the so-called Geman condition. It requires that  $r$ , the covariance function of  $X$ , is  $C^2$  and satisfies  $\int_0^\delta \frac{r''(t) - r''(0)}{t} dt < +\infty$  for some positive  $\delta$ .

## 2.2 Asymptotic variance

In the next proposition, we let the cube  $T$  grow to  $\mathbb{R}^d$  and we give a closed formula for the asymptotic variance of  $\varphi(X, T, u)$ . Actually, for any positive integer  $N$ , we consider

$$T^{(N)} = \{Nt : t \in T\}$$

the image of a fixed cube  $T$  by the dilation  $t \mapsto Nt$  and we let  $N$  go to  $+\infty$ .

**Proposition 2.** Assume that  $X$  is Gaussian and satisfies Assumption **(A)** and let  $T$  be a cube in  $\mathbb{R}^d$ . Then for any  $u$  in  $\mathbb{R}$ ,

$$\lim_{N \rightarrow +\infty} \text{Var}[|T^{(N)}|^{-1/2} \varphi(X, T^{(N)}, u)] = V(u) < +\infty$$

with

$$V(u) = \int_{\mathbb{R}^d} (G(u, t) D(t)^{-1/2} - C(u)^2) dt + (2\pi\lambda)^{-d/2} g(u), \quad (8)$$

where  $C(u)$ ,  $g(u)$ ,  $G(u, t)$ ,  $D(t)$  have been defined in (6) and (7).

Note that a one can obtain explicit formulas for functions  $G(u, t)$  and  $g(u)$  by using Gaussian regressions. This is actually performed in Section 4 for the one dimensional case (see Proposition 7).

**Proof.** From Proposition 1, for a fixed cube  $T$ ,

$$\begin{aligned} & \text{Var}[|T|^{-1/2} \varphi(X, T, u)] \\ &= \int_{\mathbb{R}^d} \frac{|T \cap (T - t)|}{|T|} G(u, t) D(t)^{-1/2} dt + (2\pi\lambda)^{-d/2} g(u) - |T|^{-1} (\mathbb{E} \varphi(X, T, u))^2 \\ &= \int_{\mathbb{R}^d} \frac{|T \cap (T - t)|}{|T|} (G(u, t) D(t)^{-1/2} - C(u)^2) dt + (2\pi\lambda)^{-d/2} g(u), \end{aligned}$$

where we have used the relation  $|T|^2 = \int_{\mathbb{R}^d} |T \cap (T - t)| dt$  to get the last line. Hence, the asymptotic formula can easily be derived using Lebesgue dominated convergence Theorem conditionally to the fact that  $t \mapsto G(u, t) D(t)^{-1/2} - C(u)^2$  belongs to  $L^1(\mathbb{R}^d)$ . This point is proved in the next lemma.

**Lemma 3.** The map  $t \mapsto G(u, t) D(t)^{-1/2} - C(u)^2$  belongs to  $L^1(\mathbb{R}^d)$ .

**Proof of Lemma 3.**

The level  $u$  is fixed, so we do not refer to it in the following lines. For  $t \in \mathbb{R}^d \setminus \{0\}$ , we define  $h(t) = G(u, t) D(t)^{-1/2} - C(u)^2$ . We have to establish that  $h$  is integrable on  $\mathbb{R}^d$ . We already know that  $h$  is integrable on any compact set of  $\mathbb{R}^d$ , so it remains to study the behavior of  $h(t)$  for large  $\|t\|$ .

We write  $K = \frac{1}{2}d(d+1) + 1$  and we introduce the function

$$(z, x) \in \mathbb{R}^K = \mathbb{R}^{\frac{1}{2}d(d+1)} \times \mathbb{R} \mapsto f(z, x) = \tilde{\det}(z) \mathbf{1}_{[u, \infty)}(x),$$

where  $\tilde{\det}(z)$  stands for the determinant of the  $d \times d$  symmetric matrix whose upper coefficient are given by the  $\frac{1}{2}d(d+1)$  dimensional vector  $z$ .

Recall that we denote by  $\mathbf{X}(\cdot)$  the stationary random vector field  $(X'(t), \nabla^2 X(t), X(t))$ , which has dimension  $D = d + \frac{1}{2}d(d+1) + 1 = d + K$ . We have

$$\begin{aligned} h(t) &= \int_{\mathbb{R}^{2K}} f(\ddot{x}_1, x_1) f(\ddot{x}_2, x_2) \\ &\quad (p_{\mathbf{X}(0), \mathbf{X}(t)}(0, \ddot{x}_1, x_1, 0, \ddot{x}_2, x_2) - p_{\mathbf{X}(0)}(0, \ddot{x}_1, x_1) p_{\mathbf{X}(t)}(0, \ddot{x}_2, x_2)) dx_1 d\ddot{x}_1 dx_2 d\ddot{x}_2. \end{aligned}$$

Let  $\Sigma$  be the covariance matrix of  $\mathbf{X}(0)$  and, for any  $t \in \mathbb{R}^d$ , let  $\mathbf{\Gamma}^{\mathbf{X}}(t)$  be the covariance matrix of  $(\mathbf{X}(0), \mathbf{X}(t))$ . The latter can be written by blocks, each of one having dimension  $D \times D$ , as follows

$$\mathbf{\Gamma}^{\mathbf{X}}(t) = \begin{pmatrix} \Sigma & \Gamma_{12}(t) \\ \Gamma_{12}(t)^T & \Sigma \end{pmatrix}.$$

Moreover we know that all the terms of matrix  $\Gamma^{\mathbf{X}}(t)$  are uniformly bounded in absolute value by  $\psi(t)$ . Using Theorem 3.1 of [11] we get, for any  $\mathbf{z}_1, \mathbf{z}_2$  in  $\mathbb{R}^D$ ,

$$\begin{aligned} & p_{\mathbf{X}(0), \mathbf{X}(t)}(\mathbf{z}_1, \mathbf{z}_2) - p_{\mathbf{X}(0)}(\mathbf{z}_1)p_{\mathbf{X}(t)}(\mathbf{z}_2) \\ &= p_{\mathbf{X}(0)}(\mathbf{z}_1)p_{\mathbf{X}(0)}(\mathbf{z}_2) \left( \sum_{\mathbf{J} \in \mathbb{N}^{D \times D}, |\mathbf{J}| > 0} H_{A_{\mathbf{J}}}(\mathbf{z}_1, \Sigma) H_{B_{\mathbf{J}}}(\mathbf{z}_2, \Sigma) \frac{\Gamma_{12}(t)^{\mathbf{J}}}{\mathbf{J}!} \right), \end{aligned}$$

where

- the sum runs over all  $D \times D$  matrices  $\mathbf{J} = (J_{jk})_{1 \leq j, k \leq D}$  with non negative integer entries and  $|\mathbf{J}| = \sum_{1 \leq j, k \leq D} J_{jk} > 0$ ,
- $A_{\mathbf{J}}$  and  $B_{\mathbf{J}}$  are  $D$ -dimensional indices defined by  $(A_{\mathbf{J}})_j = \sum_{1 \leq k \leq D} J_{jk}$  and  $(B_{\mathbf{J}})_k = \sum_{1 \leq j \leq D} J_{jk}$ ,
- $H_{\mathbf{k}}(\mathbf{z}, \Sigma) = \phi_{\Sigma}(\mathbf{z})^{-1} (-\frac{\partial}{\partial \mathbf{z}})^{\mathbf{k}} \phi_{\Sigma}(\mathbf{z})$  for any  $D$ -dimensional index  $\mathbf{k}$  and any  $\mathbf{z} \in \mathbb{R}^D$ , with  $\phi_{\Sigma}$  the probability density function of any  $D$ -dimensional Gaussian vector  $\mathcal{N}(0, \Sigma)$ ,
- $M^{\mathbf{J}} = \prod_{1 \leq j, k \leq D} (M_{jk})^{J_{jk}}$  for any matrix  $M = (M_{jk})_{1 \leq j, k \leq D}$ ,
- and  $\mathbf{J}! = \prod_{1 \leq j, k \leq D} (J_{jk})!$ .

