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► **To cite this version:**

Emeric Bouin, Vincent Calvez, Emmanuel Grenier, Grégoire Nadin. Large deviations for velocity-jump processes and non-local Hamilton-Jacobi equations. 2016. hal-01344939v1

HAL Id: hal-01344939

<https://hal.science/hal-01344939v1>

Preprint submitted on 12 Jul 2016 (v1), last revised 20 Sep 2019 (v2)

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Large deviations for velocity-jump processes and non-local Hamilton-Jacobi equations

Emeric Bouin ^{*} Vincent Calvez [†] Emmanuel Grenier [‡] Grégoire Nadin [§]

July 20, 2016

Abstract

We establish a large deviation theory for a velocity jump process, where new random velocities are picked at a constant rate from a Gaussian distribution. The Kolmogorov forward equation associated with this process is a linear kinetic transport equation in which the BGK operator accounts for the changes in velocity. We analyse its asymptotic limit after a suitable rescaling compatible with the WKB expansion. This yields a new type of Hamilton Jacobi equation which is non local with respect to velocity variable. We introduce a dedicated notion of viscosity solution for the limit problem, and we prove well-posedness in the viscosity sense. The fundamental solution is explicitly computed, yielding quantitative estimates for the large deviations of the underlying velocity-jump process *à la Freidlin-Wentzell*. As an application of this theory, we conjecture exact rates of acceleration in some nonlinear kinetic reaction-transport equations.

Key-Words: Large deviations, Piecewise Deterministic Markov Processes, Hamilton-Jacobi equations, Viscosity solutions, Scaling limits, Front acceleration.

AMS Class. No:

1 Introduction

This paper is mainly concerned with the asymptotic limit of the following linear kinetic transport equation as $\varepsilon \rightarrow 0$,

$$\partial_t f^\varepsilon(t, x, v) + v \cdot \nabla_x f^\varepsilon(t, x, v) = \frac{1}{\varepsilon} (M_\varepsilon(v) \rho^\varepsilon(t, x) - f^\varepsilon(t, x, v)), \quad t > 0, x \in \mathbb{R}^n, v \in \mathbb{R}^n. \quad (1.1) \quad \boxed{\text{eq:main}}$$

Here, $f^\varepsilon(t, x, v)$ denotes the density of particles at time $t > 0$ in the phase space \mathbb{R}^{2n} , and $\rho^\varepsilon(t, x)$ is the macroscopic density,

$$\rho^\varepsilon(t, x) = \int_{\mathbb{R}^n} f^\varepsilon(t, x, v) dv.$$

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The velocity distribution $M_\varepsilon(v)$ is given. For the sake of clarity, we focus here on the case of the Gaussian distribution with variance ε ,

$$\forall v \in \mathbb{R}, \quad M_\varepsilon(v) = \frac{1}{(2\pi\varepsilon)^{n/2}} \exp\left(-\frac{|v|^2}{2\varepsilon}\right).$$

Our methodology can be applied to a wide range of distributions, as discussed at the end of this introduction.

The time renormalizing factor ε^{-1} in front of the BGK velocity operator in (??) is chosen so as to obtain a nontrivial limit in the asymptotic regime of large deviations, *i.e.* in order to capture the vanishing exponential tails of the density f^ε . We emphasize that the classical diffusive limit would involve an additional factor ε in front of the time derivative $\partial_t f^\varepsilon$.

Scaling. Starting from the dimensionalized kinetic transport equation

$$\partial_{t'} f + v' \cdot \nabla_{x'} f = \frac{1}{\tau} (M_{\sigma^2}(v')\rho - f), \quad (1.2) \quad \text{eq:BGK-dim}$$

where τ is the rate of reorientation, and σ^2 is the variance of the velocity distribution, we can always reduce to $\tau = \sigma = 1$ after the change of variables $\tilde{t} = t'/\tau$, $\tilde{x} = x'/(\tau\sigma)$, and $\tilde{v} = v'/\sigma$. Then, we are interested in the asymptotics $(\tilde{t}, \tilde{x}, \tilde{v}) \rightarrow \infty$ with the appropriate scaling. It appears that the correct scaling in order to capture the small exponential tails of the distribution is such that $\tilde{v} \approx \tilde{t}^{1/2}$ and $\tilde{x} \approx \tilde{t}^{3/2}$. Accordingly, we set

$$(\tilde{t}, \tilde{x}, \tilde{v}) = \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{3/2}}, \frac{v}{\varepsilon^{1/2}}\right), \quad (1.3) \quad \text{eq:rescaling}$$

for some (small) parameter $\varepsilon > 0$ that drives the large scale asymptotics when it vanishes. We recover (??) in the new variables (t, x, v) . Note that the appropriate rescaling (??) is specific to the choice of a Gaussian velocity distribution.

Diffusive limit. To draw an instructive parallel, let us consider the diffusive limit of (??), namely the heat equation (see [?] and the references therein)

$$\partial_{t'} \rho(t, x) - (\tau\sigma^2) \Delta_{x'} \rho(t, x) = 0. \quad (1.4) \quad \text{eq:diffusion}$$

When the effective diffusion coefficient $\tau\sigma^2$ is small (say $\varepsilon = \tau\sigma^2$) or, equivalently, when investigating the large scale asymptotics $(t, x) = (\varepsilon t'/\tau, \varepsilon x'/(\tau\sigma))$ for small ε , one deals with the same equation with vanishing viscosity,

$$\partial_t \rho^\varepsilon(t, x) - \varepsilon \Delta_x \rho^\varepsilon(t, x) = 0. \quad (1.5) \quad \text{eq:diffusion}$$

It is well-known (see *e.g.* [?, ?, ?]) that, under appropriate conditions, $u^\varepsilon(t, x) = -\varepsilon \log \rho^\varepsilon(t, x)$ converges uniformly locally towards a viscosity solution of the following Hamilton-Jacobi equation

$$\partial_t u(t, x) + |\nabla_x u(t, x)|^2 = 0. \quad (1.6) \quad \text{eq:HJ-heat}$$

Our main purpose here is to obtain a similar result for the kinetic transport equation (??).

Large deviations. Our work can be viewed as a preliminary contribution to the theory of large deviations for simple velocity jump processes. We follow the lines of Evans and Ishii [?] (see also [?, ?]), in which PDE techniques were successfully applied to reformulate the ideas of Fleming [?] in the context of viscosity solutions. Fleming's work was devoted to applying the logarithmic transformation, and ideas from stochastic control, to the Freidlin-Wentzell theory for stochastic differential equations with a small noise.

Logarithmic transformation. As in the diffusive case, we perform the following Hopf-Cole transformation,

$$u^\varepsilon(t, x, v) = -\varepsilon \log f^\varepsilon(t, x, v).$$

The function u^ε satisfies

$$\partial_t u^\varepsilon(t, x, v) + v \cdot \nabla_x u^\varepsilon(t, x, v) = 1 - \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{\mathbb{R}^n} \exp\left(\frac{u^\varepsilon(t, x, v) - u^\varepsilon(t, x, v') - |v|^2/2}{\varepsilon}\right) dv'. \quad (1.7)$$

WKB1

Limit system. Our first result can be stated informally as follows: under suitable conditions (see Theorem ?? below), u^ε converges locally uniformly towards a viscosity solution of the following non-local Hamilton-Jacobi equation,

$$\begin{cases} \max\left(\partial_t u(t, x, v) + v \cdot \nabla_x u(t, x, v) - 1, u(t, x, v) - \min_{w \in \mathbb{R}^n} u(t, x, w) - \frac{|v|^2}{2}\right) = 0, \\ \partial_t \left(\min_{w \in \mathbb{R}^n} u(t, x, w)\right) \leq 0, \\ \partial_t \left(\min_{w \in \mathbb{R}^n} u(t, x, w)\right) = 0, \quad \text{if } \mathcal{S}(u)(t, x) = \{0\}, \\ u(0, x, v) = u_0(x, v). \end{cases} \quad (1.8)$$

eq:limit

where we have used the following notation,

$$\mathcal{S}(u)(t, x) = \left\{ v \in \mathbb{R}^n \mid u(t, x, v) = \min_{w \in \mathbb{R}^n} u(t, x, w) \right\}. \quad (1.9)$$

To avoid possible boundary layers at $t = 0$ as $\varepsilon \rightarrow 0$, we assume that the initial condition for f^ε is of the form $f^\varepsilon(0, \cdot) = \exp(-\varepsilon^{-1}u_0(\cdot))$. This set $u^\varepsilon(0, \cdot) = u_0(\cdot)$ as the initial condition for (??) and for the limit problem (??).

It is worth making some comments concerning the structure of the system (??). First of all, it is not a standard Hamilton-Jacobi equation as (??), and the one obtained in the case of bounded velocities [?]. Moreover, we notice that the first equation of (??) does not contain enough information due to the apparition of $\min_w u$ for which extra dynamics are required. Although it seems somehow sparse, the two additional (in)equations $\partial_t(\min_w u) \leq 0$ ($= 0$) are sufficient to determine a unique solution of the Cauchy problem, as stated in the comparison principle below (Theorem ??).

To the best of our knowledge, system (??) is of a new kind. We refer to it as a Hamilton-Jacobi problem by analogy with (??) which was obtained via a similar procedure. Moreover, in the case of a compactly supported velocity distribution $M(v)$, the same procedure leads to a standard Hamilton-Jacobi problem [?, ?].

We should mention that non-local Hamilton-Jacobi equations of a very different type have been studied in the context of dislocations by G. Barles, P. Cardaliaguet, O. Ley, R. Monneau, and A. Monteillet in a series of papers, see *e.g.* [?, ?, ?] and the references therein.

Viscosity solutions. Equation (??) should be read as a coupled system of Hamilton-Jacobi equations on u and $\min_w u$. Accordingly, we define viscosity solutions of (??) using a couple of test functions [?]. In the sequel, we denote by z^* (*resp.* z_*) the upper (*resp.* lower) semi continuous envelope of a given locally bounded function z .

def:subsol

Definition 1.1 (Sub-solution). Let $T > 0$. A upper semi-continuous function \underline{u} is a **viscosity sub-solution** of (??) on $[0, T) \times \mathbb{R}^{2n}$ if and only if:

(i) $\underline{u}(0, \cdot, \cdot) \leq (u_0)^*$.

(ii) It satisfies the constraint

$$(\forall (t, x, v) \in (0, T) \times \mathbb{R}^{2n}) \quad \underline{u}(t, x, v) - \min_{w \in \mathbb{R}^n} \underline{u}(t, x, w) - \frac{|v|^2}{2} \leq 0.$$

(iii) For all pair of test functions $(\phi, \psi) \in \mathcal{C}^1((0, T) \times \mathbb{R}^{2n}) \times \mathcal{C}^1((0, T) \times \mathbb{R}^n)$, if (t_0, x_0, v_0) is such that both $\underline{u}(\cdot, \cdot, v_0) - \phi(\cdot, \cdot, v_0)$ and $\min_w \underline{u}(\cdot, \cdot, w) - \psi(\cdot, \cdot)$ have a local maximum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, then

$$\begin{cases} \partial_t \phi(t_0, x_0, v_0) + v_0 \cdot \nabla_x \phi(t_0, x_0, v_0) - 1 \leq 0, \\ \partial_t \psi(t_0, x_0) \leq 0. \end{cases} \quad (1.10) \quad \text{eq:S1}$$

def:supersol

Definition 1.2 (Super-solution). Let $T > 0$. A lower semi-continuous function \bar{u} is a **viscosity super-solution** of (??) on $[0, T) \times \mathbb{R}^{2n}$ if and only if:

(i) $\bar{u}(0, \cdot, \cdot) \geq (u_0)_*$.

(ii) For all $(t, x) \in (0, T) \times \mathbb{R}^n$, $v = 0$ is a global minimum of $\bar{u}(t, x, \cdot)$. Moreover, $v = 0$ is locally uniformly isolated: for any compact set $K \subset (0, T) \times \mathbb{R}^n$, there exists $r > 0$ such that $\mathcal{S}(\bar{u})(t, x) \cap B_r(0) = \{0\}$ for all $(t, x) \in K$.

(iii) For all pair of test functions $(\phi, \psi) \in \mathcal{C}^1((0, T) \times \mathbb{R}^{2n}) \times \mathcal{C}^1((0, T) \times \mathbb{R}^n)$, if (t_0, x_0, v_0) is such that both $\bar{u}(\cdot, \cdot, v_0) - \phi(\cdot, \cdot, v_0)$ and $\min_w \bar{u}(\cdot, \cdot, w) - \psi(\cdot, \cdot)$ have a local minimum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, then

$$\begin{cases} \partial_t \phi(t_0, x_0, v_0) + v_0 \cdot \nabla_x \phi(t_0, x_0, v_0) - 1 \geq 0, & \text{if } \bar{u}(t_0, x_0, v_0) - \min_{w \in \mathbb{R}^n} \bar{u}(t_0, x_0, w) - \frac{|v_0|^2}{2} < 0, \\ \partial_t \psi(t_0, x_0) \geq 0, & \text{if } \mathcal{S}(\bar{u})(t_0, x_0) = \{0\}. \end{cases} \quad (1.11) \quad \text{eq:S2 intro}$$

Let mention that the minimality (resp. maximality) in the definition of the super- (resp. sub-) solution arises with respect to variables (t, x) only. This is consistent with the fact that there is no derivative in the velocity variable in (??).

def:sol

Definition 1.3 (Solution). Let $T > 0$. A function u is a **viscosity solution** of (??) on $[0, T) \times \mathbb{R}^{2n}$ if its upper (resp. lower) semi-continuous envelope is a sub- (resp. super-) solution in the sense of definitions ?? and ?? above.

Convergence and uniqueness for the limit system. The following theorem states a comparison principle for viscosity (sub/super-)solutions of the system (??). This establishes uniqueness of viscosity solutions as a corollary. The proof is contained in Section ??.

theo:comp

Theorem 1.4 (Comparison principle). Let \underline{u} (resp. \bar{u}) be a viscosity sub-solution (resp. super-solution) of (??) on $[0, T) \times \mathbb{R}^{2n}$. Assume that \underline{u} and \bar{u} are such that

$$\bar{b} = \bar{u} - |v|^2/2 \in L^\infty([0, T) \times \mathbb{R}^{2n}), \quad \underline{b} = \underline{u} - |v|^2/2 \in L^\infty([0, T) \times \mathbb{R}^{2n}). \quad (1.12) \quad \text{eq:v2 plus 1}$$

Then $\underline{u} \leq \bar{u}$ on $[0, T) \times \mathbb{R}^{2n}$.

This result is extended for sub- and super- solutions with quadratic spatial growth at infinity, in Section ???. This growth condition is compatible with the fundamental solution of the limit system, as we shall see below.

In Section ??, we prove the convergence of the sequence $(u^\varepsilon)_\varepsilon$ as $\varepsilon \rightarrow 0$, towards the unique viscosity solution of (??).

HJlimit

Theorem 1.5 (Convergence). *Assume that the initial condition u_0 satisfies the following properties:*

$$[A] \quad b_0 = u_0 - \frac{|v|^2}{2} \in W^{1,\infty}(\mathbb{R}^{2n}), \quad (1.13)$$

$$[B] \quad -\det(\text{Hess}_v(u_0(x, v))) \neq 0, \quad D_v^{(3)}u_0 \in L_{loc}^\infty. \quad (1.14)$$

Let u^ε be the solution of (??), with the initial data $u^\varepsilon(0, \cdot) = u_0$. Then, u^ε converges locally uniformly towards u , which is the unique viscosity solution of (??), as $\varepsilon \rightarrow 0$. In particular, for all $(t, x) \in (0, \infty) \times \mathbb{R}^n$, $v = 0$ is an isolated global minimum of $u(t, x, \cdot)$.

However, we were not able to extend this convergence result to unbounded initial data with respect to space variable. We leave this issue for future work.

Heuristics. It is useful to give some heuristics, in order to understand how the supplementary condition

$$\partial_t \left(\min_{w \in \mathbb{R}^n} u(t, x, w) \right) \leq 0 \quad (= 0),$$

appears in the limit $\varepsilon \rightarrow 0$ (??), with equality if $\mathcal{S}(u)(t, x)$ is reduced to the singleton $\{0\}$. First, notice that the constraint

$$u \leq \min_w u + \frac{|v|^2}{2}, \quad (1.15)$$

immediately follows from (??) if the left-hand-side is bounded with respect to ε . As a consequence, the minimum of u with respect to velocity is necessarily attained at $v = 0$. Then, integrating (??) against the probability measure

$$d\mu_\varepsilon = \left(\int_{\mathbb{R}^n} \exp\left(-\frac{u^\varepsilon}{\varepsilon}\right) dv \right)^{-1} \exp\left(-\frac{u^\varepsilon}{\varepsilon}\right) dv = \frac{f^\varepsilon}{\rho^\varepsilon} dv,$$

we obtain the following continuity equation,

$$\int_{\mathbb{R}^n} (\partial_t u^\varepsilon + v \cdot \nabla_x u^\varepsilon) d\mu_\varepsilon = 0. \quad (1.16)$$

The probability measure $d\mu_\varepsilon$ is expected to concentrate on the minimum points of u as $\varepsilon \rightarrow 0$. Let assume that we do have in some sense,

$$d\mu_\varepsilon \rightharpoonup \sum_{w \in \mathcal{S}(u)(t, x)} p_w \delta_{v=w} = p_0 \delta_{v=0} + \sum_{v_0 \in \mathcal{S}(u)(t, x) \setminus \{0\}} p_{v_0} \delta_{v=v_0}, \quad (1.17)$$

where the weights satisfy $\sum p_{v_0} = 1$. We notice that the constraint (??) at each $v_0 \in \mathcal{S}(u)(t, x) \setminus \{0\}$ is clearly unsaturated, in the sense that $u < \min_w u + |v_0|^2/2$. There, we expect to see the last contribution of (??) vanish. This would lead to $\partial_t u(t, x, v_0) + v_0 \cdot \nabla_x u(t, x, v_0) = 1$ for each such v_0 . Plugging this into (??), and using (??), we obtain successively,

$$0 = \sum_{v_0 \in \mathcal{S}(u)(t, x)} p_{v_0} (\partial_t u + v_0 \cdot \nabla_x u) = p_0 \partial_t u(t, x, 0) + \sum_{v_0 \in \mathcal{S}(u)(t, x) \setminus \{0\}} p_{v_0} = p_0 \partial_t u(t, x, 0) + 1 - p_0.$$

As we have formally $\partial_t u(t, x, 0) = \partial_t (\min_w u(t, x, w))$ by the chain rule, we expect eventually that $\partial_t \min_w u \leq 0$ and even $\partial_t \min_w u = 0$ if $p_0 = 1$, that is, somehow $\mathcal{S}(u)(t, x) = \{0\}$. Obviously, all this reasoning is formal, but we shall make it rigorous in Section ??.

g:min intro

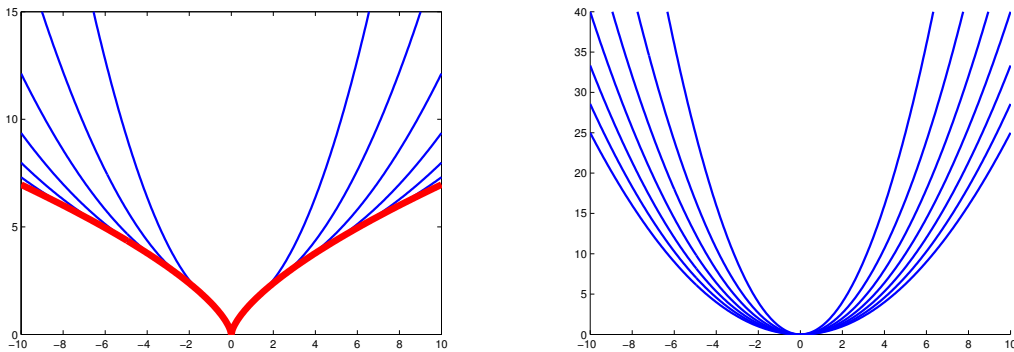


Figure 1: (Left) To illustrate the behaviour of the fundamental solution of (??), the minimum value $\mu(t, \cdot; 0)$ is plotted for successive values of time; (Right) For the sake of comparison, the fundamental solution of (??) is plotted for successive values of time.

Comparison with the case of bounded velocities. We extend previous results obtained in [?, ?] to the case of unbounded velocities. The main difference lies of course in the velocity rescaling (??) which is not admissible in the case of bounded velocities. This has important consequences. Firstly, in the case of bounded velocities, the asymptotic limit u does not depend on v , due to some averaging process arising in the velocity variable. Secondly, the limit equation is a standard Hamilton-Jacobi equation, associated with an effective Hamiltonian. In fact, the Hamilton-Jacobi equation obtained in the limit reads as

$$\int_{\mathbb{R}^n} \frac{M(v)}{1 - \partial_t u(t, x) - v \cdot \nabla_x u(t, x)} dv = 1, \quad (1.18) \quad \text{eq:HJbdd}$$

under appropriate integrability conditions¹. The case of unbounded velocities is by far more subtle, since there is no averaging with respect to velocity. High velocities play a prominent role in the dispersion process.

The case of bounded velocities is analog to large deviations estimates for slow-fast systems as in [?, ?, ?, ?, ?], and references therein. In our case, the role of the fast variable is played by velocity, whereas the space variable is the slow one.

In this work, we follow the Hamiltonian viewpoint, focusing on the value function u , solution of the Hamilton-Jacobi equation (??). There is a dual viewpoint, focusing on the trajectories of the underlying Piecewise Deterministic Markov Process (PDMP). We refer to [?, ?] for further reading.

The fundamental solution. In Sections ?? and ??, we compute explicitly the fundamental solution of the non-local Hamilton-Jacobi equation (??) *in space dimension one*, for $(x, v) \in \mathbb{R}^2$. We follow a time discrete iteration scheme based on the Duhamel formulation of (??). Since the problem is not translation invariant with respect to velocity, it is necessary to compute the solution for all initial data of the form

$$u(0, x, v) = \mathbf{0}_{x=0} + \mathbf{0}_{v=w},$$

¹Namely, it is required that $\lim \int_{\mathbb{R}^n} \frac{M(v)}{1+h-v \cdot p} dv > 1$, where the limit is taken as $h \rightarrow \max_{v \in V} (v \cdot p) - 1$ from above (in order to preserve the positivity of the denominator). Caillerie has extended this result to the general case, without such integrability condition [?].

where

$$\mathbf{0}_{x=0}(y) = \begin{cases} +\infty & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

We obtain the following expression for the fundamental solution,

$$\phi(t, x, v; w) = \min \left(\mathbf{0}_{x=tv} + \min \left(\mathbf{0}_{v=w}, \frac{|v|^2}{2} \right) + t, \frac{|v|^2}{2} + \min_{\substack{s_1, s_2, s_3 \geq 0 \\ 0 \leq s_1 + s_2 + s_3 \leq t}} \left(\frac{(x - s_1 w - s_3 v)^2}{2s_2^2} + s_1 + s_2 + s_3 \right) \right).$$

The minimum value with respect to velocity is given by

$$\mu(t, x; w) = \min_{v \in \mathbb{R}^n} \phi(t, x, v; w) = \min_{\substack{s_1, s_2 \geq 0 \\ 0 \leq s_1 + s_2 \leq t}} \left(\frac{(x - s_1 w)^2}{2s_2^2} + s_1 + s_2 \right).$$

The function $\mu(t, \cdot; 0)$ is plotted in Figure ?? for successive values of time. A striking feature is that the solution does not converge to zero as $t \rightarrow +\infty$, as compared to the fundamental solution of the Hamilton-Jacobi equation associated with the heat equation (??).

Comparison with the heat equation. At first glance, the asymptotics of the kinetic equation (??), is linked with the asymptotics of the heat equation (??) with vanishing viscosity, as the latter is the limit of the former under appropriate diffusive rescaling, yet different from (??): $(\tilde{t}, \tilde{x}, \tilde{v}) = (t/\varepsilon^2, x/\varepsilon, v)$. However, as shown in in Figure ??, the two behaviours are radically different. We may informally present our results as follows: the scaling $(\tilde{t}, \tilde{x}, \tilde{v}) = (t/\varepsilon, x/\varepsilon, v)$ leads to the heat equation with vanishing viscosity $\varepsilon > 0$ after a simple Chapman-Enskog expansion². The fundamental solution of the associated Hamilton-Jacobi equation, associated with the initial data $\mathbf{0}_{x=0}$, is $x^2/(4t)$. In particular, it converges to zero in long time, uniformly on compact intervals.

On the other hand, we have computed the fundamental solution of the limit system (??), obtained after the more appropriate rescaling $(\tilde{t}, \tilde{x}, \tilde{v}) = (t/\varepsilon, x/\varepsilon^{3/2}, v/\varepsilon^{1/2})$ (??). For the sake of comparison, it is better to describe $\min_{w \in \mathbb{R}^n} u(t, x, v)$ (which corresponds to the macroscopic density ρ via the Hopf-Cole transform). Surprisingly enough, it does not converge to zero, but to the function $(3/2)|x|^{2/3}$. We interpret this as follows: at a larger scale than the standard hyperbolic rescaling $(\tilde{t}, \tilde{x}, \tilde{v}) = (t/\varepsilon, x/\varepsilon, v)$ (both space and velocity are larger in (??)), we get non trivial asymptotics, but the density f^ε remains uniformly exponentially small far from the origin, of the order $\asymp \exp(-\mathcal{O}(|x|^{2/3})/\varepsilon)$.

