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LIPSCHITZ REGULARITY RESULTS FOR NONLINEAR STRICTLY ELLIPTIC EQUATIONS AND APPLICATIONS

OLIVIER LEY AND VINH DUC NGUYEN

ABSTRACT. Most of Lipschitz regularity results for nonlinear strictly elliptic equations are obtained for a suitable growth power of the nonlinearity with respect to the gradient variable (subquadratic for instance). For equations with superquadratic growth power in gradient, one usually uses weak Bernstein-type arguments which require regularity and/or convex-type assumptions on the gradient nonlinearity. In this article, we obtain new Lipschitz regularity results for a large class of nonlinear strictly elliptic equations with possibly arbitrary growth power of the Hamiltonian with respect to the gradient variable using some ideas coming from Ishii-Lions' method. We use these bounds to solve an ergodic problem and to study the regularity and the large time behavior of the solution of the evolution equation.

1. INTRODUCTION

The main goal of this work is to obtain gradient bounds, which are uniform in $\epsilon > 0$ and t respectively, for the viscosity solutions of a large class of nonlinear strictly elliptic equations

$$(1.1) \quad \epsilon v^\epsilon - \text{trace}(A(x)D^2v^\epsilon) + H(x, Dv^\epsilon) = 0, \quad x \in \mathbb{T}^N,$$

and

$$(1.2) \quad \begin{cases} \frac{\partial u}{\partial t} - \text{trace}(A(x)D^2u) + H(x, Du) = 0, & (x, t) \in \mathbb{T}^N \times (0, +\infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{T}^N. \end{cases}$$

We work in the periodic setting (\mathbb{T}^N denotes the flat torus $\mathbb{R}^N/\mathbb{Z}^N$) and assume for simplicity that $A(x) = \sigma(x)\sigma(x)^T$ with $\sigma \in W^{1,\infty}(\mathbb{T}^N; \mathcal{M}_N)$. Let us mention that all the results of this paper hold true if $\sigma \in C^{0,1/2}(\mathbb{T}^N; \mathcal{M}_N)$.

We recall that a diffusion matrix A is called strictly elliptic if

$$(1.3) \quad \text{there exists } \nu > 0 \text{ such that } A(x) \geq \nu I, \quad x \in \mathbb{T}^N.$$

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Most of Lipschitz regularity results for elliptic equations are obtained for a suitable growth power with respect to the gradient variable (subquadratic for instance, see Frehse [14], Gilbarg-Trudinger [15]). In this article, we establish some gradient bounds

$$(1.4) \quad |Dv^\epsilon|_\infty \leq K, \text{ where } K \text{ is independent of } \epsilon,$$

$$(1.5) \quad |Du(\cdot, t)|_\infty \leq K, \text{ where } K \text{ is independent of } t,$$

for strictly elliptic equations whose Hamiltonians H have arbitrary growth power in the gradient variable, which is unusual.

An important feature of our work is that we look for uniform gradient bounds in ϵ or t . In many results, the bounds depend crucially on the L^∞ norm of the solution (which looks like $O(\epsilon^{-1})$ or $O(t)$), something we want to avoid in order to be able to solve some ergodic problems by sending $\epsilon \rightarrow 0$ or to study the large time behavior of $u(x, t)$ when $t \rightarrow +\infty$. These applications are discussed more in details below and are done in Section 4. We focus now on the more delicate part, i.e., the Lipschitz bounds for (1.1).

Let us start by recalling the existing results when H is superquadratic and coercive. Hölder regularity of the solution is proved under the very general assumption

$$H(x, p) \geq \frac{1}{C}|p|^k - C, \quad \text{with } k > 2,$$

see Capuzzo Dolcetta et al. [10], Barles [7], Cardaliaguet-Silvestre [11], Armstrong-Tran [3]. But there are only few results as far as Lipschitz regularity is concerned. In general they are established using Bernstein method [15, 19] or the adaptation of this method in the context of viscosity solutions, see Barles [5], Barles-Souganidis [8], Lions-Souganidis [21], Capuzzo Dolcetta et al. [10]. This approach requires some structural assumptions on H which are often close to “convexity-type assumptions”. They appear naturally when differentiating the equation, a drawback of the original Bernstein method. Even if the weak Bernstein method [5] is less restrictive as far as the regularity of the data is concerned (Lipschitz continuity is enough), we do not consider this approach here to be able to deal with Hamiltonians having few regularity like Hölder continuous Hamiltonians for instance. Actually most of our assumptions do not even require the Hamiltonian to be continuous as soon as a continuous solution to the equation exists. However, let us mention that the weak Bernstein method has also several advantages: the method may be used for degenerate equations in some cases and the Hamiltonian may have arbitrary growth, see for instance [8, 10].

