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# *A class of symmetric difference-closed sets related to commuting involutions*

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Recent research on the combinatorics of finite sets has explored the structure of symmetric difference-closed sets, and recent research in combinatorial group theory has concerned the enumeration of commuting involutions in  $S_n$  and  $A_n$ . In this article, we consider an interesting combination of these two subjects, by introducing classes of symmetric difference-closed sets of elements which correspond in a natural way to commuting involutions in  $S_n$  and  $A_n$ . We consider the natural combinatorial problem of enumerating symmetric difference-closed sets consisting of subsets of a set consisting of pairwise disjoint 2-subsets of  $[n]$ . We prove an explicit combinatorial formula for the number of symmetric difference-closed 4-sets satisfying this property, and we prove an explicit combinatorial formula for an analogous class of symmetric difference-closed sets consisting of elements which correspond to commuting involutions in  $A_n$ .

**Keywords:** symmetric difference-closed set, symmetric group, alternating group

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## 1 Introduction

Combinatorial properties concerning symmetric difference-closed ( $\Delta$ -closed) sets were explored recently in Gamble and Simpson (2015) and Buck and Godbole (2014). In this article, we consider an interesting class of  $\Delta$ -closed sets related to commuting involutions in the symmetric group  $S_n$  and the alternating group  $A_n$ .

A formula for the number of pairs of commuting involutions (up to isomorphism) in  $S_n$  and  $A_n$  was proven in Kiefer and Leemans (2013). Combinatorial properties associated with pairs of commuting involutions in  $S_n$  and  $A_n$  is an interesting subject in part because this area is related to the classifications of abstract regular polytopes for fixed automorphism groups Kiefer and Leemans (2013).

Inspired in part by Kiefer and Leemans (2013), Buck and Godbole (2014), and Gamble and Simpson (2015), we introduce new classes of  $\Delta$ -closed sets consisting of elements which correspond in a natural way to commuting involutions in  $S_n$  and  $A_n$ , and we prove new combinatorial formulas for these classes of  $\Delta$ -closed sets.

### *1.1 An enumerative problem concerning symmetric difference-closed sets*

Suppose that  $n$  people arrive at a meeting, and suppose that the  $n$  people arrange themselves into pairs (except for a loner if  $n$  is odd) and that the pairs then form various organizations. What is the total number, taken over all possible pairings, of possible collections  $C$  of three distinct organizations such that given

two organizations in  $C$ , if a pair  $P$  of people belongs to only one of these two organizations, then  $P$  must be a member of the remaining (third) organization?

The number of possible collections  $C$  as given above is also equal to the number of  $\Delta$ -closed 4-subsets  $S \subseteq 2^{2^{[n]}}$  such that there exists a set  $T$  consisting of pairwise disjoint 2-subsets of  $[n]$  such that  $S \subseteq 2^T$ . A set  $S$  of this form endowed with the  $\Delta$  operation forms a group which is isomorphic to the Klein four-group  $C_2 \times C_2$ , and the elements consisting of pairwise disjoint 2-sets in  $S$  may be regarded in an obvious way as commuting involutions in  $S_n$ . We thus have that the total number of collections  $C$  as given above is also equal to the number of subgroups  $G$  of the symmetric group  $S_n$  such that  $G$  is isomorphic to the Klein four-group, and there exists a set  $T \subseteq S_n$  of pairwise disjoint transpositions such that each element in  $G$  is a product of elements in  $T$ . Note that this is not the same as the number of Klein four-subgroups of  $S_n$ : for example, the permutation subgroup

$$\{\text{id}, (12)(34), (13)(24), (14)(23)\}$$

forms a Klein four-subgroup of  $S_4$ , but the 2-sets  $\{3, 4\}$  and  $\{1, 3\}$  are not pairwise disjoint.

The sequence labeled A267840 which we contributed to the On-Line Encyclopedia of Integer Sequences (OEIS) OEIS Foundation Inc. (2011) enumerates  $\Delta$ -closed sets of this form. Accordingly, let  $A267840_n$  denote the number of  $\Delta$ -closed subsets  $S \subseteq 2^{2^{[n]}}$  satisfying the above property. In the OEIS sequence labeled A267840, we provided the following intricate and exotic triple sum for  $A267840_n$  for  $n \in \mathbb{N}$  (letting  $\delta$  denote the Kronecker delta function):

$$n! \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=1}^i \sum_{k=\max(\lceil \frac{i}{2} \rceil, i+j-\lfloor \frac{n}{2} \rfloor)}^{\min(j, \lfloor \frac{1}{4}(2i+2j-1) \rfloor)} \frac{2^{k-i-j}}{k!(i-k)!(j-k)!(n-2i-2j+2k)!(\delta_{i,j} + \delta_{i,2k} + 1)!}. \quad (1)$$

In this article, we offer an elegant proof of the formula indicated above. Amazingly, the integer sequence  $(A267840_n)_{n \in \mathbb{N}}$  seems to have a surprisingly simple exponential generating function, in stark contrast to the intricacy of the above triple summation. In 2016, Václav Kotěšovec conjectured that the e.g.f. for  $(A267840_n)_{n \in \mathbb{N}}$  is equal to the following expression OEIS Foundation Inc. (2011).

$$\frac{e^x}{3} - \frac{e^{\frac{x(x+2)}{2}}}{2} + \frac{e^{\frac{x(3x+2)}{2}}}{6}. \quad (2)$$

Václav Kotěšovec also provided a conjectural polynomial recurrence formula for  $(A267840_n)_{n \in \mathbb{N}}$ , and also provided a conjectural asymptotic formula for this integer sequence OEIS Foundation Inc. (2011), as we discuss in Section 3.

## 1.2 A class of symmetric difference-closed sets related to commuting even involutions

Since the number of pairs of commuting involutions in the alternating group  $A_n$  up to isomorphism is also considered in Kiefer and Leemans (2013), it is also natural to consider analogues of the results given in Section 1.1 for even products of transpositions.

The sequence labeled A266503 which we contributed to OEIS Foundation Inc. (2011) enumerates subgroups  $G$  of the alternating group  $A_n$  such that  $G$  is isomorphic to the Klein four-subgroup  $C_2 \times$

$C_2$ , and each element in  $G$  is the product of the elements in a subset of a fixed set of pairwise disjoint transpositions in  $A_n$ . It is important to note that this is not equal to the number of Klein four-subgroups of  $A_n$ , as is easily verified. Letting  $A266503_n$  denote the  $n^{\text{th}}$  term in the OEIS sequence A266503,  $A266503_n$  is also equal to the number of  $\Delta$ -closed subsets  $S \subseteq 2^{2^{[n]}}$  such that there exists a set  $T$  consisting of pairwise disjoint 2-subsets of  $[n]$  such that  $S \subseteq 2^T$ , and each element in  $S$  is of even order.