As compared to the heat equation, (??) lacks scaling invariance (with respect to velocity). This is emphasized by the fact that the fundamental solution of (??) does not have a single-line expression, see (??) below. Furthermore, picking a high velocity from a Gaussian distribution is a rare event, that can be completely reset at the next velocity jump. This leads to the predominance of low velocities that slows down the dispersion. This has to be compared with the Fokker-Planck equation having the same stationary velocity distribution,

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = \nabla_v \cdot (\nabla_v f(t, x, v) + v f(t, x, v)), \quad (1.19) \quad \boxed{\text{eq:FP}}$$

for which we expect that the large deviations potential in the scaling $(t, x, v) \rightarrow (\varepsilon^{-1}t, \varepsilon^{-1}x, v)$ behaves like a solution of (??). Technically speaking, we expect that the limit function is independent of the velocity variable due to the additional drift that compactifies the velocity space. This is under investigation by the first author.

²This is essentially due to the fact that time should be speed up by a factor t/ε^2 in order to preserve the diffusive scaling $x^2 = \mathcal{O}(t)$

Accelerated fronts in reaction-transport equations. As an application of this work, we investigate quantitatively front propagation in reaction-transport equations in Section ???. We focus on (??) with an additional monostable reaction term:

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = (M(v)\rho(t, x) - f(t, x, v)) + r\rho(t, x) (M(v) - f(t, x, v)). \quad (1.20) \quad \text{eq:kinreac}$$

This models a population of particles that change velocity at rate one, pick up a random new velocity following a Gaussian distribution, and divide at rate $r > 0$. Moreover, new particles pick up their initial velocity from the same Gaussian distribution. This type of model has been studied in [?, ?, ?] in the case of bounded velocities, and in [?] in the case of possible unbounded velocities. In the case of bounded velocities, there exist traveling waves with constant speed [?, ?, ?]. Moreover, any solution to the Cauchy problem with sufficiently decaying initial data spreads with the minimal speed.

The classical Fisher-KPP equation is obtained in the diffusive limit, provided that the rate of division is assumed to be of order ε^2 ,

$$\partial_t \rho(t, x) - \theta \Delta \rho(t, x) = r\rho(t, x)(1 - \rho(t, x)). \quad (1.21) \quad \text{eq:FKPP}$$

It is well known that front propagation occurs at constant speed, here $2\sqrt{r\theta}$ [?, ?], for any suitably decaying initial data (otherwise, see [?, ?]).

Beyond this classical problem, there has been some recent interest for describing accelerating fronts in reaction-diffusion equations.

Garnier has investigated integro-differential equations, where the spreading operator is given by the convolution with a fat-tailed kernel [?],

$$\partial_t \rho(t, x) + \left(- \int_{\mathbb{R}} J(x - y)\rho(t, y) dy + \rho(t, x) \right) = r\rho(t, x)(1 - \rho(t, x)). \quad (1.22) \quad \text{eq:garnier}$$

Here, fat-tailed means that the kernel J decays slower than exponentially. There, the level lines of the solution spread super linearly, depending on the decay of the convolution kernel J .

Cabr e and Roquejoffre have studied the Fisher-KPP equation where the diffusion operator is replaced with a fractional diffusion operator [?, ?],

$$\partial_t \rho(t, x) + (-\Delta)^\alpha \rho(t, x) = r\rho(t, x)(1 - \rho(t, x)), \quad (1.23) \quad \text{eq:cabre}$$

for some exponent $\alpha \in (0, 1)$. They describe quantitatively the acceleration of the front, which occurs at exponential rate, namely $X(t) \approx \exp(rt/(n + 2\alpha))$ in a weak sense. This seminal work was continued by Coulon and Roquejoffre in [?]. More recently, Mirrahimi and M el eard have described the limit of the reaction-diffusion with a fractional diffusion operator in dimension $n = 1$, after suitable rescaling and the Hopf-Cole transform [?]. As opposed to our results, the limit function has the following simple expression, with separation of variables: $u(t, x) = \max(0, (1 + 2\alpha) \log |x| - t)$.

Recently, spreading in the so-called cane toads equation has been studied intensively.

$$\partial_t n(t, x, \theta) - \theta \partial_x^2 n(t, x, \theta) - \partial_\theta^2 n(t, x, \theta) = rn(t, x, \theta)(1 - \rho(t, x)), \quad \rho(t, x) = \int n(t, x, \theta) d\theta, \quad (1.24) \quad \text{eq:canetoad}$$

When the variable θ is unbounded, accelerated propagation has been proved independently by Berestycki, Mouhot and Raoul [?], and by the first author, Henderson and Ryzhik [?]. There is a formal analogy between (??), and our problem (??), or rather (??). Indeed, acceleration also happens due to the influence of the microscopic variable θ , which plays a similar role as the

velocity variable in this paper. This is another example of a nonlinear acceleration phenomena appearing in a structured model.

Both equations (??) and (??) describe spatial jumps of particles, with a focus on the distribution of long-range jumps. Equation (??) describes a velocity-jump process, with a focus on the distribution of high velocities, that correspond somehow to long-range spatial jumps. It was established in [?] that solutions to (??) behave in the long-time asymptotics as accelerating fronts due to the (rare) occurrence of high velocities that send particles far from the bulk. Furthermore, the location of the front is of the order of $t^{3/2}$, in accordance with the scaling limit (??). However, the exact location of the front was not determined accurately, but it was estimated as

$$\left(\frac{r}{r+2}\right)^{3/2} \leq \frac{X(t)}{t^{3/2}} \leq \sqrt{2r}, \quad (1.25) \quad \text{eq:bounds e}$$

in a weak sense (see [?, Theorem 1.11] for details).

From the knowledge of the quantitative scaling limit procedure in the kinetic dispersion operator (Sections 2 to 5), we aim to apply the same procedure as in [?, ?], respectively for the classical Fisher-KPP equation, and for the fractional Fisher-KPP equation. However, we face several issues, two of them are still open.

Starting from (??), after appropriate rescaling, we prove that $u^\varepsilon(t, x, v) = -\varepsilon \log f^\varepsilon(t, x, v)$ converges towards a non-local Hamilton-Jacobi equation, very similar to (??), complemented with the additional constraint,

$$(\forall(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n) \quad \min_{w \in \mathbb{R}^n} u(t, x, w) \geq 0. \quad (1.26) \quad \text{eq:nonlocal}$$

So far, the limit is rigorously proven for bounded data u_0 only. However, this rules out compactly supported initial data $f_0 = \exp(-u_0/\varepsilon)$. On the other hand, in order to estimate quantitatively the location of the front, as the boundary of the support of the limit function $\min_v u(t, x, v)$, some condition on the extremal trajectories is needed: the so-called Freidlin condition [?, ?]. We propose an extension of this condition in our context, taking into account the non local feature of the constraint (??). We verify that this condition is verified, but only on a subset of $\mathbb{R}^+ \times \mathbb{R}^{2n}$. This rules out an immediate application of our comparison principle.

We conjecture that the edge of the front is located around

$$X(t) = \left(\frac{((2/3)r)^{3/2}}{1+r}\right) t^{3/2}.$$

Naturally, this lies between the two bounds (??) obtained in [?].

Finally, we claim that our work can be extended in a straightforward way to include the velocity distribution

$$M(v) = C_\gamma \exp\left(-\frac{|v|^\gamma}{\gamma}\right),$$

for $\gamma \geq 1$ (this last assumption appearing for convexity reasons). The rate of expansion should be given in the limit $\varepsilon \rightarrow 0$ by the following expression,

$$X(t) = \frac{((\gamma/(1+\gamma)r)^{1+1/\gamma})}{1+r} t^{1+1/\gamma}.$$

2 The comparison principle

sec:Comp

In this Section, we prove the comparison principle stated in Theorem ???. We perform a classical doubling of variables argument. However, much attention has to be paid to the velocity variable.

This is the main concern of this proof. In particular, the velocity variable is not doubled, which is consistent with the fact that there is no gradient with respect to velocity in the limit system (??). However, due to the unboundedness of the velocity space, it is mandatory to confine the velocities in a way to get absolute extrema.

Proof of Theorem ??. Let us define, for $(t, x) \in [0, T) \times \mathbb{R}^n$, the minimum values with respect to the velocity variable:

$$\underline{m}(t, x) := \min_{w \in \mathbb{R}^n} \underline{u}(t, x, w), \quad \overline{m}(t, x) := \min_{w \in \mathbb{R}^n} \overline{u}(t, x, w).$$

Let $\alpha > 0$, $R > 0$. Let $\delta > 0$ to be suitably chosen below. Since the limit system requires two test functions, we shall define

$$\hat{\chi}(t, x) = \underline{m}(t, x) - \overline{m}(t, x) - \frac{\delta}{2}|x|^2 - \frac{\alpha}{T-t}, \quad (2.1) \quad \text{eq:chi hat}$$

$$\tilde{\chi}(t, x, v) = \underline{u}(t, x, v) - \overline{u}(t, x, v) - \frac{\delta}{2}|x|^2 - \frac{\alpha}{T-t} - (|v|^2 - R^2)_+. \quad (2.2) \quad \text{eq:chi tild}$$

Notice that $\tilde{\chi}$ penalizes high velocities. We denote $B = \max(\|b\|_\infty, \|\bar{b}\|_\infty)$. The assumption (??) will give a confinement on the minima of $\underline{u}, \overline{u}$ as follows.

lem:minconf

Lemma 2.1. *Let $R_0^2 = 2B + 1$. For all $R > R_0$, for all $(t, x) \in [0, T) \times \mathbb{R}^n$, we have $\mathcal{S}(\overline{u})(t, x) \subset \overline{B}_R(0)$.*

Proof of Lemma ??. Fix $(t, x) \in [0, T) \times \mathbb{R}^n$. Suppose that there exists $v \in \mathcal{S}(\overline{u})(t, x)$ such that $|v| > R$. Then at (t, x) ,

$$\overline{m}(t, x) = \frac{v^2}{2} + \bar{b}(t, x, v) \geq \frac{v^2}{2} - \|\bar{b}\|_\infty > \|\bar{b}\|_\infty \geq \bar{b}(t, x, 0) = \overline{u}(t, x, 0) \geq \overline{m}(t, x),$$

and this cannot hold. Therefore we have $\mathcal{S}(\overline{u})(t, x) \subset \overline{B}_R(0)$. □

Let $R > R_0$. We can now introduce the following maximum values,

$$\omega = \max_{[0, T) \times \mathbb{R}^n} \hat{\chi}, \quad \Omega = \max_{[0, T) \times \mathbb{R}^n \times \mathbb{R}^n} \tilde{\chi}. \quad (2.3) \quad \text{eq:def delt}$$

These two quantities are clearly finite from (??).

lem:Omega

Lemma 2.2. *We have $\omega \leq \Omega$.*

Proof of Lemma ??. Let $(t^*, x^*) \in [0, T) \times \mathbb{R}^n$ be a maximum point of $\hat{\chi}$. Let $v^* \in \mathcal{S}(\overline{u})(t^*, x^*) \subset \overline{B}_R(0)$. The following sequence of inequalities hold true,

$$\begin{aligned} \Omega &\geq \tilde{\chi}(t^*, x^*, v^*) \\ &= \underline{u}(t^*, x^*, v^*) - \overline{u}(t^*, x^*, v^*) - \frac{\delta}{2}|x^*|^2 - \frac{\alpha}{T-t^*} - (|v^*|^2 - R^2)_+ \\ &\geq \underline{m}(t^*, x^*) - \overline{u}(t^*, x^*, v^*) - \frac{\delta}{2}|x^*|^2 - \frac{\alpha}{T-t^*} \\ &= \underline{m}(t^*, x^*) - \overline{m}(t^*, x^*) - \frac{\delta}{2}|x^*|^2 - \frac{\alpha}{T-t^*} \\ &\geq \omega. \end{aligned}$$

□

We now prove that for suitably chosen parameters α, δ , the supremum Ω of $\tilde{\chi}$ is attained at $t = 0$. For that, we distinguish between two cases: $\omega < \Omega$, and $\omega = \Omega$.

Case 1: $\omega < \Omega$.

We denote by $(\tilde{t}_0, \tilde{x}_0, v_0)$ a maximum point of $\tilde{\chi}$, such that $\tilde{\chi}(\tilde{t}_0, \tilde{x}_0, v_0) = \Omega$. This point exists thanks to the penalization in the three variables and the fact that $\tilde{\chi}$ is upper semi-continuous. We emphasize that this point depends on the parameters α, R and δ , but for legibility we omit this dependency. We aim to prove that $\tilde{t}_0 = 0$. We argue by contradiction, and assume that $\tilde{t}_0 > 0$.

We know show that v_0 necessarily satisfies *a priori* a bound independent of δ .

lem:velconf

Lemma 2.3. *We have*

$$(|v_0|^2 - R^2)_+ \leq 4B.$$

Proof of Lemma ??. The evaluation $\tilde{\chi}(\tilde{t}_0, \tilde{x}_0, v_0) \geq \tilde{\chi}(0, 0, 0)$ gives

$$\begin{aligned} 4B &\geq -\underline{u}(0, 0, 0) + \bar{u}(0, 0, 0) + \underline{u}(\tilde{t}_0, \tilde{x}_0, v_0) - \bar{u}(\tilde{t}_0, \tilde{x}_0, v_0) \\ &\geq \frac{\delta}{2} |\tilde{x}_0|^2 + \frac{\alpha}{T - \tilde{t}_0} + (|v_0|^2 - R^2)_+ - \frac{\alpha}{T} \geq (|v_0|^2 - R^2)_+. \end{aligned}$$

□

We define some auxiliary function with twice the number of variables, except for the velocity, as follows

$$\begin{aligned} \tilde{\chi}_\varepsilon(t, x, s, y, v) &= \underline{u}(t, x, v) - \bar{u}(s, y, v) - \frac{\delta}{2} |x|^2 \\ &\quad - \frac{\alpha}{T - t} - \frac{1}{2\varepsilon} (|t - s|^2 + |x - y|^2) \\ &\quad - \frac{1}{2} (|s - \tilde{t}_0|^2 + |y - \tilde{x}_0|^2) - (|v|^2 - R^2)_+. \end{aligned}$$

Let $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon)$ which realizes the maximum of $\tilde{\chi}_\varepsilon(\cdot, v_0)$.

em:limit t0

Lemma 2.4. *The following limit holds true,*

$$\lim_{\varepsilon \rightarrow 0} (\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon) = (\tilde{t}_0, \tilde{x}_0, \tilde{t}_0, \tilde{x}_0).$$

Proof of Lemma ??. The evaluation $\Omega = \tilde{\chi}_\varepsilon(\tilde{t}_0, \tilde{x}_0, \tilde{t}_0, \tilde{x}_0, v_0) \leq \tilde{\chi}_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0)$ yields

$$\frac{\delta}{2} |\tilde{x}_\varepsilon|^2 + \frac{\alpha}{T - \tilde{t}_\varepsilon} + \frac{1}{2\varepsilon} (|\tilde{t}_\varepsilon - \tilde{s}_\varepsilon|^2 + |\tilde{x}_\varepsilon - \tilde{y}_\varepsilon|^2) + \frac{1}{2} (|\tilde{s}_\varepsilon - \tilde{t}_0|^2 + |\tilde{y}_\varepsilon - \tilde{x}_0|^2) \leq 2B - \Omega.$$

We deduce the following estimates,

$$\delta |\tilde{x}_\varepsilon| \leq C\delta^{1/2}, \quad |\tilde{t}_\varepsilon - \tilde{s}_\varepsilon|, |\tilde{x}_\varepsilon - \tilde{y}_\varepsilon| \leq C\varepsilon^{1/2}.$$

Therefore, the sequence $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon)$ converges as $\varepsilon \rightarrow 0$, up to a subsequence. A closer look shows that the limit is necessarily $(\tilde{t}_0, \tilde{x}_0, \tilde{x}_0, \tilde{x}_0)$. We denote by $(\tilde{t}'_0, \tilde{x}'_0, \tilde{s}'_0, \tilde{y}'_0)$ an accumulation point. We have $\tilde{t}'_0 = \tilde{s}'_0$ and $\tilde{x}'_0 = \tilde{y}'_0$. On the other hand, passing to the limit $\varepsilon \rightarrow 0$ in the inequality $\tilde{\chi}_\varepsilon(\tilde{t}_0, \tilde{x}_0, \tilde{t}_0, \tilde{x}_0, v_0) \leq \tilde{\chi}_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0)$, we deduce

$$\begin{aligned} \tilde{\chi}(\tilde{t}_0, \tilde{x}_0, v_0) &+ \frac{1}{2} (|\tilde{t}'_0 - \tilde{t}_0|^2 + |\tilde{x}'_0 - \tilde{x}_0|^2) \\ &\leq \underline{u}(\tilde{t}'_0, \tilde{x}'_0, v_0) - \bar{u}(\tilde{t}'_0, \tilde{x}'_0, v_0) - \frac{\delta}{2} |\tilde{x}'_0|^2 - \frac{\alpha}{T - \tilde{t}'_0} \leq \tilde{\chi}(\tilde{t}_0, \tilde{x}_0, v_0). \end{aligned}$$

Therefore, we have necessarily $\tilde{t}'_0 = \tilde{t}_0$, and $\tilde{x}'_0 = \tilde{x}_0$. □

:constraint

Lemma 2.5. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,*

$$\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) - \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) - \frac{|v_0|^2}{2} < 0.$$

Proof of Lemma ??. The evaluation $\tilde{\chi}_\varepsilon(\tilde{t}_0, \tilde{x}_0, \tilde{t}_0, \tilde{x}_0, v_0) \leq \tilde{\chi}_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0)$ gives the following piece of information,

$$\tilde{\chi}_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) \geq \tilde{\chi}(\tilde{t}_0, \tilde{x}_0, v_0) = \Omega. \quad (2.4) \quad \text{eq:chi eps}$$

We rewrite (??) as

$$\begin{aligned} \bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) - \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) - \frac{|v_0|^2}{2} &\leq -\Omega + \left(\underline{u}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v_0) - \underline{m}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon) - \frac{|v_0|^2}{2} \right) \\ &\quad + \left(\underline{m}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon) - \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) - \frac{\delta}{2} |\tilde{x}_\varepsilon|^2 - \frac{\alpha}{T - \tilde{t}_\varepsilon} \right). \end{aligned}$$

Since \underline{u} is a subsolution and $\tilde{t}_\varepsilon > 0$ is verified for ε small enough, we get that

$$\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) - \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) - \frac{|v_0|^2}{2} \leq -\Omega + \left(\underline{m}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon) - \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) - \frac{\delta}{2} |\tilde{x}_\varepsilon|^2 - \frac{\alpha}{T - \tilde{t}_\varepsilon} \right). \quad (2.5) \quad \text{supersolutio}$$

As $\varepsilon \rightarrow 0$, by upper semi-continuity, the last contribution in the r.h.s. of (??) is such that

$$\limsup_{\varepsilon \rightarrow 0} \left(\underline{m}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon) - \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) - \frac{\delta}{2} |\tilde{x}_\varepsilon|^2 - \frac{\alpha}{T - \tilde{t}_\varepsilon} \right) \leq \underline{m}(\tilde{t}_0, \tilde{x}_0) - \bar{m}(\tilde{t}_0, \tilde{x}_0) - \frac{\delta}{2} |\tilde{x}_0|^2 - \frac{\alpha}{T - \tilde{t}_0} \leq \omega.$$

Therefore, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) - \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) - \frac{|v_0|^2}{2} \leq -\Omega + \frac{\omega + \Omega}{2} = \frac{\omega - \Omega}{2} < 0.$$

since we have assumed in this case that $\Omega > \omega$. □

We now use the test function

$$\begin{aligned} \phi_2(s, y, v) &= \underline{u}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v) - \frac{\delta}{2} |\tilde{x}_\varepsilon|^2 - \frac{\alpha}{T - \tilde{t}_\varepsilon} \\ &\quad - \frac{1}{2\varepsilon} (|\tilde{t}_\varepsilon - s|^2 + |\tilde{x}_\varepsilon - y|^2) - \frac{1}{2} (|s - \tilde{t}_0|^2 + |y - \tilde{x}_0|^2) - (|v|^2 - R^2)_+, \end{aligned}$$

associated to the supersolution \bar{u} at the point $(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0)$. Notice that the condition $\tilde{s}_\varepsilon > 0$ is verified for ε small enough. By using the definition ?? of a super-solution, this yields

$$-\frac{1}{\varepsilon} (\tilde{s}_\varepsilon - \tilde{t}_\varepsilon) - (\tilde{s}_\varepsilon - \tilde{t}_0) + v_0 \cdot \left(-\frac{1}{\varepsilon} (\tilde{y}_\varepsilon - \tilde{x}_\varepsilon) - (\tilde{y}_\varepsilon - \tilde{x}_0) \right) - 1 \geq 0. \quad (2.6) \quad \text{eq:chain eps}$$

On the other hand, using the test function

$$\begin{aligned} \phi_1(t, x, v) &= \bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v) + \frac{\delta}{2} |x|^2 + \frac{\alpha}{T - t} \\ &\quad + \frac{1}{2\varepsilon} (|t - \tilde{s}_\varepsilon|^2 + |x - \tilde{y}_\varepsilon|^2) + \frac{1}{2} (|\tilde{s}_\varepsilon - \tilde{t}_0|^2 + |\tilde{y}_\varepsilon - \tilde{x}_0|^2) + (|v|^2 - R^2)_+, \end{aligned}$$

associated to the subsolution \underline{u} at the point $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v_0)$, we obtain

$$\frac{\alpha}{(T - \tilde{t}_\varepsilon)^2} + \frac{1}{\varepsilon}(\tilde{t}_\varepsilon - \tilde{s}_\varepsilon) + v_0 \cdot \left(\delta \tilde{x}_\varepsilon + \frac{1}{\varepsilon}(\tilde{x}_\varepsilon - \tilde{y}_\varepsilon) \right) - 1 \leq 0, \quad (2.7) \quad \boxed{\text{eq:chain ru}}$$

by using the definition ?? of a sub-solution. By subtracting (??) to (??), we obtain

$$\frac{\alpha}{T^2} \leq \frac{\alpha}{(T - \tilde{t}_\varepsilon)^2} \leq -\delta v_0 \cdot \tilde{x}_\varepsilon + (\tilde{s}_\varepsilon - \tilde{t}_0) - v_0 \cdot (\tilde{y}_\varepsilon - \tilde{x}_0).$$

Letting $\varepsilon \rightarrow 0$, this gives $\alpha/T^2 \leq -\delta v_0 \cdot \tilde{x}_0$. Recall that from the estimates in the proof of Lemma ??, we have that $\delta|\tilde{x}_0| \leq C\delta^{1/2}$ holds true for some constant $C > 0$, and for $\varepsilon > 0$ small enough. Recall also Lemma ?. We obtain

$$\alpha/T^2 \leq C\delta^{1/2}R,$$

where C depends only on B . Hence, the choice $\delta = (\alpha T^{-2} R^{-1} C^{-1})^2/2$ establishes a contradiction. We conclude that $\tilde{t}_0 = 0$ in this case.

Case 2: $\omega = \Omega$. We denote by (\hat{t}_0, \hat{x}_0) a maximum point of $\hat{\chi}$. We aim to prove that $\hat{t}_0 = 0$. We argue again by contradiction and we suppose that $\hat{t}_0 > 0$. The evaluation $\hat{\chi}(0, 0) \leq \hat{\chi}(\hat{t}_0, \hat{x}_0)$ yields immediately the following bound:

$$\frac{\delta}{2}|\hat{x}_0|^2 + \frac{\alpha}{T - \hat{t}_0} \leq 4B + \frac{\alpha}{T}. \quad (2.8) \quad \boxed{\text{eq:bound t_}}$$

Case 2.1: We first consider the case where $\mathcal{S}(\bar{u})(\hat{t}_0, \hat{x}_0) = \{0\}$. We introduce the following auxiliary function

$$\begin{aligned} \hat{\chi}_\varepsilon(t, x, s, y) = & \underline{m}(t, x) - \overline{m}(s, y) - \frac{\delta}{2}|x|^2 \\ & - \frac{\alpha}{T - t} - \frac{1}{2\varepsilon} (|t - s|^2 + |x - y|^2) - \frac{1}{2} (|s - \hat{t}_0|^2 + |y - \hat{x}_0|^2). \end{aligned}$$

Let $(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{s}_\varepsilon, \hat{y}_\varepsilon)$ which realizes the maximum value of $\hat{\chi}_\varepsilon$. We can prove as in Lemma ??, that the following limit holds true,

$$\lim_{\varepsilon \rightarrow 0} (\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{s}_\varepsilon, \hat{y}_\varepsilon) = (\hat{t}_0, \hat{x}_0, \hat{t}_0, \hat{x}_0).$$

We use the test function

$$\psi_2(s, y) = \underline{m}(\hat{t}_\varepsilon, \hat{x}_\varepsilon) - \frac{\delta}{2}|\hat{x}_\varepsilon|^2 - \frac{\alpha}{T - \hat{t}_\varepsilon} - \frac{1}{2\varepsilon} (|\hat{t}_\varepsilon - s|^2 + |\hat{x}_\varepsilon - y|^2) - \frac{1}{2} (|s - \hat{t}_0|^2 + |y - \hat{x}_0|^2),$$

associated to the supersolution \overline{m} at the point $(\hat{s}_\varepsilon, \hat{y}_\varepsilon)$. Notice that the condition $\hat{s}_\varepsilon > 0$ is verified for ε small enough.