Instead, in this work, we use the Ishii-Lions’ method introduced in [16], see also [12, 6]. This method allows to take profit of the strict ellipticity of the equation to control the strong nonlinearities of the Hamiltonian. In Ishii-Lions [16] and Barles [4], weak regularity assumptions are assumed over H , merely a kind of balance between some Hölder continuity in x and the growth size of H with respect to the gradient, namely

$$(1.6) \quad |H(x, p) - H(y, p)| \leq \omega(|x - y|)|x - y|^\tau |p|^{2+\tau} + C \quad \text{in [16, Assumption (3.2)]},$$

or

$$(1.7) \quad |H(x, p) - H(y, p)| \leq C|x - y||p|^3 + C(1 + |p|^2) \quad \text{in [4, Assumption (3.4)]},$$

where $x, y \in \mathbb{T}^N$, $p \in \mathbb{R}^N$, $\tau \in [0, 1]$, ω is a modulus of continuity and $C > 0$. These assumptions are designed for subquadratic (or growing at most like $|p|^3$) Hamiltonians. This is not surprising since it is known that, in general, the ellipticity is not powerful enough to control nonlinearities which are more than quadratic [10]. Under these assumptions, the authors prove a Lipschitz bound, which depends however of the L^∞ norm of the solution.

Our results consists in improving the previous ones in the periodic setting. We give two new results, the first one being a slight generalization of of [16, 4] while the second one takes profit of the strong coercivity of H and allows arbitrary growth of H with respect to the gradient.

Theorem 1.1. *Assume (1.3) and H satisfies*

$$(1.8) \quad \left\{ \begin{array}{l} \text{there exists } L > 1 \text{ such that for all } x, y \in \mathbb{T}^N, \\ \text{if } |p| = L, \text{ then } H(x, p) \geq |p| \left[H(y, \frac{p}{|p|}) + |H(\cdot, 0)|_\infty + N|x - y| |\sigma_x|_\infty^2 \right]. \end{array} \right.$$

and

$$(1.9) \quad \left\{ \begin{array}{l} \text{There are constants } \alpha > 0, C \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N, p \in \mathbb{R}^N, |H(x, p) - H(y, p)| \leq C|x - y|^\alpha |p|^{\alpha+2} + C(1 + |p|^2). \end{array} \right.$$

Then, there exists $K > 0$ such that for all $\epsilon > 0$, any continuous solution v^ϵ of (1.1) satisfies (1.4).

Theorem 1.2. *Assume (1.3) and H satisfies*

$$(1.10) \quad \text{there exist constants } k > 2, C > 0 \text{ such that } H(x, p) \geq \frac{1}{C}|p|^k - C$$

and

$$(1.11) \quad \left\{ \begin{array}{l} \text{there exist a modulus of continuity } \omega \text{ and constants } \alpha \in [0, 1], \beta < k - 1 \\ \text{such that for all } x, y \in \mathbb{T}^N, p \in \mathbb{R}^N, \\ |H(x, p) - H(y, p)| \leq \omega((1 + |p|^\beta)|x - y|) |x - y|^\alpha |p|^{(k-1)\alpha+k} + o(|p|^k), \end{array} \right.$$

where $o(|p|^k)/|p|^k \rightarrow 0$ as $|p| \rightarrow +\infty$, uniformly with respect to $x \in \mathbb{T}^N$. Then, there exists $K > 0$ such that for all $\epsilon > 0$, any continuous solution v^ϵ of (1.1) satisfies (1.4).

Before giving some comments about these results, let us explain in a formal way the strategy to establish them. The proof follows roughly the same lines as the one in [4]. We aim at proving that the maximum

$$\max_{x, y \in \mathbb{T}^N} \{v^\epsilon(x) - v^\epsilon(y) - \psi(|x - y|)\}$$

is nonnegative, choosing in a first step $\psi(r) = Lr^\alpha$, $\alpha \in (0, 1)$, to obtain a Hölder bound, and, in a second step, $\psi(r) = L(r - r^{1+\alpha})$, to improve the Hölder bound into a Lipschitz one. To do this, we use in a crucial way the strict concave behavior of ψ near 0 to take profit of the strict ellipticity of the equation as usual in Ishii-Lions' method.

The first notable difference with the previous works is that we are able to force the maximum to be achieved at (\bar{x}, \bar{y}) with $r := |\bar{x} - \bar{y}|$ enough close to 0 without increasing L in terms of the L^∞ norm of v^ϵ . This is a consequence of an a priori oscillation bound

$$(1.12) \quad \text{osc}(v^\epsilon) := \sup_{\mathbb{T}^N} v^\epsilon - \inf_{\mathbb{T}^N} v^\epsilon \leq K, \quad \text{where } K \text{ is independent of } \epsilon,$$

obtained by the authors [18] for any continuous solution of (1.1) when merely (1.8) holds. Let us underline that this oscillation bound is a crucial tool in our work and that the assumption (1.8) is very general; it is satisfied as soon as

$$(1.13) \quad \limsup_{|p| \rightarrow +\infty} \frac{H(x, p)}{|p|} = +\infty \text{ uniformly with respect to } x.$$

We extend the oscillation bound in the parabolic setting, see Lemma 4.5, and give an application.