Thus

$$\begin{aligned} h(t) &= \int_{\mathbb{R}^{2K}} f(\mathbf{x}_1) f(\mathbf{x}_2) p_{\mathbf{X}(0)}(0, \mathbf{x}_1) p_{\mathbf{X}(0)}(0, \mathbf{x}_2) \\ &\quad \left( \sum_{\mathbf{J}, |\mathbf{J}| > 0} H_{A_{\mathbf{J}}}((0, \mathbf{x}_1), \Sigma) H_{B_{\mathbf{J}}}((0, \mathbf{x}_2), \Sigma) \frac{\Gamma_{12}(t)^{\mathbf{J}}}{\mathbf{J}!} \right) d\mathbf{x}_1 d\mathbf{x}_2. \end{aligned}$$

We can bound  $|h(t)|$  in the following form

$$|h(t)| \leq \sum_{\mathbf{J}, |\mathbf{J}| > 0} \frac{\psi(t)^{|\mathbf{J}|}}{\mathbf{J}!} D(A_{\mathbf{J}}) D(B_{\mathbf{J}}), \quad (9)$$

where

$$\begin{aligned} D(A_{\mathbf{J}}) &= \int_{\mathbb{R}^K} |f(\mathbf{x}_1) H_{A_{\mathbf{J}}}((0, \mathbf{x}_1), \Sigma)| p_{\mathbf{X}(0)}(0, \mathbf{x}_1) d\mathbf{x}_1 \\ &\leq \mathbb{E} (|\det(X''(0)) H_{A_{\mathbf{J}}}((0, X''(0), X(0)), \Sigma)|) \\ &\leq \mathbb{E} (\det(X''(0))^2)^{1/2} \mathbb{E} (H_{A_{\mathbf{J}}}((0, X''(0), X(0)), \Sigma)^2)^{1/2} \\ &\leq \psi(0)^d \mathbb{E} (H_{A_{\mathbf{J}}}((0, X''(0), X(0)), \Sigma)^2)^{1/2}, \end{aligned} \quad (10)$$

where the last inequality is a consequence of expanding  $\det(X''(0))$  as a multivariate polynomial function of degree  $d$  evaluated at the coordinates of the matrix  $X''(0)$ .

We now concentrate on  $H_{A_{\mathbf{J}}}((0, \nabla^2 X(0), X(0)), \Sigma)$ . Following [11], we have

$$H_{A_{\mathbf{J}}}((0, \mathbf{x}), \Sigma) = \mathbb{E} \left[ (\Sigma^{-1}((0, \mathbf{x}) + iZ))^{A_{\mathbf{J}}} \right],$$

where  $Z$  is any  $D$ -dimensional centered Gaussian random vector with covariance matrix  $\Sigma$ . Remember that  $\Sigma$  is the covariance matrix of  $\mathbf{X}(0) = (X'(0), X''(0), X(0))$ , so it can be factorized as  $\Sigma = \begin{pmatrix} \lambda I_d & 0 \\ 0 & \Sigma_1 \end{pmatrix}$  where  $\Sigma_1$  has size  $K \times K$ . Hence  $\Sigma^{-1} = \begin{pmatrix} \lambda^{-1} I_d & 0 \\ 0 & \Sigma_1^{-1} \end{pmatrix}$  and  $Z$  can be expanded as  $Z = (Z_0, Z_1)$  where  $Z_0$  is a  $d$ -dimensional centered Gaussian vector with covariance

matrix  $\lambda I_d$  and  $Z_1$  is a  $K$ -dimensional centered Gaussian vector with covariance matrix  $\Sigma_1$ ,  $Z_0$  and  $Z_1$  being independent. Then

$$\Sigma^{-1}((0, \mathbf{x}) + iZ) = (i\lambda^{-1}Z_0, \Sigma_1^{-1}(\mathbf{x} + iZ_1)) ,$$

where the two blocks of coordinates are independent. So

$$H_{A_{\mathbf{J}}}((0, \mathbf{x}), \Sigma) = H_{A_{\mathbf{J}}^{(0)}}(0, \lambda I_d) H_{A_{\mathbf{J}}^{(1)}}(\mathbf{x}, \Sigma_1),$$

where the  $D$ -dimensional vector  $A_{\mathbf{J}}$  is equal to  $(A_{\mathbf{J}}^{(0)}, A_{\mathbf{J}}^{(1)})$  with blocks of respective size  $d$  and  $K$ .

For the first term, we note that  $H_{A_{\mathbf{J}}^{(0)}}(0, \lambda I_d) = \lambda^{-\frac{1}{2}|A_{\mathbf{J}}^{(0)}|} H_{A_{\mathbf{J}}^{(0)}}(0)$  where  $H_{\mathbf{k}}(\cdot)$  is the usual multidimensional Hermite polynomial of multi-order  $\mathbf{k}$ .

Concerning the second term, we compute  $\mathbb{E}\left(H_{A_{\mathbf{J}}^{(1)}}((X''(0), X(0)), \Sigma_1)^2\right)$  thanks to a formula for multivariate Gaussian integrals that can be found in [6] for instance (see equation (35) in this reference),

$$\begin{aligned} \mathbb{E}\left(H_{A_{\mathbf{J}}^{(1)}}((X''(0), X(0)), \Sigma_1)^2\right) &= \int_{\mathbb{R}^K} H_{A_{\mathbf{J}}^{(1)}}(\mathbf{x}, \Sigma_1)^2 p_{X''(0), X(0)}(\mathbf{x}) d\mathbf{x} \\ &= 2^{|A_{\mathbf{J}}^{(1)}|-D} (A_{\mathbf{J}}^{(1)})! \text{diag}(\Sigma_1^{-1})^{A_{\mathbf{J}}^{(1)}} \end{aligned}$$

where  $\text{diag}(B)$  denotes the vector containing the diagonal coefficients of any square matrix  $B$ . Finally, since  $H_{A_{\mathbf{J}}^{(0)}}(0)^2 \leq (A_{\mathbf{J}}^{(0)})!$ ,

$$\mathbb{E}\left(H_{A_{\mathbf{J}}}((0, X''(0), X(0)), \Sigma)^2\right) \leq (2\|\Sigma^{-1}\|)^{|\mathbf{J}|} (A_{\mathbf{J}})!$$

where we have used that  $|A_{\mathbf{J}}^{(0)}| + |A_{\mathbf{J}}^{(1)}| = |\mathbf{J}|$  and  $(A_{\mathbf{J}}^{(0)})! (A_{\mathbf{J}}^{(1)})! = (A_{\mathbf{J}})!$ .

Note that for any  $D$ -dimensional index  $\mathbf{k} = (k_1, \dots, k_D)$ , we have  $|\mathbf{k}|! \leq D^{|\mathbf{k}|} (\mathbf{k})!$  (actually,  $\frac{|\mathbf{k}|!}{(\mathbf{k})!}$  is a multinomial coefficient). Hence  $(A_{\mathbf{J}})_j! \leq D^{\sum_{1 \leq k \leq D} J_{jk}} \prod_{1 \leq k \leq D} J_{jk}!$  and so

$$(A_{\mathbf{J}})! = \prod_{1 \leq j \leq D} (A_{\mathbf{J}})_j! \leq D^{|\mathbf{J}|} \mathbf{J}!$$

Coming back to inequality (10) yields

$$D(A_{\mathbf{J}}) \leq \psi(0)^d (2D\|\Sigma^{-1}\|)^{|\mathbf{J}|/2} \sqrt{\mathbf{J}}!$$

The same inequality can be established for  $D(B_{\mathbf{J}})$ . Hence, from (9) we get

$$|h(t)| \leq \psi(0)^{2d} \psi(t) \sum_{\mathbf{J}, |\mathbf{J}| > 0} (2D\|\Sigma^{-1}\| \psi(t))^{|\mathbf{J}|-1}.$$

The series in the right hand is convergent for  $\|t\|$  large enough. Indeed we can choose  $A$  such that for  $\|t\| > A$ , we have  $2D\|\Sigma^{-1}\| \psi(t) \leq \eta < 1$  and so the series is bounded by  $(1 - \eta)^{-1}$ .