In order to apply the second criterion in (??), it is required that the set $\mathcal{S}(\bar{u})(\hat{s}_\varepsilon, \hat{y}_\varepsilon)$ is reduced to $\{0\}$, provided ε is sufficiently small.

lem:26

Lemma 2.6. *There exists $\varepsilon_0 > 0$ such that the set $\mathcal{S}(\bar{u})(\hat{s}_\varepsilon, \hat{y}_\varepsilon)$ is reduced to $\{0\}$, provided that $\varepsilon < \varepsilon_0$.*

Proof of Lemma ??. We argue by contradiction. Assume that there exists a sequence $\varepsilon_n \searrow 0$ such that $\mathcal{S}(\bar{u})(\hat{s}_{\varepsilon_n}, \hat{y}_{\varepsilon_n})$ contains some nonzero v_n^* . It is required in the definition ?? of a viscosity super-solution that $v = 0$ is a locally uniformly isolated minimum over \mathbb{R}^d . Hence, there exists $r > 0$ such that $r < |v_n^*| < R$ for all n . Up to extraction of a subsequence, we can assume that (v_n^*) converges to some nonzero v^* . The following inequalities are satisfied,

$$\bar{u}(\hat{t}_0, \hat{x}_0, v^*) \leq \liminf_{n \rightarrow +\infty} \bar{u}(\hat{s}_{\varepsilon_n}, \hat{y}_{\varepsilon_n}, v_n^*) = \liminf_{n \rightarrow +\infty} \bar{m}(\hat{s}_{\varepsilon_n}, \hat{y}_{\varepsilon_n}). \quad (2.9) \quad \boxed{\text{eq:m bar li}}$$

There is some subtlety here, because the lower semi continuity of \bar{m} would only imply $\liminf \bar{m}(\hat{s}_{\varepsilon_n}, \hat{y}_{\varepsilon_n}) \geq \bar{m}(\hat{t}_0, \hat{x}_0)$ in full generality. However, the following argument establishes that such a lim inf is in fact a true limit.

From the maximality of $(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{s}_\varepsilon, \hat{y}_\varepsilon)$, we have

$$\begin{aligned} \underline{m}(\hat{t}_\varepsilon, \hat{x}_\varepsilon) - \bar{m}(\hat{s}_\varepsilon, \hat{y}_\varepsilon) &= \hat{\chi}_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{s}_\varepsilon, \hat{y}_\varepsilon) \\ &+ \frac{\delta}{2} |\hat{x}_\varepsilon|^2 + \frac{\alpha}{T - \hat{t}_\varepsilon} + \frac{1}{2\varepsilon} (|\hat{t}_\varepsilon - \hat{s}_\varepsilon|^2 + |\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2) + \frac{1}{2} (|\hat{s}_\varepsilon - \hat{t}_0|^2 + |\hat{y}_\varepsilon - \hat{x}_0|^2) \\ &\geq \hat{\chi}_\varepsilon(\hat{t}_0, \hat{x}_0, \hat{t}_0, \hat{x}_0) + \frac{\delta}{2} |\hat{x}_\varepsilon|^2 + \frac{\alpha}{T - \hat{t}_\varepsilon} \\ &= \underline{m}(\hat{t}_0, \hat{x}_0) - \bar{m}(\hat{t}_0, \hat{x}_0) + \frac{\delta}{2} |\hat{x}_\varepsilon|^2 + \frac{\alpha}{T - \hat{t}_\varepsilon} - \frac{\delta}{2} |\hat{x}_0|^2 - \frac{\alpha}{T - \hat{t}_0}. \end{aligned}$$

Now using the upper semi-continuity of \underline{m} and the lower semi-continuity of \bar{m} , we deduce that

$$\begin{aligned} 0 &= \underline{m}(\hat{t}_0, \hat{x}_0) - \underline{m}(\hat{t}_0, \hat{x}_0) \\ &\geq \limsup_{\varepsilon \rightarrow 0} (\underline{m}(\hat{t}_\varepsilon, \hat{x}_\varepsilon) - \underline{m}(\hat{t}_0, \hat{x}_0)) \\ &\geq \limsup_{\varepsilon \rightarrow 0} \left(\bar{m}(\hat{s}_\varepsilon, \hat{y}_\varepsilon) - \bar{m}(\hat{t}_0, \hat{x}_0) + \frac{\delta}{2} (|\hat{x}_\varepsilon|^2 - |\hat{x}_0|^2) + \frac{\alpha}{T - \hat{t}_\varepsilon} - \frac{\alpha}{T - \hat{t}_0} \right) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \left(\bar{m}(\hat{s}_\varepsilon, \hat{y}_\varepsilon) - \bar{m}(\hat{t}_0, \hat{x}_0) + \frac{\delta}{2} (|\hat{x}_\varepsilon|^2 - |\hat{x}_0|^2) + \frac{\alpha}{T - \hat{t}_\varepsilon} - \frac{\alpha}{T - \hat{t}_0} \right) \\ &\geq \bar{m}(\hat{t}_0, \hat{x}_0) - \bar{m}(\hat{t}_0, \hat{x}_0) = 0. \end{aligned}$$

As a consequence of the previous inequalities, all inequalities are equalities. Thus, the following limit is well-defined,

$$\lim_{\varepsilon \rightarrow 0} (\bar{m}(\hat{s}_\varepsilon, \hat{y}_\varepsilon) - \bar{m}(\hat{t}_0, \hat{x}_0)) = 0.$$

We conclude from (??) that

$$\bar{u}(\hat{t}_0, \hat{x}_0, v^*) \leq \bar{m}(\hat{t}_0, \hat{x}_0).$$

This yields the existence of $v^* \neq 0$ such that $\mathcal{S}(\bar{u})(\hat{t}_0, \hat{x}_0) \supset \{0, v^*\}$, which is a contradiction. \square

The second criterion in (??) writes as follows,

$$-\frac{1}{\varepsilon} (\hat{s}_\varepsilon - \hat{t}_\varepsilon) - (\hat{s}_\varepsilon - \hat{t}_0) \geq 0. \quad (2.10) \quad \boxed{\text{eq:chain ru}}$$

On the other hand, using the test function

$$\psi_1(t, x) = \bar{m}(\hat{s}_\varepsilon, \hat{y}_\varepsilon) + \frac{\delta}{2} |x|^2 + \frac{\alpha}{T - t} + \frac{1}{2\varepsilon} (|t - \hat{s}_\varepsilon|^2 + |x - \hat{y}_\varepsilon|^2) + \frac{1}{2} (|\hat{s}_\varepsilon - \hat{t}_0|^2 + |\hat{y}_\varepsilon - \hat{x}_0|^2),$$

associated to the subsolution \underline{m} at the point $(\hat{t}_\varepsilon, \hat{x}_\varepsilon)$, we obtain

$$\frac{\alpha}{(T - \hat{t}_\varepsilon)^2} + \frac{1}{\varepsilon} (\hat{t}_\varepsilon - \hat{s}_\varepsilon) \leq 0. \quad (2.11) \quad \boxed{\text{eq:chain ru}}$$

By subtracting (??) to (??), we obtain

$$-(\hat{s}_\varepsilon - \hat{t}_0) \geq \frac{\alpha}{(T - \hat{t}_\varepsilon)^2} \geq \frac{\alpha}{T^2}.$$

Passing to the limit $\varepsilon \rightarrow 0$, we get a contradiction.

Case 2.2: There exists some nonzero $v_0 \in \mathcal{S}(\bar{u})(\hat{t}_0, \hat{x}_0)$.

The following lemma describes the equality cases in Lemma ??.

constraint2

Lemma 2.7. *Assume $\omega = \Omega$, and let $v_0 \in \mathcal{S}(\bar{u})(\hat{t}_0, \hat{x}_0) \setminus \{0\}$. Then,*

1. $(\hat{t}_0, \hat{x}_0, v_0)$ realizes the supremum of $\tilde{\chi}$,
2. v_0 is a minimum velocity also for $\underline{u}(\hat{t}_0, \hat{x}_0, \cdot)$.

Proof of Lemma ??. We repeat the proof of Lemma ?? with $(t^*, x^*, v^*) = (\hat{t}_0, \hat{x}_0, v_0)$. By examining the case of equality, we realize that

$$\begin{cases} \Omega = \tilde{\chi}(\hat{t}_0, \hat{x}_0, v_0), \\ \underline{u}(\hat{t}_0, \hat{x}_0, v_0) = \underline{m}(\hat{t}_0, \hat{x}_0). \end{cases}$$

□

Similarly as in Case 1, we define some auxiliary function with twice the number of variables, except for the velocity, as follows

$$\begin{aligned} \tilde{\chi}_\varepsilon(t, x, s, y, v) = & \underline{u}(t, x, v) - \bar{u}(s, y, v) - \frac{\delta}{2}|x|^2 \\ & - \frac{\alpha}{T-t} - \frac{1}{2\varepsilon} (|t-s|^2 + |x-y|^2) \\ & - \frac{1}{2} (|s-\hat{t}_0|^2 + |y-\hat{x}_0|^2) - (|v|^2 - R^2)_+ . \end{aligned}$$

Let $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon)$ which realizes the maximum value of $\tilde{\chi}_\varepsilon(\cdot, v_0)$. We can prove as in Lemma ??, that the following limit holds true,

$$\lim_{\varepsilon \rightarrow 0} (\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon) = (\hat{t}_0, \hat{x}_0, \hat{t}_0, \hat{x}_0).$$

One key observation is that the following strict inequality holds true

$$\bar{u}(\hat{t}_0, \hat{x}_0, v_0) - \min_{w \in \mathbb{R}^n} \bar{u}(\hat{t}_0, \hat{x}_0, w) - \frac{|v_0|^2}{2} < 0, \quad (2.12)$$

by the very definition of $v_0 \neq 0$.

As $(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) \rightarrow (\hat{t}_0, \hat{x}_0)$ as $\varepsilon \rightarrow 0$, we expect that this inequality is also strict for ε small enough. However, this is not compatible with the *a priori* lower semi-continuity of \bar{u} as it may exhibit some negative jump when passing to the limit $\varepsilon \rightarrow 0$. The following lemma resolves this difficulty (see also the proof of Lemma ??).

constraint2

Lemma 2.8. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,*

$$\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) - \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) - \frac{|v_0|^2}{2} < 0.$$

Proof of Lemma ??. From the maximality of $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0)$, we have

$$\begin{aligned} \underline{u}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v_0) - \bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) &= \tilde{\chi}_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) \\ &+ \frac{\delta}{2} |\tilde{x}_\varepsilon|^2 + \frac{\alpha}{T - \tilde{t}_\varepsilon} + \frac{1}{2\varepsilon} (|\tilde{t}_\varepsilon - \tilde{s}_\varepsilon|^2 + |\tilde{x}_\varepsilon - \tilde{y}_\varepsilon|^2) + \frac{1}{2} (|\tilde{s}_\varepsilon - \hat{t}_0|^2 + |\tilde{y}_\varepsilon - \hat{x}_0|^2) + (|v_0|^2 - R^2)_+ \\ &\geq \tilde{\chi}_\varepsilon(\hat{t}_0, \hat{x}_0, \hat{t}_0, \hat{x}_0, v_0) + \frac{\delta}{2} |\tilde{x}_\varepsilon|^2 + \frac{\alpha}{T - \tilde{t}_\varepsilon} + (|v_0|^2 - R^2)_+ \\ &= \underline{u}(\hat{t}_0, \hat{x}_0, v_0) - \bar{u}(\hat{t}_0, \hat{x}_0, v_0) + \frac{\delta}{2} |\tilde{x}_\varepsilon|^2 + \frac{\alpha}{T - \tilde{t}_\varepsilon} - \frac{\delta}{2} |\hat{x}_0|^2 - \frac{\alpha}{T - \hat{t}_0}. \end{aligned}$$

Using the upper semi-continuity of \underline{u} and the lower semi-continuity of \bar{u} , we deduce that

$$\begin{aligned} 0 &= \underline{u}(\hat{t}_0, \hat{x}_0, v_0) - \underline{u}(\hat{t}_0, \hat{x}_0, v_0) \\ &\geq \limsup_{\varepsilon \rightarrow 0} (\underline{u}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v_0) - \underline{u}(\hat{t}_0, \hat{x}_0, v_0)) \\ &\geq \limsup_{\varepsilon \rightarrow 0} \left(\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) - \bar{u}(\hat{t}_0, \hat{x}_0, v_0) + \frac{\delta}{2} (|\tilde{x}_\varepsilon|^2 - |\hat{x}_0|^2) + \frac{\alpha}{T - \tilde{t}_\varepsilon} - \frac{\alpha}{T - \hat{t}_0} \right) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \left(\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) - \bar{u}(\hat{t}_0, \hat{x}_0, v_0) + \frac{\delta}{2} (|\tilde{x}_\varepsilon|^2 - |\hat{x}_0|^2) + \frac{\alpha}{T - \tilde{t}_\varepsilon} - \frac{\alpha}{T - \hat{t}_0} \right) \\ &= \bar{u}(\hat{t}_0, \hat{x}_0, v_0) - \bar{u}(\hat{t}_0, \hat{x}_0, v_0) = 0. \end{aligned}$$

As a consequence of the previous inequalities, all inequalities are equalities. Thus, the following limit is well-defined,

$$\lim_{\varepsilon \rightarrow 0} (\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) - \bar{u}(\hat{t}_0, \hat{x}_0, v_0)) = 0. \quad (2.13) \quad \boxed{\text{eq:continui}}$$

Now define $d = -\bar{u}(\hat{t}_0, \hat{x}_0, v_0) + \bar{m}(\hat{t}_0, \hat{x}_0) + \frac{|v_0|^2}{2} > 0$ and take ε sufficiently small such that both inequalities hold true,

$$\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) \leq \bar{u}(\hat{t}_0, \hat{x}_0, v_0) + \frac{d}{4}, \quad \bar{m}(\hat{t}_0, \hat{x}_0) \leq \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) + \frac{d}{4}$$

The former is a consequence of (??), while the latter is deduced from lower semi-continuity of \bar{m} . Then, one has

$$\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) \leq \bar{u}(\hat{t}_0, \hat{x}_0, v_0) + \frac{d}{4} = \frac{d}{4} - d + \bar{m}(\hat{t}_0, \hat{x}_0) + \frac{|v_0|^2}{2} \leq \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) + \frac{|v_0|^2}{2} - \frac{d}{2}$$

and the lemma follows. \square

Therefore, we can use the test function

$$\begin{aligned} \phi_2(s, y, v) &= \underline{u}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v) - \frac{\delta}{2} |\tilde{x}_\varepsilon|^2 - \frac{\alpha}{T - \tilde{t}_\varepsilon} \\ &\quad - \frac{1}{2\varepsilon} (|\tilde{t}_\varepsilon - s|^2 + |\tilde{x}_\varepsilon - y|^2) \\ &\quad - \frac{1}{2} (|s - \hat{t}_0|^2 + |y - \hat{x}_0|^2) - (|v_0|^2 - R^2)_+, \end{aligned}$$

associated to the supersolution \bar{u} at the point $(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0)$. Notice that the condition $\tilde{s}_\varepsilon > 0$ is verified for ε small enough. This yields

$$-\frac{1}{\varepsilon} (\tilde{s}_\varepsilon - \tilde{t}_\varepsilon) - (\tilde{s}_\varepsilon - \hat{t}_0) + v_0 \cdot \left(-\frac{1}{\varepsilon} (\tilde{y}_\varepsilon - \tilde{x}_\varepsilon) - (\tilde{y}_\varepsilon - \hat{x}_0) \right) - 1 \geq 0. \quad (2.14) \quad \boxed{\text{eq:chain ru}}$$

On the other hand, using the test function

$$\begin{aligned} \phi_1(t, x, v) = & \bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v) + \frac{\delta}{2}|x|^2 + \frac{\alpha}{T-t} \\ & + \frac{1}{2\varepsilon} (|t - \tilde{s}_\varepsilon|^2 + |x - \tilde{y}_\varepsilon|^2) \\ & + \frac{1}{2} (|\tilde{s}_\varepsilon - \hat{t}_0|^2 + |\tilde{y}_\varepsilon - \hat{x}_0|^2) + (|v_0|^2 - R^2)_+, \end{aligned}$$

associated to the subsolution \underline{u} at the point $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v_0)$, we obtain

$$\frac{\alpha}{(T - \tilde{t}_\varepsilon)^2} + \frac{1}{\varepsilon}(\tilde{t}_\varepsilon - \tilde{s}_\varepsilon) + v_0 \cdot \left(\delta \tilde{x}_\varepsilon + \frac{1}{\varepsilon}(\tilde{x}_\varepsilon - \tilde{y}_\varepsilon) \right) - 1 \leq 0. \quad (2.15) \quad \boxed{\text{eq:chain ru}}$$

By subtracting (??) to (??), we obtain

$$-(\tilde{s}_\varepsilon - \hat{t}_0) - v_0 \cdot (\tilde{y}_\varepsilon - \hat{x}_0) - \delta v_0 \cdot \tilde{x}_\varepsilon \geq \frac{\alpha}{(T - \tilde{t}_\varepsilon)^2} \geq \frac{\alpha}{T^2}.$$

Passing to the limit $\varepsilon \rightarrow 0$, we get

$$\frac{\alpha}{T^2} \leq -\delta v_0 \cdot \hat{x}_0 \leq C\delta^{\frac{1}{2}}R_0.$$

Similarly as in Case 1, the choice $\delta = (\alpha T^{-2}R_0^{-1}C^{-1})^2/2$ yields a contradiction. We conclude that $\tilde{t}_0 = 0$ in this case also.

We are in position to conclude. By using that $\underline{u}(0, \cdot, \cdot) \leq u_0 \leq \bar{u}(0, \cdot, \cdot)$, we deduce that $\Omega \leq 0$, that is

$$(\forall (t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n) \quad \underline{u}(t, x, v) - \bar{u}(t, x, v) \leq \frac{1}{4}(\alpha T^{-2}R^{-1}C^{-1})^2|x|^2 + \frac{\alpha}{T-t} + (|v|^2 - R^2)_+.$$

We obtain the comparison by taking limits $\alpha \rightarrow 0$, and $R \rightarrow +\infty$. □

3 Convergence of u^ε when $\varepsilon \rightarrow 0$.

sec:Conv

In this Section, we shall prove that u^ε converges locally uniformly towards the unique viscosity solution of the system (??). We first establish Lipschitz bounds that allow to extract a locally uniformly converging subsequence. Then, we prove that this limit is a solution of (??).

estimates **Proposition 3.1. (Uniform estimates).** *Let $u^\varepsilon \in \mathcal{C}_b^1(\mathbb{R}_+ \times \mathbb{R}^{2n})$ be a solution of equation (??) and define b^ε as $b^\varepsilon := u^\varepsilon - |v|^2/2$. If $b_0^\varepsilon = b_0$ satisfies condition (??), then b^ε is uniformly locally Lipschitz. Precisely the following a priori bounds hold for all $t \in \mathbb{R}_+$:*

- (i) $\|b^\varepsilon(t, \cdot)\|_\infty \leq \|b_0\|_\infty,$
- (ii) $\|\nabla_x b^\varepsilon(t, \cdot)\|_\infty \leq \|\nabla_x b_0\|_\infty,$
- (iii) $\|\partial_t b^\varepsilon(t, \cdot)\|_{L^\infty(B(0,R) \times B(0,R'))} \leq C(R, R'),$
- (iv) $\|\nabla_v b^\varepsilon(t, \cdot)\|_\infty \leq \|\nabla_v b_0\|_\infty + t\|\nabla_x b_0\|_\infty.$

Proof of Proposition ??. The function b^ε satisfies

$$\partial_t b^\varepsilon + v \cdot \nabla_x b^\varepsilon = 1 - \int_{\mathbb{R}^n} M_\varepsilon(v') \exp\left(\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}\right) dv'. \quad (3.1) \quad \boxed{\text{eq:b}}$$

Let us notice that we obtain a unique solution $b^\varepsilon \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^{2n})$, and thus a unique u^ε , from a fixed point method on the Duhamel formulation of (??):

$$b^\varepsilon(t, x, v) = b_0(x - tv, v) + \int_0^t \left(1 - \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{b^\varepsilon(t-s, x-sv, v) - b^\varepsilon(t-s, x-sv, v')}{\varepsilon}} dv'\right) ds. \quad (3.2) \quad \boxed{\text{eq:duhamel}}$$

Proof of the bound (i). We define, for $C, \delta > 0$ to be chosen later:

$$\psi_\delta^\varepsilon(t, x, v) = b^\varepsilon(t, x, v) - C\delta t - \delta^4|x|^2 - \delta|v|.$$

For any $\delta > 0$, ψ_δ^ε attains a maximum at point $(t_\delta, x_\delta, v_\delta)$. Suppose that $t_\delta > 0$. Then, we have

$$\partial_t b^\varepsilon(t_\delta, x_\delta, v_\delta) \geq C\delta, \quad \nabla_x b^\varepsilon(t_\delta, x_\delta, v_\delta) = 2\delta^4 x_\delta.$$

As a consequence, we have at the maximum point $(t_\delta, x_\delta, v_\delta)$:

$$\begin{aligned} C\delta + 2v_\delta \delta^4 x_\delta &\leq 1 - \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{\delta|v_\delta| - \delta|v'|}{\varepsilon}} e^{\frac{\psi_\delta^\varepsilon(t_\delta, x_\delta, v_\delta) - \psi_\delta^\varepsilon(t_\delta, x_\delta, v')}{\varepsilon}} dv', \\ &\leq 1 - \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{\delta|v_\delta| - \delta|v'|}{\varepsilon}} dv', \\ &\leq 1 - \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{-\delta|v'|}{\varepsilon}} dv', \end{aligned} \quad (3.3) \quad \boxed{\text{ppmax}}$$

Moreover, the maximal character of $(t_\delta, x_\delta, v_\delta)$ also implies

$$\sup b^\varepsilon - \delta^4|x_\delta|^2 - \delta|v_\delta| \geq b^\varepsilon(t_\delta, x_\delta, v_\delta) - C\delta t_\delta - \delta^4|x_\delta|^2 - \delta|v_\delta| \geq b^\varepsilon(0, 0, 0) \geq \inf b^\varepsilon.$$

It yields that $|x_\delta| \leq \frac{(\sup b^\varepsilon)^{\frac{1}{2}}}{\delta^2}$ and $|v_\delta| \leq \frac{\sup b^\varepsilon}{\delta}$. Introducing these last inequalities in (??) yields

$$C\delta - 2\delta(\sup b^\varepsilon)^{\frac{3}{2}} \leq 1 - \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{-\delta|v'|}{\varepsilon}} dv',$$

and thus

$$C - 2(\sup b^\varepsilon)^{\frac{3}{2}} \leq \frac{1}{\delta} \left(1 - \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{-\delta|v'|}{\varepsilon}} dv'\right).$$

One can choose C such that, for sufficiently small δ , this last inequality is impossible since the r.h.s is $\mathcal{O}(1)$ when $\delta \rightarrow 0$:

$$\frac{1}{\delta} \left(1 - \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{-\delta|v'|}{\varepsilon}} dv'\right) \xrightarrow{\delta \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^n} |v'| M_\varepsilon(v') dv'.$$

As a consequence $t_\delta = 0$, and we have,

$$b^\varepsilon(t, x, v) \leq b^0(x_\delta, v_\delta) + C\delta t + \delta^4|x|^2 + \delta|v| \leq \sup_{\mathbb{R}^{2n}} b_0 + C\delta t + \delta^4|x|^2 + \delta|v|.$$

Passing to the limit $\delta \rightarrow 0$, we obtain $b^\varepsilon(t, x, v) \leq \sup_{\mathbb{R}^{2n}} b_0$. One can carry out the same argument with $-b^\varepsilon$ to get the lower bound.

Proof of the bound (ii). To find Lipschitz bounds, we use the same ideas on the difference $b_h^\varepsilon(t, x, v) = b^\varepsilon(t, x + h, v) - b^\varepsilon(t, x, v)$. The equation for b_h^ε reads as follows,

$$\partial_t b_h^\varepsilon + v \cdot \nabla_x b_h^\varepsilon = \int_{\mathbb{R}^n} M_\varepsilon(v') \exp\left(\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}\right) \left(1 - \exp\left(\frac{b_h^\varepsilon(v) - b_h^\varepsilon(v')}{\varepsilon}\right)\right) dv'. \quad (3.4) \quad \boxed{\text{WKB2}}$$

Using the same argument as above with the correction function $-C\delta t - \delta^4|x|^2 - \delta|v|$, we end up with

$$C - 2(\sup b_h^\varepsilon)^{\frac{3}{2}} \leq \frac{1}{\delta} \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}} \left(1 - e^{\frac{-\delta|v'|}{\varepsilon}}\right) dv'.$$

Using the L^∞ bound on b^ε , we find

$$C - 2(\sup b_h^\varepsilon)^{\frac{3}{2}} \leq \frac{1}{\delta} \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{2\|b^\varepsilon\|_\infty}{\varepsilon}} \left(1 - e^{\frac{-\delta|v'|}{\varepsilon}}\right) dv'.$$

Taking again C suitably, we get that the previous inequality is impossible for small δ . As a consequence,

$$\forall(t, x, v) \in [0, T] \times \mathbb{R}^{2n}, \quad b_h^\varepsilon(t, x, v) \leq \sup_{(x, v) \in \mathbb{R}^{2n}} (b_0(x + h, v) - b_0(x, v))$$

The same argument will apply to $-b_h^\varepsilon$. Indeed, one can write an equation on $-b_h^\varepsilon$ on the following form,

$$\partial_t(-b_h^\varepsilon) + v \cdot \nabla_x(-b_h^\varepsilon) = - \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}} \left(1 - \exp\left(-\frac{(-b_h^\varepsilon)(v) - (-b_h^\varepsilon)(v')}{\varepsilon}\right)\right) dv', \quad (3.5)$$

such the maximum principle works since the r.h.s has the right sign when $-b_h^\varepsilon$ attains a maximum. Finally,

$$\forall(t, x, v) \in [0, T] \times \mathbb{R}^{2n}, \quad |b_h^\varepsilon(t, x, v)| \leq \sup_{(x, v) \in \mathbb{R}^{2n}} |b_0(x + h, v) - b_0(x, v)| \leq \|\nabla_x b_0\|_\infty |h|.$$

from which the estimate follows.