The second step starts by noticing that, once we have on hands a Hölder bound, then the strength of the nonlinearity is weakened. We can apply again Ishii-Lions' method in a context where the ellipticity is reinforced compared to the nonlinearity, even when the Hamiltonian has a large growth with respect to the gradient. It allows to improve the regularity up to Lipschitz continuity. This is one of the main novelty to obtain the gradient bounds. Then, a careful study of the balance between both terms finally gives the best exponents.

Let us comment our results. Theorem 1.1 reduces to [4, III.1] when $\alpha = 1$. But notice that our Lipschitz bound does not depend on the L^∞ bound of the solution and we are able to deal with Hamiltonians having less regularity with respect to x . For instance, our result applies when

$$(1.14) \quad H(x, p) = |\Sigma(x)p|^m + G(x, p), \quad m \leq \alpha + 2, \Sigma \in C^{0,\alpha}(\mathbb{T}^N; \mathcal{M}_N),$$

and G satisfies (1.13) (superlinearity) and $|G(x, p)| \leq C(1 + |p|^2)$ (subquadratic) without any regularity condition on G .

In Theorem 1.2, the coercivity assumption (1.10) is the one needed to obtain the Hölder regularity with exponent $\frac{k-2}{k-1}$ in [10]. Notice that, this estimate being independent of ϵ , we get for free the oscillation bound (1.12). The first step in this case consists in showing that the solution is γ -Hölder continuous for any $\gamma \in (\frac{k-2}{k-1}, 1)$. It then allows us to improve the regularity up to Lipschitz continuity. In (1.11), the growth power with respect to the gradient variable can be much greater than $k > 2$, which enlarges the class of Hamiltonians under which our result applies. Let us emphasize that the situation is very different comparing to Theorem 1.1 where we can start with any Hölder exponent to get the Lipschitz regularity. Here, starting with a Hölder exponent equal to $\frac{k-2}{k-1}$ seems crucial to be able to improve the regularity when H has a strong growth with respect to the gradient.

As examples of applications of Theorem 1.2, we can deal with some new classes of Hamiltonians for which the existing regularity theory does not apply. We can first consider

again (1.14), where now there exists $k > 2$ such that

$$k \leq m \leq (k-1)\alpha + k, \quad \Sigma > 0 \quad \text{and} \quad \frac{G(x, p)}{|p|^k} \xrightarrow{|p| \rightarrow +\infty} 0.$$

Notice that even if Σ is now assumed to be nondegenerate, this Hamiltonian is not necessarily convex.

The Hamiltonian

$$H(x, p) = a(x)h(p) + G(x, p), \quad k > 2, \quad \frac{|p|^k}{C} \leq h(p) \leq C|p|^k, \quad \frac{G(x, p)}{|p|^k} \xrightarrow{|p| \rightarrow +\infty} 0,$$

where a is merely continuous and positive, satisfies all the assumptions of Theorem 1.2 and is not convex in general.

Let us give another example which will be used in Section 4.2 to extend the results to the parabolic case (1.2) and in Section 4.4 to prove an existence result in a quite surprising situation. Let K be any continuous function satisfying $K(x, p) \leq C(|p|^M + 1)$, for any $x \in \mathbb{T}^N, p \in \mathbb{R}^N, M > 2$. Then, the function

$$H(x, p) = K(x, p) + \alpha|p|^{M+\delta}, \quad \alpha > 0, \delta > 0$$

satisfies all the assumptions of Theorem 1.2. These examples also illustrate the few regularity assumptions on the datas which are needed.

Our work takes place in the periodic setting to take profit of the compactness and the absence of boundary of \mathbb{T}^N . The issue of extending our results in a bounded set is very interesting and not obvious. In the case of Neumann boundary conditions, it should be true but the case of Dirichlet boundary conditions faces the problem of loss of boundary conditions when H is superquadratic [9]. Notice that we cannot expect such general results to be true in a general bounded set since it is known [10] that $\frac{k-2}{k-1}$ -Hölder continuity is optimal in general. Our results can be extended for $A = \sigma\sigma^T$ with $\sigma \in C^{0,1/2}(\mathbb{T}^N; \mathcal{M}_N)$, for quasilinear equations when $A = A(x, p)$ and for fully nonlinear equations of Bellman-Isaacs type, see Section 2.5 for a discussion.

To study the well-posedness of (1.1) under the assumptions of Theorems 1.1 and 1.2, we have first to prove a comparison principle (Theorem 3.2) whose proof is not classical since the Hamiltonian is not Lipschitz continuous with respect to the gradient. Instead, we use the same ideas as for the proof of the Lipschitz bounds. As a consequence, we obtain the existence and uniqueness of a continuous viscosity solution to (1.1). Moreover this solution is Lipschitz continuous and, if the datas are C^∞ , then the solution is C^∞ thanks to the classical elliptic regularity theory. Let us mention that our approach also allows to construct Hölder continuous solutions to (1.1) (Theorem 4.9) under the general assumption (4.20) which is not sufficient to provide a comparison principle.