In the OEIS sequence labeled A266503, we provided the following beautiful expression for  $A266503_n$  for  $n \in \mathbb{N}$ .

$$n! \sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \sum_{j=1}^i \sum_{k=\max(i, 2i+2j-\lfloor \frac{n}{2} \rfloor)}^{\min(2j, \lfloor \frac{1}{4}(4i+4j-1) \rfloor)} \frac{2^{k-2i-2j}}{k!(2i-k)!(2j-k)!(n-4i-4j+2k)! (\delta_{i,j} + \delta_{i,k} + 1)!}.$$

The integer sequence  $(A266503_n)_{n \in \mathbb{N}}$  also seems to have a surprisingly simple exponential generating function. Václav Kotěšovec conjectured that the e.g.f. for  $(A267840_n)_{n \in \mathbb{N}}$  is equal to the following expression OEIS Foundation Inc. (2011).

$$\frac{e^x}{3} - \frac{e^{-\frac{x(x-2)}{2}}}{8} - \frac{e^{\frac{x(x+2)}{2}}}{4} + \frac{e^{\frac{x(3x+2)}{2}}}{24}.$$

Václav Kotěšovec also provided a conjectural polynomial recurrence formula for  $(A267840_n)_{n \in \mathbb{N}}$ , and also provided a conjectural asymptotic formula for this integer sequence OEIS Foundation Inc. (2011). We also discuss these conjectural formulas in Section 3.

## 2 Main results

Let  $n \geq 4$ . Then  $\Delta$ -closed 4-sets consisting of the empty set together with subsets of a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, n\}$  either: (1) are of the form

$$\{\emptyset, \{t_1, t_2, \dots, t_i\}, \{t_{i+1}, t_{i+2}, \dots, t_j\}, \{t_1, t_2, \dots, t_i, t_{i+1}, t_{i+2}, \dots, t_j\}\}$$

where  $\{t_1, t_2, \dots, t_j\}$  is a set consisting of  $j$  distinct pairwise disjoint 2-sets in  $\{1, 2, \dots, n\}$ , or (2) consist of non-empty elements of the following form, letting  $\{t_1, t_2, \dots, t_k\}$  be a set consisting of  $k$  distinct pairwise disjoint 2-sets in  $\{1, 2, \dots, n\}$ :

$$\begin{aligned} &\{t_1, t_2, \dots, t_i, t_{i+1}, t_{i+2}, \dots, t_j\}, \\ &\{t_{i+1}, t_{i+2}, \dots, t_j, t_{j+1}, t_{j+2}, \dots, t_k\}, \\ &\{t_1, t_2, \dots, t_i, t_{j+1}, t_{j+2}, \dots, t_k\}. \end{aligned}$$

Given an arbitrary  $\Delta$ -closed 4-set  $S$  consisting of subsets of a set consisting of pairwise disjoint 2-subsets of  $[n]$ , we define the *partition type*  $\text{type}(S)$  of  $S$  as the unique partition  $\lambda = \text{type}(S)$  of length 3 such that a largest set of 2-sets in  $S$  consists of  $\frac{\lambda_1}{2}$  (pairwise disjoint) 2-sets consisting of a total of  $\lambda_1$  elements, a second-largest set of 2-sets in  $S$  consists of  $\frac{\lambda_2}{2}$  2-sets, and a third-largest set of 2-sets in  $S$  consists of  $\frac{\lambda_3}{3}$  2-sets. For example,

$$\text{type}(\{\{\{1, 3\}, \{2, 4\}\}, \{\{1, 3\}\}, \{\{2, 4\}\}, \emptyset\}) = (4, 2, 2) \vdash 8,$$

$$\text{type}(\{\{\{1, 3\}, \{2, 4\}\}, \{\{2, 4\}, \{5, 6\}\}, \{\{1, 3\}, \{5, 6\}\}, \emptyset\}) = (4, 4, 4) \vdash 12,$$

$$\text{type}(\{\{\{1, 3\}, \{2, 4\}, \{5, 6\}\}, \{\{1, 3\}, \{5, 6\}\}, \{2, 4\}, \emptyset\}) = (6, 4, 2) \vdash 12.$$

We define a *Klein partition* for  $n \in \mathbb{N}$  as a partition  $\lambda$  such that  $\lambda = \text{type}(S)$  for some set  $S$  of the form described above.

**Lemma 2.1.** *A partition  $\lambda$  is a Klein partition for  $n \in \mathbb{N}$  if and only if*

(a) *The length  $\ell(\lambda)$  of  $\lambda$  is 3;*

(b) *Each entry of  $\lambda$  is even;*

(c) *The first entry  $\lambda_1$  of  $\lambda$  satisfies  $\lambda_1 \leq 2\lfloor \frac{n}{2} \rfloor$ ; and*

(d) *There exists an index  $i$  in  $[0, \frac{\lambda_2}{2}]$  such that  $\lambda_1 + \lambda_2 - 4i = \lambda_3$  and  $\lambda_1 + \lambda_2 - 2i \leq n$ .*

**Proof:** ( $\implies$ ) Suppose that  $\lambda$  is a Klein partition for  $n \in \mathbb{N}$ . We thus have that  $\lambda = \text{type}(S)$  for some  $\Delta$ -closed 4-set  $S$  consisting of  $\emptyset$  together with subsets of a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, n\}$ . By definition of the partition type of a set of this form, we have that  $\lambda = \text{type}(S)$  must be of length 3 and must have even entries. The first entry  $\lambda_1$  of  $\lambda$  is equal to the total number of elements among all 2-sets in a largest set of 2-sets in  $S$ . If  $n$  is even, then the maximal total number of elements among all 2-sets in a largest set of 2-sets in  $S$  is  $n$ , and otherwise,  $\lambda_1$  is at most  $n - 1$ . We thus have that  $\lambda_1 \leq 2\lfloor \frac{n}{2} \rfloor$ . Let  $p_1, p_2$ , and  $p_3$  be (pairwise) distinct nontrivial sets of 2-sets such that  $p_1$  is a largest set of 2-sets in  $S$ ,  $p_2$  is a second-largest set of 2-sets in  $S$ , and  $p_3$  is a smallest set of 2-sets in  $S$ . Note that it is possible that  $p_1, p_2$ , and  $p_3$  are all sets of equal cardinality. Also observe that  $p_1 \Delta p_2 = p_3 \in S$ . Suppose that  $p_1$  and  $p_2$  share exactly  $j \in \mathbb{N}_0$  2-sets in common. It is easily seen that  $j > 0$  since  $S$  forms a group under the binary operation  $\Delta: S \times S \rightarrow S$ , and since the number of 2-sets of  $p_1$  is greater than or equal to the number of 2-sets of  $p_2$  and greater than or equal to the number of 2-sets of  $p_3$ . Since  $\lambda_2 \leq \lambda_1$  it is thus clear that  $j \in [0, \frac{\lambda_2}{2}]$ . Since  $p_1$  and  $p_2$  share exactly  $j$  2-sets, we thus have that total number  $\lambda_3$  of elements among all 2-sets in  $p_3$  is  $(\lambda_1 - 2j) + (\lambda_2 - 2j)$ . Now consider the total number of elements among the 2-sets in either  $p_1$  or  $p_2$ . Since  $p_1$  and  $p_2$  share exactly  $j$  2-sets, by the principle of inclusion-exclusion, we have that the total number of elements among the 2-sets in either  $p_1$  or  $p_2$  is equal to  $\lambda_1 + \lambda_2 - 2j$ . We thus have that  $\lambda_1 + \lambda_2 - 2j \leq n$ , and we thus have that there exists an index  $i$  in  $[0, \frac{\lambda_2}{2}]$  such that