Proof of the bound (iii). Let $R > 16\|b_0\|_\infty$ and $\mathcal{B}_R := B(0, R)$. To obtain the local bound on the time derivative on $\mathbb{R}_+^* \times \mathbb{R}^n \times \mathcal{B}_R$, let us differentiate with respect to time,

$$(\partial_t + v \cdot \nabla_x)(\partial_t b^\varepsilon) = -\frac{1}{\varepsilon} \int_{\mathbb{R}^n} M_\varepsilon(v') (\partial_t b^\varepsilon(v) - \partial_t b^\varepsilon(v')) e^{\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}} dv'.$$

We shall multiply by $\text{sgn}(\partial_t b^\varepsilon)$ and split the r.h.s into two parts :

$$\begin{aligned} (\partial_t + v \cdot \nabla_x)(|\partial_t b^\varepsilon|) &= -\frac{1}{\varepsilon} \int_{\mathcal{B}_R} M_\varepsilon(v') (|\partial_t b^\varepsilon(v)| - \text{sgn}(\partial_t b^\varepsilon(v)) \partial_t b^\varepsilon(v')) e^{\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}} dv' \\ &\quad - \frac{1}{\varepsilon} \int_{\mathbb{R}^n \setminus \mathcal{B}_R} M_\varepsilon(v') (|\partial_t b^\varepsilon(v)| - \text{sgn}(\partial_t b^\varepsilon(v)) \partial_t b^\varepsilon(v')) e^{\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}} dv', \end{aligned}$$

and estimate both parts separately. First, re-using the equation on b^ε and the estimate on the space derivative, we get

$$\begin{aligned} &\left| \frac{1}{\varepsilon} \int_{\mathbb{R}^n \setminus \mathcal{B}_R} M_\varepsilon(v') \text{sgn}(\partial_t b^\varepsilon(v)) \partial_t b^\varepsilon(v') e^{\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}} dv' \right| \\ &\leq \frac{1}{\varepsilon} \int_{\mathbb{R}^n \setminus \mathcal{B}_R} M_\varepsilon(v') \left[|v'| \|\nabla_x b^\varepsilon\|_\infty + 1 + e^{\frac{2\|b^\varepsilon\|_\infty}{\varepsilon}} \right] e^{\frac{2\|b^\varepsilon\|_\infty}{\varepsilon}} dv' := C(\varepsilon), \end{aligned}$$

where $C(\varepsilon)$ is uniformly bounded (and vanishes when $\varepsilon \rightarrow 0$) since the ball \mathcal{B}_R contains the centered ball of radius $16\|b_0\|_\infty$.

Assume now that $|\partial_t b^\varepsilon| - C(\varepsilon)t$ has a positive local maximum at $(t_0, x_0, v_0) \in \mathbb{R}_+^* \times \mathbb{R}^n \times \mathcal{B}_R$. Then at this point

$$C(\varepsilon) < -\frac{1}{\varepsilon} \int_{\mathcal{B}_R} M_\varepsilon(v') (\partial_t b^\varepsilon(v) - \partial_t b^\varepsilon(v')) e^{\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}} dv' + \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^n \setminus \mathcal{B}_R} M_\varepsilon(v') \operatorname{sgn}(\partial_t b^\varepsilon(v)) \partial_t b^\varepsilon(v') e^{\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}} dv' \right| \leq C(\varepsilon).$$

We thus deduce that $t_0 = 0$, and thus

$$\forall (t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathcal{B}_R, \quad |\partial_t b^\varepsilon| \leq C(\varepsilon)t + \sup_{(x, v) \in \mathbb{R}^n \times \mathcal{B}_R} |\partial_t b^\varepsilon|(0, x, v).$$

We now recall (??) to compute $\partial_t b^\varepsilon(0, x, v) = \partial_t u^\varepsilon(0, x, v)$. Indeed, we have

$$\partial_t u^\varepsilon(0, x, v) + v \cdot \nabla_x u^\varepsilon(0, x, v) = 1 - \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}^n} \exp\left(\frac{u_0(x, v) - u_0(x, v') - |v|^2/2}{\varepsilon}\right) dv'.$$

The Laplace method gives that

$$\frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}^n} \exp\left(\frac{u_0(x, v) - u_0(x, v') - |v|^2/2}{\varepsilon}\right) dv'$$

is uniformly bounded in ε , locally in $(x, v) \in \mathbb{R}^n \times \mathcal{B}_R$ as soon as any minimum point in velocity v of u_0 is non degenerate (in the sense that $-\det(\operatorname{Hess}_v(u_0(x, v))) \neq 0$) and that the third order derivatives in velocities are locally uniformly bounded, which is exactly hypothesis [B]. As a consequence, we may bound $\partial_t u^\varepsilon(0, x, v)$ by a uniform constant in ε , locally uniformly in (x, v) .

Proof of the bound (iv). To obtain regularity in the velocity variable, we differentiate (??) with respect to v ,

$$(\partial_t + v \cdot \nabla_x)(\nabla_v b^\varepsilon) = -g_\varepsilon(b^\varepsilon) \nabla_v b^\varepsilon - \nabla_x b^\varepsilon,$$

where $g_\varepsilon(b^\varepsilon) = \frac{1}{\varepsilon} \int_{\mathbb{R}^n} M_\varepsilon(v') e^{\frac{b^\varepsilon - b^{\varepsilon'}}{\varepsilon}} dv' \geq 0$. Multiplying by $\frac{\nabla_v b^\varepsilon}{|\nabla_v b^\varepsilon|}$, we obtain

$$\begin{aligned} (\partial_t + v \cdot \nabla_x)(|\nabla_v b^\varepsilon|) &= -g_\varepsilon(b^\varepsilon) |\nabla_v b^\varepsilon| - \left(\nabla_x b^\varepsilon \cdot \frac{\nabla_v b^\varepsilon}{|\nabla_v b^\varepsilon|} \right) \\ &\leq \|\nabla_x b_0\|_\infty. \end{aligned}$$

from which we deduce

$$\forall t > 0, \quad \|\nabla_v b^\varepsilon(t, \cdot)\|_\infty \leq \|\nabla_v b_0\|_\infty + t \|\nabla_x b_0\|_\infty.$$

□

Proof of Theorem ??. Thanks to Proposition ??, the sequence u^ε converges locally uniformly towards u^0 . We first start with a lemma that shows how the supplementary equation giving the evolution of the minimum in velocity in the system appears in the limit $\varepsilon \rightarrow 0$.

Lemma 3.2. *Let $I \subset \mathbb{R}^n$. One has, locally uniformly in $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,*

$$\lim_{\varepsilon \rightarrow 0} \left(-\varepsilon \log \left(\int_I e^{-\frac{u^\varepsilon(t, x, v')}{\varepsilon}} dv' \right) \right) = \min_{w \in I} u^0(t, x, w).$$

Proof of Lemma ??. We shall find the limit of

$$-\varepsilon \log \left(\int_I e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv' \right) = \min_{w \in \mathbb{R}^n} u^\varepsilon - \varepsilon \log \left(\int_I e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' \right). \quad (3.6) \quad \boxed{\text{eq:estimate}}$$

Thanks to the local uniform convergence of u^ε , we know that $\min_{w \in \mathbb{R}^n} u^\varepsilon$ converges locally uniformly to $\min_{w \in \mathbb{R}^n} u$. We shall prove that the remaining part of the r.h.s converges to 0. For this purpose, we will prove the following estimate:

$$\forall \delta > 0, \exists \delta' > 0, \forall K > 0, \quad -\varepsilon \log \left(\omega_n K^n + \sqrt{2\pi\varepsilon} \right) \leq -\varepsilon \log \left(\int_I e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' \right) \leq \delta - \varepsilon \log(\delta'),$$

for some δ, δ', K suitably chosen. We start with the l.h.s.. For $\delta > 0$ (possibly depending on ε at the end), we write

$$\int_I e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' = \int_{I_\delta} e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' + \int_{I \setminus I_\delta} e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' \quad (3.7) \quad \boxed{\text{eq:intI}}$$

where $I_\delta := \{v \in I \mid \min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v) > -\delta\}$. Thanks to the Lipschitz bound on the derivatives in v , I_δ has non empty interior. Indeed, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, for all $v^\varepsilon \in \mathcal{S}(u^\varepsilon)(t, x)$, one has

$$\left| \min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v) \right| \leq \left| b^\varepsilon(v^\varepsilon) - b^\varepsilon(v) + \frac{|v^\varepsilon|^2}{2} - \frac{|v|^2}{2} \right| \leq \left(\|\nabla_v b^\varepsilon\|_\infty + \frac{1}{2}|v - v^\varepsilon| + |v^\varepsilon| \right) |v^\varepsilon - v|.$$

Next, as $v^\varepsilon \in \mathcal{S}(u^\varepsilon)(t, x)$, one has $\nabla_v u^\varepsilon(t, x, v^\varepsilon) = 0$, and thus $\nabla_v b^\varepsilon(t, x, v^\varepsilon) = -v^\varepsilon$. Hence, $|v^\varepsilon| \leq \|\nabla_v b^\varepsilon\|_\infty$, we deduce

$$\forall (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2n}, \forall v^\varepsilon \in \mathcal{S}(u^\varepsilon)(t, x), \quad \left| \min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v) \right| \leq \left(\|\nabla_v b^\varepsilon\|_\infty + \frac{1}{2}|v - v^\varepsilon| \right) |v^\varepsilon - v|.$$

Take $\delta' > 0$ such that $(\|\nabla_v b^\varepsilon\|_\infty + \frac{1}{2}\delta')\delta' < \delta$. The previous inequality gives that

$$\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \forall v^\varepsilon \in \mathcal{S}(u^\varepsilon)(t, x), \quad B(v^\varepsilon, \delta') \subset I_\delta.$$

Coming back to (??), we get

$$\int_I e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' > \int_{I_\delta} e^{-\frac{\delta}{\varepsilon}} dv' > |I_\delta| e^{-\frac{\delta}{\varepsilon}} > \delta' e^{-\frac{\delta}{\varepsilon}}, \quad (3.8) \quad \boxed{\text{eq:boundI1}}$$

which gives the first bound that we want to prove. Let us come to the second part, namely the r.h.s. of (??). For all $v \in V$, one has

$$\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v) = \left(b^\varepsilon(v^\varepsilon) - b^\varepsilon(v) + \frac{|v^\varepsilon|^2}{2} \right) - \frac{|v|^2}{2} \leq 2\|b^\varepsilon\|_\infty + \frac{1}{2}\|\nabla_v b^\varepsilon\|_\infty^2 - \frac{|v|^2}{2}.$$

We write $\frac{K^2}{2} := \frac{1}{2}\|\nabla_v b^\varepsilon\|_\infty^2 + 2\|b^\varepsilon\|_\infty$ for legibility. We deduce

$$\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v) \leq \min \left(0, \frac{K^2}{2} - \frac{|v|^2}{2} \right). \quad (3.9) \quad \boxed{\text{eq:K/2}}$$

As a consequence,

$$\begin{aligned} \int_I e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v)}{\varepsilon}} dv &= \int_{|v| \leq K} 1 dv + \int_{|v| \geq K} e^{\frac{K^2 - |v|^2}{2\varepsilon}} dv \\ &< \omega_n K^n + \sqrt{2\pi\varepsilon} e^{\frac{K^2}{2\varepsilon}} \left(1 - \sqrt{1 - e^{-\frac{K^2}{2\varepsilon}}} \right) < \omega_n K^n + \sqrt{2\pi\varepsilon}, \end{aligned}$$

(where ω_n denotes the volume of the unit ball in \mathbb{R}^n). Gathering the inequalities, we find

$$-\varepsilon \log \left(\omega_n K^n + \sqrt{2\pi\varepsilon} \right) \leq -\varepsilon \log \left(\int_I e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' \right) \leq -\varepsilon \log \left(\delta' e^{-\frac{\delta}{\varepsilon}} \right) = \delta - \varepsilon \log(\delta'),$$

which gives the conclusion. \square

We shall start by the following lemma:

Lemma 3.3. *For $\varepsilon > 0$, $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, consider the probability measure on \mathbb{R}^n :*

$$d\mu_{t_0, x_0}^\varepsilon(v) := \frac{e^{-\frac{u^\varepsilon(v)}{\varepsilon}} dv'}{\int_{\mathbb{R}^n} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv'}.$$

Assume that $\mathcal{S}(u^0)(t_0, x_0) = \{0\}$. Then for any $\delta > 0$, one has

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B(0, \delta)} |v| d\mu_{t_0, x_0}^\varepsilon(v) = 0.$$

Proof of Lemma ??. Let us rewrite

$$d\mu_{t, x}^\varepsilon(v) := \frac{e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv'}{\int_{\mathbb{R}^n} e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv'}$$

One has:

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B(0, \delta)} |v'| e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' &= \int_{\mathbb{R}^n \setminus B(0, K)} |v'| e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' \\ &\quad + \int_{B(0, K) \setminus B(0, \delta)} |v'| e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv', \end{aligned}$$

where K comes from (??). We estimate again

$$\int_{\mathbb{R}^n \setminus B(0, K)} |v'| e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' \leq \int_{\mathbb{R}^n \setminus B(0, K)} |v'| e^{\frac{K^2 - |v|^2}{2\varepsilon}} dv' = \mathcal{O}\left(\varepsilon^{1 + \frac{n}{2}}\right).$$

The estimation of the remaining part will use the condition on the minimum points. Since u^0 only achieves its minimum at $v = 0$, since u^ε converges locally uniformly towards u^0 and the minimas of u^ε are uniformly localized, we have necessarily some $\gamma > 0$ such that

$$\forall v \in B(0, K) \setminus B(0, \delta), \quad \min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon \leq -\gamma.$$

Thus,

$$\int_{B(0, K) \setminus B(0, \delta)} |v'| e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v')}{\varepsilon}} dv' \leq \mathcal{O}\left(e^{-\frac{\gamma}{\varepsilon}}\right).$$

Gathering everything, the result follows. \square

To prove the Theorem ??, we need to go through the two following steps.

Step 1 : Viscosity sub-solution.

Let $T > 0$. One wants to prove that u^0 is a viscosity sub-solution of (??) on $[0, T]$, in the sense of Definition ??. Let $(\phi, \psi) \in \mathcal{C}^2([0, T] \times \mathbb{R}^{2n}) \times \mathcal{C}^2([0, T] \times \mathbb{R}^n)$, such that $u^0 - \phi$ and $(\min_{v \in \mathbb{R}^n} u^0) - \psi$ have a local maximum in the (t, x) variables at the point $(t_0, x_0, v_0) \in \mathbb{R}_+ \times \mathbb{R}^{2n}$, with $t_0 > 0$. We shall prove that the following conditions are fulfilled at (t_0, x_0, v_0) :

$$\boxed{\text{(i)}} \quad \text{(i)} \quad \partial_t \phi + v \cdot \nabla_x \phi - 1 \leq 0,$$

$$\boxed{\text{(ii)}} \quad \text{(ii)} \quad u^0 - \min_{v \in \mathbb{R}^n} u^0 - \frac{|v|^2}{2} \leq 0,$$

$$\boxed{\text{(iii)}} \quad \text{(iii)} \quad \partial_t \psi \leq 0.$$

Inequality (??) comes from the fact that one can pass to the limit in the viscosity sense in the following inequality

$$\forall \varepsilon > 0, \quad \forall (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^{2n}, \quad \partial_t u^\varepsilon + v \cdot \nabla_x u^\varepsilon - 1 \leq 0.$$

To prove the constraint (??), assume by contradiction that

$$\left(u^0 - \min_{w \in \mathbb{R}^n} u^0 - \frac{|v|^2}{2} \right) (t_0, x_0, v_0) = \delta > 0.$$

Thanks to the local uniform convergence of u^ε , for ε sufficiently small,

$$\left(u^\varepsilon - \min_{w \in \mathbb{R}^n} u^\varepsilon - \frac{|v|^2}{2} \right) (t_0, x_0, v_0) > \frac{\delta}{2}.$$

Since $\nabla_v u^\varepsilon$ is locally bounded uniformly in ε , there exists a uniform interval I centered at a minimal point such that

$$\forall v' \in I, \quad u^\varepsilon(t_0, x_0, v_0) - u^\varepsilon(t_0, x_0, v') - \frac{|v_0|^2}{2} > \frac{\delta}{4}.$$

Forgetting the dependence in (t, x) in order to shorten the notations, we deduce that at the point (t_0, x_0, v_0) ,

$$\int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(\frac{u^\varepsilon(v_0) - u^\varepsilon(v') - \frac{|v_0|^2}{2}}{\varepsilon}\right) dv' \geq \frac{|I|}{\sqrt{2\pi\varepsilon}} \exp\left(\frac{\delta}{4\varepsilon}\right) \xrightarrow{\varepsilon \rightarrow 0} +\infty.$$

which contradicts the estimates of Proposition ?? since $(\partial_t u^\varepsilon + v_0 \cdot \nabla_x u^\varepsilon)(t_0, x_0, v_0)$ is bounded with respect to ε .

It remains to prove inequality (??). This is the main originality in this procedure. Let $\delta > 0$ be any positive number. Let us fix $(t_0, x_0) \in \mathbb{R}_+^* \times \mathbb{R}^n$.

For all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, let us integrate (??) over $B(0, \delta)$ to find:

$$\begin{aligned} \partial_t \left(-\varepsilon \int_{B(0, \delta)} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv' \right) + \int_{B(0, \delta)} (v \cdot \nabla_x u^\varepsilon) e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv' \\ = \int_{B(0, \delta)} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv' - \int_{B(0, \delta)} M_\varepsilon(v') dv' \int_{\mathbb{R}^n} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv' \end{aligned}$$

One can write, using $\int_{\mathbb{R}^n} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv' \geq \int_{B(0, \delta)} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv'$:

$$\partial_t \left(-\varepsilon \int_{B(0, \delta)} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv' \right) + \int_{B(0, \delta)} (v \cdot \nabla_x u^\varepsilon) e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv' \leq \left(1 - \int_{B(0, \delta)} M_\varepsilon(v') dv' \right) \int_{B(0, \delta)} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv'.$$

As a consequence, introducing the probability measure $d\mu_{t,x}^\varepsilon$, we obtain:

$$\partial_t \left(-\varepsilon \log \left(\int_{B(0, \delta)} e^{-\frac{u^\varepsilon(v)}{\varepsilon}} dv \right) \right) + \int_{B(0, \delta)} (v \cdot \nabla_x u^\varepsilon) d\mu_{t,x}^\varepsilon(v) \leq \int_{\mathbb{R}^n \setminus B(0, \delta)} M_\varepsilon(v) dv. \quad (3.10)$$

We conclude by investigating the following integral

$$\left| \int_{B(0,\delta)} (v \cdot \nabla_x u^\varepsilon) d\mu_{t,x}^\varepsilon(v) \right| \leq \int_{B(0,\delta)} |v| \|\nabla_x u^\varepsilon\|_\infty d\mu_{t,x}^\varepsilon(v) \leq \delta \|\nabla_x u^\varepsilon\|_\infty \int_{B(0,\delta)} d\mu_{t,x}^\varepsilon(v) \leq \delta \|\nabla_x u^\varepsilon\|_\infty \quad (3.11)$$

We can now pass to the limit in the viscosity sense with Lemma ??, to conclude that there exists $C > 0$ such that for all $\delta > 0$,

$$\partial_t \left(\min_{w \in B(0,\delta)} u^0(\cdot, w) \right) \leq C\delta$$

We now recall the crucial fact that for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, any sub-solution achieves its minimum in velocity at $v = 0$, since $u - \min_{w \in \mathbb{R}^n} u - |v|^2/2 \leq 0$. As a consequence, for any $\delta > 0$, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\min_{w \in B(0,\delta)} u^0(t, x, w) = \min_{w \in \mathbb{R}^n} u^0(t, x, w).$$

Finally, we can pass to the limit $\delta \rightarrow 0$, and get

$$\partial_t \left(\min_{w \in \mathbb{R}^n} u^0(t, x, w) \right) \leq 0, \quad (3.12) \quad \boxed{\text{eq:m decrea}}$$

in the viscosity sense.

Step 2 : Viscosity super-solution.

We now need to prove that u^0 is a super solution of (??) in the viscosity sense

$$\begin{cases} \partial_t \phi + v \cdot \nabla_x \phi - 1 \geq 0, & \text{if } u^0 - \min_{w \in \mathbb{R}^n} u^0 - |v|^2/2 < 0, \\ \partial_t \psi(t_0, x_0) \geq 0, & \text{if } \mathcal{S}(u^0)(t_0, x_0) = \{0\}, \end{cases} \quad (3.13) \quad \boxed{\text{eq:suptop}}$$

To prove the first line of (??), let us take $(t_0, x_0, v_0) \in \mathbb{R}_+^* \times \mathbb{R}^{2n}$ such that $-\delta := u^0(t_0, x_0, v_0) - \min_{w \in \mathbb{R}^n} u^0(t_0, x_0) - |v_0|^2/2 < 0$. Thus, thanks to the locally uniform convergence and the confinement of the minima of u^ε , for ε sufficiently small, one has $u^\varepsilon(t_0, x_0, v_0) - \min_{w \in \mathbb{R}^n} u^\varepsilon(t_0, x_0) - |v_0|^2/2 < -\delta/2$. Thus, recalling $\min_{w \in \mathbb{R}^n} u^\varepsilon - u^\varepsilon(v) \leq \min(0, K^2/2 - |v|^2/2)$, we have

$$\begin{aligned} (2\pi\varepsilon)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{u^\varepsilon(t_0, x_0, v_0) - u^\varepsilon(t_0, x_0, v') - |v_0|^2/2}{\varepsilon}} dv' &< (2\pi\varepsilon)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{-u^\varepsilon(t_0, x_0, v') - \delta/2 + \min_{w \in \mathbb{R}^n} u^\varepsilon(t_0, x_0, w)}{\varepsilon}} dv' \\ &< e^{-\frac{\delta}{2\varepsilon}} (2\pi\varepsilon)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{\min_{w \in \mathbb{R}^n} u^\varepsilon(t_0, x_0, w) - u^\varepsilon(t_0, x_0, v')}{\varepsilon}} dv' \rightarrow 0. \end{aligned}$$

Next, assume that $\mathcal{S}(\bar{u})(t, x) = \{0\}$, that is, 0 is the only minimum of $v \mapsto u(t, x, v)$. For all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, let us integrate (??) over \mathbb{R}^n to find:

$$\partial_t \left(-\varepsilon \log \left(\int_{\mathbb{R}^n} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv' \right) \right) = - \int_{\mathbb{R}^n} (v' \cdot \nabla_x u^\varepsilon) d\mu_{t,x}^\varepsilon(v'), \quad (3.14) \quad \boxed{\text{eq:supersol}}$$

For any fixed $\delta > 0$, we now split the last r.h.s into two parts:

$$\int_{\mathbb{R}^n} (v' \cdot \nabla_x u^\varepsilon) d\mu_{t,x}^\varepsilon(v') = \int_{B(0,\delta)} (v' \cdot \nabla_x u^\varepsilon) d\mu_{t,x}^\varepsilon(v') + \int_{\mathbb{R}^n \setminus B(0,\delta)} (v' \cdot \nabla_x u^\varepsilon) d\mu_{t,x}^\varepsilon(v').$$

We estimate the first part exactly as in the sub-solution case:

$$\left| \int_{B(0,\delta)} (v' \cdot \nabla_x u^\varepsilon) d\mu_{t,x}^\varepsilon(v') \right| \leq \delta \|\nabla_x u^\varepsilon\|_\infty. \quad (3.15)$$

To estimate the second part, we use Lemma ??:

$$\int_{\mathbb{R}^n \setminus B(0, \delta)} (v' \cdot \nabla_x u^\varepsilon) d\mu_{t_0, x_0}^\varepsilon(v') \leq \|\nabla_x u^\varepsilon\|_\infty \int_{\mathbb{R}^n \setminus B(0, \delta)} |v'| d\mu_{t_0, x_0}^\varepsilon(v') = o_{\varepsilon \rightarrow 0}(1). \quad (3.16)$$

As a consequence, passing to the limit in ε and then in δ , and recalling again Lemma ??, we get

$$\partial_t \left(\min_{w \in \mathbb{R}^n} u^0(t_0, x_0, w) \right) = 0,$$

in the viscosity sense.

It remains to check that $v = 0$ is an isolated minimum for $t > 0$. This is the purpose of the final Lemma,

em:isolated

Lemma 3.4. *For all $t > 0$ and $x \in \mathbb{R}^d$, $v = 0$ is an isolated minimum point of the function $u(t, x, \cdot)$.*

Proof of Lemma ??. We introduce the minimal value function $m(t, x) = \min_w u(t, x, w)$. We shall prove that for all (t, x) such that $t > 0$, there exists a neighbourhood $\mathcal{V}(t, x)$ of 0, such that

$$(\forall v \in \mathcal{V}(t, x)) \quad u(t, x, v) = m(t, x) + \frac{|v|^2}{2}. \quad (3.17)$$

eq:saturated

This would immediately imply that 0 is an isolated minimum point.