We then give several applications of our results. A straightforward consequence to the bound (1.4) is the solvability of the ergodic problem associated with (1.1), see [20, 2] and Theorem 4.1: there exists $(c, v^0) \in \mathbb{R} \times W^{1,\infty}(\mathbb{T}^N)$ solution to

$$(1.15) \quad -\text{trace}(A(x)D^2v^0) + H(x, Dv^0) = c, \quad x \in \mathbb{T}^N.$$

The next application is the study of the parabolic equation (1.2). The natural idea to extend the gradient bound for (1.1) to (1.2) is to prove first a bound for the time derivative $|\frac{\partial u}{\partial t}|_\infty$ and then to apply the results obtained for the stationary equation. This approach does not work directly for several reasons. On the one side, the bound for the time derivative is usually obtained as a consequence of the comparison principle which is not available here. On the other side, our a priori stationary gradient bounds are valid for continuous solutions and not for subsolutions. We overcome these difficulties by considering a tricky approximate equation where H is replaced by

$$H_{nq}(x, p) = \frac{1}{q}|p|^M + H_n(x, p),$$

with a bounded uniformly continuous approximation H_n of H . A crucial point is that, since the coercive term $\frac{1}{q}|p|^M$ does not depend on x , the comparison principle holds for this new equation allowing us to build a continuous viscosity solution. Moreover, the approximate Hamiltonian satisfies the key assumptions (1.8)–(1.9) or (1.10)–(1.11) with the same constants as the original H . So we can build a solution of the ergodic problem in this case. This solution allows us to control the L^∞ , oscillation and time derivative bounds of the solution of the parabolic problem. We therefore can prove a parabolic version of the Hölder regularity result of [10] using the strong coercivity of H_{nq} (Lemma 4.3). By this way, we are in position to mimic the proofs of gradient bounds in the stationary case and to conclude to the existence of a unique Lipschitz continuous solution to (1.2), see Theorem 4.2.

We finally apply all the previous results to prove the large time behavior of the solution of (1.2). Having on hands the gradient bound (1.5), a solution of the ergodic problem (1.15) and the strong maximum principle, the proof is classical [8].

The paper is organized as follows. In Section 2, we prove the stationary gradient bounds, Theorems 1.1 and 1.2. Section 3 is devoted to establish the well-posedness of (1.1). Finally, the applications are presented in Section 4. We start by solving the ergodic problem, then a study of the parabolic equation (1.2) is provided. We end with the long-time behavior of the solution of (1.2) and the construction of Hölder continuous solutions to equations with Hamiltonians of arbitrary growth without the use of comparison principle.

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2. GRADIENT BOUND FOR THE STATIONARY EQUATION (1.1)

2.1. Oscillation bound.

Lemma 2.1. *Assume (1.8). Let v^ϵ be a continuous solution of (1.1) and let $v^\epsilon(x_\epsilon) = \min v^\epsilon$. Then*

$$v^\epsilon(x) - v^\epsilon(x_\epsilon) \leq L|x - x_\epsilon| \quad \text{for all } x \in \mathbb{T}^N,$$

where L is the constant (independent of ϵ) which appears in (1.8).

An immediate consequence is

$$\text{osc}(v^\epsilon) := \max v^\epsilon - \min v^\epsilon \leq \sqrt{NL}.$$

To make the article self-contained, we present the proof of this result in Appendix.

2.2. Preliminary lemma for Ishii-Lions's method. The following technical lemma is a key tool in this article.

Lemma 2.2. *Suppose v^ϵ is a continuous viscosity solution of (1.1) in some open subset Ω with $A(x) = \sigma(x)\sigma^T(x)$, $\sigma \in W^{1,\infty}(\overline{\Omega})$. Let $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an increasing concave function such that $\Psi(0) = 0$ and the maximum of*

$$\max_{x,y \in \overline{\Omega}} \{v^\epsilon(x) - v^\epsilon(y) - \Psi(|x - y|)\},$$

is achieved at (\bar{x}, \bar{y}) . If we can write the viscosity inequalities for v^ϵ at \bar{x} and \bar{y} , then for every $\varrho > 0$, there exists $(p, X) \in \overline{J}^{2,+}v^\epsilon(\bar{x})$, $(p, Y) \in \overline{J}^{2,-}v^\epsilon(\bar{y})$ such that

$$(2.1) \quad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varrho A^2,$$

with

$$(2.2) \quad p = \Psi'(|\bar{x} - \bar{y}|)q, \quad q = \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}, \quad B = \frac{1}{|\bar{x} - \bar{y}|}(I - q \otimes q),$$

$$(2.3) \quad A = \Psi'(|\bar{x} - \bar{y}|) \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \Psi''(|\bar{x} - \bar{y}|) \begin{pmatrix} q \otimes q & -q \otimes q \\ -q \otimes q & q \otimes q \end{pmatrix}$$

and the following estimate holds

$$(2.4) \quad -\text{trace}(A(\bar{x})X - A(\bar{y})Y) \geq -N|\sigma_x|_\infty^2|\bar{x} - \bar{y}|\Psi'(|\bar{x} - \bar{y}|) + O(\varrho).$$