$$\lambda_1 + \lambda_2 - 4i = \lambda_3, \lambda_1 + \lambda_2 - 2i \leq n$$

as desired.

( $\impliedby$ ) Conversely, suppose that  $\lambda$  is a partition such that (a) The length of  $\lambda$  is 3; (b) Each entry of  $\lambda$  is even; (c) The first entry  $\lambda_1$  of  $\lambda$  satisfies  $\lambda_1 \leq 2\lfloor \frac{n}{2} \rfloor$ ; and (d) There exists an index  $i$  in  $[0, \frac{\lambda_2}{2}]$  such that  $\lambda_1 + \lambda_2 - 4i = \lambda_3, \lambda_1 + \lambda_2 - 2i \leq n$ . Let  $p_1$  denote the following set of (pairwise disjoint) 2-sets:

$$p_1 = \{\{1, 2\}, \{3, 4\}, \dots, \{\lambda_1 - 1, \lambda_1\}\}.$$

Since  $\lambda_1$  is even (since each entry of  $\lambda$  is even), our above definition of  $p_1$  is well-defined. Since  $\lambda_1 \leq 2\lfloor \frac{n}{2} \rfloor$ , we thus have that  $p_1$  is a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, n\}$ . Since there exists an integer  $i$  in the interval  $[0, \frac{\lambda_2}{2}]$  such that

$$\lambda_1 + \lambda_2 - 4i = \lambda_3, \lambda_1 + \lambda_2 - 2i \leq n$$

by assumption, let  $j \in [0, \frac{\lambda_2}{2}]$  denote a fixed integer satisfying  $\lambda_1 + \lambda_2 - 4j = \lambda_3$  and  $\lambda_1 + \lambda_2 - 2j \leq n$ . Now let  $p_2$  denote the following set of (pairwise disjoint) 2-sets:

$$p_2 = \{ \{ \lambda_1 - 2j + 1, \lambda_1 - 2j + 2 \}, \{ \lambda_1 - 2j + 3, \lambda_1 - 2j + 4 \}, \dots, \\ \{ \lambda_1 - 2j + \lambda_2 - 1, \lambda_1 - 2j + \lambda_2 \} \}.$$

The total number of elements among the distinct 2-sets in  $p_2$  is thus

$$(\lambda_1 - 2j + \lambda_2) - (\lambda_1 - 2j + 1) + 1 = \lambda_2$$

and since  $\lambda_1$  and  $\lambda_2$  are both even, the above definition of  $p_2$  is thus well-defined. Furthermore, since  $\lambda_1 + \lambda_2 - 2j \leq n$ , we thus have that  $p_2$  is a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, n\}$ . Now consider the expression  $p_1 \Delta p_2 \in S$ . The total number of elements among the 2-sets in the expression  $p_1 \Delta p_2$  is equal to

$$(\lambda_1 - 2j) + (-2j + \lambda_2) = \lambda_1 + \lambda_2 - 4j$$

and thus since  $\lambda_1 + \lambda_2 - 4j = \lambda_3$  we have that the expression  $p_1 \Delta p_2 \in S$  consists of  $\frac{\lambda_3}{2}$  (pairwise disjoint) 2-cycles consisting of a total of  $\lambda_3$  entries. Now consider the expression  $\text{type}(S)$ . The set  $p_1$  consists of  $\frac{\lambda_1}{2}$  2-sets, the set  $p_2$  consists of  $\frac{\lambda_2}{2}$  2-sets, and the set  $p_3$  consists of  $\frac{\lambda_3}{2}$  2-sets, with

$$\frac{\lambda_1}{2} \geq \frac{\lambda_2}{2} \geq \frac{\lambda_3}{2}$$

since  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$  is a partition. We thus have that  $\text{type}(S) = \lambda$ , thus proving that  $\lambda$  is a Klein partition for  $n \in \mathbb{N}$ .  $\square$

**Lemma 2.2.** For  $n \in \mathbb{N}$ , the Klein partitions for  $n$  are precisely tuples of the form

$$(2a, 2b, 2a + 2b - 4i)$$

such that:

1.  $1 \leq a \leq \lfloor \frac{n}{2} \rfloor$ ;
2.  $1 \leq b \leq a$ ; and
3.  $\max(a + b - \lfloor \frac{n}{2} \rfloor, \lceil \frac{a}{2} \rceil) \leq i \leq \min(b, \lfloor \frac{2a+2b-1}{4} \rfloor)$ .

**Proof:** Let  $n \in \mathbb{N}$ . Let  $\lambda = (2a, 2b, 2a + 2b - 4i)$  be a tuple satisfying the conditions (1), (2), and (3) given above. We have that  $1 \leq \lambda_2 \leq \lambda_1$  from condition (2). We have that  $2a + 2b - 4i \leq 2b$  since  $\frac{a}{2} \leq i$  since  $\lceil \frac{a}{2} \rceil \leq i$  from condition (3), and we thus have that  $\lambda_3 \leq \lambda_2 \leq \lambda_1$  as desired. We have that  $1 \leq 2a + 2b - 4i$  since  $i \leq \frac{2a+2b-1}{4}$  since  $i \leq \lfloor \frac{2a+2b-1}{4} \rfloor$  from condition (3), and we thus have that

$$1 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1$$

thus proving that that the tuple

$$\lambda = (2a, 2b, 2a + 2b - 4i)$$

is in fact an integer partition. We will now make use of Lemma 2.1. Certainly,  $\lambda$  is of length 3, and each entry of  $\lambda$  is even. The first entry  $\lambda_1 = 2a$  of  $\lambda$  satisfies  $\lambda_1 = 2a \leq 2 \lfloor \frac{n}{2} \rfloor$  since  $a \leq \lfloor \frac{n}{2} \rfloor$  from condition (1). Certainly,  $\lambda_1 + \lambda_2 - 4i = \lambda_3 = 2a + 2b - 4i$ . Furthermore, we have that  $\lambda_1 + \lambda_2 - 2i \leq n$  since  $a + b - \frac{n}{2} \leq i$  since  $a + b - \lfloor \frac{n}{2} \rfloor \leq i$  from condition (3). By Lemma 2.1, we thus have that the partition  $\lambda$  is a Klein partition for  $n \in \mathbb{N}$ .