Passing to the limit $\varepsilon \rightarrow 0$ in the Duhamel formula,

$$f^\varepsilon(t, x, v) = f_0^\varepsilon(x - tv, v)e^{-t/\varepsilon} + M_\varepsilon(v) \int_0^t \rho^\varepsilon(t - s, x - sv)e^{-s/\varepsilon} ds,$$

we deduce that the functions u , and m satisfy the following identity,

$$u(t, x, v) = \min \left(u_0(x - tv, v) + t, \min_{0 \leq s \leq t} (m(t - s, x - sv) + s) + \frac{|v|^2}{2} \right). \quad (3.18)$$

eq:duhamel

The claim (??) is equivalent to proving the two following estimates:

$$(\forall v \in \mathcal{V}(t, x)) \quad \begin{cases} u_0(x - tv, v) + t \geq m(t, x) + \frac{|v|^2}{2} & (\star), \\ (\forall s \in [0, t]) \quad m(t - s, x - sv) + s + \frac{|v|^2}{2} \geq m(t, x) + \frac{|v|^2}{2} & (\star\star) \end{cases} \quad (3.19)$$

The proof of $(\star\star)$ goes as follows. First recall that m is time decreasing (??). Thus, it is sufficient to establish that $m(t, x - sv) + s \geq m(t, x)$ for all s and small v . This holds for $v = 0$. On the other hand, the uniform spatial Lipschitz estimate in Proposition ??(ii) guarantees that the minimal value m is Lipschitz continuous with respect to x , with the same bound, *i.e.* $Lip_x m \leq \|\nabla_x u_0\|_\infty$. We deduce that

$$m(t, x - sv) - m(t, x) + s \geq (1 - \|\nabla_x u_0\|_\infty |v|) s.$$

As a conclusion, $(\star\star)$ holds true for $|v| \leq (\|\nabla_x u_0\|_\infty)^{-1}$.

The proof of (\star) is similar. Firstly, we have

$$\begin{aligned} u_0(x - tv, v) + t - m(t, x) - \frac{|v|^2}{2} &\geq m(0, x - tv) + t - m(0, x) - \frac{|v|^2}{2} \\ &\geq t - \|\nabla_x u_0\|_\infty |v| t - \frac{|v|^2}{2}. \end{aligned}$$

The latter is nonnegative, provided that

$$|v| \leq \|\nabla_x u_0\|_\infty t \left(-1 + \left(1 + \frac{2}{\|\nabla_x u_0\|_\infty^2 t} \right)^{1/2} \right) \leq \frac{1}{\|\nabla_x u_0\|_\infty}.$$

The latter condition suitably defines the neighbourhood $\mathcal{V}(t, x)$. □

□

4 Derivation of the fundamental solution of the limit system

sec:fund

Our aim in this Section is to compute some particular solutions of the Cauchy problem (??). For this purpose we first present a recursive numerical scheme which is based on the representation of the solution as an iterated semi-group. We propose a time-discretization of this formulation which enables to compute the solution. We justify a posteriori that the solution that we exhibit via the discretization procedure is the solution of the Cauchy problem.

4.1 Interlude: The spatially homogeneous case as a toy problem

sec:homog

Let us focus for a little while on the spatially homogeneous problem:

$$\begin{cases} \partial_t f = M(v)\rho(t) - f, \\ \rho(t) = \int_{\mathbb{R}^n} f(t, v) dv. \end{cases}$$

Its solution is given by

$$f(t, v) = \int_{w \in \mathbb{R}^n} K(t, v, w) f^0(w) dw,$$

where $K(t, v, w)$ is the fundamental solution of the system, starting from the measure $f^0(v) = \delta_{v=w}$. Since in this case $\rho \equiv 1$, one can compute this fundamental solution straightforwardly:

$$K(t, v, w) = e^{-t} \delta_{v=w} + (1 - e^{-t}) M(v).$$

After performing the Hopf-Cole transform $\psi^\varepsilon = -\varepsilon \log K^\varepsilon$ in the scaling $\tilde{t} = \varepsilon^{-1}t, \tilde{v} = \varepsilon^{-1/2}v$, we obtain with the fundamental solution in the HJ framework,

$$\psi^0(t, v, w) = \min \left(t + \mathbf{0}_{v=w}; \frac{|v|^2}{2} \right),$$

where $\mathbf{0}_{v=w}(v) := +\infty$ if $v \neq w$, 0 if $v = w$. We thus have the following representation for the associated Cauchy problem:

$$\phi(t, v) = \min_{w \in \mathbb{R}^n} (\psi^0(t, v, w) + \phi_0(w)) = \min \left(t + \phi^0(v); \frac{|v|^2}{2} + \min_w \phi^0(w) \right)$$

4.2 The discrete time numerical scheme.

We now come back to equation (??). The Duhamel formula for a positive $\varepsilon > 0$ writes

$$f^\varepsilon(t, x, v) = f_0^\varepsilon(x - tv, v) e^{-t/\varepsilon} + M_\varepsilon(v) \int_0^t \rho^\varepsilon(t - s, x - sv) e^{-s/\varepsilon} ds. \quad (4.1)$$

eq:Duhamel

We deduce that the functions $\phi_\varepsilon = -\varepsilon \log f^\varepsilon$, $\mu_\varepsilon = -\varepsilon \log \rho^\varepsilon$ satisfy formally in the limit $\varepsilon \rightarrow 0$,

$$\begin{cases} \phi(t, x, v) = \min \left(\phi^0(x - tv, v) + t, \frac{|v|^2}{2} + \min_{0 \leq s \leq t} (\mu(t - s, x - sv) + s) \right) \\ \mu(t, x) = \min_{v \in \mathbb{R}^n} \phi(t, x, v). \end{cases} \quad (4.2) \quad \text{eq:phi recur}$$

The minimal value function μ satisfies a closed equation:

$$\mu(t, x) = \min \left(\min_{v \in \mathbb{R}^n} (\phi^0(x - tv, v) + t), \min_{0 \leq s \leq t} \left(\min_{v \in \mathbb{R}^n} \left(\frac{|v|^2}{2} + \mu(t - s, x - sv) \right) + s \right) \right) \quad (4.3) \quad \text{eq:mu}$$

The time discretization of this implicit semi-group formulation is not trivial. We propose to use an explicit method for the last recursive contribution in (??). We approximate the function μ by μ_n at time $t_n = n\Delta t$, where $\Delta t > 0$ is a given time step. Our recursive formula for $(\mu_n)_{n \geq 0}$ writes

$$\mu_{n+1}(x) = \min \left(\min_{v \in \mathbb{R}^n} (\phi^0(x - t_{n+1}v, v)) + t_{n+1}, \min_{0 \leq k \leq n} \left(\min_{v \in \mathbb{R}^n} \left(\frac{|v|^2}{2} + \mu_{n-k}(x - t_k v) \right) + t_k \right) \right) \quad (4.4) \quad \text{eq:mu_n}$$

4.3 The fundamental solution starting from $(x, v) = (0, 0)$.

We consider the initial data $\phi^0(x, v) = \mathbf{0}_{x=0} + \frac{|v|^2}{2}$. This corresponds in fact to the initial condition $\mathbf{0}_{x=0} + \mathbf{0}_{v=0}$ which has been instantaneously projected in order to satisfy the constraint $\phi^0 \leq \min_{v \in \mathbb{R}^n} \phi^0 + \frac{|v|^2}{2}$, see the spatially homogeneous case of Section ?? for a discussion.

em:minvalue **Lemma 4.1.** *The minimum value satisfies*

$$\mu_n(x) = \min_{0 \leq k \leq n} \left(\min_{v \in \mathbb{R}^n} (\phi^0(x - t_k v, v)) + t_k \right). \quad (4.5) \quad \text{eq:formula}$$

Proof of Lemma ??. The formula (??) is clearly true for $n = 0$. To proceed with the recursion at range $n + 1$, it is sufficient to establish the following identity:

$$\min_{0 \leq k \leq n} \left(\min_{v \in \mathbb{R}^n} \left(\frac{|v|^2}{2} + \mu_{n-k}(x - t_k v) \right) + t_k \right) = \min_{0 \leq k \leq n} \left(\min_{v \in \mathbb{R}^n} (\phi^0(x - t_k v, v)) + t_n \right). \quad (4.6) \quad \text{eq:recursion}$$

By the induction hypothesis (??), we have

$$\begin{aligned} & \min_{0 \leq k \leq n} \left(\min_{v \in \mathbb{R}^n} \left(\frac{|v|^2}{2} + \mu_{n-k}(x - t_k v) \right) + t_k \right) \\ &= \min_{0 \leq k \leq n} \left(\min_{v \in \mathbb{R}^n} \left(\frac{|v|^2}{2} + \min_{0 \leq j \leq n-k} \left(\min_w (\phi^0(x - t_k v - t_j w, w)) + t_j \right) \right) + t_k \right) \\ &= \min_{0 \leq j+k \leq n} \left(\min_{(v,w)} \left(\frac{|v|^2}{2} + \phi^0(x - t_k v - t_j w, w) \right) + t_j + t_k \right). \end{aligned}$$

The key convexity inequality is simply the following,

$$\frac{1}{2} \left| \frac{t_k v + t_j w}{t_{k+j}} \right|^2 \leq \left(\frac{t_k}{t_{k+j}} \right) \frac{|v|^2}{2} + \left(\frac{t_j}{t_{k+j}} \right) \frac{|w|^2}{2} \leq \frac{|v|^2}{2} + \frac{|w|^2}{2}.$$

The second inequality is an equality if and only if $j = 0$ and $w = 0$, or $k = 0$ and $v = 0$. We deduce that

$$\begin{aligned} \min_{(k,j)} \left(\min_{(v,w)} \left(\frac{|v|^2}{2} + \phi^0(x - t_k v - t_j w, w) \right) + t_j + t_k \right) \\ = \min_{(k,j)} \min_{(v,w)} \left(\phi^0 \left(x - t_{k+j} \left(\frac{t_k v + t_j w}{t_{k+j}} \right), \frac{t_k v + t_j w}{t_{k+j}} \right) + t_{k+j} \right) \\ = \min_{0 \leq k' \leq n} \min_{v'} (\phi^0(x - t_{k'} v', v') + t_{k'}) \end{aligned}$$

□

We deduce from (??) the more explicit formula:

$$\mu_n(x) = \min_{0 \leq k \leq n} \left(\frac{|x|^2}{2t_k^2} + t_k \right), \quad (4.7)$$

eq:mu_n exp

with the convention that $\frac{|x|^2}{2t_k^2} = \mathbf{0}_{x=0}$ if $k = 0$. Therefore, the solution is an envelope of parabolas. For the sake of convenience, we prove directly that (??) is the solution of the recursive identity (??). It is sufficient to compute the minimum value for (k, j) such that $0 \leq j \leq n - k$,

$$\min_{v \in \mathbb{R}^n} \left(\frac{|v|^2}{2} + \frac{|x - t_k v|^2}{2t_j^2} + t_j + t_k \right) = \frac{|x|^2}{2(t_j^2 + t_k^2)} + t_j + t_k \geq \frac{|x|^2}{2(t_{j+k})^2} + t_{j+k}, \quad (4.8)$$

with equality if and only if $j = 0$ or $k = 0$.

4.4 The fundamental solution starting from $(x, v) = (0, w)$.

Let $w \in \mathbb{R}$. We consider the initial data $\phi^0(x, v) = \mathbf{0}_{x=0} + \min \left(\mathbf{0}_{v=w}, \frac{|v|^2}{2} \right)$. This corresponds in fact to the initial condition $\mathbf{0}_{x=0} + \mathbf{0}_{v=w}$ which has been instantaneously projected in order to satisfy the constraint $\phi^0 \leq \min_{v \in \mathbb{R}^n} \phi^0 + \frac{|v|^2}{2}$, as in the previous case. Let us compute first the primary evolution of the initial data, which is essential for the comprehension of the dynamics. It amounts to compute the following minimum value:

$$\min_{v \in \mathbb{R}^n} (\phi^0(x - t_n v, v)) + t_n = \min \left(\mathbf{0}_{x=t_n w} + t_n, \frac{|x|^2}{2t_n^2} + t_n \right).$$

The first contribution is new in comparison with the case $w = 0$. This corresponds to the deposition of mass at point $x = t_n w$, with an initial level of t_n . It is expected that this mass will create secondary parabolas, as the ones which are already located around $x = 0$. Therefore, the solution should consist of the envelope of parabolas centered at $x = 0$, and secondary parabolas centered at $x = t_n w$, having some delay with respect to the central parabolas. We infer from this reasoning the following Lemma. Let us introduce the set of indices $A_n \subset \mathbb{N}^2$ defined by

$$A_n = \{(i, k) : 0 \leq i + k \leq n - 1\} \cup \{(n, 0)\} \cup \{(0, n)\}$$

Lemma 4.2. *The solution of the recursive identity (??) starting from the initial condition $\phi^0(x, v) = \mathbf{0}_{x=0} + \min \left(\mathbf{0}_{v=w}, \frac{|v|^2}{2} \right)$ is given by the formula*

$$\mu_n(x) = \min_{(i,k) \in A_n} \left(\frac{|x - t_i w|^2}{2t_k^2} + t_{i+k} \right), \quad (4.9)$$

eq:mu_n w

with the convention that $\frac{|x - t_i w|^2}{2t_k^2} = \mathbf{0}_{x=t_i w}$ if $k = 0$.

Proof of Lemma ??. The formula (??) coincides with the initial condition for $n = 0$, namely $\mu_0(x) = \mathbf{0}_{x=0}$. We inject (??) into the recursive formula (??). The first contribution gives:

$$\begin{aligned} \min_{v \in \mathbb{R}^n} (\phi^0(x - t_{n+1}v, v)) + t_{n+1} &= \min \left(\mathbf{0}_{x=t_{n+1}w} + t_{n+1}, \frac{|x|^2}{2t_{n+1}^2} + t_{n+1} \right) \\ &= \min \left(\frac{|x - t_{n+1}w|^2}{2t_0^2} + t_{n+1}, \frac{|x|^2}{2t_{n+1}^2} + t_{n+1} \right). \end{aligned}$$

This corresponds to the two extreme cases in the formula (??), namely $(i, k) = (n + 1, 0)$ and $(i, k) = (0, n + 1)$. On the other hand, the second contribution when (??) is plugged into (??) is the following,

$$\begin{aligned} \min_{0 \leq k \leq n} \min_{v \in \mathbb{R}^n} \left(\frac{|v|^2}{2} + \min_{(i,j) \in A_{n-k}} \left(\frac{|x - t_k v - t_i w|^2}{2t_j^2} + t_{i+j} \right) + t_k \right) \\ = \min_{0 \leq k \leq n} \left(\min_{(i,j) \in A_{n-k}} \frac{|x - t_i w|^2}{2(t_j^2 + t_k^2)} + t_{i+j+k} \right) \\ \geq \min_{0 \leq k \leq n} \left(\min_{(i,j) \in A_{n-k}} \frac{|x - t_i w|^2}{2(t_{j+k})^2} + t_{i+j+k} \right), \end{aligned}$$

with equality if and only if $j = 0$ or $k = 0$. It remains to verify that the rule of induction on the sets $A_n \rightarrow A_{n+1}$ is compatible with this recursion. We have already seen that the two extreme points $(n + 1, 0)$ and $(0, n + 1)$ are taken into account. For $k = 0$ in the preceding equality, we obtain the set A_n . Then, for all $0 \leq k \leq n$, we obtain (i, k) from the choice $j = 0$, where $(i, 0) \in A_{n-k}$, *i.e.* $i \leq n - k$. This gives the additional set of indices (i, k) such that $i + k \leq n$. This completes the set A_{n+1} . \square

4.5 The continuous time fundamental solution.

We infer from the preceding recursive numerical scheme that the fundamental solution should write in the continuous setting as follows,

$$\mu(t, x; w) = \min_{\substack{s_1, s_2 \geq 0 \\ 0 \leq s_1 + s_2 \leq t}} \left(\frac{|x - s_1 w|^2}{2s_2^2} + s_1 + s_2 \right) \quad (4.10) \quad \boxed{\text{eq:mu total}}$$

For the sake of clarity, we notify the dependency of μ with respect to the parameter w (initial concentration in the velocity space). It is the envelope of a two-parameters family of parabolas. To conclude, we plug the expression (??) into the formula for the full fundamental solution ϕ (??),

$$\begin{aligned} \phi(t, x, v; w) \\ = \min \left(\mathbf{0}_{x=tv} + \min \left(\mathbf{0}_{v=w}, \frac{|v|^2}{2} \right) + t, \frac{|v|^2}{2} + \min_{\substack{s_1, s_2, s_3 \geq 0 \\ 0 \leq s_1 + s_2 + s_3 \leq t}} \left(\frac{|x - s_1 w - s_3 v|^2}{2s_2^2} + s_1 + s_2 + s_3 \right) \right) \end{aligned} \quad (4.11) \quad \boxed{\text{eq:kernel}}$$

We deduce from the knowledge of the fundamental solution, and the comparison principle Theorem ?? a useful representation formula.

Theorem 4.3. Let $u^0 : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a bounded continuous function. Then, the viscosity solution of (??) is given by the following formula:

$$u(t, x, v) = \inf_{(y, w)} (\phi(t, x - y, v; w) + u^0(y, w)) . \quad (4.12) \quad \text{eq:hopflax}$$

Interestingly enough, by exchanging the minima, the minimum value has a similar representation:

$$\min_{v \in \mathbb{R}^n} u(t, x, v) = \min_{(y, w)} (\mu(t, x - y; w) + u^0(y, w)) . \quad (4.13) \quad \text{eq:hopflax}$$

Notice that (??) is not written as an inf-convolution with respect to velocity. This reflects the lack of invariance of the problem with respect to this variable.

5 Explicit computations around the fundamental solution in dimension 1 in space and velocity.

ec:explicit

In this section, we now stick to dimension 1 in space and velocity. This is to make explicit computations without having a too big computational cost, which appears to increase when the dimension is greater. Moreover, this is reasonable for our final purpose which is the study of propagation fronts.

5.1 Computation of the macroscopic fundamental solution $\mu(t, x; w)$.

We can derive a more explicit formulation of the minimization problem (??). This is done after a discussion of several possible cases.

In particular, if $w = 0$, it is better to choose $s_1 = 0$, and we have

$$\mu(t, x; 0) = \min_{0 \leq s \leq t} \left(\frac{|x|^2}{2s^2} + s \right) = \begin{cases} \frac{3}{2}|x|^{2/3}, & \text{if } |x| \leq t^{3/2}, \\ \frac{|x|^2}{2t^2} + t, & \text{if } |x| \geq t^{3/2}. \end{cases} \quad (5.1) \quad \text{eq:calcul m}$$

We now assume that $w \neq 0$. We introduce some notations for the sake of convenience,

$$\forall (s_1, s_2) \in [0, t]^2, 0 \leq s_1 + s_2 \leq t, \quad J(s_1, s_2) = \frac{|x - s_1 w|^2}{2s_2^2} + s_1 + s_2 .$$

(i) Assume that the minimum is attained at an interior point (s_1, s_2) , that is such that

$$s_1 > 0, \quad s_2 > 0, \quad s_1 + s_2 < t .$$

Then, computing the gradient yields the first order condition and the possible points of minimum:

$$\begin{cases} s_2^2 = w \cdot (x - s_1 w) \\ s_2^3 = |x - s_1 w|^2 \end{cases}, \quad \begin{cases} s_1 = \frac{x}{w} - |w|^2 \\ s_2 = |w|^2. \end{cases} \quad (5.2)$$

Therefore, under the conditions to be an interior point, which read equivalently,

$$|w|^2 < \frac{x}{w} < t, \quad (5.3) \quad \text{eq:condition}$$

the first candidate for the minimum value is

$$J\left(\frac{x}{w} - |w|^2, |w|^2\right) = \frac{|w|^2}{2} + \frac{x}{w} . \quad (5.4) \quad \text{eq:candidat}$$

- (ii) Assume that the minimum is attained on the edge $(s_1, 0)$, $s_1 \in [0, t]$. Recall that we have by convention

$$J(s_1, 0) = \mathbf{0}_{x=s_1w} + s_1.$$

Therefore, under the condition $0 \leq s_1 = \frac{x}{w} \leq t$, including the equality $x = tw$, the second candidate for the minimum value is

$$J\left(\frac{x}{w}, 0\right) = \frac{x}{w}. \quad (5.5) \quad \text{eq:candidat}$$

We notice immediately that this candidate is always better than (??).

- (iii) Assume that the minimum is attained on the edge $(0, s_2)$, $s_2 \in [0, t]$. We have from (??)

$$\min_{s_2 \in [0, t]} J(0, s_2) = \min_{s_2 \in [0, t]} \left(\frac{|x|^2}{2s_2^2} + s_2 \right) = \begin{cases} \frac{3}{2}|x|^{2/3}, & \text{if } |x| \leq t^{3/2} \\ \frac{|x|^2}{2t^2} + t, & \text{if } |x| \geq t^{3/2} \end{cases}, \quad (5.6) \quad \text{eq:candidat}$$

and the optimal s_2 is given by $s_2 = \min\left(t, |x|^{2/3}\right)$.

- (iv) Finally, assume that the minimum is attained on the edge $s_1 + s_2 = t$, with $(s_1, s_2) \in]0, t[^2$. Then minimizing $J(s_1, s_2)$ reduces to minimizing on $]0, t[$ the following function

$$\tilde{J}(s_2) := \frac{|x - tw + s_2w|^2}{2s_2^2} + t.$$

If $x = tw$, then \tilde{J} is constant and its value is always worse than (??). We can now assume that $x \neq tw$ since we are done if not. We have

$$\tilde{J}'(s_2) = -\frac{(x - tw) \cdot (x - tw + s_2w)}{s_2^3}.$$

The minimizing s_2 is $t - \frac{x}{w}$, if and only if $t - \frac{x}{w} \in]0, t[$, that is $\frac{x}{w} \in]0, t[$. The value of the minimum is $\tilde{J}(s_2) = t$ in this case. Else, the minimum can not be achieved on the interior of the edge. In any case, we notice that this choice is always worse than (??) under the same restriction $\frac{x}{w} \in]0, t[$.

Therefore, only three possible candidates remain after this discussion,

$$\mu(t, x; w) := \begin{cases} \frac{x}{w}, & \text{if } 0 \leq \frac{x}{w} \leq t \\ \frac{3}{2}|x|^{2/3}, & \text{if } |x| \leq t^{3/2} \\ \frac{|x|^2}{2t^2} + t, & \text{if } |x| \geq t^{3/2}. \end{cases} \quad (5.7) \quad \text{eq:candidat}$$

We plot in Figure ?? the final result that we obtain after minimizing between these three candidates.

fig:fund

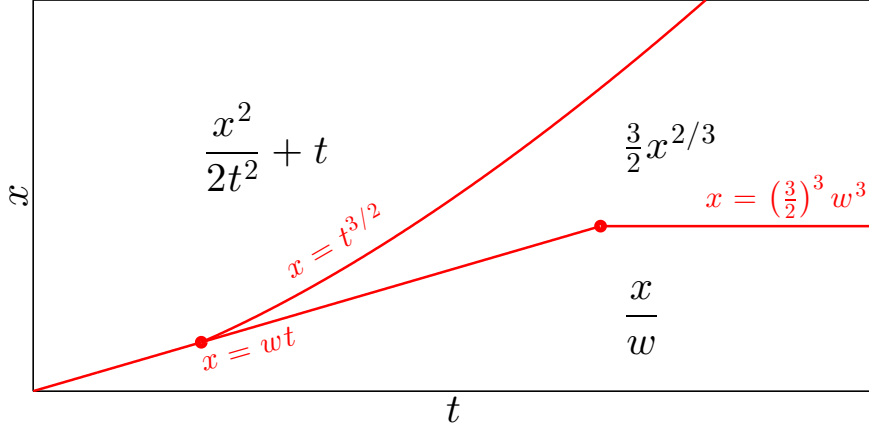


Figure 2: Zones of the fundamental solution

5.2 Computation of the full fundamental kernel $\phi(t, x, v; w)$ and of the characteristic trajectories.

Starting from an initial data of the form $\mathbf{0}_{x=0} + \mathbf{0}_{v=w}$, a particle arriving at (x, v) at time t follows the following trajectory in the phase plane

$$x(\tau) := \begin{cases} w\tau, & \text{if } 0 \leq \tau \leq s_1 \\ \left(\frac{x - s_1 w - s_3 v}{s_2} \right) (\tau - s_1) + s_1 w, & \text{if } s_1 \leq \tau \leq s_1 + s_2 \\ x - s_3 v, & \text{if } s_1 + s_2 \leq \tau \leq t - s_3 \\ x - s_3 v + (\tau - (t - s_3)) v, & \text{if } t - s_3 \leq \tau \leq t, \end{cases}$$

$$v(\tau) := \begin{cases} w, & \text{if } 0 \leq \tau \leq s_1 \\ \frac{x - s_1 w - s_3 v}{s_2}, & \text{if } s_1 \leq \tau \leq s_1 + s_2 \\ 0, & \text{if } s_1 + s_2 \leq \tau \leq t - s_3 \\ v, & \text{if } t - s_3 \leq \tau \leq t. \end{cases}$$

In order to compute the fundamental kernel ϕ given by (??), we first fully determine:

$$K(s_1, s_2, s_3) := \frac{|x - s_1 w - s_3 v|^2}{2s_2^2} + s_1 + s_2 + s_3,$$

on the set $\{(s_1, s_2, s_3) \in [0, t]^3 : 0 \leq s_1 + s_2 + s_3 \leq t\}$.