If, in addition, (1.3) holds, then there exists $\tilde{C} = \tilde{C}(N, \nu, |\sigma|_\infty, |\sigma_x|_\infty)$ (given by (5.3)) such that

$$(2.5) \quad -\text{trace}(A(\bar{x})X - A(\bar{y})Y) \geq -4\nu\Psi''(|\bar{x} - \bar{y}|) - \tilde{C}\Psi'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}| + O(\varrho)$$

and, if the maximum is positive, then

$$(2.6) \quad -4\nu\Psi''(|\bar{x} - \bar{y}|) - \tilde{C}\Psi'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}| + H(\bar{x}, \Psi'(|\bar{x} - \bar{y}|)q) - H(\bar{y}, \Psi'(|\bar{x} - \bar{y}|)q) < 0.$$

The first part of the result is a basic application of Ishii's Lemma in viscosity theory, see [12]. The trace estimates can be found in [16, 4, 8] and (2.6) takes benefit of the ellipticity of the equation and allows to apply Ishii-Lion's method introduced in [16]. For reader's convenience, we provide a proof in the Appendix.

2.3. Proof of Theorem 1.1. The proof relies on some ideas of [4]. The main difference is that, thanks to the uniform oscillation bound presented in Lemma 2.1, we can obtain a gradient bound independent of the L^∞ norm of the solution.

Step 1. Hölder continuity. We claim that there exist some constants $\gamma \in (0, 1]$, $K > 0$ independent of ϵ such that

$$|v^\epsilon|_{C^{0,\gamma}} \leq K_0.$$

We skip the ϵ superscript in v^ϵ hereafter for sake of notations.

Thanks to Lemma 2.1, the oscillation of v is uniformly (in ϵ) bounded by a constant \mathcal{O} . Consider

$$\max_{x,y \in \mathbb{T}^N} \{v(x) - v(y) - \Psi(|x - y|)\},$$

where $\Psi(s) = K_0 s^\gamma$. Our goal is to choose $\gamma \in (0, 1]$, $K_0 > 0$, which depend only on C, α given by the hypothesis (1.9) such that the above maximum is nonnegative. To do so, we assume by contradiction that the maximum is positive and hence, it is achieved at (\bar{x}, \bar{y}) with $\bar{x} \neq \bar{y}$ thanks to the continuity of v . We next choose r depending on K_0 such that $K_0 r^\gamma = \mathcal{O} + 1$.

With such a choice of r , it is clear that $|\bar{x} - \bar{y}| < r$. Denote $s := |\bar{x} - \bar{y}|$. From Lemma 2.2 and (1.9), we will have a contradiction if we can choose K, γ such that

$$-4\nu\Psi''(s) - \tilde{C}s\Psi'(s) \geq Cs^\alpha\Psi'(s)^{\alpha+2} + C\Psi'(s)^2 + C.$$

Computing $\Psi'(s) = K_0\gamma s^{\gamma-1}$ and $\Psi''(s) = K_0\gamma(\gamma-1)s^{\gamma-2}$, we have to prove

$$4\nu K_0\gamma(1-\gamma)s^{\gamma-2} - \tilde{C}K_0\gamma s^\gamma \geq Cs^\alpha(K_0\gamma s^{\gamma-1})^{\alpha+2} + C(K_0\gamma s^{\gamma-1})^2 + C.$$

It is clear that $\nu K_0\gamma(1-\gamma)s^{\gamma-2} \geq \tilde{C}K_0\gamma s^\gamma + C$ when r is small enough. Hence, the above inequality holds true if we can choose K_0, γ such that the two following inequalities hold,

$$\nu K_0\gamma(1-\gamma)s^{\gamma-2} \geq Cs^\alpha(K_0\gamma s^{\gamma-1})^{\alpha+2} \Leftrightarrow \nu(1-\gamma) \geq C(K_0s^\gamma)^{\alpha+1}\gamma^{\alpha+1},$$

and

$$\nu K_0\gamma(1-\gamma)s^{\gamma-2} \geq C\Psi'^2 \Leftrightarrow \nu(1-\gamma) \geq CK_0s^\gamma\gamma.$$

Since $K_0s^\gamma \leq K_0r^\gamma \leq \mathcal{O} + 1$, both inequalities hold true when γ is small enough depending on the oscillation \mathcal{O} (but not on K_0). This proves the claim.