Conversely, let  $\lambda$  be a Klein partition for  $n$ . By Lemma 2.1, we thus have that:

- (a) The length  $\ell(\lambda)$  of  $\lambda$  is 3;
- (b) Each entry of  $\lambda$  is even;
- (c) The first entry  $\lambda_1$  of  $\lambda$  satisfies  $\lambda_1 \leq 2 \lfloor \frac{n}{2} \rfloor$ ; and
- (d) There exists an integer  $i$  in such that:

$$\left[0, \frac{\lambda_2}{2}\right] \lambda_1 + \lambda_2 - 4i = \lambda_3, \lambda_1 + \lambda_2 - 2i \leq n.$$

Begin by rewriting the entries of  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$  as follows. By condition (b) we may thus write  $\lambda_1 = 2a$  and  $\lambda_2 = 2b$ , letting  $a, b \in \mathbb{N}$ . Let  $i \in [0, \frac{\lambda_2}{2}]$  be as given condition (d) above. We thus have that  $\lambda_3 = 2a + 2b - 4i$  and we thus have that the integer partition  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$  is a tuple of the following form:

$$\lambda = (2a, 2b, 2a + 2b - 4i).$$

Since  $\lambda$  is an integer partition, we have that  $1 \leq a$ . Since the first entry  $\lambda_1$  of  $\lambda$  satisfies  $\lambda_1 \leq 2 \lfloor \frac{n}{2} \rfloor$  by condition (c) above, we thus have that  $a \leq \lfloor \frac{n}{2} \rfloor$  and we thus have that the first condition given in Lemma 2.2 holds. Since  $\lambda$  is an integer partition, we have that  $1 \leq b \leq a$ , and we thus have that the second condition given in Lemma 2.2 holds.

From condition (d), we have that

$$\lambda_1 + \lambda_2 - 2i \leq n$$

and we thus have that  $2a + 2b - 2i \leq n$  and thus  $i \geq a + b - \frac{n}{2}$  and we thus have that  $a + b - \lfloor \frac{n}{2} \rfloor \leq i$ . Since  $\lambda$  is a partition, we have that

$$2a + 2b - 4i \leq 2b$$

and we thus have that  $\frac{a}{2} \leq i$  and therefore:  $\lceil \frac{a}{2} \rceil \leq i$ . From the inequality  $a + b - \lfloor \frac{n}{2} \rfloor \leq i$  together with the inequality  $\lceil \frac{a}{2} \rceil \leq i$  we thus have that:

$$\max\left(a + b - \lfloor \frac{n}{2} \rfloor, \lceil \frac{a}{2} \rceil\right) \leq i.$$

Since  $i \in [0, \frac{\lambda_2}{2}]$ , we thus have that  $i \leq b$ . Since  $\lambda$  is an integer partition, we have that  $1 \leq \lambda_3$ . Therefore,  $1 \leq 2a + 2b - 4i$ . We thus have that  $i \leq \frac{2a+2b-1}{4}$ , and we thus have that  $i \leq \lfloor \frac{2a+2b-1}{4} \rfloor$ . From the inequality  $i \leq \lfloor \frac{2a+2b-1}{4} \rfloor$  together with the inequality  $i \leq b$ , we thus have that

$$i \leq \min\left(b, \left\lfloor \frac{2a + 2b - 1}{4} \right\rfloor\right)$$

thus proving that condition (3) given in Lemma 2.2 holds.  $\square$

**Lemma 2.3.** *Letting  $\lambda$  be a fixed Klein partition, the number of  $\Delta$ -closed 4-sets consisting of  $\emptyset$  together with subsets of a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, n\}$  of partition type  $\lambda$  is*

$$\frac{1}{(\text{repeat}(\lambda))!} \frac{\prod_{j=0}^{\frac{\lambda_1}{2}-1} \binom{n-2j}{2}}{\left(\frac{\lambda_1}{2}\right)!} \binom{\frac{\lambda_1}{2}}{\frac{\lambda_1+\lambda_2-\lambda_3}{4}} \frac{\prod_{j=0}^{\frac{\lambda_2-\lambda_1+\lambda_3}{4}-1} \binom{n-\lambda_1-2j}{2}}{\left(\frac{\lambda_2-\lambda_1+\lambda_3}{4}\right)!}$$

letting  $\text{repeat}(\lambda)$  denote the maximum repetition length of  $\lambda$ .

**Proof:** There are

$$\frac{\prod_{j=0}^{\frac{\lambda_1}{2}-1} \binom{n-2j}{2}}{\left(\frac{\lambda_1}{2}\right)!}$$

distinct sets of 2-sets in  $S_n$  of length  $\frac{\lambda_1}{2}$ . Let  $i$  denote the unique index in  $[0, \frac{\lambda_2}{2}]$  such that  $\lambda_1 + \lambda_2 - 4i = \lambda_3$  and  $\lambda_1 + \lambda_2 - 2i \leq n$ . We thus have that there are precisely  $i$  ‘‘overlap’’ 2-sets shared among the largest set of sets in a 4-set  $S$  of partition type  $\lambda$  and the second-largest set of sets in  $S$ . There are

$$\binom{\frac{\lambda_1}{2}}{\frac{\lambda_1+\lambda_2-\lambda_3}{4}}$$

choices for  $i$  ‘‘overlap’’ 2-sets, and for each such choice there are

$$\frac{\prod_{j=0}^{\frac{\lambda_2-\lambda_1+\lambda_3}{4}-1} \binom{n-\lambda_1-2j}{2}}{\left(\frac{\lambda_2-\lambda_1+\lambda_3}{4}\right)!}$$

remaining choices for the remaining 2-sets for the second-largest set in  $S$ . □

**Theorem 2.4.** *The number of  $\Delta$ -closed 4-sets  $S$  such that there exists a set  $T$  consisting of pairwise disjoint 2-subsets of  $[n]$  such that each element in  $S$  is contained in  $T$  is*

$$n! \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=1}^i \sum_{k=\max(\lceil \frac{i}{2} \rceil, i+j-\lfloor \frac{n}{2} \rfloor)}^{\min(j, \lfloor \frac{1}{4}(2i+2j-1) \rfloor)} \frac{2^{k-i-j}}{k!(i-k)!(j-k)!(n-2i-2j+2k)!(\delta_{i,j} + \delta_{i,2k} + 1)!}$$

for arbitrary  $n \in \mathbb{N}$ .