The result follows since on the other hand, by assumption **(A)**,  $\int_{\|t\| > A} \psi(t) dt < +\infty$ . Lemma 3, and hence Proposition 2 also, are proved.  $\square$

Beyond the existence of a finite asymptotic variance as established in the previous proposition,  $\varphi(X, T^{(N)}, u)$  satisfies a central limit theorem. Actually, Theorem 2.6 of [4] states that the next convergence holds in distribution

$$|T^{(N)}|^{-1/2} (\varphi(X, T^{(N)}, u) - \mathbb{E}[\varphi(X, T^{(N)}, u)]) \xrightarrow[N \rightarrow \infty]{distr} \mathcal{N}(0, V(u)), \quad (11)$$

where  $\mathcal{N}(0, V(u))$  stands for the centered Gaussian distribution with variance  $V(u)$ .

## 2.3 Disjoint domains and various levels

We now consider domains  $T_1$  and  $T_2$  that are disjoint and levels  $u_1$  and  $u_2$  that can be equal or not.

**Proposition 4.** *Assume that  $X$  is Gaussian and satisfies Assumption **(A)**.*

**(a)** *Let  $T_1$  and  $T_2$  be two cubes in  $\mathbb{R}^d$  such that  $|T_1| = |T_2|$  and  $\text{dist}(T_1, T_2) > 0$  and let  $u_1$  and  $u_2$  belong to  $\mathbb{R}$ . For any integer  $N > 0$ , we introduce*

$$Z_i^{(N)} = |T_i^{(N)}|^{-1/2} (\varphi(X, T_i^{(N)}, u_i) - \mathbb{E}[\varphi(X, T_i^{(N)}, u_i)]) \quad \text{for } i = 1, 2.$$

As  $N \rightarrow +\infty$ ,  $(Z_1^{(N)}, Z_2^{(N)})$  converges in distribution to a centered Gaussian vector with diagonal covariance matrix  $\begin{pmatrix} V(u_1) & 0 \\ 0 & V(u_2) \end{pmatrix}$  where  $V(u_i)$  is prescribed by (8).

**(b)** *Let  $T$  be a cube in  $\mathbb{R}^d$  and let  $u_1$  and  $u_2$  belong to  $\mathbb{R}$ . For any integer  $N > 0$ , we introduce*

$$\zeta_i^{(N)} = |T^{(N)}|^{-1/2} (\varphi(X, T^{(N)}, u_i) - \mathbb{E}[\varphi(X, T^{(N)}, u_i)]) \quad \text{for } i = 1, 2.$$

As  $N \rightarrow +\infty$ ,  $(\zeta_1^{(N)}, \zeta_2^{(N)})$  converges in distribution to a centered Gaussian vector with covariance matrix  $\begin{pmatrix} V(u_1) & V(u_1, u_2) \\ V(u_1, u_2) & V(u_2) \end{pmatrix}$ .

**Proof.** Point **(a)**.

It is proved in [4] Proposition 1.3 that, for any cube  $T$  in  $\mathbb{R}^d$  and any level  $u$  in  $\mathbb{R}$ , the following expansion holds both *a.s.* and in  $L^2(\Omega)$ .

$$\tilde{\varphi}(X, T, u) = (-1)^d \sum_{q=1}^{\infty} \sum_{\mathbf{n} \in \mathbb{N}^D; |\mathbf{n}|=q} a(\mathbf{n}, u) \int_T H_{\mathbf{n}}(Y(t)) dt \quad (12)$$

where

$$\tilde{\varphi}(X, T, u) = \varphi(X, T, u) - \mathbb{E}[\varphi(X, T, u)],$$

$H_{\mathbf{n}}$  denotes the  $D$  dimensional Hermite coefficient of multi-order  $\mathbf{n} \in \mathbb{N}^D$  and  $Y(t) = \Lambda^{-1} \mathbf{X}(t)$  with  $\Lambda$  any square root of matrix  $\Sigma$ . Moreover the Hermite coefficients  $a(\mathbf{n}, u)$  are such that for any  $q \geq 1$ ,

$$\sum_{\mathbf{n} \in \mathbb{N}^D; |\mathbf{n}|=q} a(\mathbf{n}, u)^2 \mathbf{n}! \leq C q^d \quad (13)$$

where  $C$  is some positive constant that only depends on  $d, \Lambda$  and  $u$ .

The latter results will be useful to study the asymptotic covariance of  $Z_1^{(N)}$  and  $Z_2^{(N)}$  in the next lemma.

**Lemma 5.**

$$\text{Cov} \left( Z_1^{(N)}, Z_2^{(N)} \right) \xrightarrow{N \rightarrow +\infty} 0$$

**Proof of Lemma 5.** Without loss of generality, we can assume that  $|T_1| = |T_2| = 1$  and that there exists  $a$  and  $b$  in  $\mathbb{R}^d$  such that  $T_1 = a + [0, 1]^d$  and  $T_2 = b + [0, 1]^d$ . Moreover, since the distance between  $T_1$  and  $T_2$  is non negative, we have  $\|b - a\|_{\infty} > 1$ . In the following lines, we denote  $\delta = \|b - a\|_{\infty} - 1$ . We use the additivity of the Euler characteristic to expand  $Z_1^{(N)}$  as

$$Z_1^{(N)} = N^{-d/2} \sum_{\mathbf{s} \in \mathcal{I}_N} \tilde{\varphi}(\theta_{Na+\mathbf{s}} X, [0, 1]^d, u_1),$$



where  $\theta_{\mathbf{s}}$  stands for the shift operator and  $\mathcal{I}_N$  for the set of integer valued indices  $[[0, N - 1]]^d$ . The same holds for  $Z_2^{(N)}$ . Hence, by stationarity

$$\begin{aligned} |\text{Cov}(Z_1^{(N)}, Z_2^{(N)})| &= N^{-d} \left| \sum_{\mathbf{s}, \mathbf{s}' \in \mathcal{I}_N} \mathbb{E}(\tilde{\varphi}(X, [0, 1]^d, u_1) \tilde{\varphi}(\theta_{N(b-a)+\mathbf{s}'-\mathbf{s}}X, [0, 1]^d, u_2)) \right| \\ &\leq \sum_{\mathbf{s} \in \mathcal{J}_N} |\mathbb{E}(\tilde{\varphi}(X, [0, 1]^d, u_1) \tilde{\varphi}(\theta_{N(b-a)+\mathbf{s}}X, [0, 1]^d, u_2))|, \end{aligned}$$

where  $\mathcal{J}_N = [[-N + 1, N - 1]]^d$ . Besides, thanks to (12) and to the orthogonality of Hermite polynomials, for any  $\tau \in \mathbb{R}^d$ ,

$$\begin{aligned} &\mathbb{E}(\tilde{\varphi}(X, [0, 1]^d, u_1) \tilde{\varphi}(\theta_{\tau}X, [0, 1]^d, u_2)) \\ &= \sum_{q=1}^{\infty} \int_{[0,1]^d} \int_{[0,1]^d} \mathbb{E}(F_q(Y(t_1), u_1) F_q(Y(\tau + t_2), u_2)) dt_1 dt_2, \end{aligned}$$

where  $F_q(y, u) = \sum_{\mathbf{n} \in \mathbb{N}^D; |\mathbf{n}|=q} a(\mathbf{n}, u) H_{\mathbf{n}}(y)$ ,  $y \in \mathbb{R}^D$ . One can adapt Arcones inequality, for instance following the proof of [2] Lemma 1 step by step, to get

$$\begin{aligned} &|\mathbb{E}(F_q(Y(t_1), u_1) F_q(Y(\tau + t_2), u_2))| \\ &\leq (DK)^q \psi(\tau + t_2 - t_1)^q \sum_{\mathbf{n} \in \mathbb{N}^D; |\mathbf{n}|=q} \frac{1}{2} (a(\mathbf{n}, u_1)^2 + a(\mathbf{n}, u_2)^2) \mathbf{n}!. \end{aligned}$$

Let now  $\eta > 0$  be fixed. We choose  $N$  large enough such that  $DK\psi(\tau) \leq 1/2$  as soon as  $\|\tau\| \geq N\delta$ ,  $\int_{\|t\| \geq N} \psi(t) dt < \eta$  and  $N\delta > 2$ . Hence, using (13), we deduce that for  $\|\tau\| \geq N\delta$ ,

$$\begin{aligned} |\mathbb{E}(\tilde{\varphi}(X, [0, 1]^d, u_1) \tilde{\varphi}(\theta_{\tau}X, [0, 1]^d, u_2))| &\leq C \sum_{q=1}^{\infty} q^d (DK)^q \int_{[0,1]^d} \int_{[0,1]^d} \psi(\tau + t_2 - t_1)^q dt_1 dt_2 \\ &\leq C \left( \sum_{q=1}^{\infty} q^d (1/2)^{q-1} \right) \int_{[-1,1]^d} \psi(\tau + t) dt. \end{aligned}$$