Step 2. Improvement of the Hölder regularity to Lipschitz regularity. From the previous step, v is γ Hölder continuous (γ is possibly small) and the Hölder constant K_0 can be chosen to be independent of ϵ . We fix such a γ . We also recall that, from Lemma 2.1, the oscillation of v is bounded by a constant \mathcal{O} independent of ϵ .

We first construct a concave function $\Psi : [0, r] \rightarrow \mathbb{R}_+$ by

$$(2.7) \quad \Psi(s) = A_1[A_2s - (A_2s)^{1+\gamma}],$$

where $r, A_1, A_2 > 0$, which depend only on C, α, β given by the hypothesis (1.9), will be precised later. We extend Ψ into \mathbb{R}_+ by defining $\Psi(s) = \Psi(r)$ for $s \geq r$.

We compute, for $0 \leq s < r$,

$$\Psi'(s) = A_1 A_2 [1 - A_2^\gamma (1 + \gamma) s^\gamma], \quad \Psi''(s) = -A_1 A_2^{1+\gamma} \gamma (1 + \gamma) s^{\gamma-1} < 0.$$

We then choose r depending on A_2 (A_2 may vary in the next arguments) such that

$$(2.8) \quad A_2 r = \frac{1}{3} \quad \text{and} \quad \Psi(r) = \mathcal{O} + 1$$

A consequence of this choice is that A_1 is now fixed since $A_1(3^{-1} - 3^{-1-\gamma}) = \mathcal{O} + 1$. It is straightforward to see that Ψ is a smooth concave increasing function on $[0, r)$ satisfying $\Psi(0) = 0$ and, for all $s \in [0, r]$,

$$(2.9) \quad A_1 A_2 \left[1 - \frac{1 + \gamma}{3^\gamma}\right] = \Psi'(r) \leq \Psi'(s) \leq \Psi'(0) = A_1 A_2.$$

Consider

$$M := \max_{x, y \in \mathbb{T}^N} \{v(x) - v(y) - \Psi(|x - y|)\}.$$

If $M \leq 0$ then the theorem holds with $K = A_1 A_2$. The rest of the proof consists in proving that M is indeed nonpositive for A_2 big enough. We argue by contradiction assuming that $M > 0$. This maximum is achieved at (\bar{x}, \bar{y}) with $\bar{x} \neq \bar{y}$. With the choice of r in the condition (2.8) and the fact that Ψ is non-decreasing, it is clear that $|\bar{x} - \bar{y}| < r$.

Denote $s := |\bar{x} - \bar{y}|$. From (1.9) and Lemma 2.2, we have

$$-4\nu\Psi''(s) - \tilde{C}s\Psi'(s) < Cs^\alpha\Psi'(s)^{\alpha+2} + C + C\Psi'(s)^2,$$

which gives us

$$4\nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) s^{\gamma-1} - \tilde{C} A_1 A_2 s [1 - A_2^\gamma (1 + \gamma) s^\gamma] < Cs^\alpha\Psi'(s)^{\alpha+2} + C + C\Psi'(s)^2.$$

The goal now is to have a contradiction in the above inequality for large A_2 .

We first note that it is possible to increase A_2 in order that

$$(2.10) \quad \nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) s^{\gamma-1} - \tilde{C} A_1 A_2 s [1 - A_2^\gamma (1 + \gamma) s^\gamma] \geq 0.$$

Indeed, the inequality is true for all $A_2 \geq 1$ if $s \geq 1$ and, when $s \leq 1$, it is sufficient to take $A_2 \geq (\nu\gamma)^{-1}\tilde{C}$.

Therefore, it is enough to show that we may choose A_2 such that the following inequalities hold true,

$$(2.11) \quad \nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) s^{\gamma-1} \geq Cs^\alpha\Psi'(s)^{\alpha+2} = C(s\Psi'(s))^{\frac{1}{1-\gamma}} \Psi'(s)^{\alpha+2-\frac{\alpha}{1-\gamma}},$$

and

$$(2.12) \quad 2\nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) s^{\gamma-1} \geq C + C\Psi'(s)^2.$$

We first prove that it is possible to choose A_2 such that (2.11) holds true. We know that Ψ is concave and γ -Hölder continuous, so we have

$$s\Psi'(s) \leq \Psi(s) < v(\bar{x}) - v(\bar{y}) \leq K_0 s^\gamma,$$

Hence

$$(2.13) \quad s\Psi'(s)^{\frac{1}{1-\gamma}} \leq K_0^{\frac{1}{1-\gamma}}$$

and it follows that (2.11) is true provided

$$(2.14) \quad \nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) s^{\gamma-1} \geq C K_0^{\frac{\alpha}{1-\gamma}} (A_1 A_2)^{\alpha+2-\frac{\alpha}{1-\gamma}}.$$

Recalling that $1/s \geq 1/r > A_2$ from (2.8), we have

$$(2.15) \quad A_2^{1+\gamma} s^{\gamma-1} \geq A_2^2$$

and (2.14) is true if

$$(2.16) \quad \nu A_1 A_2^2 \gamma (1 + \gamma) \geq C K_0^{\frac{\alpha}{1-\gamma}} (A_1 A_2)^{\alpha+2-\frac{\alpha}{1-\gamma}}.$$

Finally, (2.16) indeed holds for A_2 big enough since $\alpha + 2 - \frac{\alpha}{1-\gamma} < 2$.