**Proof:** From the above lemma, we have that the number of  $\Delta$ -closed 4-sets consisting of the empty set together with subsets of a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, n\}$  is

$$\sum_{\lambda} \frac{1}{(\text{repeat}(\lambda))!} \frac{\prod_{j=0}^{\frac{\lambda_1}{2}-1} \binom{n-2j}{2}}{\left(\frac{\lambda_1}{2}\right)!} \binom{\frac{\lambda_1}{2}}{\frac{\lambda_1+\lambda_2-\lambda_3}{4}} \frac{\prod_{j=0}^{\frac{\lambda_2-\lambda_1+\lambda_3}{4}-1} \binom{n-\lambda_1-2j}{2}}{\left(\frac{\lambda_2-\lambda_1+\lambda_3}{4}\right)!}$$

where the above sum is over all Klein partitions  $\lambda$  for  $n$ . By Lemma 2.2, we thus have that the above summation may be rewritten as:

$$\sum_{a=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{b=1}^a \sum_{i=\max(a+b-\lfloor \frac{n}{2} \rfloor, \lceil \frac{a}{2} \rceil)}^{\min(b, \lfloor \frac{2a+2b-1}{4} \rfloor)} \frac{1}{(\delta_{0,a-b} + \delta_{0,2i-a} + 1)!} \frac{\prod_{j=0}^{a-1} \binom{n-2j}{2}}{a!} \binom{a}{i} \frac{\prod_{j=0}^{b-i-1} \binom{n-2a-2j}{2}}{(b-i)!}$$



Rewriting the above expression by evaluating the products in the summand yields the desired result.  $\square$

The integer sequence

$$(0, 0, 0, 3, 15, 105, 525, 3255, 17703, 112455, 669735, 4485195, 29023995, 205768563, \dots)$$

given by the number of  $\Delta$ -closed 4-sets consisting of the empty set together with subsets of a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, n\}$  is given in the On-Line Encyclopedia of Integer Sequences sequence A267840 which we contributed. For example, there are 15  $\Delta$ -closed 4-sets of this form in the case whereby  $n = 5$ :

$$\begin{aligned} & \{\emptyset, \{\{1, 2\}\}, \{\{3, 4\}\}, \{\{1, 2\}, \{3, 4\}\}, \{\emptyset, \{\{1, 2\}\}, \{\{3, 5\}\}, \{\{1, 2\}, \{3, 5\}\}\}, \\ & \{\emptyset, \{\{1, 2\}\}, \{\{4, 5\}\}, \{\{1, 2\}, \{4, 5\}\}, \{\emptyset, \{\{1, 3\}\}, \{\{2, 4\}\}, \{\{1, 3\}, \{2, 4\}\}\}, \\ & \{\emptyset, \{\{1, 3\}\}, \{\{2, 5\}\}, \{\{1, 3\}, \{2, 5\}\}, \{\emptyset, \{\{1, 3\}\}, \{\{4, 5\}\}, \{\{1, 3\}, \{4, 5\}\}\}, \\ & \{\emptyset, \{\{1, 4\}\}, \{\{2, 3\}\}, \{\{1, 4\}, \{2, 3\}\}, \{\emptyset, \{\{1, 4\}\}, \{\{2, 5\}\}, \{\{1, 4\}, \{2, 5\}\}\}, \\ & \{\emptyset, \{\{1, 4\}\}, \{\{3, 5\}\}, \{\{1, 4\}, \{3, 5\}\}, \{\emptyset, \{\{1, 5\}\}, \{\{2, 3\}\}, \{\{1, 5\}, \{2, 3\}\}\}, \\ & \{\emptyset, \{\{1, 5\}\}, \{\{2, 4\}\}, \{\{1, 5\}, \{2, 4\}\}, \{\emptyset, \{\{1, 5\}\}, \{\{3, 4\}\}, \{\{1, 5\}, \{3, 4\}\}\}, \\ & \{\emptyset, \{\{2, 3\}\}, \{\{4, 5\}\}, \{\{2, 3\}, \{4, 5\}\}, \{\emptyset, \{\{2, 4\}\}, \{\{3, 5\}\}, \{\{2, 4\}, \{3, 5\}\}\}, \\ & \{\emptyset, \{\{2, 5\}\}, \{\{3, 4\}\}, \{\{2, 5\}, \{3, 4\}\}\}. \end{aligned}$$

An illustration of the 105 symmetric difference-closed 4-sets consisting of subsets of a set of pairwise disjoint 2-subsets of  $\{1, 2, \dots, 6\}$  is given in Section 4.

It is natural to use Lemma 2.2 to determine “even” analogues of the above results.

**Lemma 2.5.** *For  $n \in \mathbb{N}$ , the Klein partitions for  $n$  corresponding to 4-sets consisting of  $\emptyset$  together with subsets of a set consisting of pairwise disjoint 2-subsets of  $[n]$  are precisely tuples of the form  $(4d, 4e, 4d + 4e - 4i)$  such that:*

1.  $1 \leq d \leq \lfloor \frac{n}{4} \rfloor$ ;
2.  $1 \leq e \leq d$ ; and
3.  $\max\{2d + 2e - \lfloor \frac{n}{2} \rfloor, d\} \leq i \leq \min\{2e, \lfloor \frac{4d+4e-1}{4} \rfloor\}$ .

**Proof:** The above lemma follows immediately from Lemma 2.2 by letting  $a = 2d$  and  $b = 2e$ .  $\square$

**Theorem 2.6.** *The number of  $\Delta$ -closed 4-sets consisting of even-order subsets of a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, n\}$  is*

$$n! \sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \sum_{j=1}^i \sum_{k=\max(i, 2i+2j-\lfloor \frac{n}{2} \rfloor)}^{\min(2j, \lfloor \frac{1}{4}(4i+4j-1) \rfloor)} \frac{2^{k-2i-2j}}{k!(2i-k)!(2j-k)!(n-4i-4j+2k)!(\delta_{i,j} + \delta_{i,k} + 1)!}$$

for arbitrary  $n \in \mathbb{N}$ .