Coming back to the covariance of  $Z_1^{(N)}$  and  $Z_2^{(N)}$  yields

$$\begin{aligned} |\text{Cov}(Z_1^{(N)}, Z_2^{(N)})| &\leq C' \sum_{\mathbf{s} \in \mathcal{J}_N} \int_{[-1,1]^d} \psi(N(b-a) + \mathbf{s} + t) dt \\ &\leq 2C' \int_{\|t\| \geq N} \psi(t) dt \leq C'' \eta. \end{aligned}$$

Lemma 5 is proved.  $\square$

Thanks to the lemma and the convergence in (11), we know that the covariance matrix of the random vector  $(Z_1^{(N)}, Z_2^{(N)})$  tends to  $\begin{pmatrix} V(u_1) & \mathbf{0} \\ \mathbf{0} & V(u_2) \end{pmatrix}$ . One can use the same arguments as

those in [4] (which were inspired by [8]) to establish that any linear combination  $xZ_1^{(N)} + yZ_2^{(N)}$  has a Gaussian limit in distribution. Point (a) is now proved.

Point (b) is proved in [4] Theorem 2.5.  $\square$

### 3 Under $\chi^2$ hypothesis

In this section, we deal with  $\chi^2$  distributions instead of Gaussian ones. Let us fix  $s$ , a non negative integer, as the degrees of freedom.

We start with  $\{X_i(\cdot)\}_{i=1}^s$ , an independent sample of centered stationary Gaussian fields on  $\mathbb{R}^d$  that satisfy Assumption **(A)** of Section 1. We denote by  $r_X$  their covariance function and recall that  $r_X(0) = 1$ . Consider now the following stationary fields,

$$\chi_s^2(\cdot) = \sum_{i=1}^s (X_i(\cdot))^2 \quad \text{and} \quad Z^{(s)}(\cdot) = \frac{1}{\sqrt{2s}} \left( \sum_{i=1}^s (X_i(\cdot))^2 - s \right).$$

Note that for any  $t \in \mathbb{R}^d$ ,  $\chi_s^2(t)$  is a Chi-square random variable with  $s$  degrees of freedom. One gets readily that

$$\mathbb{E}[Z^{(s)}(0)] = 0, \quad \text{Var}[Z^{(s)}(0)] = 1, \quad \mathbb{E}[Z^{(s)}(t)Z^{(s)}(0)] = r_X(t)^2.$$

Moreover, the second spectral moment of  $Z^{(s)}$  is equal to  $\lambda$  with  $\lambda = -((r_X)^2)''(0) = -2r_X''(0)$ .

We are first interested in the expectation of the Euler characteristic of

$$\{t \in T : Z^{(s)}(t) \geq u\} = \{t \in T : \chi_s^2(t) \geq s + u\sqrt{2s}\},$$

for a fixed cube  $T \subset \mathbb{R}^d$  and a fixed level  $u$  in  $\mathbb{R}$ . A formula for the mean EC of excursion sets of  $\chi^2$  fields is given in [12], Theorem 3.5 (see also Theorem 15.10.1 in [1]). We can then establish that

$$\begin{aligned} & \mathbb{E}[\varphi(Z^{(s)}, T, u)] \\ &= |T| \left( \frac{\lambda}{\pi} \right)^{d/2} \frac{e^{-(s+u\sqrt{2s})/2} (s + u\sqrt{2s})^{(s-d)/2}}{2^{(s-2+2d)/2} \Gamma(s/2)} P_{d,s}(s + u\sqrt{2s}) \mathbf{1}_{[0,\infty)}(s + u\sqrt{2s}), \end{aligned} \quad (14)$$

where  $P_{d,s}(\cdot)$  is a polynomial of degree  $d-1$  with integer coefficients. In particular,  $P_{1,s}(u) = 1$  and  $P_{2,s}(u) = u - s + 1$ . Let us quote that we have to handle carefully with the second spectral moment of the  $X_i$ 's, which is equal to  $\lambda/2$  in our context.

Let us recall that, in dimension one, the mean Euler characteristic of the excursion above the level  $u$  is equal to the mean number of upcrossings at level  $u$ , see Equation (3). With this point of view, Formula (14) can also be found in [9] for instance. Next proposition is concerned with the second moment in dimension one, in the same spirit as Proposition 2.

**Proposition 6.** *Let  $d = 1$ . Let us assume that the  $X_i$ 's are one dimensional i.i.d. Gaussian processes that satisfy Assumption **(A)** for  $i = 1, \dots, s$ .*

*Then, for any  $u$  in  $\mathbb{R}$  and  $T$  in  $[0, +\infty)$ ,  $\varphi(Z^{(s)}, [0, T], u)$  admits a finite second moment.*

*Moreover, there exists  $v_s(u) \in [0, +\infty)$  such that*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \text{Var}[\varphi(Z^{(s)}, [0, T], u)] = v_s(u).$$

**Proof.** Firstly, let us remark that, since  $Z^{(s)}(\cdot) = \frac{1}{\sqrt{2s}}(\chi_s^2(\cdot) - s)$ , it is sufficient to prove the statement of Proposition 6 for the process  $\chi_s^2$  instead of  $Z^{(s)}$ . Secondly, let us recall the relation (3), so that we can replace  $\varphi(Z^{(s)}, [0, T], u)$  by  $U(\chi_s^2, [0, T], u)$  in the proposition.

The finiteness of

$$V_s(T, u) := \mathbb{E}[U(\chi_s^2, [0, T], u)(U(\chi_s^2, [0, T], u) - 1)]$$

is proved in [9] under the following conditions,

$$r_X(t) = 1 - \lambda_2 \frac{t^2}{2} + \lambda_4 \frac{t^4}{4} + o(t^4) \quad \text{as } t \rightarrow 0 \quad \text{and} \quad r_X(t) = o(t^{-\alpha}) \quad \text{as } t \rightarrow \infty \quad (15)$$

for some positive finite constants  $\lambda_2, \lambda_4, \alpha$  (see (4.1) and (4.2) in the cited paper). They are clearly satisfied in our case since Assumption **(A)** is in force. This establishes that  $U(\chi_s^2, [0, T], u)$

admits a finite variance.

We now focus on the asymptotic variance as  $T$  goes to infinity. We will actually prove that

$$\lim_{T \rightarrow +\infty} \frac{1}{T} (V_s(T, u) - (\mathbb{E}[U(\chi_s^2, [0, T], u)])^2) < +\infty.$$

First, in a similar way as (14), let us remark that the expectation of  $U(\chi_s^2, [0, T], u)$  is proportional to  $T$ ,

$$K_s(u) := \frac{1}{T} \mathbb{E}[U(\chi_s^2, [0, T], u)] = \left(\frac{\lambda}{\pi}\right)^{1/2} \frac{e^{-u/2} u^{(s-1)/2}}{2^{s/2} \Gamma(s/2)}. \quad (16)$$

On the other hand, introducing the function  $\mathcal{G}(x) = x \mathbf{1}_{[0, \infty)}(x)$ , Rice formula for the above factorial moment gives

$$V_s(T, u) = 2 \int_0^T (T-t) \mathcal{I}(t) dt,$$

where

$$\mathcal{I}(t) = \mathbb{E}[\mathcal{G}((\chi_s^2)'(0)) \mathcal{G}((\chi_s^2)'(t)) \mid \chi_s^2(0) = \chi_s^2(t) = u] p_{\chi_s^2(0), \chi_s^2(t)}(u, u).$$

For computing this conditional expectation, we begin with the following one that is easier to handle with. Let us define  $\vec{X}(t) = (X_1(t), \dots, X_s(t))$  and, for  $\mathbf{z} = (z_1, \dots, z_s)$  and  $\mathbf{w} = (w_1, \dots, w_s)$  in  $\mathbb{R}^s$ ,

$$J(t, \mathbf{z}, \mathbf{w}) := \mathbb{E}[\mathcal{G}((\chi_s^2)'(0)) \mathcal{G}((\chi_s^2)'(t)) \mid \vec{X}(0) = \mathbf{z}, \vec{X}(t) = \mathbf{w}, \chi_s^2(0) = \chi_s^2(t) = u].$$