Proposition 5.1 (Expression of the characteristic trajectories). *We have*

$$\min_{\substack{(s_1, s_2, s_3): \\ 0 \leq s_1 + s_2 + s_3 \leq t}} K(s_1, s_2, s_3) = \min(\mu(t, x; v), \mu(t, x; w)).$$

Define $s(t, x) := \min\left(t, |x|^{2/3}\right)$. The Lagrangian trajectories are given in the phase space by:

- If $\mu(t, x; v) = \min(\mu(t, x; v), \mu(t, x; w))$, then depending on the position on the (t, x) -plane (see Figure ??), we have either

1.

$$x(\tau) := \begin{cases} \frac{\tau x}{s}, & \text{if } 0 \leq \tau \leq s \\ x, & \text{if } s \leq \tau \leq t, \end{cases} \quad v(\tau) := \begin{cases} \frac{x}{s}, & \text{if } 0 \leq \tau \leq s \\ 0, & \text{if } s \leq \tau \leq t, \end{cases}$$

2.

$$x(\tau) := \begin{cases} 0, & \text{if } 0 \leq \tau \leq t - \frac{x}{v} \\ x - (t - \tau)v, & \text{if } t - \frac{x}{v} \leq \tau \leq t, \end{cases} \quad v(\tau) := \begin{cases} 0, & \text{if } 0 \leq \tau \leq t - \frac{x}{v} \\ v, & \text{if } t - \frac{x}{v} \leq \tau \leq t. \end{cases}$$

- If $\mu(t, x; w) = \min(\mu(t, x; v), \mu(t, x; w))$, then depending on the position on the (t, x) -plane (see Figure ??), we have either

1.

$$x(\tau) := \begin{cases} \frac{\tau x}{s}, & \text{if } 0 \leq \tau \leq s \\ x, & \text{if } s \leq \tau \leq t, \end{cases} \quad v(\tau) := \begin{cases} \frac{x}{s}, & \text{if } 0 \leq \tau \leq s \\ 0, & \text{if } s \leq \tau \leq t. \end{cases}$$

2.

$$x(\tau) := \begin{cases} w\tau, & \text{if } 0 \leq \tau \leq \frac{x}{w} \\ x, & \text{if } \frac{x}{w} \leq \tau \leq t, \end{cases} \quad v(\tau) := \begin{cases} w, & \text{if } 0 \leq \tau \leq \frac{x}{w} \\ 0, & \text{if } \frac{x}{w} \leq \tau \leq t. \end{cases}$$

In particular, it shows that to reach its final point, it is better for a particle to redistribute its velocity only once.

Remark 1. If v and w have the same sign, positive say, then

$$\min(\mu(t, x; v), \mu(t, x; w)) = \mu(t, x; \max(v, w)).$$

Proof of Proposition ??. We should now discuss the values of the three times s_1, s_2, s_3 that minimize the travelling energy. We can make a bunch of reductions before computing the minimum. We may assume that $w \neq 0$ and $v \neq 0$, as well as $v \neq w$. Indeed, in the three opposite cases, the work is already done through the minimization of the functional J above. The values of the minimum are then respectively given by $\mu(t, x; v)$, $\mu(t, x; w)$ and $\mu(t, x; v = w)$ and the travel times are obtained through the minimization of the functional J above.

- (i) Assume that the minimum is attained at an interior point (s_1, s_2, s_3) :

$$s_1 > 0, \quad s_2 > 0, \quad s_3 > 0, \quad s_1 + s_2 + s_3 < t.$$

Then, computing the gradient yields the first order condition,

$$\begin{cases} s_2^2 = w \cdot (x - s_1 w - s_3 v), \\ s_2^2 = v \cdot (x - s_1 w - s_3 v), \\ s_2^3 = |x - s_1 w - s_3 v|^2. \end{cases} \quad (5.8)$$

Since we work with $v \neq w$, there is no solution to the first order condition, so the minimizer can not be an interior point.

- (ii) Assume that the minimum is attained on the edge $(s_1, 0, s_3)$, $s_1 + s_3 \in [0, t]$. We have by convention

$$K(s_1, 0, s_3) = \mathbf{0}_{x=s_1w+s_3v} + s_1 + s_3.$$

The latter quantity is finite for values of $(s_1, s_3) \in [0, t]^2$ and $s_1 + s_3 \in [0, t]$ such that $x = s_1w + s_3v$ can be achieved. The minimum of the quantity $s_1 + s_3$ on this set is necessarily attained on the boundary of the triangle. Thus,

- The minimizer $(0, 0, \frac{x}{v})$ is available only if $\frac{x}{v} \in [0, t]$, and then a candidate is $\frac{x}{v}$.
- The minimizer $(\frac{x}{w}, 0, 0)$ is available only if $\frac{x}{w} \in [0, t]$, and then a candidate is $\frac{x}{w}$.
- The minimizer $(\frac{x-tv}{v-w}, 0, \frac{x-tw}{v-w})$ is available only if $x \in [\min(v, w)t, \max(v, w)t]$, and then a candidate is t .

We check that the latest possibility is always worse than the two others. As a consequence, one has to take the minimum between the two remaining of these possibilities,

- The minimizer $(0, 0, \frac{x}{v})$ is available only if $\frac{x}{v} \in [0, t]$, and then a candidate is $\frac{x}{v}$,
- The minimizer $(\frac{x}{w}, 0, 0)$ is available only if $\frac{x}{w} \in [0, t]$, and then a candidate is $\frac{x}{w}$,

and there is no possible minimum else. We see that the value of the minimum in these zones is nevertheless given by $\min(\mu(t, x; v), \mu(t, x; w))$.

- (iii) Assume that the minimum is attained on the edge $(0, s_2, s_3)$, $s_2 + s_3 \in [0, t]$. We have

$$K(0, s_2, s_3) = J(s_2, s_3) = \mu(t, x; v),$$

and the optimal times are given through μ .

- (iv) Assume that the minimum is attained on the edge $(s_1, s_2, 0)$, $s_1 + s_2 \in [0, t]$. We have

$$K(s_1, s_2, 0) = J(s_1, s_2) = \mu(t, x; w).$$

and the optimal times are given through μ .

- (v) Finally, assume that the minimum is attained in the interior of the edge $s_1 + s_2 + s_3 = t$. We have to minimize for

$$(s_1, s_3) \in]0, t[^2, \quad 0 < s_1 + s_3 < t,$$

the quantity

$$K(s_1, s_2, s_3) = K(s_1, t - (s_1 + s_3), s_3).$$

The first order condition reads

$$\begin{aligned} \exists (s_1, s_3) \in]0, t[^2, \quad 0 < s_1 + s_3 < t, \quad x - s_1w - s_3v = 0, \\ \iff \exists u \in]0, t[, \quad \exists s_1 \in]0, u[, \quad x = u \left(\frac{s_1}{u}w + \left(1 - \frac{s_1}{u}\right)v \right), \\ \iff \exists u \in]0, t[, \quad x \in u \cdot [v, w]. \end{aligned}$$

Then the corresponding value of the minimum is t . We observe that this possibility is always worse than $\mu(t, x; v)$ or $\mu(t, x; w)$.

Discussing all the different cases, we see that

$$\min_{\substack{(s_1, s_2, s_3): \\ 0 \leq s_1 + s_2 + s_3 \leq t}} K(s_1, s_2, s_3) = \min(\mu(t, x; v), \mu(t, x; w)).$$

and that the optimal travel times and trajectories are given by the statement of Proposition ??.

□

With the knowledge of the trajectories in hand, we can now simplify the expression of the fundamental kernel $\phi(t, x, v; w)$.

prop:kernel

Proposition 5.2 (Expression of $\phi(t, x, v; w)$). *For all $(t, x, v, w) \in \mathbb{R}_+ \times \mathbb{R}^{3n}$, one has*

$$\phi(t, x, v; w) = \frac{|v|^2}{2} + \min(\mu(t, x; v), \mu(t, x; w)),$$

except in the very special case when $v = w = x/t$, for which

$$\phi(t, x, v; w) = \frac{x}{w}.$$

We point out the discontinuity in the fundamental kernel due to the initial condition when $w \neq 0$.

Proof of Proposition ??. From Proposition ??, recall that we have

$$\min_{\substack{(s_1, s_2, s_3): \\ 0 \leq s_1 + s_2 + s_3 \leq t}} K(s_1, s_2, s_3) = \min(\mu(t, x; v), \mu(t, x; w)).$$

Thus,

$$\phi(t, x, v; w) = \min\left(\mathbf{0}_{x=tv} + \min\left(\mathbf{0}_{v=w}, \frac{|v|^2}{2}\right) + t, \frac{|v|^2}{2} + \min(\mu(t, x; v), \mu(t, x; w))\right)$$

We can now distinguish various cases, depending on the value of $\mathbf{0}_{x=tv}$ and $\mathbf{0}_{v=w}$.

1. First, if $x \neq vt$, we have directly $\phi(t, x, v; w) = \frac{|v|^2}{2} + \min(\mu(t, x; v), \mu(t, x; w))$.
2. Now assuming that $x = vt$, one has

$$\phi(t, x, v; w) = \min\left(\min\left(\mathbf{0}_{v=w}, \frac{|v|^2}{2}\right) + \frac{x}{v}, \frac{|v|^2}{2} + \min(\mu(t, x; v), \mu(t, x; w))\right)$$

(a) If $w = v$, then

$$\phi(t, x, v; w) = \min\left(\frac{x}{v}, \frac{|v|^2}{2} + \mu(t, x; v)\right) = \frac{x}{v}.$$

(b) Now, if $w \neq v$, then

$$\begin{aligned} \phi(t, x, v; w) &= \min\left(\frac{|v|^2}{2} + \frac{x}{v}, \frac{|v|^2}{2} + \min(\mu(t, x; v), \mu(t, x; w))\right) \\ &= \frac{|v|^2}{2} + \min\left(\min\left(\frac{x}{v}, \mu(t, x; v)\right), \mu(t, x; w)\right), \\ &= \frac{|v|^2}{2} + \min(\mu(t, x; v), \mu(t, x; w)). \end{aligned}$$

Finally, $\phi(t, x, v; w) = |v|^2/2 + \mu(t, x; \max(v, w))$ unless $v = w = x/t$ for which $\phi(t, x, v; w) = x/w$.

□

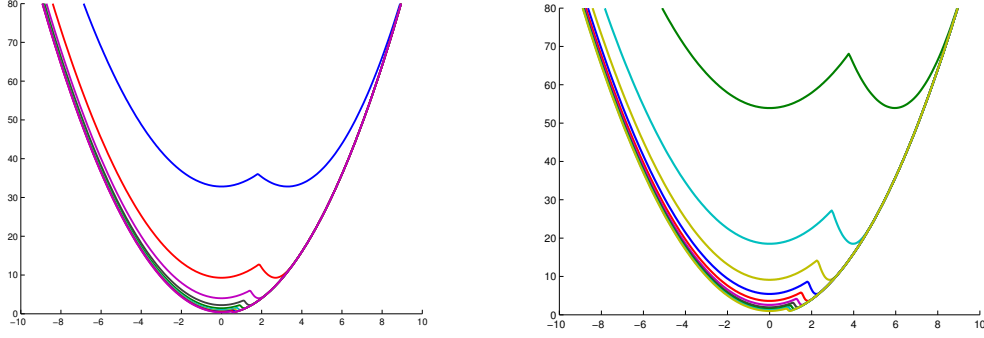


fig:min

Figure 3: Plot of $v \mapsto \phi(t, x, v; 0)$: (left) $x = 5$ for $t = (1 : 10)$, (right) $x = 10$ for $t = (1 : 10)$.

6 Rate of acceleration in kinetic reaction-transport equations

sec:acc

We consider a kinetic reaction-transport problem. We focus on equation (??) after rescaling (??) [?, ?, ?]:

$$\begin{aligned} \forall(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2n}, \quad & \varepsilon (\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v)) \\ & = M_\varepsilon(v) \rho^\varepsilon(t, x) - f(t, x, v) + r \rho^\varepsilon(t, x) \left(M_\varepsilon(v) - \varepsilon^{-\frac{n}{2}} f(t, x, v) \right). \end{aligned} \quad (6.1) \quad \text{eq:kinKPPR}$$

The latter equation satisfies a maximum principle as soon as the initial data f_0^ε is measurable and satisfies

$$\forall(t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \quad 0 \leq f_0^\varepsilon(x, v) \leq \varepsilon^{\frac{n}{2}} M_\varepsilon(v). \quad (6.2) \quad \text{eq:estf0}$$

This readily implies the following estimate on u_0 :

$$\forall(t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \quad u_0^\varepsilon(x, v) \geq \frac{|v|^2}{2} + \frac{\varepsilon n}{2} \log(2\pi\varepsilon).$$

We recall the following existence result from [?]:

Proposition 6.1 (Global existence: Theorem 4 in [?]). *Suppose that f_0^ε satisfies (??). Then the Cauchy problem (??) has a unique solution $f^\varepsilon \in \mathcal{C}_b^0(\mathbb{R}_+ \times \mathbb{R}^{2n})$ in the sense of distributions, satisfying*

$$\forall(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2n}, \quad 0 \leq f^\varepsilon(t, x, v) \leq \varepsilon^{\frac{n}{2}} M_\varepsilon(v). \quad (6.3) \quad \text{eq:maxpcple}$$

After performing the Hopf-Cole transform $u^\varepsilon := -\varepsilon \log f^\varepsilon$, the equation satisfied by u^ε is the following

$$\forall(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2n}, \quad \partial_t u^\varepsilon + v \cdot \nabla_x u^\varepsilon = 1 - (1+r) \int_{\mathbb{R}^n} \frac{1}{(2\pi\varepsilon)^{\frac{n}{2}}} e^{\frac{u^\varepsilon - u'^\varepsilon - |v|^2/2}{\varepsilon}} dv' + \frac{r}{\varepsilon^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{u'^\varepsilon}{\varepsilon}} dv'. \quad (6.4) \quad \text{eq:kinreacW}$$

In this section, we extend the comparison principle (Theorem ??), and the convergence of (u^ε) to a viscosity solution of the corresponding non-local Hamilton-Jacobi equation (Theorem ??) for bounded initial data. We further extend Theorem ?? for solutions having locally uniform quadratic growth both in space and velocity (Section ??). This class of solutions appears naturally in our problem (Section ??). However, we were not able to prove the convergence Theorem ?? for unbounded initial data. In particular, we were not able to adapt the proof of uniformly isolated minima (Lemma ??) due to the lack of uniform Lipschitz regularity in space.

Then, we extend the representation formula (??) to the case $r > 0$, but ignoring the quadratic nonlinearity in (??). This nonlinear term yields a constraint in the limit problem. Usually, this additional constraint is handled by means of the so-called Freidlin condition which asserts that the solution to the constrained problem can be obtained by truncating the representation formula of the unconstrained problem so that it fulfils the constraint $\min u \geq 0$. The latter condition is an immediate consequence of (??).

We now define the non-local Hamilton-Jacobi system obtained in the limit $\varepsilon \rightarrow 0$. Formally, under suitable conditions (see Theorem ?? below), u^ε converges locally uniformly towards a viscosity solution of the following non-local Hamilton-Jacobi system,

1. If $\min_{w \in \mathbb{R}^n} \underline{u}(t, x, w) = 0$ then, for all $v \in \mathbb{R}^n$, one has $\underline{u}(t, x, v) = \frac{|v|^2}{2}$.

2. If $\min_{w \in \mathbb{R}^n} \underline{u}(t, x, w) > 0$,

$$\begin{cases} \max \left(\partial_t \underline{u}(t, x, v) + v \cdot \nabla_x \underline{u}(t, x, v) - 1, \underline{u}(t, x, v) - \min_{w \in \mathbb{R}^n} \underline{u}(t, x, w) - \frac{|v|^2}{2} \right) = 0, \\ \partial_t \left(\min_{w \in \mathbb{R}^n} \underline{u}(t, x, w) \right) \leq -r, \\ \partial_t \left(\min_{w \in \mathbb{R}^n} \underline{u}(t, x, w) \right) = -r, \quad \text{if } \mathcal{S}(u)(t, x) = \{0\}. \end{cases}$$

(6.5) eq:nlsys

3. $u(0, \cdot, \cdot) = u_0(\cdot, \cdot)$.

A function u is a solution of the limit system if, and only if it is both a sub and a super-solution as described in the following definitions.

def:nlsup **Definition 6.2** (Sub-solution). *Let $T > 0$. A upper semi-continuous function \underline{u} is a **viscosity sub-solution** of equation (??) on $[0, T)$ if and only if for all $(t_0, x_0) \in [0, T) \times \mathbb{R}^n$:*

1. $\underline{u}(0, \cdot, \cdot) \leq (u_0)^*$

2. If $\min_{w \in \mathbb{R}^n} \underline{u}(t_0, x_0, w) > 0$,

(i) *It satisfies the constraint*

$$(\forall v \in \mathbb{R}^n) \quad \underline{u}(t_0, x_0, v) - \min_{w \in \mathbb{R}^n} \underline{u}(t_0, x_0, w) - \frac{|v|^2}{2} \leq 0.$$

(ii) *For all pair of test functions $(\phi, \psi) \in \mathcal{C}^1((0, T) \times \mathbb{R}^{2n}) \times \mathcal{C}^1((0, T) \times \mathbb{R}^n)$, if (t_0, x_0, v_0) is such that both $\underline{u}(\cdot, \cdot, v_0) - \phi(\cdot, \cdot, v_0)$ and $\min_w \underline{u}(\cdot, \cdot, w) - \psi(\cdot, \cdot)$ have a local maximum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, then*

$$\begin{cases} \partial_t \phi(t_0, x_0, v_0) + v_0 \cdot \nabla_x \phi(t_0, x_0, v_0) - 1 \leq 0, \\ \partial_t \psi(t_0, x_0) \leq -r. \end{cases}$$

(6.6) eq:S1

3. If $\min_{w \in \mathbb{R}^n} \underline{u}(t_0, x_0, w) \leq 0$ then, for all $v \in \mathbb{R}^n$, one has $\underline{u}(t_0, x_0, v) \leq \frac{|v|^2}{2}$.

def:nlsup **Definition 6.3** (Super-solution). *A lower semi-continuous function \bar{u} is a **viscosity super-solution** of equation (??) on $[0, T)$ if and only if for all $(t_0, x_0) \in [0, T) \times \mathbb{R}^n$,*

1. $\bar{u}(0, \cdot, \cdot) \geq (u_0)_*$
2. For all $v \in \mathbb{R}^n$, one has $\bar{u}(t_0, x_0, v) \geq \frac{|v|^2}{2}$.
3. For all $(t, x) \in (0, T) \times \mathbb{R}^n$, $v = 0$ is a global minimum of $\bar{u}(t, x, \cdot)$. Moreover, $v = 0$ is locally uniformly isolated: for any compact set $K \subset (0, T) \times \mathbb{R}^n$, there exists $r > 0$ such that $\mathcal{S}(\bar{u})(t, x) \cap B_r(0) = \{0\}$ for all $(t, x) \in K$.
4. For all pair of test functions $(\phi, \psi) \in \mathcal{C}^1((0, T) \times \mathbb{R}^{2n}) \times \mathcal{C}^1((0, T) \times \mathbb{R}^n)$, if (t_0, x_0, v_0) is such that both $\bar{u}(\cdot, \cdot, v_0) - \phi(\cdot, \cdot, v_0)$ and $\min_w \bar{u}(\cdot, \cdot, w) - \psi(\cdot, \cdot)$ have a local minimum at $(t_0, x_0) \in [0, T) \times \mathbb{R}^n$, then

$$\begin{cases} \partial_t \phi(t_0, x_0, v_0) + v_0 \cdot \nabla_x \phi(t_0, x_0, v_0) - 1 \geq 0, & \text{if } \bar{u}(t_0, x_0, v_0) - \min_{w \in \mathbb{R}^n} \bar{u}(t_0, x_0, w) - \frac{|v_0|^2}{2} < 0, \\ \partial_t \psi(t_0, x_0) \geq -r, & \text{if } \mathcal{S}(\bar{u})(t_0, x_0) = \{0\}. \end{cases} \quad (6.7)$$

We consider the linearized version of (??), ignoring the quadratic nonlinearity,

$$\varepsilon \left(\partial_t \tilde{f}(t, x, v) + v \cdot \nabla_x \tilde{f}(t, x, v) \right) = M_\varepsilon(v) \tilde{\rho}^\varepsilon(t, x) - \tilde{f}(t, x, v) + r \tilde{\rho}^\varepsilon(t, x) M_\varepsilon(v). \quad (6.8)$$

eq:kinKPPR

We point out that (??) and (??) are equivalent after the following change of variables: Let g defined by $g(t', x', v) = e^{rt} \tilde{f}(t'/(1+r), x'/(1+r), v)$, then g solves (??). As a consequence, the corresponding Hamilton-Jacobi system for \tilde{u} is written

$$\begin{cases} \max \left(\partial_t \tilde{u}(t, x, v) + v \cdot \nabla_x \tilde{u}(t, x, v) - 1, \tilde{u}(t, x, v) - \min_{w \in \mathbb{R}^n} \tilde{u}(t, x, w) - \frac{|v|^2}{2} \right) = 0, \\ \partial_t \left(\min_{w \in \mathbb{R}^n} \tilde{u}(t, x, w) \right) \leq -r, \\ \partial_t \left(\min_{w \in \mathbb{R}^n} \tilde{u}(t, x, w) \right) = -r, & \text{if } \mathcal{S}(\tilde{u})(t, x) = \{0\}, \\ \tilde{u}(0, x, v) = u_0(x, v). \end{cases} \quad (6.9)$$

eq:limit r

We define accordingly the fundamental solution ϕ_r by performing the same change of variables in (??).

We refer to it as the *unconstrained problem*, because the definition of viscosity sub-solution (resp. super-solution) coincide with Definition ?? (resp. Definition ??) except that the constraint $\min u \geq 0$ is ignored, meaning that Item 2 is ignored in Definition ?? (resp. in Definition ??).

By opposition, we refer to the combination of Definition ?? and Definition ?? as the *constrained problem*.

For the rest of this preliminary discussion, we restrict to dimension $n = 1$.

We conjecture that a solution to the constrained problem can be built following a procedure analogous to [?].

conj

Conjecture 6.4. *First, solve the unconstrained problem,*

$$\tilde{u}(t, x, v) = \min_{(y, w)} (\phi_r(t, x - y, v; w) + u_0(y, w)). \quad (6.10)$$

eq:hopf lax

Then, truncate the solution by taking into account the constraint $\min u \geq 0$ in the following way: for (t, x) such that $\min \tilde{u} \leq 0$, replace \tilde{u} by $v^2/2$ on the zone $v < x/t$ (for $x > 0$), resp. on the zone $v > x/t$ (for $x < 0$). This would determine the minimal value $\min u$ for all (t, x) as the latter is necessarily attained for $v = 0$.

Although we were not able to prove that this truncation procedure is valid, this would have the following consequences on the spatial spreading rate of accelerating fronts for (??). In fact, we can compute the exact location of the level sets associated with the minimal value of the fundamental solution ϕ_r .

Proposition 6.5 (Level lines of μ_r). *Assume that Conjecture ?? holds. Let $w \in \mathbb{R}$. There exists $T_0(w)$ such that for all $t \geq T_0(w)$, the negative part of the minimum value μ_r is located as follows,*

$$\left\{ x \in \mathbb{R} \mid \mu_r(t, x; w) \leq 0 \right\} = \left\{ x \in \mathbb{R} \mid |x| \leq \frac{\left(\frac{2}{3}r\right)^{\frac{3}{2}}}{1+r} t^{\frac{3}{2}} \right\}.$$

This yields formally the rate of spreading for initial data of the form $u_0(x, v) = \mathbf{0}_{x=0} + \min\left(\mathbf{0}_{v=w}, \frac{|v|^2}{2}\right)$. We conjecture that the same rate of spreading holds for initial data which are growing sufficiently fast at infinity.

Proof of Proposition ??. The spreading rate comes straightforwardly from the expression of μ_r . We can infer that

$$\begin{aligned} \mu_r(t, x; w) \leq 0 &\iff \begin{cases} \left((1+r)\frac{x}{w} - rt \right) \mathbf{1}_{\frac{x}{w} \leq \min(t, (\frac{3}{2})^3 w^2)} \leq 0, \\ \text{or} \\ \left(\frac{3}{2} \left((1+r)|x| \right)^{\frac{2}{3}} - rt \right) \mathbf{1}_{\frac{x}{w} > \min(t, (\frac{3}{2})^3 w^2)} \leq 0, \end{cases} \\ &\iff |x| \leq \frac{\left(\frac{2}{3}r\right)^{\frac{3}{2}}}{1+r} t^{\frac{3}{2}} \quad \text{for } t \text{ sufficiently large.} \end{aligned}$$

□

Although we were not able to prove this Conjecture, we discuss in the rest of this section several intermediate results supporting it. First, in order to prove that the truncation procedure might yield a representation of the solution of equation (??), one needs to check that this solution is unique. We thus first show a comparison principle for bounded initial data in Subsection ??. We then extend in Subsection ?? this comparison principle to unbounded initial data with an appropriate quadratic growth rate, this rate being satisfied by the fundamental solution after positive time. We then discuss in Subsection ?? the validity of the truncation by checking a Freidlin-type condition for equation (??), although we are not able to prove that this condition yields the Conjecture.

6.1 Uniformly bounded initial data

Here, we restrict to bounded initial data, as in Sections ?? and ??.