**Proof:** The above theorem follows from Lemma 2.3 by analogy with Theorem 2.4.  $\square$

The corresponding integer sequence is given below, and is given in the sequence A266503 which we contributed to OEIS Foundation Inc. (2011).

$$(0, 0, 0, 0, 0, 15, 105, 735, 4095, 26775, 162855, 1105335, 7187895, 51126075, 356831475, \dots).$$

### 3 Conclusion

The number of Klein partitions for  $n = 1, 2, \dots$  is given by the following integer sequence:

$$(0, 0, 0, 1, 1, 3, 3, 6, 6, 10, 10, 16, 16, 23, 23, 32, 32, 43, 43, 56, 56, 71, 71, 89, \dots).$$

We have previously noted that the corresponding integer sequence

$$(0, 1, 3, 6, 10, 16, 23, 32, 43, 56, 71, 89, 109, 132, 158, \dots)$$

coincides with the sequence A034198 given by the number of binary codes of a given length with 3 words OEIS Foundation Inc. (2011). We currently leave it as an open problem to use Lemma 2.2 to prove this. Interestingly, there are known connections between the OEIS sequence A034198 and Klein four-subgroups. In particular,  $A034198(n)$  is the number of orbits of Klein subgroups of  $C_2^n$  under automorphisms of  $C_2^n$ , and  $A034198(n)$  is the number of faithful representations of  $K_4 = C_2^2$  of dimension  $n$  up to equivalence by automorphisms of  $C_2^2$  OEIS Foundation Inc. (2011).

In 2016, Václav Kotěšovec discovered a variety of interesting formulas for the integer sequence

$$(A267840_n : n \in \mathbb{N})$$

enumerating  $\Delta$ -closed subsets  $S \subseteq 2^{[n]}$  such that there exists a set  $T$  consisting of pairwise disjoint 2-subsets of  $[n]$  such that  $S \subseteq 2^T$ . Kotěšovec discovered the following unexpected linear recurrence with a polynomial coefficients for this sequence using the Mathematica function `plinrec`:

$$\begin{aligned} (n-4)(n-2)A267840_n &= 3(n^2 - 5n + 5)A267840_{n-1} \\ &+ (n-1)(4n^2 - 27n + 41)A267840_{n-2} \\ &- (n-2)(n-1)(8n - 29)A267840_{n-3} \\ &- (n-3)(n-2)(n-1)(3n - 16)A267840_{n-4} \\ &+ 3(n-4)(n-3)(n-2)(n-1)A267840_{n-5}. \end{aligned}$$

We currently leave it as an open problem to use Theorem 2.4 to prove the above recurrence. This recurrence may be used to prove the exponential generating function for the sequence  $(A267840_n)_{n \in \mathbb{N}}$  discovered by Kotěšovec given in (2). Kotěšovec also discovered the following interesting asymptotic formula for this sequence:

$$A267840_n \sim 2^{-\frac{3}{2}} 3^{\frac{n}{2}-1} e^{\sqrt{\frac{n}{3}} - \frac{n}{2} - \frac{1}{12}n^{\frac{3}{2}}}. \quad (3)$$

We leave it as an open problem to use Theorem 2.4 to prove (3), for example through the use of the Birkhoff-Trjitznisky method. Kotěšovec also discovered the following interesting recursive formula for the sequence  $(A266503_n)_{n \in \mathbb{N}}$ :

$$\begin{aligned} (n-6)(n-4)(n-2)A266503_n &= (2n-7)(2n^2 - 14n + 15)A266503_{n-1} \\ &+ 3(n-7)(n-1)(n^2 - 7n + 11)A266503_{n-2} \\ &- (n-2)(n-1)(9n^2 - 85n + 189)A266503_{n-3} \\ &+ (n-3)(n-2)(n-1)(n^2 - n - 22)A266503_{n-4} \end{aligned}$$

$$\begin{aligned}
& - 2(n-4)^2(n-3)(n-2)(n-1)A266503_{n-5} \\
& - (n-5)(n-4)(n-3)(n-2)(n-1)(3n-19)A266503_{n-6} \\
& + 3(n-6)(n-5)(n-4)(n-3)(n-2)(n-1)A266503_{n-7}.
\end{aligned}$$

Kotěšovec discovered the following exponential generating function for the sequence  $(A266503_n)_{n \in \mathbb{N}}$ :

$$\frac{e^x}{3} - \frac{e^{-\frac{x(x-2)}{2}}}{8} - \frac{e^{\frac{x(x+2)}{2}}}{4} + \frac{e^{\frac{x(3x+2)}{2}}}{24}.$$

Kotěšovec also discovered a similar asymptotic formula for this sequence. We leave it as an open problem to use Theorem 2.6 to prove these results.

## Acknowledgments

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## 4 Appendix

Symmetric difference-closed 4-sets  $S$  consisting of subsets of a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, 6\}$  of partition type  $(4, 2, 2)$  are listed below.