Let us fix  $\mathbf{z}$  and  $\mathbf{w}$  such that  $\|\mathbf{z}\|^2 = \|\mathbf{w}\|^2 = u$ . In that case, we have

$$J(t, \mathbf{z}, \mathbf{w}) = 4 \mathbb{E}[\mathcal{G}(\sum_{j=1}^s X_j(0) X_j'(0)) \mathcal{G}(\sum_{j=1}^s X_j(t) X_j'(t)) \mid \vec{X}(0) = \mathbf{z}, \vec{X}(t) = \mathbf{w}].$$

We will use, as is classical, the following regression model. For each  $j \in \{1, \dots, s\}$ , we write

$$\begin{aligned} X_j'(0) &= \alpha_1(t) X_j(0) + \beta_1(t) X_j(t) + \varepsilon_{1j}(t) \\ X_j'(t) &= \alpha_2(t) X_j(0) + \beta_2(t) X_j(t) + \varepsilon_{2j}(t). \end{aligned}$$

It is important to point out that, because of the independence, the coefficients above do not depend on  $j$ . We readily obtain that

$$\alpha_1(t) = \frac{r'_X(t) r_X(t)}{1 - r_X^2(t)}, \beta_1(t) = \frac{-r'_X(t)}{1 - r_X^2(t)}, \beta_2(t) = -\alpha_1(t), \alpha_2(t) = -\beta_1(t).$$

Thus

$$\begin{aligned} J(t, \mathbf{z}, \mathbf{w}) &= 4 \mathbb{E}[\mathcal{G}(\sum_{j=1}^s z_j (\alpha_1(t) z_j + \beta_1(t) w_j + \varepsilon_{1j}(t))) \mathcal{G}(\sum_{j=1}^s w_j (\beta_1(t) z_j + \alpha_1(t) w_j + \varepsilon_{2j}(t)))] \\ &= 4 \mathbb{E}[\mathcal{G}(\alpha_1(t) u + \beta_1(t) \langle \mathbf{w}, \mathbf{z} \rangle + \sqrt{u} \sigma_\varepsilon W(t)) \mathcal{G}(\alpha_1(t) u + \beta_1(t) \langle \mathbf{w}, \mathbf{z} \rangle + \sqrt{u} \sigma_\varepsilon Z(t))] \\ &= 4 \mathbb{E}[\mathcal{G}(d_u(t) + \sqrt{u} \sigma_\varepsilon(t) W(t)) \mathcal{G}(d_u(t) + \sqrt{u} \sigma_\varepsilon(t) Z(t))], \end{aligned}$$

where

$$\sigma_\varepsilon^2(t) = -r''_X(t) - (\alpha_1^2(t) + \beta_1^2(t) + 2\alpha_1(t)\beta_1(t)r_X(t))$$

is the variance of each  $\varepsilon_{1,j}(t)$  or  $\varepsilon_{2,j}(t)$ ,

$$W(t) = \frac{1}{\sqrt{u} \sigma_\varepsilon(t)} \sum_{i=1}^s w_i \varepsilon_{1,i}(t) \quad \text{and} \quad Z(t) = \frac{1}{\sqrt{u} \sigma_\varepsilon(t)} \sum_{i=1}^s z_i \varepsilon_{2,i}(t)$$

are independent  $N(0, 1)$  random variables, and  $d_u(t) = \alpha_1(t)u + \beta_1(t)\langle \mathbf{w}, \mathbf{z} \rangle$ .

In particular, the latter shows that  $J(t, \mathbf{z}, \mathbf{w})$  only depends on  $\mathbf{w}$  and  $\mathbf{z}$  through their scalar product. So, from now on, for ease of notation, we will write  $J(t, \langle \mathbf{z}, \mathbf{w} \rangle)$ .

We have  $\beta_1(t) \rightarrow \infty$  when  $t \rightarrow 0$  and  $d_u(t) = (\alpha_1(t) + \beta_1(t))u - \beta_1(t)u(1 - \frac{1}{u}\langle \mathbf{w}, \mathbf{z} \rangle) \xrightarrow[t \rightarrow 0]{} -\infty$ ,

for all  $\mathbf{w} \neq \mathbf{z}$ . The latter convergence is equivalent to  $-\frac{1}{t}$ .

Moreover,  $d_u(t) \rightarrow 0$  whenever  $t \rightarrow \infty$ , because of Assumption **(A)**.

Then we can write

$$\begin{aligned} J(t, \langle \mathbf{z}, \mathbf{w} \rangle) &= 4u\sigma_\varepsilon^2(t) \mathbb{E}[\mathcal{G}(\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)} + W(t)) \mathcal{G}(\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)} + Z(t))] \\ &= 4u\sigma_\varepsilon^2(t) \sum_{k=0}^{\infty} d_{uk}^2(t) k! (\text{Corr}(W(t), Z(t)))^k, \end{aligned} \quad (17)$$

where the last equality is a consequence of Mehler's formula. The coefficients  $d_{uk}(t)$  are given by

$$\begin{aligned} d_{uk}(t) &= \frac{1}{k!} \int_{\mathbb{R}} \mathcal{G}(\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)} + y) H_k(y) \phi(y) dy \\ &= \frac{1}{k!} \int_{-\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)}}^{\infty} (\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)} + y) H_k(y) \phi(y) dy, \end{aligned}$$

with  $\phi$  the standard Gaussian density, and

$$\text{Corr}(W(t), Z(t)) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle}{u\sigma_\varepsilon^2(t)} (-r_X''(t) - 2\alpha_1(t)\beta_1(t) - (\alpha_1^2 + \beta_1^2(t))r_X(t)).$$

Let us continue our study with the behaviour of  $J(t, \langle \mathbf{z}, \mathbf{w} \rangle)$  for  $t$  close to 0. Let us start with the series. It holds

$$\begin{aligned} \sum_{k=0}^{\infty} d_{uk}^2(t) k! &= \int_{-\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)}}^{\infty} (\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)} + y)^2 \phi(y) dy, \\ &\leq 2((\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)})^2 \int_{-\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)}}^{\infty} \phi(y) dy + \int_{-\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)}}^{\infty} y^2 \phi(y) dy). \end{aligned}$$

For the last integral, we use the upper bound

$$\int_a^{\infty} y^2 \phi(y) dy \leq a\phi(a) + \frac{1}{a}\phi(a),$$

that yields, as  $t \rightarrow 0$ ,

$$\sum_{k=0}^{\infty} d_{uk}^2(t) k! \leq 2(2(-\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)})\phi(\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)}) + (-\frac{\sqrt{u}\sigma_\varepsilon(t)}{d_u(t)})\phi(\frac{d_u(t)}{\sqrt{u}\sigma_\varepsilon(t)})) = O(1). \quad (18)$$

Thus, for  $0 < t < \delta$ , we have  $J(t, \langle \mathbf{z}, \mathbf{w} \rangle) \leq \mathbf{C} u \sigma_\varepsilon^2(t)$ . Moreover, as this bound only depends on  $u$ , it yields immediately that,

$$\mathcal{I}(t) \leq \mathbf{C} u \sigma_\varepsilon^2(t), \quad \text{for } 0 < t < \delta.$$