6.1.1 The comparison principle

Theorem 6.6 (Comparison principle). *Let \underline{u} and \bar{u} be respectively a viscosity sub-solution and a viscosity super-solution of equation (??) in the sense of Definitions ?? and ??. Assume that \underline{u} and \bar{u} satisfy (??). Then $\underline{u} \leq \bar{u}$.*

Proof of Theorem ??. We follow the same lines as for the proof of Theorem ??. For the sake of clarity, we do not reproduce all the details, but we focus on the arguments specific to the constrained problem. This is the discussion on the minimum $\min_{w \in \mathbb{R}^n} \underline{u}$.

Define $\hat{\chi}$ and $\tilde{\chi}$ as in (??)-(??), and define ω, Ω accordingly. For the sake of clarity, we omit the correcting terms in the following discussion.

Suppose by contradiction that the maximum value Ω is positive. Let (t_0, x_0, v_0) be either $(\tilde{t}_0, \tilde{x}_0, v_0)$ or $(\hat{t}_0, \hat{x}_0, v_0)$.

Case 1: $\omega < \Omega$. If $\min_w u(t_0, x_0, w) > 0$, then argue as in Section ??. On the other hand, if $\min_w \underline{u}(t_0, x_0, w) \leq 0$, then by definition we have $(\forall v) \underline{u}(t_0, x_0, v) \leq \frac{|v|^2}{2}$. But we also have $\bar{u}(t_0, x_0, v) \geq \frac{|v|^2}{2}$. Therefore, we have a contradiction at $v = v_0$.

Case 2.1: $\omega = \Omega$ and $\mathcal{S}(\bar{u})(t_0, x_0) = \{0\}$. If $\underline{m}(t_0, x_0) > 0$, then argue as in Section ??. On the other hand, if $\underline{m}(t_0, x_0) \leq 0$, then by definition $\bar{m}(t_0, x_0) \geq 0$. This is a contradiction.

Case 2.2: $\omega = \Omega$ and $\mathcal{S}(\bar{u})(t_0, x_0) \neq \{0\}$. Same as **Case 1**. \square

6.1.2 Convergence of solutions to the kinetic equation

Theorem 6.7 (Convergence). *Let u^ε be the solution of (??), with the initial data $u^\varepsilon = u_0$. We assume that the initial data satisfies conditions (??) and (??). Then, as $\varepsilon \rightarrow 0$, u^ε converges towards u , the unique viscosity solution of the limit system, characterized by Definitions ?? and ??.*

Proof of Theorem ??. We now show how to pass to the limit in the presence of the nonlinearity. We again follow the same lines as for the linear case, and do not reproduce the technicalities to focus only on the main features of the argument. We start with reproducing the Lipschitz estimates. Then, we derive the viscosity procedure.

Lipschitz estimates. The function b^ε satisfies :

$$(\partial_t + v \cdot \nabla_x)(b^\varepsilon)(v) = 1 - (1+r) \int_{\mathbb{R}^n} (2\pi\varepsilon)^{-\frac{n}{2}} e^{\frac{b^\varepsilon(v) - b^\varepsilon(v') - |v'|^2/2}{\varepsilon}} dv' + \frac{r}{(2\pi\varepsilon)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|v'|^2}{2\varepsilon}} e^{-\frac{b^\varepsilon(v')}{\varepsilon}} dv'.$$

We shall rewrite this equation on the form :

$$(\partial_t + v \cdot \nabla_x)(b^\varepsilon)(v) = (1+r) \left(1 - \int_{\mathbb{R}^n} \frac{1}{(2\pi\varepsilon)^{\frac{n}{2}}} e^{\frac{b^\varepsilon(v) - b^\varepsilon(v') - |v'|^2/2}{\varepsilon}} dv' \right) + r(\rho^\varepsilon - 1).$$

The fact that $\rho^\varepsilon \leq 1$ combined with a maximum principle argument for b^ε gives the uniform bound. For the space derivative, we write :

$$\begin{aligned} (\partial_t + v \cdot \nabla_x)(\nabla_x u^\varepsilon)(v) &= -\frac{1}{\varepsilon} \int_{\mathbb{R}^n} M_\varepsilon(v) [\nabla_x u^\varepsilon(v) - \nabla_x u^\varepsilon(v')] e^{\frac{u^\varepsilon(v) - u^\varepsilon(v')}{\varepsilon}} dv' \\ &\quad - \frac{r}{\varepsilon} \int_{\mathbb{R}^n} [\nabla_x u^\varepsilon(v) - \nabla_x u^\varepsilon(v')] \left(M_\varepsilon(v) e^{\frac{u^\varepsilon(v)}{\varepsilon}} - 1 \right) e^{-\frac{u^\varepsilon(v')}{2\varepsilon}} dv' - \frac{r}{\varepsilon} \left(\int_{\mathbb{R}^n} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv' \right) \nabla_x u^\varepsilon, \end{aligned}$$

and the result follows from a maximum principle argument. To bound the time derivative, we write the same type of equation as for the space derivative and redo the argument used for the linear case.

Convergence procedure.

Step 1 : Viscosity super-solution.

We start with the viscosity super-solution step. We first show that u^0 is a super solution of (??) that is

$$\begin{cases} \partial_t u + v \cdot \nabla_x u - 1 \geq 0 & \text{if } \{(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2n} \mid u - \min_{w \in \mathbb{R}^n} u - \frac{|v|^2}{2} < 0\}, \\ \partial_t (\min_{w \in \mathbb{R}^n} u) \geq -r & \text{if } \mathcal{S}(u)(t, x) = \{0\}. \end{cases} \quad (6.11)$$

in the viscosity sense, and $u(t, x, v) \geq \frac{|v|^2}{2}$.

The first inequality comes from the fact that $\rho^\varepsilon \geq 0$ and then the same argument as for the linear case works. Let us set $v^\varepsilon = u^\varepsilon + rt$, we get

$$\partial_t v^\varepsilon + v \cdot \nabla_x v^\varepsilon \geq (1+r) \left(1 - \int_{\mathbb{R}^n} \frac{1}{(2\pi\varepsilon)^{\frac{n}{2}}} e^{\frac{v^\varepsilon - v'^\varepsilon - |v|^2/2}{\varepsilon}} dv' \right).$$

Since the r.h.s is now mass conservative, the same procedure as for the linear case applied to v^ε gives

$$\partial_t \left(\min_{w \in \mathbb{R}^n} u + rt \right) \geq 0 \quad \text{if } \mathcal{S}(u^0)(t, x) = \{0\}$$

and thus the result follows.

From the fact that $f^\varepsilon(t, x, v) \leq \varepsilon^{\frac{n}{2}} M_\varepsilon(v)$ for all $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2n}$, we deduce

$$u^\varepsilon(t, x, v) = -\varepsilon \log f^\varepsilon(t, x, v) \geq \frac{|v|^2}{2} + \frac{\varepsilon n}{2} \log(2\pi\varepsilon),$$

and thus $u(t, x, v) \geq \frac{|v|^2}{2}$ passing to the limit $\varepsilon \rightarrow 0$.

Step 2 : Viscosity sub-solution.

Since

$$\frac{r}{\varepsilon^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv'$$

is bounded uniformly, the same proof as in the linear case gives

$$(\forall (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2n}), \quad u(t, x, v) \leq \min_{w \in \mathbb{R}^n} u(t, x, w) + |v|^2/2.$$

The maximum principle on f^ε yields that in the limit, one has necessarily $u \geq |v|^2/2$. We deduce

$$\frac{|v|^2}{2} \leq u(t, x, v) \leq \min_{w \in \mathbb{R}^n} u(t, x, w) + \frac{|v|^2}{2}.$$

Take first a point (t, x) such that $\min_{w \in \mathbb{R}^n} u \leq 0$. The previous inequality gives $u(t, x, v) = |v|^2/2$. Take finally a point (t, x) such that $\min_{w \in \mathbb{R}^n} u > 0$. Then in this case ρ^ε goes to zero. Thus, passing to the limit in the viscosity sense in

$$\partial_t u^\varepsilon + v \cdot \nabla_x u^\varepsilon \leq 1 + \frac{r}{\varepsilon^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{u^\varepsilon}{\varepsilon}} dv'.$$

gives the first inequality of the limit system. The proof in the linear case can be reproduced to give

$$\partial_t \left(\min_{w \in \mathbb{R}^n} u \right) \geq -r \quad \text{if } \mathcal{S}(u)(t, x) = \{0\}$$

□

subsec:unbdd

6.2 Unbounded initial data

sec:621

6.2.1 The comparison principle

Here, we explain how to extend the comparison principle Section ?? beyond Hypothesis (??), *i.e.* $u = v^2/2 + L^\infty$. Specifically, we consider sub- and super- solutions having locally uniform quadratic growth at infinity both in space and velocity. This is motivated by the fact that the fundamental solution that was derived in Section ?? has precisely this large scale behaviour after positive time (see also Figure ??). The method cannot handle asymptotic behaviour bigger than quadratic.

theo:comp2

Theorem 6.8 (Comparison principle). *Let \underline{u} (resp. \bar{u}) be a viscosity sub-solution (resp. super-solution) of (??) on $[0, T) \times \mathbb{R}^{2n}$. Assume that \underline{u} and \bar{u} are such that there exist constants $A > 1, B > 0, C > 0$ such that for all (x, v) ,*

$$\frac{1}{A}|v|^2 - B|x|^2 - C \leq \underline{u}(t, x, v), \bar{u}(t, x, v) \leq A|v|^2 + B|x|^2 + C. \quad (6.12)$$

eq:growth c

Then $\underline{u} \leq \bar{u}$ on $[0, T) \times \mathbb{R}^{2n}$.

An immediate consequence of (??) is that any minimum with respect to velocity, say attained at v_0 , satisfies

$$|v_0|^2 \leq 2A(B|x|^2 + C). \quad (6.13)$$

eq:localiza

Proof of Theorem ??. We shall not reproduce all the details of proof of the comparison principle, but only indicate the major changes that have to be made in comparison with Section ?. The major discrepancy concerns the localization of minima with respect to the velocity variable. Indeed, they are no more confined uniformly with respect to x as in Section ?, see (??). We have to adapt the penalization terms in $\hat{\chi}$ and $\tilde{\chi}$ accordingly. A side effect is that a short time condition $T < T_0$ is required. However, this does not affect the conclusion since we can iterate the comparison principle on time intervals of length T_0 . Let us define

$$T_0 = \frac{1}{4\sqrt{AB}}. \quad (6.14)$$

eq:shorttim

Let $T < T_0$. Let $\alpha > 0, R > 0$. Let $\delta > 0$ to be suitably chosen below. We define

$$\hat{\chi}(t, x) = \underline{m}(t, x) - \bar{m}(t, x) - \frac{\delta}{4}e^{\gamma(t-T)}|x|^4 - \frac{\alpha}{T-t} \quad (6.15)$$

eq:chiapp

$$\tilde{\chi}(t, x, v) = \underline{u}(t, x, v) - \bar{u}(t, x, v) - \frac{\delta}{4}e^{\gamma(t-T)}|x|^4 - \frac{\alpha}{T-t} - \Lambda(|v|^2 - B_1|x - tv|^2 - R^2)_+, \quad (6.16)$$

for some suitable constants γ, B_1 to be chosen below. The quadratic penalty term in (??) has been turned into a quartic one in order to ensure the existence of a minimum with respect to space variable. The exponential prefactor is inspired from [?, page 72]. Also, the penalization with respect to velocity has been extended in order to take into account the nonuniform velocity confinement.

We define ω, Ω as in (??). To establish that $\omega \leq \Omega$, it is enough to check that the penalty term $(|v|^2 - B_1|x - tv|^2 - R^2)_+$ vanishes at the minima of \bar{u} with respect to velocity. This is indeed a consequence of (??) and the short time condition. The following lemma replaces the confinement Lemma ??.

em:minconf A

Lemma 6.9. Let $B_1 \geq \frac{4AB}{1-4ABT^2}$ and $R \geq \sqrt{2AC(1+T^2B_1)}$. For all $(t, x) \in [0, T] \times \mathbb{R}^n$, the additionnal penalization vanishes on $\mathcal{S}(\bar{u})(t, x)$,

$$(\forall v_0 \in \mathcal{S}(\bar{u})(t, x)) \quad (|v_0|^2 - B_1|x - tv_0|^2 - R^2)_+ = 0. \quad (6.17)$$

Proof of Lemma ??. Let v_0 be a minimum, we have:

$$\begin{aligned} |v_0|^2 - B_1|x - tv_0|^2 - R^2 &\leq (1 - t^2B_1) |v_0|^2 + 2B_1tx \cdot v_0 - B_1|x|^2 - R^2 \\ &\leq (1 - t^2B_1) |v_0|^2 + \frac{1}{2}B_1|x|^2 + 2B_1t^2|v_0|^2 - B_1|x|^2 - R^2 \\ &\leq (1 + t^2B_1) |v_0|^2 - \frac{1}{2}B_1|x|^2 - R^2 \\ &\leq A(1 + T^2B_1) (2B|x|^2 + 2C) - \frac{1}{2}B_1|x|^2 - R^2. \end{aligned}$$

The last quantity is certainly nonpositive due to our choice of B_1 and R . □

em:Omega A

Lemma 6.10. We have $\omega \leq \Omega$.

The proof goes the same as in Lemma ??.

Case 1: $\omega < \Omega$. We denote by $(\tilde{t}_0, \tilde{x}_0, v_0)$ a maximum point of $\tilde{\chi}$, such that $\tilde{\chi}(\tilde{t}_0, \tilde{x}_0, v_0) = \Omega$.

em:velconf A

Lemma 6.11. There exist two constants C_1, C_2 such that

$$|v_0| \leq C_1|\tilde{x}_0| + C_2R + C_3,$$

Proof of Lemma ??. The evaluation $\tilde{\chi}(\tilde{t}_0, \tilde{x}_0, v_0) \geq \tilde{\chi}(0, 0, 0)$ gives

$$\begin{aligned} \Lambda (|v_0|^2 - B_1|\tilde{x}_0 - \tilde{t}_0v_0|^2 - R^2)_+ &\leq \underline{u}(\tilde{t}_0, \tilde{x}_0, v_0) - \bar{u}(\tilde{t}_0, \tilde{x}_0, v_0) - \underline{u}(0, 0, 0) + \bar{u}(0, 0, 0) \\ \Lambda (|v_0|^2 - B_1|\tilde{x}_0 - \tilde{t}_0v_0|^2 - R^2) &\leq \left(A - \frac{1}{A}\right) |v_0|^2 + 2B|\tilde{x}_0|^2 + C \end{aligned}$$

We expand the left-hand-side, to obtain

$$\begin{aligned} \Lambda (1 - 2B_1\tilde{t}_0^2) |v_0|^2 &\leq 2\Lambda B_1|\tilde{x}_0|^2 + \Lambda R^2 + \left(A - \frac{1}{A}\right) |v_0|^2 + 2B|\tilde{x}_0|^2 + C \\ \left(\frac{1}{3}\Lambda - \left(A - \frac{1}{A}\right)\right) |v_0|^2 &\leq B|\tilde{x}_0|^2 + C + \Lambda R^2 \end{aligned}$$

We conclude by taking $\Lambda = 3(A - 1/A) + 1$. □

According to the changes performed in the definitions of $\hat{\chi}, \tilde{\chi}$, we change the definition of $\tilde{\chi}_\varepsilon$ to

$$\begin{aligned} \tilde{\chi}_\varepsilon(t, x, s, y, v) &= \underline{u}(t, x, v) - \bar{u}(s, y, v) - \frac{\delta}{4}e^{\gamma(t-T)}|x|^4 \\ &\quad - \frac{\alpha}{T-t} - \frac{1}{2\varepsilon} (|t-s|^2 + |x-y|^2) - \frac{1}{2} (|s-\tilde{t}_0|^2 + |y-\tilde{x}_0|^2) \\ &\quad - \Lambda (|v|^2 - B_1|x - tv|^2 - R^2)_+. \end{aligned}$$

Let $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon)$ which realizes the maximum of $\tilde{\chi}_\varepsilon(\cdot, v_0)$.

limit to A

Lemma 6.12. *The following limit holds true,*

$$\lim_{\varepsilon \rightarrow 0} (\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon) = (\tilde{t}_0, \tilde{x}_0, \tilde{t}_0, \tilde{x}_0).$$

constraint A

Lemma 6.13. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,*

$$\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) - \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) - \frac{|v_0|^2}{2} < 0.$$

We now present the test function step

$$\begin{aligned} \phi_2(s, y, v) = & \underline{u}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v) - \frac{\delta}{4} e^{\gamma(\tilde{t}_\varepsilon - T)} |\tilde{x}_\varepsilon|^4 \\ & - \frac{\alpha}{T - \tilde{t}_\varepsilon} - \frac{1}{2\varepsilon} (|\tilde{t}_\varepsilon - s|^2 + |\tilde{x}_\varepsilon - y|^2) - \frac{1}{2} (|s - \tilde{t}_0|^2 + |y - \tilde{x}_0|^2) \\ & - \Lambda (|v|^2 - B_1|x - tv|^2 - R^2)_+, \end{aligned}$$

associated to the supersolution \bar{u} at the point $(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0)$. Notice that the condition $\tilde{s}_\varepsilon > 0$ is verified for ε small enough. This yields

$$-\frac{1}{\varepsilon}(\tilde{s}_\varepsilon - \tilde{t}_\varepsilon) - (\tilde{s}_\varepsilon - \tilde{t}_0) + v_0 \cdot \left(-\frac{1}{\varepsilon}(\tilde{y}_\varepsilon - \tilde{x}_\varepsilon) - (\tilde{y}_\varepsilon - \tilde{x}_0) \right) - 1 \geq 0. \quad (6.18)$$

eq:chain ru

We strongly emphasize a crucial cancellation. Namely, the free transport operator cancels the velocity penalization $(|v|^2 - B_1|x - tv|^2 - R^2)_+$. This is the main motivation for considering such an expression.

On the other hand, using the test function

$$\begin{aligned} \phi_1(t, x, v) = & \bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v) + \frac{\delta}{4} e^{\gamma(t - T)} |x|^4 \\ & + \frac{\alpha}{T - t} + \frac{1}{2\varepsilon} (|t - \tilde{s}_\varepsilon|^2 + |x - \tilde{y}_\varepsilon|^2) + \frac{1}{2} (|\tilde{s}_\varepsilon - \tilde{t}_0|^2 + |\tilde{y}_\varepsilon - \tilde{x}_0|^2) \\ & + \Lambda (|v|^2 - B_1|x - tv|^2 - R^2)_+, \end{aligned}$$

associated to the subsolution \underline{u} at the point $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v_0)$, we obtain

$$\frac{\alpha}{(T - \tilde{t}_\varepsilon)^2} + \frac{\delta}{4} \gamma e^{\gamma(\tilde{t}_\varepsilon - T)} |\tilde{x}_\varepsilon|^4 + \frac{1}{\varepsilon} (\tilde{t}_\varepsilon - \tilde{s}_\varepsilon) + v_0 \cdot \left(\delta e^{\gamma(\tilde{t}_\varepsilon - T)} |\tilde{x}_\varepsilon|^2 \tilde{x}_\varepsilon + \frac{1}{\varepsilon} (\tilde{x}_\varepsilon - \tilde{y}_\varepsilon) \right) - 1 \leq 0. \quad (6.19)$$

eq:chain ru

By subtracting (6.19) to (6.18), we obtain

$$\frac{\alpha}{T^2} \leq \frac{\alpha}{(T - \tilde{t}_\varepsilon)^2} \leq \delta e^{\gamma(\tilde{t}_\varepsilon - T)} \left(-\frac{1}{4} \gamma |\tilde{x}_\varepsilon|^4 - |\tilde{x}_\varepsilon|^2 v_0 \cdot \tilde{x}_\varepsilon \right) + (\tilde{s}_\varepsilon - \tilde{t}_0) - v_0 \cdot (\tilde{y}_\varepsilon - \tilde{x}_0).$$

We deduce from Lemma 6.13 and $\lim_{\varepsilon \rightarrow 0} \tilde{x}_\varepsilon = \tilde{x}_0$ that

$$|v_0| \leq C_1 |\tilde{x}_\varepsilon| + C_2 R + C_3,$$

for ε small enough. Now observe that

$$-\frac{1}{4} \gamma |\tilde{x}_\varepsilon|^4 - |\tilde{x}_\varepsilon|^2 v_0 \cdot \tilde{x}_\varepsilon \leq -\frac{1}{4} C |\tilde{x}_\varepsilon|^4 + |\tilde{x}_\varepsilon|^3 |v_0| \leq \left(C_1 - \frac{1}{4} \gamma \right) |\tilde{x}_\varepsilon|^4 + C_2 R |\tilde{x}_\varepsilon|^3 + C_3 |\tilde{x}_\varepsilon|^3 \leq K(R),$$

for some explicit constant $K(R)$, provided that $\gamma > 4C$. By taking the limit $\varepsilon \rightarrow 0$, we deduce that

$$\frac{\alpha}{T^2} \leq \delta e^{\gamma(\hat{t}_\varepsilon - T)} K \leq \delta K(R).$$

The choice $\delta = (\alpha T^{-2} K(R)^{-1})/2$ yields a contradiction.

Case 2: $\omega = \Omega$. We denote by (\hat{t}_0, \hat{x}_0) a maximum point of $\hat{\chi}$.

Case 2.1: We first consider the case where $\mathcal{S}(\bar{u})(\hat{t}_0, \hat{x}_0) = \{0\}$.

In this case, the spatial derivative of the transport equation is not used, however the time derivative of the test function being different, one has to perform slight changes. The new function $\hat{\chi}_\varepsilon$ is

$$\begin{aligned} \hat{\chi}_\varepsilon(t, x, s, y) = & \underline{m}(t, x) - \bar{m}(s, y) - \frac{\delta}{4} e^{\gamma(t-T)} |x|^4 \\ & - \frac{\alpha}{T-t} - \frac{1}{2\varepsilon} (|t-s|^2 + |x-y|^2) - \frac{1}{2} (|s-\hat{t}_0|^2 + |y-\hat{x}_0|^2). \end{aligned}$$

lem:26 A

Lemma 6.14. *There exists $\varepsilon_0 > 0$ such that the set $\mathcal{S}(\bar{u})(\hat{s}_\varepsilon, \hat{y}_\varepsilon)$ is reduced to $\{0\}$, provided that $\varepsilon < \varepsilon_0$.*

We now use the test function

$$\begin{aligned} \psi_2(s, y) = & \underline{m}(\hat{t}_\varepsilon, \hat{x}_\varepsilon) - \frac{\delta}{4} e^{\gamma(\hat{t}_\varepsilon - T)} |\hat{x}_\varepsilon|^4 \\ & - \frac{\alpha}{T - \hat{t}_\varepsilon} - \frac{1}{2\varepsilon} (|\hat{t}_\varepsilon - s|^2 + |\hat{x}_\varepsilon - y|^2) - \frac{1}{2} (|s - \hat{t}_0|^2 + |y - \hat{x}_0|^2), \end{aligned}$$

associated to the supersolution \bar{m} at the point $(\hat{s}_\varepsilon, \hat{y}_\varepsilon)$. Notice that the condition $\hat{s}_\varepsilon > 0$ is verified for ε small enough.

The second criterion in (??) writes as follows,

$$-\frac{1}{\varepsilon} (\hat{s}_\varepsilon - \hat{t}_\varepsilon) - (\hat{s}_\varepsilon - \hat{t}_0) \geq 0. \quad (6.20)$$

eq:chain ru

On the other hand, using the test function

$$\begin{aligned} \psi_1(t, x) = & \bar{m}(\hat{s}_\varepsilon, \hat{y}_\varepsilon) + \frac{\delta}{4} e^{\gamma(t-T)} |x|^4 \\ & + \frac{\alpha}{T-t} + \frac{1}{2\varepsilon} (|t-\hat{s}_\varepsilon|^2 + |x-\hat{y}_\varepsilon|^2) + \frac{1}{2} (|\hat{s}_\varepsilon - \hat{t}_0|^2 + |\hat{y}_\varepsilon - \hat{x}_0|^2), \end{aligned}$$

associated to the subsolution \underline{m} at the point $(\hat{t}_\varepsilon, \hat{x}_\varepsilon)$, we obtain

$$\frac{\delta}{4} \gamma e^{\gamma(\hat{t}_\varepsilon - T)} |\hat{x}_\varepsilon|^4 + \frac{\alpha}{(T - \hat{t}_\varepsilon)^2} + \frac{1}{\varepsilon} (\hat{t}_\varepsilon - \hat{s}_\varepsilon) \leq 0. \quad (6.21)$$

eq:chain ru

By subtracting (??) to (??), we obtain

$$-(\hat{s}_\varepsilon - \hat{t}_0) \geq \frac{\alpha}{(T - \hat{t}_\varepsilon)^2} + \frac{\delta}{4} \gamma e^{\gamma(\hat{t}_\varepsilon - T)} |\hat{x}_\varepsilon|^4 \geq \frac{\alpha}{T^2}.$$

Passing to the limit $\varepsilon \rightarrow 0$, we get a contradiction.

Case 2.2: There exists some nonzero $v_0 \in \mathcal{S}(\bar{u})(\hat{t}_0, \hat{x}_0)$.

Omega eq A

Lemma 6.15. *Assume $\omega = \Omega$, and let $v_0 \in \mathcal{S}(\bar{u})(\hat{t}_0, \hat{x}_0) \setminus \{0\}$. Then,*

1. $(\hat{t}_0, \hat{x}_0, v_0)$ realizes the supremum of $\tilde{\chi}$,
2. v_0 is a minimum velocity also for $\underline{u}(\hat{t}_0, \hat{x}_0, \cdot)$.