$$\begin{aligned}
& \{\{1, 2\}, \{3, 4\}\}, \{\{1, 2\}\}, \{\{3, 4\}\}, \emptyset & \{\{1, 2\}, \{3, 5\}\}, \{\{1, 2\}\}, \{\{3, 5\}\}, \emptyset \\
& \{\{1, 2\}, \{3, 6\}\}, \{\{1, 2\}\}, \{\{3, 6\}\}, \emptyset & \{\{1, 2\}, \{4, 5\}\}, \{\{1, 2\}\}, \{\{4, 5\}\}, \emptyset \\
& \{\{1, 2\}, \{4, 6\}\}, \{\{1, 2\}\}, \{\{4, 6\}\}, \emptyset & \{\{1, 2\}, \{5, 6\}\}, \{\{1, 2\}\}, \{\{5, 6\}\}, \emptyset \\
& \{\{1, 3\}, \{2, 4\}\}, \{\{1, 3\}\}, \{\{2, 4\}\}, \emptyset & \{\{1, 3\}, \{2, 5\}\}, \{\{1, 3\}\}, \{\{2, 5\}\}, \emptyset \\
& \{\{1, 3\}, \{2, 6\}\}, \{\{1, 3\}\}, \{\{2, 6\}\}, \emptyset & \{\{1, 3\}, \{4, 5\}\}, \{\{1, 3\}\}, \{\{4, 5\}\}, \emptyset \\
& \{\{1, 3\}, \{4, 6\}\}, \{\{1, 3\}\}, \{\{4, 6\}\}, \emptyset & \{\{1, 3\}, \{5, 6\}\}, \{\{1, 3\}\}, \{\{5, 6\}\}, \emptyset \\
& \{\{1, 4\}, \{2, 3\}\}, \{\{1, 4\}\}, \{\{2, 3\}\}, \emptyset & \{\{1, 4\}, \{2, 5\}\}, \{\{1, 4\}\}, \{\{2, 5\}\}, \emptyset \\
& \{\{1, 4\}, \{2, 6\}\}, \{\{1, 4\}\}, \{\{2, 6\}\}, \emptyset & \{\{1, 4\}, \{3, 5\}\}, \{\{1, 4\}\}, \{\{3, 5\}\}, \emptyset \\
& \{\{1, 4\}, \{3, 6\}\}, \{\{1, 4\}\}, \{\{3, 6\}\}, \emptyset & \{\{1, 4\}, \{5, 6\}\}, \{\{1, 4\}\}, \{\{5, 6\}\}, \emptyset \\
& \{\{1, 5\}, \{2, 3\}\}, \{\{1, 5\}\}, \{\{2, 3\}\}, \emptyset & \{\{1, 5\}, \{2, 4\}\}, \{\{1, 5\}\}, \{\{2, 4\}\}, \emptyset \\
& \{\{1, 5\}, \{2, 6\}\}, \{\{1, 5\}\}, \{\{2, 6\}\}, \emptyset & \{\{1, 5\}, \{3, 4\}\}, \{\{1, 5\}\}, \{\{3, 4\}\}, \emptyset \\
& \{\{1, 5\}, \{3, 6\}\}, \{\{1, 5\}\}, \{\{3, 6\}\}, \emptyset & \{\{1, 5\}, \{4, 6\}\}, \{\{1, 5\}\}, \{\{4, 6\}\}, \emptyset \\
& \{\{1, 6\}, \{2, 3\}\}, \{\{1, 6\}\}, \{\{2, 3\}\}, \emptyset & \{\{1, 6\}, \{2, 4\}\}, \{\{1, 6\}\}, \{\{2, 4\}\}, \emptyset \\
& \{\{1, 6\}, \{2, 5\}\}, \{\{1, 6\}\}, \{\{2, 5\}\}, \emptyset & \{\{1, 6\}, \{3, 4\}\}, \{\{1, 6\}\}, \{\{3, 4\}\}, \emptyset \\
& \{\{1, 6\}, \{3, 5\}\}, \{\{1, 6\}\}, \{\{3, 5\}\}, \emptyset & \{\{1, 6\}, \{4, 5\}\}, \{\{1, 6\}\}, \{\{4, 5\}\}, \emptyset \\
& \{\{2, 3\}, \{4, 5\}\}, \{\{2, 3\}\}, \{\{4, 5\}\}, \emptyset & \{\{2, 3\}, \{4, 6\}\}, \{\{2, 3\}\}, \{\{4, 6\}\}, \emptyset \\
& \{\{2, 3\}, \{5, 6\}\}, \{\{2, 3\}\}, \{\{5, 6\}\}, \emptyset & \{\{2, 4\}, \{3, 5\}\}, \{\{2, 4\}\}, \{\{3, 5\}\}, \emptyset \\
& \{\{2, 4\}, \{3, 6\}\}, \{\{2, 4\}\}, \{\{3, 6\}\}, \emptyset & \{\{2, 4\}, \{5, 6\}\}, \{\{2, 4\}\}, \{\{5, 6\}\}, \emptyset
\end{aligned}$$

$\{\{\{2, 5\}, \{3, 4\}\}, \{\{2, 5\}\}, \{\{3, 4\}\}, \emptyset\}$     $\{\{\{2, 5\}, \{3, 6\}\}, \{\{2, 5\}\}, \{\{3, 6\}\}, \emptyset\}$   
 $\{\{\{2, 5\}, \{4, 6\}\}, \{\{2, 5\}\}, \{\{4, 6\}\}, \emptyset\}$     $\{\{\{2, 6\}, \{3, 4\}\}, \{\{2, 6\}\}, \{\{3, 4\}\}, \emptyset\}$   
 $\{\{\{2, 6\}, \{3, 5\}\}, \{\{2, 6\}\}, \{\{3, 5\}\}, \emptyset\}$     $\{\{\{2, 6\}, \{4, 5\}\}, \{\{2, 6\}\}, \{\{4, 5\}\}, \emptyset\}$   
 $\{\{\{3, 4\}, \{5, 6\}\}, \{\{3, 4\}\}, \{\{5, 6\}\}, \emptyset\}$     $\{\{\{3, 5\}, \{4, 6\}\}, \{\{3, 5\}\}, \{\{4, 6\}\}, \emptyset\}$   
 $\{\{\{3, 6\}, \{4, 5\}\}, \{\{3, 6\}\}, \{\{4, 5\}\}, \emptyset\}$

Symmetric difference-closed 4-sets  $S$  consisting of subsets of a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, 6\}$  of partition type  $(4, 4, 4)$  are listed below.

$\{\{\{1, 2\}, \{3, 4\}\}, \{\{3, 4\}, \{5, 6\}\}, \{\{1, 2\}, \{5, 6\}\}, \emptyset\}$   
 $\{\{\{1, 2\}, \{3, 5\}\}, \{\{3, 5\}, \{4, 6\}\}, \{\{1, 2\}, \{4, 6\}\}, \emptyset\}$   
 $\{\{\{1, 2\}, \{3, 6\}\}, \{\{3, 6\}, \{4, 5\}\}, \{\{1, 2\}, \{4, 5\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 4\}\}, \{\{2, 4\}, \{5, 6\}\}, \{\{1, 3\}, \{5, 6\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 5\}\}, \{\{2, 5\}, \{4, 6\}\}, \{\{1, 3\}, \{4, 6\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 6\}\}, \{\{2, 6\}, \{4, 5\}\}, \{\{1, 3\}, \{4, 5\}\}, \emptyset\}$   
 $\{\{\{1, 4\}, \{2, 3\}\}, \{\{2, 3\}, \{5, 6\}\}, \{\{1, 4\}, \{5, 6\}\}, \emptyset\}$   
 $\{\{\{1, 4\}, \{2, 5\}\}, \{\{2, 5\}, \{3, 6\}\}, \{\{1, 4\}, \{3, 6\}\}, \emptyset\}$   
 $\{\{\{1, 4\}, \{2, 6\}\}, \{\{2, 6\}, \{3, 5\}\}, \{\{1, 4\}, \{3, 5\}\}, \emptyset\}$   
 $\{\{\{1, 5\}, \{2, 3\}\}, \{\{2, 3\}, \{4, 6\}\}, \{\{1, 5\}, \{4, 6\}\}, \emptyset\}$   
 $\{\{\{1, 5\}, \{2, 4\}\}, \{\{2, 4\}, \{3, 6\}\}, \{\{1, 5\}, \{3, 6\}\}, \emptyset\}$   
 $\{\{\{1, 5\}, \{2, 6\}\}, \{\{2, 6\}, \{3, 4\}\}, \{\{1, 5\}, \{3, 4\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 3\}\}, \{\{2, 3\}, \{4, 5\}\}, \{\{1, 6\}, \{4, 5\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 4\}\}, \{\{2, 4\}, \{3, 5\}\}, \{\{1, 6\}, \{3, 5\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 5\}\}, \{\{2, 5\}, \{3, 4\}\}, \{\{1, 6\}, \{3, 4\}\}, \emptyset\}$

Symmetric difference-closed 4-sets  $S$  consisting of subsets of a set consisting of pairwise disjoint 2-subsets of  $\{1, 2, \dots, 6\}$  of partition type  $(6, 4, 2)$  are listed below.