We now focus on the behaviour of  $\mathcal{I}(t)$  for  $t$  close to  $+\infty$ . We have

$$\begin{aligned} \mathcal{I}(t) &= \int_{\mathbb{R}^s \times \mathbb{R}^s} J(t, \langle \mathbf{z}, \mathbf{w} \rangle) p_{\vec{X}(0), \vec{X}(t), \chi_s^2(0), \chi_s^2(t)}(\mathbf{z}, \mathbf{w}, u, u) d\mathbf{z} d\mathbf{w} \\ &= \frac{u^{s-2}}{(2\pi)^s (1 - r_X^2(t))^{s/2}} \int_{\mathbb{S}^{s-1} \times \mathbb{S}^{s-1}} J(t, u\langle \tau_1, \tau_2 \rangle) e^{-u(\frac{1-r_X(t)\langle \tau_1, \tau_2 \rangle}{1-r_X^2(t)})} d\sigma_{s-1}(\tau_1) d\sigma_{s-1}(\tau_2), \end{aligned}$$

where  $\sigma_{s-1}$  stands for the surface Lebesgue measure on  $\mathbb{S}^{s-1}$ . A consequence of Funk-Hecke formula says that, for any reasonable function  $F$ ,

$$\int_{\mathbb{S}^{s-1} \times \mathbb{S}^{s-1}} F(\langle \tau_1, \tau_2 \rangle) d\sigma_{s-1}(\tau_1) d\sigma_{s-1}(\tau_2) = \frac{\sigma_{s-1}(\mathbb{S}^{s-1})^2}{2W_{s-2}} \int_0^\pi F(\cos \psi) \sin^{s-2} \psi d\psi,$$

with  $W_s$  the Wallis integral, *i.e.*  $W_s = \int_0^{\pi/2} \sin^s(\psi) d\psi$ . It implies

$$\mathcal{I}(t) = \frac{4u^{s-2}}{(2\pi)^s (1-r_X^2(t))^{s/2}} \frac{\sigma_{s-1}(\mathbb{S}^{s-1})^2}{2W_{s-2}} \int_0^\pi J(t, u \cos \psi) e^{-u \left( \frac{1-r_X(t) \cos \psi}{1-r_X^2(t)} \right)} \sin^{s-2} \psi d\psi.$$

Following (17), we now expand  $J(t, x)$  as  $J(t, x) := 4u\sigma_\varepsilon^2(t) \sum_{k=0}^\infty Q_k(t, x)$ , so that

$$\begin{aligned} \frac{1}{T} V_s(u, T) &= 2 \int_0^T \left(1 - \frac{t}{T}\right) \mathcal{I}(t) dt \\ &= \int_0^T \left(1 - \frac{t}{T}\right) \frac{4u^{s-1} \sigma_\varepsilon^2(t)}{(2\pi)^s (1-r_X^2(t))^{s/2}} \frac{\sigma_{s-1}(\mathbb{S}^{s-1})^2}{W_{s-2}} \\ &\quad \times \int_0^\pi \sum_{k=0}^\infty Q_k(t, u \cos \psi_1) e^{-u \left( \frac{1-r_X(t) \cos \psi}{1-r_X^2(t)} \right)} \sin^{s-2} \psi d\psi dt. \end{aligned}$$

In first place, let us study the limit when  $t \rightarrow \infty$  of the ( $k=0$ )-term in the integrand. We introduce the following notation,

$$\mathcal{I}_0(t) = \frac{4u^{s-1} \sigma_\varepsilon^2(t)}{(2\pi)^s (1-r_X^2(t))^{s/2}} \frac{\sigma_{s-1}(\mathbb{S}^{s-1})^2}{W_{s-2}} \int_0^\pi Q_0(t, u \cos \psi_1) e^{-u \left( \frac{1-r_X(t) \cos \psi}{1-r_X^2(t)} \right)} \sin^{s-2} \psi d\psi.$$

We have

$$\mathcal{I}_0(t) \xrightarrow{t \rightarrow \infty} \frac{4u^{s-1} (-r_X''(0))}{(2\pi)^{s+1}} e^{-u} \sigma_{s-1}(\mathbb{S}^{s-1})^2 = K_s(u)^2,$$

given that  $d_{u_0}^2(t) \xrightarrow{t \rightarrow \infty} \frac{1}{2\pi}$ ,  $\sigma_{s-1}(\mathbb{S}^{s-1}) = \frac{2\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}$  and using (16). Therefore, recalling the identity

$2 \int_0^T \left(1 - \frac{t}{T}\right) dt = T$ , the first term of the asymptotic variance is

$$\begin{aligned} &\lim_{T \rightarrow \infty} 2 \int_0^T \left(1 - \frac{t}{T}\right) \mathcal{I}_0(t) dt - \frac{1}{T} (\mathbb{E}[U(\chi_s^2, T, u)])^2 \\ &= \lim_{T \rightarrow \infty} 2 \int_0^T \left(1 - \frac{t}{T}\right) \mathcal{I}_0(t) dt - T K_s(u)^2 \\ &= 2 \int_0^\infty (\mathcal{I}_0(t) - K_s(u)^2) dt. \end{aligned} \tag{19}$$

This last equality holds true if we show that the difference appearing into the integrand is bounded by an  $L^1(\mathbb{R})$  function, outside of a compact interval.

Actually, under **(A)**, it is easy to prove that for  $t$  large enough,  $|\mathcal{I}_0(t) - K_s(u)^2| \leq \mathbf{C} \left| \frac{d_u(t)}{\sqrt{u} \sigma_\varepsilon(t)} \right| \leq \mathbf{C} \psi(t)$ . Hence, the limit (19) is established.

In the sequel we are going to study the asymptotic behaviour of the remaining terms. Let us introduce

$$\begin{aligned} \mathcal{R}(t) &:= \frac{4u^{s-1} \sigma_\varepsilon^2(t)}{(2\pi)^s (1-r^2(t))^{s/2}} \frac{\sigma_{s-1}(\mathbb{S}^{s-1})^2}{2W_{s-2}} \\ &\quad \times \int_0^\pi \sum_{k=1}^\infty Q_k(t, u \cos \psi_1) e^{-u \left( \frac{1-r_X(t) \cos \psi}{1-r_X^2(t)} \right)} \sin^{s-2} \psi d\psi. \end{aligned}$$

So, by using (18), we obtain that for  $t$  large enough,

$$\left| \sum_{k=1}^{\infty} Q_k(t, u \cos \psi_1) \right| \leq |(\text{Corr}(W(t), Z(t)))| \sum_{k=1}^{\infty} d_{uk}^2(t) k! \leq \mathbf{C}\psi(t).$$

This bound and Assumption **(A)** entail

$$\lim_{T \rightarrow \infty} 2 \int_0^T \left(1 - \frac{t}{T}\right) \mathcal{R}(t) dt = 2 \int_0^{\infty} \mathcal{R}(t) dt. \quad (20)$$

Summing up (14), (19) and (20) yield

$$\lim_{T \rightarrow \infty} \frac{1}{T} (V_s(T, u) - (\mathbb{E}[U(\chi_s^2, [0, T], u)])^2) = 2 \int_0^{\infty} (\mathcal{I}_0(t) - K_s(u)^2) dt + 2 \int_0^{\infty} \mathcal{R}(t) dt < \infty.$$

□

**Comment concerning Assumption (A).** As in the one dimensional Gaussian case, one can remark that we did not use the full strength of Assumption **(A)** in the previous proof. Actually, the result of Proposition 6 still holds under the following assumption:  $X$  is almost surely of class  $C^2$ ,  $r_X$  satisfies (15) and  $t \mapsto |r_X(t)| + |r'_X(t)| + |r''_X(t)|$  is integrable and tends to 0 as  $t$  tends to  $\infty$ .

## 4 One dimensional case

In this section, we focus on the one dimensional case ( $d = 1$ ) and we rewrite the formulas of Section 2 in this case.

Let us consider a stationary centered Gaussian process  $X$  with variance 1, covariance function  $r$  and second spectral moment  $\lambda = -r''(0)$ . Equations (5) and (8) become as follows,

$$\mathbb{E}[\varphi(X, T, u)] = |T| C(u, \lambda) \quad \text{with} \quad C(u, \lambda) = (2\pi)^{-1} \lambda^{1/2} e^{-u^2/2}, \quad (21)$$

and  $\text{Var}[\varphi(X, T, u)] \sim |T| V(u)$  as  $|T| \rightarrow +\infty$ , with

$$V(u) = \int_{\mathbb{R}} (G(u, t) D(t)^{-1/2} - C(u, \lambda)^2) dt + (2\pi\lambda)^{-1/2} g(u), \quad (22)$$

with  $D(t) = (2\pi)^2 (\lambda^2 - r''(t)^2)$  and  $g(u), G(u, t)$  are given by (7) in the case  $d = 1$ , *i.e.*

$$\begin{aligned} g(u) &= \mathbb{E}[\mathbf{1}_{[u, \infty)}(X(0)) |X''(0)|] \\ G(u, t) &= \mathbb{E}[\mathbf{1}_{[u, \infty)}(X(0)) \mathbf{1}_{[u, \infty)}(X(t)) X''(0) X''(t) | X'(0) = X'(t) = 0]. \end{aligned}$$

Next proposition provides a closed formula for the asymptotic variance  $V(u)$ . It involves Gaussian integrals that can be numerically evaluated.