We now prove that it is possible to choose A_2 such that (2.12) holds true. At first, from (2.15), we have $\nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) s^{\gamma-1} \geq \nu A_1 A_2^2 \gamma (1 + \gamma) \geq C$ when A_2 big enough. So, (2.12) holds if we can choose A_2 such that

$$(2.17) \quad \nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) \geq C s^{1-\gamma} \Psi'(s)^2.$$

From (2.9) and (2.13),

$$C s^{1-\gamma} \Psi'(s)^2 = C (s \Psi'(s))^{\frac{1}{1-\gamma}})^{1-\gamma} \Psi'(s) \leq C K_0 A_1 A_2,$$

so (2.17) holds provided

$$\nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) \geq C K_0 A_1 A_2,$$

which is obviously true if A_2 is big enough.

The proof of the theorem is complete □

2.4. Proof of Theorem 1.2. From [10], we have

$$(2.18) \quad v \text{ is } \frac{k-2}{k-1}\text{-H\"older continuous and the H\"older constant is equal to } K_0,$$

where $k > 2$ is given by the assumption (1.10). In [10], the authors prove that the H\"older constant depends only on $N, k, |\epsilon v^\epsilon|_\infty$ and, since $|\epsilon v^\epsilon|_\infty \leq |H(x, 0)|_\infty$, K_0 can be chosen independent of ϵ . A by-product of the above result (or of Lemma 2.1) is that the oscillation of v^ϵ is bounded by a constant $\mathcal{O} > 0$ independent of ϵ .

Hereafter we write v for v^ϵ .

Step 1. Improvement of the H\"older exponent. Fix any $\chi \in (\frac{k-2}{k-1}, 1)$. We show that v is χ -H\"older continuous.

We set

$$\Psi(s) = K s^\chi,$$

where $K > 0$, which depend only on C, α, β given by the hypothesis (1.11), will be precised later. We fix a constant r which depends on K as follows

$$(2.19) \quad K r^\chi = \mathcal{O} + 1.$$

Consider

$$(2.20) \quad \max_{x, y \in \mathbb{T}^N} \{v(x) - v(y) - K|x - y|^\chi\}.$$

If the maximum is nonpositive then the theorem holds. From now on, we argue by contradiction assuming that the maximum is positive. The maximum is achieved at (\bar{x}, \bar{y}) with $\bar{x} \neq \bar{y}$. With the choice of r in (2.19), it is clear that $|\bar{x} - \bar{y}| < r$.

Denote $s := |\bar{x} - \bar{y}|$. From (1.11) and Lemma 2.2, we have

$$-4\nu\Psi''(s) - \tilde{C}s\Psi'(s) < \omega\left((1 + \Psi'(s)^\beta)s\right) s^\alpha\Psi'(s)^{(k-1)\alpha+k} + o(|p|^k),$$

with $\beta < k - 1$, $\Psi'(s) = K\chi s^{\chi-1}$, $\Psi''(s) = K\chi(\chi - 1)s^{\chi-2}$ and $p = \Psi'(s)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}$. We can rewrite the above inequality as

$$4\nu K\chi(1 - \chi)s^{\chi-2} - \tilde{C}K\chi s^\chi < \omega\left((1 + \Psi'(s)^\beta)s\right) s^\alpha\Psi'(s)^{(k-1)\alpha+k} + o(|p|^k).$$

At first, from (2.19), it is possible to increase K such that r is small enough in order to have

$$2\nu K\chi(1 - \chi)s^{\chi-2} \geq \tilde{C}K\chi s^\chi, \quad \text{for } s \leq r.$$

Hence, to get a contradiction in the above inequality, we only need to choose K such that the two following inequalities hold,

$$(2.21) \quad \nu K\chi(1 - \chi)s^{\chi-2} \geq \omega\left((1 + \Psi'(s)^\beta)s\right) s^\alpha\Psi'(s)^{(k-1)\alpha+k}$$

and

$$(2.22) \quad \nu K\chi(1 - \chi)s^{\chi-2} \geq o(|p|^k).$$

Step 1.1. Choosing K large enough such that we have (2.21). Writing that the maximum (2.20) is positive and using the concavity of Ψ and (2.18), we have $s\Psi'(s) \leq Ks^\chi = \Psi(s) < v(\bar{x}) - v(\bar{y}) \leq K_0 s^{\frac{k-2}{k-1}}$, hence

$$(2.23) \quad s\Psi'(s)^{k-1} \leq K_0^{k-1} \text{ and } \frac{1}{s} \geq \left(\frac{K}{K_0}\right)^{\frac{1}{\chi - \frac{k-2}{k-1}}}.$$