$\{\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}, \{\{3, 4\}, \{5, 6\}\}, \{\{1, 2\}\}, \emptyset\}$   
 $\{\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}, \{\{1, 2\}, \{5, 6\}\}, \{\{3, 4\}\}, \emptyset\}$   
 $\{\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}, \{\{1, 2\}, \{3, 4\}\}, \{\{5, 6\}\}, \emptyset\}$   
 $\{\{\{1, 2\}, \{3, 5\}, \{4, 6\}\}, \{\{3, 5\}, \{4, 6\}\}, \{\{1, 2\}\}, \emptyset\}$   
 $\{\{\{1, 2\}, \{3, 5\}, \{4, 6\}\}, \{\{1, 2\}, \{4, 6\}\}, \{\{3, 5\}\}, \emptyset\}$   
 $\{\{\{1, 2\}, \{3, 5\}, \{4, 6\}\}, \{\{1, 2\}, \{3, 5\}\}, \{\{4, 6\}\}, \emptyset\}$   
 $\{\{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, \{\{3, 6\}, \{4, 5\}\}, \{\{1, 2\}\}, \emptyset\}$   
 $\{\{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, \{\{1, 2\}, \{4, 5\}\}, \{\{3, 6\}\}, \emptyset\}$   
 $\{\{\{1, 2\}, \{3, 6\}, \{4, 5\}\}, \{\{1, 2\}, \{3, 6\}\}, \{\{4, 5\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 4\}, \{5, 6\}\}, \{\{2, 4\}, \{5, 6\}\}, \{\{1, 3\}\}, \emptyset\}$

$\{\{\{1, 3\}, \{2, 4\}, \{5, 6\}\}, \{\{1, 3\}, \{5, 6\}\}, \{\{2, 4\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 4\}, \{5, 6\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{5, 6\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 5\}, \{4, 6\}\}, \{\{2, 5\}, \{4, 6\}\}, \{\{1, 3\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 5\}, \{4, 6\}\}, \{\{1, 3\}, \{4, 6\}\}, \{\{2, 5\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 5\}, \{4, 6\}\}, \{\{1, 3\}, \{2, 5\}\}, \{\{4, 6\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 6\}, \{4, 5\}\}, \{\{2, 6\}, \{4, 5\}\}, \{\{1, 3\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 6\}, \{4, 5\}\}, \{\{1, 3\}, \{4, 5\}\}, \{\{2, 6\}\}, \emptyset\}$   
 $\{\{\{1, 3\}, \{2, 6\}, \{4, 5\}\}, \{\{1, 3\}, \{2, 6\}\}, \{\{4, 5\}\}, \emptyset\}$   
 $\{\{\{1, 4\}, \{2, 3\}, \{5, 6\}\}, \{\{2, 3\}, \{5, 6\}\}, \{\{1, 4\}\}, \emptyset\}$   
 $\{\{\{1, 4\}, \{2, 3\}, \{5, 6\}\}, \{\{1, 4\}, \{5, 6\}\}, \{\{2, 3\}\}, \emptyset\}$   
 $\{\{\{1, 4\}, \{2, 3\}, \{5, 6\}\}, \{\{1, 4\}, \{2, 3\}\}, \{\{5, 6\}\}, \emptyset\}$   
 $\{\{\{1, 4\}, \{2, 5\}, \{3, 6\}\}, \{\{2, 5\}, \{3, 6\}\}, \{\{1, 4\}\}, \emptyset\}$   
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 $\{\{\{1, 5\}, \{2, 3\}, \{4, 6\}\}, \{\{2, 3\}, \{4, 6\}\}, \{\{1, 5\}\}, \emptyset\}$   
 $\{\{\{1, 5\}, \{2, 3\}, \{4, 6\}\}, \{\{1, 5\}, \{4, 6\}\}, \{\{2, 3\}\}, \emptyset\}$   
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 $\{\{\{1, 5\}, \{2, 4\}, \{3, 6\}\}, \{\{1, 5\}, \{2, 4\}\}, \{\{3, 6\}\}, \emptyset\}$   
 $\{\{\{1, 5\}, \{2, 6\}, \{3, 4\}\}, \{\{2, 6\}, \{3, 4\}\}, \{\{1, 5\}\}, \emptyset\}$   
 $\{\{\{1, 5\}, \{2, 6\}, \{3, 4\}\}, \{\{1, 5\}, \{3, 4\}\}, \{\{2, 6\}\}, \emptyset\}$   
 $\{\{\{1, 5\}, \{2, 6\}, \{3, 4\}\}, \{\{1, 5\}, \{2, 6\}\}, \{\{3, 4\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 3\}, \{4, 5\}\}, \{\{2, 3\}, \{4, 5\}\}, \{\{1, 6\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 3\}, \{4, 5\}\}, \{\{1, 6\}, \{4, 5\}\}, \{\{2, 3\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 3\}, \{4, 5\}\}, \{\{1, 6\}, \{2, 3\}\}, \{\{4, 5\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 4\}, \{3, 5\}\}, \{\{2, 4\}, \{3, 5\}\}, \{\{1, 6\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 4\}, \{3, 5\}\}, \{\{1, 6\}, \{3, 5\}\}, \{\{2, 4\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 4\}, \{3, 5\}\}, \{\{1, 6\}, \{2, 4\}\}, \{\{3, 5\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 5\}, \{3, 4\}\}, \{\{2, 5\}, \{3, 4\}\}, \{\{1, 6\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 5\}, \{3, 4\}\}, \{\{1, 6\}, \{3, 4\}\}, \{\{2, 5\}\}, \emptyset\}$   
 $\{\{\{1, 6\}, \{2, 5\}, \{3, 4\}\}, \{\{1, 6\}, \{2, 5\}\}, \{\{3, 4\}\}, \emptyset\}$

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