We need some extra notations.

- $\gamma := r^{(4)}(0)$  so that the covariance matrix of  $(X(0), X''(0))$  is  $\begin{pmatrix} 1 & -\lambda \\ -\lambda & \gamma \end{pmatrix}$ ,
- $\phi$  denotes the standard Gaussian density,  $\Phi$  the standard Gaussian distribution, and  $\bar{\Phi}(\cdot) = 1 - \Phi(\cdot)$ .
- For any  $(u, \alpha, m) \in \mathbb{R}^3$ , we introduce

$$\begin{cases} Q^{(0)}(u, \alpha, m) = \int_u^{\infty} \bar{\Phi}(\alpha x + m) \phi(x) dx ; & Q^{(1)}(u, \alpha, m) = - \int_u^{\infty} \phi(\alpha x + m) \phi'(x) dx \\ Q^{(2)}(u, \alpha, m) = - \int_u^{\infty} \phi'(\alpha x + m) \phi(x) dx ; & Q^{(3)}(u, \alpha, m) = \int_u^{\infty} \bar{\Phi}(\alpha x + m) \phi''(x) dx. \end{cases}$$

**Proposition 7.** *Let  $X$  be a Gaussian process that satisfies Assumption (A).*

1. *For any  $u$  in  $\mathbb{R}$ ,*

$$g(u) = \lambda\phi(u)(2\Phi(\frac{\lambda u}{(\gamma - \lambda^2)^{1/2}}) - 1) + (\frac{2\gamma}{\pi})^{1/2}\bar{\Phi}((\frac{\gamma}{\gamma - \lambda^2})^{1/2}u).$$

2. *Let  $u \in \mathbb{R}$  and  $t \in \mathbb{R}$  be fixed. Let  $\Gamma(t)$  be the covariance matrix of the conditional distribution*

$$\mathcal{L}(X(0), X(t), X''(0), X''(t) \mid X'(0) = X'(t) = 0)$$

*and let denote by  $(l_{ij})_{1 \leq i \leq j \leq 4}$  the coefficients of any lower triangular matrix  $L$  such that  $LL^T = \Gamma(t)$ . Then*

$$\begin{aligned} G(u, t) &= (l_{41}l_{31} + l_{42}l_{32} + l_{43}l_{33}) Q^{(0)}(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}) \\ &+ (l_{42}l_{31} + l_{41}l_{32}) Q^{(1)}(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}) \\ &+ l_{42}l_{32} Q^{(2)}(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}) + l_{41}l_{31} Q^{(3)}(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}). \end{aligned}$$

**Proof.**

1. **Computation of  $g(u)$ .** By Equation (7), we have

$$\begin{aligned} g(u) &= \mathbb{E}[\mathbf{1}_{[u, \infty)}(X(0)) |X''(0)|] \\ &= \frac{1}{2\pi(\gamma - \lambda^2)^{1/2}} \int_u^\infty \int_{\mathbb{R}} |\ddot{x}| e^{-\frac{1}{2(\gamma - \lambda^2)}(\gamma x^2 + 2\lambda x\ddot{x} + \ddot{x}^2)} d\ddot{x} dx \\ &= \frac{1}{2\pi(\gamma - \lambda^2)^{1/2}} \int_u^\infty e^{-\frac{1}{2}x^2} dx \int_{\mathbb{R}} |\ddot{x}| e^{-\frac{1}{2(\gamma - \lambda^2)}(\ddot{x} + \lambda x)^2} d\ddot{x} \end{aligned}$$

Let us split the inner integral into  $\ddot{x} > 0$  and  $\ddot{x} < 0$ , i.e.

$$\frac{1}{\sqrt{2\pi}(\gamma - \lambda^2)^{1/2}} \int_{\mathbb{R}} |\ddot{x}| e^{-\frac{1}{2(\gamma - \lambda^2)}(\ddot{x} + \lambda x)^2} d\ddot{x} := I_1(x, \lambda, \gamma) + I_2(x, \lambda, \gamma),$$

where

$$\begin{aligned} I_1(x, \lambda, \gamma) &= \frac{1}{\sqrt{2\pi}(\gamma - \lambda^2)^{1/2}} \int_0^\infty \ddot{x} e^{-\frac{1}{2(\gamma - \lambda^2)}(\ddot{x} + \lambda x)^2} d\ddot{x} \\ I_2(x, \lambda, \gamma) &= \frac{1}{\sqrt{2\pi}(\gamma - \lambda^2)^{1/2}} \int_0^\infty \ddot{x} e^{-\frac{1}{2(\gamma - \lambda^2)}(\ddot{x} - \lambda x)^2} d\ddot{x} = I_1(x, -\lambda, \gamma). \end{aligned}$$

We make the following change of variable,  $\frac{\ddot{x} + \lambda x}{(\gamma - \lambda^2)^{1/2}} = y$ , to get

$$\begin{aligned} I_1(x, \lambda, \gamma) &= \frac{1}{\sqrt{2\pi}} \int_{\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}}^\infty ((\gamma - \lambda^2)^{1/2}y - \lambda x) e^{-\frac{1}{2}y^2} dy \\ &= (\gamma - \lambda^2)^{1/2} \phi(\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}) - \lambda x \bar{\Phi}(\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}) \\ I_2(x, \lambda, \gamma) &= (\gamma - \lambda^2)^{1/2} \phi(\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}) + \lambda x \bar{\Phi}(-\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}). \end{aligned}$$

But it holds that  $\bar{\Phi}(x) = \Phi(-x)$ . Thus

$$I_1(x, \lambda, \gamma) + I_2(x, \lambda, \gamma) = 2(\gamma - \lambda^2)^{1/2} \phi(\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}) + \lambda x (2\Phi(\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}) - 1).$$

Summing up, we have

$$g(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{1}{2}x^2} [2(\gamma - \lambda^2)^{1/2} \phi\left(\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}\right) + \lambda x (2\Phi\left(\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}\right) - 1)] dx.$$

Integrating by parts the second integral, we get that it is equal to

$$\lambda \phi(u) (2\Phi\left(\frac{\lambda u}{(\gamma - \lambda^2)^{1/2}}\right) - 1) + \frac{2\lambda^2}{\sqrt{2\pi}(\gamma - \lambda^2)^{1/2}} \int_u^\infty e^{-\frac{1}{2}x^2} \phi\left(\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}\right) dx,$$

yielding

$$g(u) = \lambda \phi(u) (2\Phi\left(\frac{\lambda u}{(\gamma - \lambda^2)^{1/2}}\right) - 1) + \frac{2\gamma}{\sqrt{2\pi}(\gamma - \lambda^2)^{1/2}} \int_u^\infty e^{-\frac{1}{2}x^2} \phi\left(\frac{\lambda x}{(\gamma - \lambda^2)^{1/2}}\right) dx.$$

The second term is equal to

$$\begin{aligned} \frac{2\gamma}{2\pi(\gamma - \lambda^2)^{1/2}} \int_u^\infty e^{-\frac{1}{2}x^2} e^{-\frac{\lambda^2 x^2}{2(\gamma - \lambda^2)}} dx &= \frac{\gamma}{\pi(\gamma - \lambda^2)^{1/2}} \int_u^\infty e^{-\frac{\gamma x^2}{2(\gamma - \lambda^2)}} dx \\ &= \left(\frac{2\gamma}{\pi}\right)^{1/2} \bar{\Phi}\left(\left(\frac{\gamma}{(\gamma - \lambda^2)}\right)^{1/2} u\right). \end{aligned}$$

Finally,

$$g(u) = \lambda \phi(u) (2\Phi\left(\frac{\lambda u}{(\gamma - \lambda^2)^{1/2}}\right) - 1) + \left(\frac{2\gamma}{\pi}\right)^{1/2} \bar{\Phi}\left(\left(\frac{\gamma}{(\gamma - \lambda^2)}\right)^{1/2} u\right).$$