Similarly as in Case 1, we define the following auxiliary function,

$$\begin{aligned} \tilde{\chi}_\varepsilon(t, x, s, y, v) = & \underline{u}(t, x, v) - \bar{u}(s, y, v) - \frac{\delta}{4} e^{\gamma(t-T)} |\hat{x}_\varepsilon|^4 \\ & - \frac{\alpha}{T-t} - \frac{1}{2\varepsilon} (|t-s|^2 + |x-y|^2) \\ & - \frac{1}{2} (|s-\hat{t}_0|^2 + |y-\hat{x}_0|^2) - \Lambda (|v|^2 - B_1|x-tv|^2 - R^2)_+ . \end{aligned}$$

Let $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon)$ which realizes the maximum value of $\tilde{\chi}_\varepsilon(\cdot, v_0)$. We can prove as in Lemma ??, that the following limit holds true,

$$\lim_{\varepsilon \rightarrow 0} (\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{s}_\varepsilon, \tilde{y}_\varepsilon) = (\hat{t}_0, \hat{x}_0, \hat{t}_0, \hat{x}_0) .$$

constraint2 A

Lemma 6.16. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,*

$$\bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0) - \bar{m}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon) - \frac{|v_0|^2}{2} < 0 .$$

Therefore, we can use the test function

$$\begin{aligned} \phi_2(s, y, v) = & \underline{u}(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v) - \frac{\delta}{4} e^{\gamma(\tilde{t}_\varepsilon-T)} |\tilde{x}_\varepsilon|^4 - \frac{\alpha}{T-\tilde{t}_\varepsilon} \\ & - \frac{1}{2\varepsilon} (|\tilde{t}_\varepsilon - s|^2 + |\tilde{x}_\varepsilon - y|^2) - \frac{1}{2} (|s-\hat{t}_0|^2 + |y-\hat{x}_0|^2) \\ & - \Lambda (|v|^2 - B_1|\tilde{x}_\varepsilon - \tilde{t}_\varepsilon v|^2 - R^2)_+ , \end{aligned}$$

associated to the supersolution \bar{u} at the point $(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v_0)$. Notice that the condition $\tilde{s}_\varepsilon > 0$ is verified for ε small enough. This yields

$$-\frac{1}{\varepsilon} (\tilde{s}_\varepsilon - \tilde{t}_\varepsilon) - (\tilde{s}_\varepsilon - \hat{t}_0) + v_0 \cdot \left(-\frac{1}{\varepsilon} (\tilde{y}_\varepsilon - \tilde{x}_\varepsilon) - (\tilde{y}_\varepsilon - \hat{x}_0) \right) - 1 \geq 0 . \quad (6.22)$$

On the other hand, using the test function

$$\begin{aligned} \phi_1(t, x, v) = & \bar{u}(\tilde{s}_\varepsilon, \tilde{y}_\varepsilon, v) + \frac{\delta}{4} e^{\gamma(t-T)} |x|^4 + \frac{\alpha}{T-t} \\ & + \frac{1}{2\varepsilon} (|t-\tilde{s}_\varepsilon|^2 + |x-\tilde{y}_\varepsilon|^2) + \frac{1}{2} (|\tilde{s}_\varepsilon - \hat{t}_0|^2 + |\tilde{y}_\varepsilon - \hat{x}_0|^2) \\ & + \Lambda (|v|^2 - B_1|x-tv|^2 - R^2)_+ , \end{aligned}$$

associated to the subsolution \underline{u} at the point $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, v_0)$, we obtain

$$\frac{\delta}{4} \gamma e^{\gamma(\tilde{t}_\varepsilon-T)} |\tilde{x}_\varepsilon|^4 + \frac{\alpha}{(T-\tilde{t}_\varepsilon)^2} + \frac{1}{\varepsilon} (\tilde{t}_\varepsilon - \tilde{s}_\varepsilon) + v_0 \cdot \left(\delta e^{\gamma(\tilde{t}_\varepsilon-T)} |\tilde{x}_\varepsilon|^2 \tilde{x}_\varepsilon + \frac{1}{\varepsilon} (\tilde{x}_\varepsilon - \tilde{y}_\varepsilon) \right) - 1 \leq 0 . \quad (6.23)$$

We conclude as in Case 1, except that the confinement estimate on v_0 is given by (??) rather than Lemma ???. The proof of the comparison principle can be ended exactly as in Section ???. \square

Reproducing the same proof as for the proof of Theorem ??, we obtain as a corollary a comparison principle for the constrained system (??).

Corollary 6.17 (Comparison principle for the constrained system). *Let \underline{u} (resp. \bar{u}) be a viscosity sub-solution (resp. super-solution) of (??) on $[0, T] \times \mathbb{R}^{2n}$. Assume that \underline{u} and \bar{u} are such that there exist constants $A > 1, B > 0, C > 0$ such that for all (x, v) ,*

$$\frac{1}{A}|v|^2 - B|x|^2 - C \leq \underline{u}(t, x, v), \bar{u}(t, x, v) \leq A|v|^2 + B|x|^2 + C. \quad (6.24)$$

Then $\underline{u} \leq \bar{u}$ on $[0, T] \times \mathbb{R}^{2n}$.

eq: growth c

6.2.2 Quadratic bounds

sec: 622

The purpose of this section is to provide crude *a priori* bounds for both the fundamental solution in the case $r = 0$, and the solution of the Cauchy problem (??) for compactly supported initial data, in the case $r \geq 0$. We show that in both cases $u^\varepsilon = -\varepsilon \log f^\varepsilon$ is naturally bounded above by $(A/2)v^2 + Bx^2/(t^2) + C$. Thus, it would be natural to extend the uniqueness result (Theorem ??) to such unbounded solutions, in order to prove that the solution of the Cauchy problem coincides with the solution built from the representation formula. In this direction, we establish such a uniqueness result for solutions satisfying a uniform quadratic bound involving $(A/2)v^2 + Bx^2 + C$ on $[0, T] \times \mathbb{R} \times \mathbb{R}$.

We deduce immediately from the expression (??) that the fundamental solution ϕ satisfies

$$\phi(t, x, v; w) \leq \frac{v^2}{2} + \frac{x^2}{2t^2} + t. \quad (6.25)$$

For this, simply choose $s_2 = t$ in the minimization problem. Note that a similar bound can be derived for ϕ_r when $r > 0$.

Equation (??) has the property of infinite speed of propagation. We make this property quantitative in the next proposition.

For the sake of simplicity, we restrict to initial conditions compactly supported in x , with a Gaussian velocity distribution,

$$f_0^\varepsilon(x, v) = \left[\mathbf{1}_{\{G_0\}}(x) \right] \cdot \left[\varepsilon^{1/2} M_\varepsilon(v) \right], \quad (6.26)$$

for some is some open interval G_0 .

The next proposition establishes quantitative estimates of $f^\varepsilon(t, x, v)$ from below, which yields some locally uniform upper bound for $u^\varepsilon(t, x, v)$.

peestimate

Proposition 6.18. *For all $\varepsilon > 0$, for all $(t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$, one has*

$$\frac{v^2}{2} - \frac{\varepsilon}{2} \log(2\pi\varepsilon) \leq u^\varepsilon(t, x, v) \leq \frac{v^2}{2} + t + \frac{1}{2} \left(\frac{|x| + (t/2)|v| + 1}{t/2} \right)^2.$$

Proof of Proposition ??. First, we claim that we can restrict to the case without the non-linear term ($r = 0$), since the contribution $r\rho(\varepsilon^{1/2}M - f)$ is nonnegative. From the Duhamel formula, we deduce that f^ε is bounded below by the damped free transport equation,

$$f^\varepsilon(t, x, v) \geq f_0(x - tv, v)e^{-t/\varepsilon}. \quad (6.27)$$

Hence, we have

$$\begin{aligned}\rho^\varepsilon(t, x) &\geq \int_{\mathbb{R}} f_0(x - tw, w) e^{-t/\varepsilon} dw \\ &\geq e^{-t/\varepsilon} \int_{\mathbb{R}} \mathbf{1}_{\{G_0\}}(x - tw) \exp\left(-\frac{w^2}{2\varepsilon}\right) dw\end{aligned}$$

Up to elementary spatial rescaling, we assume that G_0 contains the interval $(-1, 1)$. Plugging this bound into the Duhamel formula with $r = 0$, we deduce

$$\begin{aligned}f^\varepsilon(t, x, v) &\geq M_\varepsilon(v) \int_0^t \left(e^{-(t-s)/\varepsilon} \int_{\mathbb{R}} \mathbf{1}_{\{G_0\}}(x - sv - (t-s)w) \exp\left(-\frac{w^2}{2\varepsilon}\right) dw \right) e^{-s/\varepsilon} ds \\ &\geq M_\varepsilon(v) e^{-t/\varepsilon} \int_0^t \int_{\frac{x-sv-1}{t-s}}^{\frac{x-sv+1}{t-s}} \exp\left(-\frac{w^2}{2\varepsilon}\right) dw ds \\ &\geq M_\varepsilon(v) e^{-t/\varepsilon} \int_0^t \int_{\frac{x-sv-1}{t-s}}^{\frac{x-sv+1}{t-s}} \exp\left(-\frac{1}{2\varepsilon} \max\left(\left(\frac{x-sv-1}{t-s}\right)^2, \left(\frac{x-sv+1}{t-s}\right)^2\right)\right) dw ds \\ &\geq M_\varepsilon(v) e^{-t/\varepsilon} \int_0^{\frac{t}{2}} \frac{2}{t-s} \exp\left(-\frac{1}{2\varepsilon} \left(\frac{|x| + s|v| + 1}{t-s}\right)^2\right) ds \\ &\geq M_\varepsilon(v) e^{-t/\varepsilon} \exp\left(-\frac{1}{2\varepsilon} \left(\frac{|x| + (t/2)|v| + 1}{t/2}\right)^2\right).\end{aligned}$$

This estimate from below, combined with (??) yields the following estimate on u^ε ,

$$\frac{v^2}{2} - \frac{\varepsilon}{2} \log(2\pi\varepsilon) \leq u^\varepsilon(t, x, v) \leq \frac{v^2}{2} + t + \frac{1}{2} \left(\frac{|x| + (t/2)|v| + 1}{t/2}\right)^2. \quad (6.28)$$

□

6.3 Non-local Freidlin's condition

Here, we restrict to dimension $n = 1$.

We now continue our discussion about the limit system. This limit system appears to be an obstacle problem, subject to the constraint $\min u \geq 0$. It is analogous to the following constrained Hamilton-Jacobi equation,

$$\forall(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \min(\partial_t U + D|\nabla_x U|^2 + r, U) = 0,$$

which can be derived from the Fisher-KPP equation (??) using the same methodology as described above [?] (see also [?] for earlier works using a different framework based on large deviations).

6.3.1 The case of the Fisher-KPP problem

Assume that the initial data $\tilde{U}_0 = U_0$ is an indicator function of some open bounded interval G_0 (as in [?, ?]),

$$U_0 = \mathbf{0}_{x \in G_0}. \quad (6.29)$$

The so-called Freidlin condition allows to solve this obstacle problem by the following truncation procedure: First, solve the unconstrained problem

$$\begin{cases} \partial_t \tilde{U} + D|\nabla_x \tilde{U}|^2 + r = 0 & (t, x) \in (0, +\infty) \times \mathbb{R} \\ U(0, x) = 0 & x \in G_0 \\ \lim_{t \rightarrow 0} U(t, x) = +\infty & x \in \mathbb{R} \setminus G_0 \end{cases} \quad (6.30) \quad \boxed{\text{eq:FKPP U}}$$

Then, truncate the solution \tilde{U} by taking into account the constraint $\min U \geq 0$:

$$U(t, x) = \max \left(\tilde{U}(t, x), 0 \right)$$

The solution \tilde{U} can be written using the Hopf-Lax representation formula,

$$\begin{aligned} \tilde{U}(t, x) &= \inf_{x \in \mathbf{H}_{loc}^1((0, t), \mathbb{R}^n)} \left\{ \int_0^t \left[\frac{|\dot{x}(s)|^2}{2} - r \right] ds + \tilde{U}_0(x(0)) \mid x(t) = x \right\} \\ &= \inf_{x \in \mathbf{H}_{loc}^1((0, t), \mathbb{R}^n)} \left\{ \int_0^t \left[\frac{|\dot{x}(s)|^2}{2} - r \right] ds \mid x(0) \in \overline{G_0}, x(t) = x \right\}. \end{aligned}$$

The Freidlin condition states that, starting from a final point (t, x) such that $\tilde{U}(t, x) > 0$, then tracing backward a minimizing trajectory $x(s)$, it verifies $\tilde{U}(s, x(s)) > 0$ for all $s \in (0, t)$. This implies the following modification of the Hopf-Lax formula,

$$\tilde{U}(t, x) = \inf_{x \in \mathbf{H}_{loc}^1((0, t), \mathbb{R}^n)} \left\{ \int_0^t \left[\frac{|\dot{x}(s)|^2}{2} - r \right] ds \mid x(0) \in \overline{G_0}, x(t) = x, (\forall s) \tilde{U}(s, x(s)) > 0 \right\},$$

Alternatively speaking, trajectories ending in the unconstrained area (*i.e.* $\tilde{U} > 0$) have gone through the unconstrained area only. The Freidlin condition holds true for the unconstrained problem (??).

6.3.2 Formal extension to the kinetic transport-reaction problem

Here, we assume that the initial data $\tilde{u}_0 = u_0$ is the indicator function of some open interval G_0 in the space variable,

$$u_0(x, v) = \mathbf{1}_{x \in G_0} + \frac{v^2}{2}. \quad (6.31)$$

For the sake of convenience, we set $G_0 = (-1, 0)$.

Variables are changed as previously: $(t', x', v) = (t'/(1+r), x'/(1+r), v)$.

The following representation formula of the unconstrained problem (??), analogous to the Hopf-Lax formula, is solution to the non-local Hamilton-Jacobi system,

$$\tilde{u}(t, x, v) = \inf_{(y, w) \in G_0 \times \mathbb{R}} \left(\phi_r(t, x - y, v; w) + \frac{w^2}{2} \right). \quad (6.32) \quad \boxed{\text{eq:hopf lax}}$$

In Section ??, we derived explicit formulas for extremal trajectories. Let define the following set,

$$\mathcal{Z}_+ := \left\{ (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \mid t \leq \frac{x}{v} \right\}.$$

An important observation is that extremal trajectories ending in \mathcal{Z}_+ remain in \mathcal{Z}_+ in backward time.

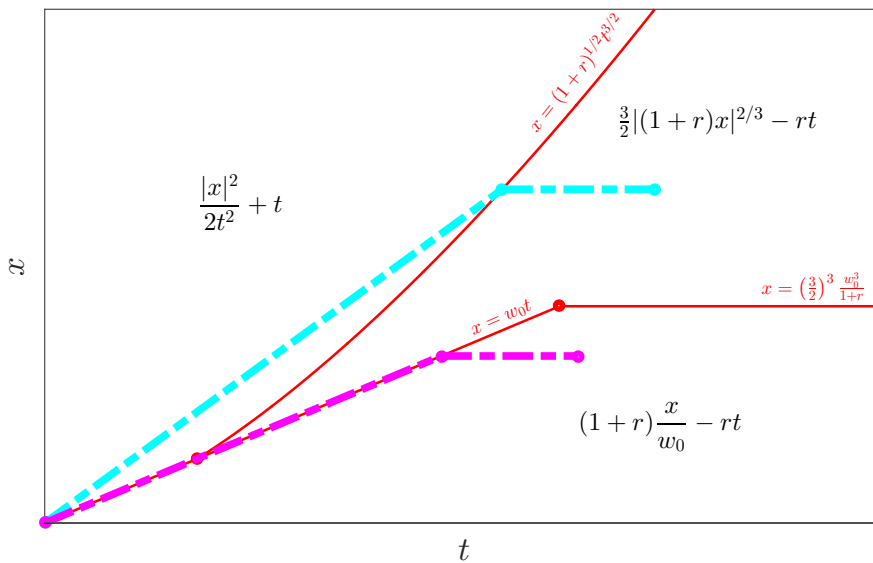


Figure 4: Extremal trajectories in the zone \mathcal{Z}_+ , superimposed with the possible values of the function $\mu_r(t, x, w)$: case 1. (blue) and case 2. (magenta).

Let $(t, x, v) \in \mathcal{Z}_+$. Clearly, $y = 0$ is an optimal choice in (??). We deduce from Proposition ?? that an extremal trajectory joining $(0, w_0)$ and (x, v) is necessarily given by one of the two following choices,

traj1

1.

$$x(\tau) := \begin{cases} \frac{\tau x}{s}, & \text{if } 0 \leq \tau \leq s \\ x, & \text{if } s \leq \tau \leq t, \end{cases} \quad v(\tau) := \begin{cases} \frac{x}{s}, & \text{if } 0 \leq \tau \leq s \\ 0, & \text{if } s \leq \tau \leq t, \end{cases}$$

where the transition occurs at time

$$s = \min \left(t, (1+r)^{-\frac{1}{3}} |x|^{\frac{2}{3}} \right).$$

traj2

2.

$$x(\tau) := \begin{cases} w_0 \tau, & \text{if } 0 \leq \tau \leq \frac{x}{w_0} \\ x, & \text{if } \frac{x}{w_0} \leq \tau \leq t, \end{cases} \quad v(\tau) := \begin{cases} w_0, & \text{if } 0 \leq \tau \leq \frac{x}{w_0} \\ 0, & \text{if } \frac{x}{w_0} \leq \tau \leq t. \end{cases}$$

The choice between the first case (referred to as trajectory 1), and the second case (referred to as trajectory 2) is set up according to Proposition ??, see also Figure ?? for a sketchy diagram.

We propose the following extension of the Freidlin condition: Let $(x, v) \in H^1(0, t)$ be an extremal trajectory such that the ending point $(t, x(t), v(t)) = (t, x, v)$ belongs to \mathcal{Z}_+ . Assume that (t, x, v) is such that $\min_{v'} \tilde{u}(t, x, v') > 0$, then for all $\tau \in (0, t)$,

$$\min_{v' \in \mathbb{R}} \tilde{u}(\tau, x(\tau), v') > 0. \quad (6.33)$$

We refer to it as a non-local Freidlin condition, as the sign condition is required for all w along the projected backward trajectory $(\tau, x(\tau))$.

We prove below that this condition holds true. Let $(\tau, x(\tau), v(\tau))$ be such an extremal trajectory belonging to \mathcal{Z}_+ . We denote by $(0, w_0)$ its starting point. The condition $\min_{v'} \tilde{u}(t, x, v') > 0$ is rewritten using (??)

$$(\forall w) \quad \mu_r(t, x; w) + \frac{w^2}{2} > 0, \quad (6.34) \quad \boxed{\text{eq:hyp w}}$$

and the requirement is:

$$(\forall \tau \in (0, t)) (\forall w) \quad \mu_r(\tau, x(\tau); w) + \frac{w^2}{2} > 0, \quad (6.35)$$

We can conclude, provided that the function $\tau \mapsto \mu_r(\tau, x(\tau); w) + w^2/2$ has the following monotonicity for $\tau \in (0, t)$: increasing, then decreasing. Indeed, it is nonnegative at $\tau = 0$ and at $\tau = t$ (??). Therefore, it is positive for $\tau \in (0, t)$.

ck Freidlin

Lemma 6.19. *The function $\tau \mapsto \mu_r(\tau, x(\tau); w)$ is increasing, then decreasing for $\tau \in (0, t)$.*

There is some subtlety here, because, the trajectory is associated with some parameter w_0 , which is generally different from w . It means that the typical trajectories shown in Figure ?? are not necessarily in phase with background areas delimited by the plain red curves.

Proof of Lemma ??. There is quite a number of cases to discuss. However, we can overcome the complexity of the picture, by remarking that the trajectory always finish with a constant part, $x(\tau) = x$. On this part, the function $\mu_r(\tau, x; w)$ is clearly decreasing.

To conclude, it remains to prove that the function $\mu(\tau) = \mu_r(\tau, x(\tau); w)$ is concave on the linear part of the trajectory. Let it be $x(\tau) = c\tau$, for some positive speed c . On each of the zones A, B, C , μ is concave: it is linear on A and C , and it is equal to $(3/2)(1+r)^{2/3}|c\tau|^{2/3} - r\tau$ on B . In addition, there are two possible junctions: the one from A to B , and the one from C to B (see Figure ??).

Junction from A to B (Figure ??a): let τ_0 be the time of transition from A to B . It satisfies $c\tau_0 = (1+r)^{1/2}\tau_0^{3/2}$. The slope at $\tau = \tau_0^-$ is equal to 1. The slope at $\tau = \tau_0^+$ is

$$(1+r)^{2/3}c^{2/3}\tau_0^{-1/3} - r = (1+r)^{2/3}c^{2/3} \left(\frac{c^2}{1+r} \right)^{-1/3} - r = 1. \quad (6.36)$$

There is no slope discontinuity at this junction.

Junction from C to B (Figure ??b): let τ_0 be the time of transition from C to B . It satisfies $(1+r)c\tau_0 = (3/2)^3w^3$. The slope at $\tau = \tau_0^-$ is

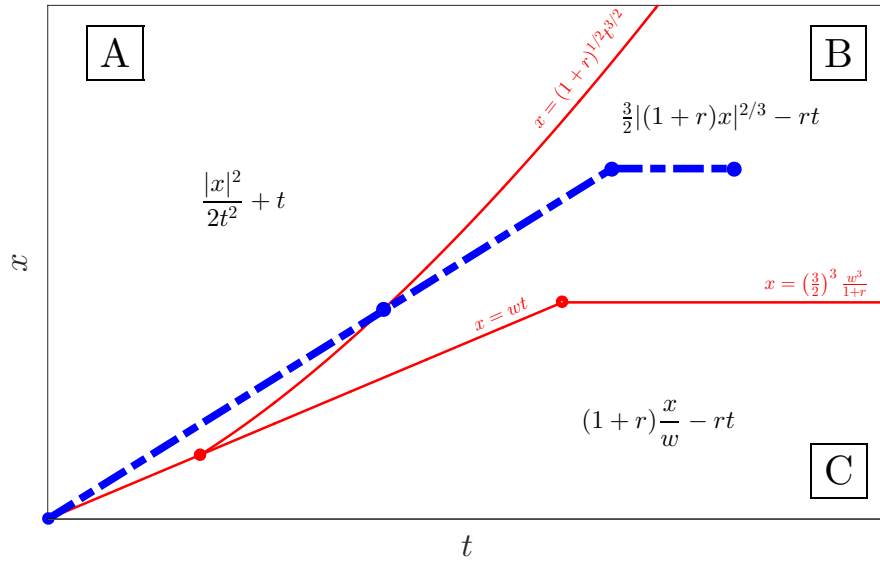
$$(1+r)\frac{c}{w} - r. \quad (6.37)$$

The slope at $\tau = \tau_0^+$ is

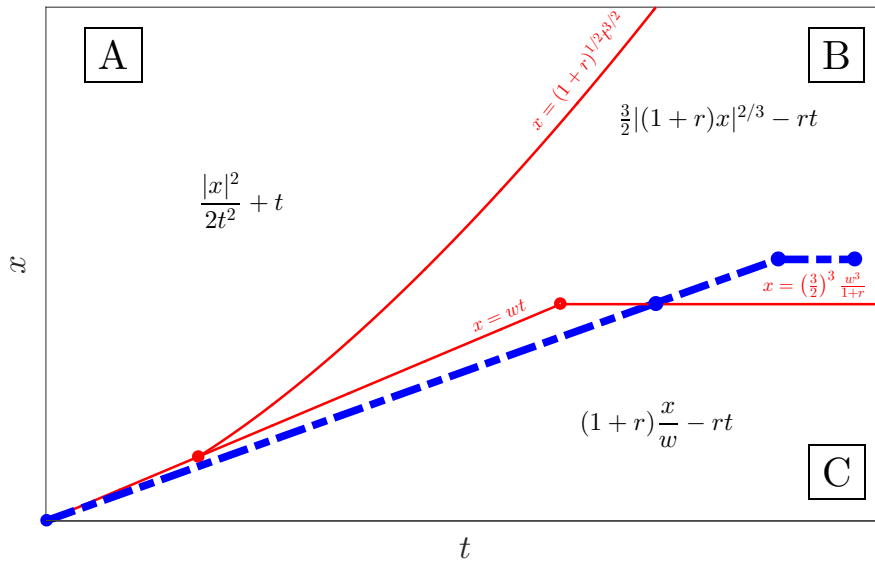
$$(1+r)^{2/3}c^{2/3}\tau_0^{-1/3} - r = (1+r)^{2/3}c^{2/3} \left(\left(\frac{3}{2} \right)^3 \frac{w^3}{(1+r)c} \right)^{-1/3} - r = \frac{2}{3}(1+r)\frac{c}{w} - r. \quad (6.38)$$

The slope is necessarily decreasing at this junction.

We conclude that μ is globally concave on the linear part of the trajectory, wherever it goes through. Then, it is decreasing on the constant part of the trajectory. Consequently, μ has the required monotonicity. \square



(a)



(b)

Figure 5: Extremal trajectories associated with the initial velocity w_0 , superimposed with the possible values of the function $\mu_r(t, x, w)$, for $w \neq w_0$. There are two possible transitions occurring on the linear part of the trajectory: (a) transition from A to B, and (b) transition from C to B. The concavity of $\mu_r(\tau, x(\tau); w)$ across each of these junctions is proven.

fig:junction

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