**2. Computation of  $G(u, t)$ .** The conditional distribution

$$\mathcal{L}(X(0), X(t), X''(0), X''(t) \mid X'(0) = X'(t) = 0)$$

is a 4-dimensional centered Gaussian distribution with covariance matrix given by

$$\Gamma(t) = C_{11}(t) - C_{12}(t)C_{22}(t)^{-1}C_{12}(t)^T, \quad (23)$$

where  $C_{11}(t)$  is the covariance matrix of the vector  $(X(0), X(t), X''(0), X''(t))$ ,  $C_{22}(t)$  is the covariance matrix of the vector  $(X'(0), X'(t))$  and  $C_{12}(t)$  is the matrix of the covariances between those two vectors. Precisely, we have

$$C_{11}(t) = \begin{pmatrix} 1 & r(t) & -\lambda & r''(t) \\ r(t) & 1 & r''(t) & -\lambda \\ -\lambda & r''(t) & \gamma & r^{(4)}(t) \\ r''(t) & -\lambda & r^{(4)}(t) & \gamma \end{pmatrix},$$

$$C_{22}(t) = \begin{pmatrix} \lambda & -r''(t) \\ -r''(t) & \lambda \end{pmatrix} \quad \text{and} \quad C_{12}(t) = \begin{pmatrix} 0 & r'(t) \\ -r'(t) & 0 \\ 0 & r^{(3)}(t) \\ -r^{(3)}(t) & 0 \end{pmatrix}$$

Then, we can write

$$G(u, t) = \mathbb{E}(\mathbf{1}_{Z_1 > u} \mathbf{1}_{Z_2 > u} Z_3 Z_4) \quad \text{where } \mathbf{Z} = (Z_1, Z_2, Z_3, Z_4) \stackrel{\mathcal{L}}{=} N(0, \Gamma(t)). \quad (24)$$

From now on, we remove the dependence on  $t$ . We begin by writing  $\mathbf{Z} = L\mathbf{Y}$ , where  $L$  is a lower triangular matrix such that  $LL^T = \Gamma(t)$ . Denoting by  $l_{ij}$  for  $i \geq j$  the elements of  $L$ , we have

$$Z_1 = l_{11}Y_1 \quad Z_2 = l_{21}Y_1 + l_{22}Y_2 \quad Z_3 = \sum_{i=1}^3 l_{3i}Y_i \quad Z_4 = \sum_{i=1}^4 l_{4i}Y_i,$$



and the expectation (24) can be written as

$$G(u, t) = \int_{\mathbb{R}^4} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} \left( \sum_{i=1}^3 l_{3i}y_i \right) \left( \sum_{i=1}^4 l_{4i}y_i \right) \phi(\mathbf{y}) d\mathbf{y}.$$

By expanding the second sum, the integral can be written as the sum of the following terms (starting with index  $i = 4$  term)

$$\begin{aligned} I_4 &= l_{44} \int_{\mathbb{R}^4} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} \left( \sum_{i=1}^3 l_{3i}y_i \right) y_4 \phi(\mathbf{y}) d\mathbf{y} = 0, \\ I_3 &= l_{43} \int_{\mathbb{R}^4} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} \left( \sum_{i=1}^3 l_{3i}y_i \right) y_3 \phi(\mathbf{y}) d\mathbf{y} \\ &= l_{43} l_{33} \int_{\mathbb{R}^2} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} \phi(y_1) \phi(y_2) dy_1 dy_2 \left( \int_{\mathbb{R}} y_3^2 \phi(y_3) dy_3 \right) \\ &= l_{43} l_{33} Q^{(0)}\left(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}\right) \end{aligned}$$

$$\begin{aligned} I_2 &= l_{42} \int_{\mathbb{R}^2} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} (l_{31}y_1 y_2 + l_{32}[(y_2^2 - 1) + 1]) \phi(y_1) \phi(y_2) dy_1 dy_2 \\ &= l_{42} l_{32} Q^{(0)}\left(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}\right) + l_{42} l_{32} \int_{\mathbb{R}^2} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} \phi(y_1) \phi''(y_2) dy_1 dy_2 \\ &\quad + l_{42} l_{31} \int_{\mathbb{R}^2} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} \phi'(y_1) \phi'(y_2) dy_1 dy_2 \\ &= l_{42} l_{32} Q^{(0)}\left(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}\right) - l_{42} l_{32} \int_{\frac{u}{l_{11}}}^{\infty} \phi'\left(\frac{u}{l_{22}} - \frac{l_{21}}{l_{22}} y_1\right) \phi(y_1) dy_1 \\ &\quad - l_{42} l_{31} \int_{\frac{u}{l_{11}}}^{\infty} \phi\left(\frac{u}{l_{22}} - \frac{l_{21}}{l_{22}} y_1\right) \phi'(y_1) dy_1 \\ &= l_{42} l_{32} \left( Q^{(0)}\left(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}\right) + Q^{(2)}\left(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}\right) \right) + l_{42} l_{31} Q^{(1)}\left(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}\right) \end{aligned}$$

$$\begin{aligned} I_1 &= l_{41} \int_{\mathbb{R}^4} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} \left( \sum_{i=1}^3 l_{3i}y_i \right) y_1 \phi(\mathbf{y}) d\mathbf{y} \\ &= l_{41} \int_{\mathbb{R}^2} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} (l_{31}y_1 + l_{32}y_2) y_1 \phi(y_1) \phi(y_2) dy_1 dy_2 \\ &= l_{41} l_{32} \int_{\mathbb{R}^2} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} \phi'(y_1) \phi'(y_2) dy_1 dy_2 + l_{41} l_{31} Q^{(0)}\left(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}\right) \\ &\quad + l_{41} l_{31} \int_{\mathbb{R}^2} \mathbf{1}_{\{l_{11}y_1 > u\}} \mathbf{1}_{\{l_{21}y_1 + l_{22}y_2 > u\}} \phi''(y_1) \phi(y_2) dy_1 dy_2 \\ &= l_{41} l_{31} \left( Q^{(0)}\left(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}\right) + Q^{(3)}\left(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}\right) \right) + l_{41} l_{32} Q^{(1)}\left(\frac{u}{l_{11}}, -\frac{l_{21}}{l_{22}}, \frac{u}{l_{22}}\right). \end{aligned}$$

In this manner, we get the result.  $\square$

## References

- [1] Adler R.J., Taylor J.E. *Random Fields and Geometry*. Springer Monographs in Mathematics. Springer (2007).
- [2] Arcones M. A. *Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors*. Ann. Probab., vol 22, 2242-2274 (1994).
- [3] Azaïs J-M., Wschebor M. *Level Sets and Extrema of Random Processes and Fields*. Wiley (2009).
- [4] Estrade A., León J.R. *A central limit theorem for the Euler characteristic of a Gaussian excursion set*. Preprint, hal-00943054 (2015). To appear in Ann. of Prob.
- [5] Geman D. *On the variance of the number of zeros of a stationary Gaussian process*. Ann. Math. Stat., 43, 977-982 (1972).
- [6] Kauderer M. *Modes in n-dimensional first-order systems*. Journal of Math. Phys. 34, p.4221 (1993).
- [7] Kratz M., León J.R. *On the second moment of the number of crossings by a stationary Gaussian process*. Ann. Probab. 34(4), 1601-1607 (2006).
- [8] Nourdin I., Peccati G. and Podolskij M. *Quantitative Breuer-Major theorems*. Stochastic Processes and their Applications, vol 121, 793-812 (2011).
- [9] Sharpe K. *Some properties of the crossings process generated by a stationary  $\chi^2$  process*. Adv. App. Prob. 10, 373-391 (1978).
- [10] Taheriyoun A.R., Shafie K., Jafari Jozani M. *A note on the higher moments of the Euler characteristic of the excursion sets of random fields*. Statist. Probab. Lett. 79, 1074-1082 (2009).
- [11] Withers C. S., Nadarajah S. *Expansions for the multivariate normal*. Journal of Multivariate Analysis 101, 1311-1316 (2010).
- [12] Worsley K.J. *Local maxima and the expected Euler characteristic of excursion sets of  $\chi^2$ ,  $F$  and  $t$  fields*. Adv. Appl. Probab. 26, 13-42 (1994).