It follows

$$\begin{aligned} \omega\left((1 + \Psi'(s)^\beta)s\right) s^\alpha\Psi'(s)^{(k-1)\alpha+k} &= \omega\left((1 + \Psi'(s)^\beta)s\right) (s\Psi'(s)^{k-1})^\alpha \Psi'(s)^k \\ &\leq \omega\left((1 + \Psi'(s)^\beta)s\right) K_0^{\alpha(k-1)} (K\chi s^{\chi-1})^k. \end{aligned}$$

Therefore, (2.21) is true provided

$$\nu K\chi(1 - \chi)s^{\chi-2} \geq \omega\left((1 + \Psi'(s)^\beta)s\right) K_0^{\alpha(k-1)} (K\chi s^{\chi-1})^k.$$

Setting $\tilde{\nu} = \nu\chi(1 - \chi)K_0^{\alpha(1-k)}\chi^{-k}$, which is a constant independent of K, s , we rewrite the above desired inequality as

$$(2.24) \quad \tilde{\nu} \left(\frac{1}{s}\right)^{k(\chi-1)-\chi+2} \geq \omega\left((1 + \Psi'(s)^\beta)s\right) K^{k-1}.$$

From (2.23) and the choice $\chi > \frac{k-2}{k-1}$, it follows that inequality (2.24) holds true if

$$\tilde{\nu} \left(\frac{K}{K_0}\right)^{\frac{k(\chi-1)-\chi+2}{\chi - \frac{k-2}{k-1}}} \geq \omega\left((1 + \Psi'(s)^\beta)s\right) K^{k-1}$$

or

$$(2.25) \quad \tilde{\nu} \left(\frac{1}{K_0} \right)^{k-1} \geq \omega \left((1 + \Psi'(s)^\beta) s \right).$$

Using (2.23) and $\beta < k - 1$, we have

$$s\Psi'(s)^\beta = s^{1-\frac{\beta}{k-1}} (s\Psi'(s)^{k-1})^{\frac{\beta}{k-1}} \leq s^{\frac{k-1-\beta}{\beta}} K_0^\beta \leq r^{\frac{k-1-\beta}{\beta}} K_0^\beta \xrightarrow{r \rightarrow 0} 0.$$

Finally (2.25) holds true for large K since $r \rightarrow 0$ as $K \rightarrow +\infty$ by (2.19). This proves (2.21).

Step 1.2. Choosing K large enough such that we have (2.22). We have

$$o(|p|^k) = \frac{o(|p|^k)}{\Psi'(s)^k} (K\chi s^{\chi-1})^k = \frac{o(|p|^k)}{\Psi'(s)^k} \chi^k \Psi(s)^{k-1},$$

so (2.22) holds if

$$\nu(1 - \chi) \geq \frac{o(|p|^k)}{\Psi'(s)^k} \chi^{k-1} s^{2-k} \Psi(s)^{k-1}.$$

Recalling that $\Psi(s) \leq K_0 s^{\frac{k-2}{k-1}}$, it is sufficient to ensure

$$\nu(1 - \chi) \geq \frac{o(|p|^k)}{\Psi'(s)^k} (\chi K_0)^{k-1}.$$

Since

$$|p| = \Psi'(s) = \chi K s^{\chi-1} \geq \chi K r^{\chi-1} = \chi(\mathcal{O} + 1)^{\frac{\chi-1}{\chi}} K^{\frac{1}{\chi}}$$

by (2.19), we have that $|p| \rightarrow +\infty$ as $K \rightarrow +\infty$. We then obtain that the above inequality holds true for large K concluding (2.22). This ends Step 1.

Step 2. Improvement of the new Hölder exponent to Lipschitz continuity. We are now ready to prove the Lipschitz continuity.

The beginning of the proof is similar to the one of Theorem 1.1. We consider the increasing concave function Ψ given by (2.7) for any $\gamma \in (0, 1)$ and $A_1, A_2, r > 0$ satisfying (2.8) and set

$$M = \max_{x, y \in \mathbb{T}^N} \{v(x) - v(y) - \Psi(|x - y|)\}.$$

We are done if the maximum is nonnegative. Assuming by contradiction that the maximum is positive, we know it is achieved at (\bar{x}, \bar{y}) with $s := |\bar{x} - \bar{y}| < r$. Applying Lemma 2.2 and (1.11), we see that we reach the desired contradiction if the following inequalities hold

$$(2.26) \quad \begin{aligned} \nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) s^{\gamma-1} &\geq \omega \left((1 + \Psi'(s)^\beta) s \right) s^\alpha \Psi'(s)^{(k-1)\alpha+k} \\ &= \omega \left((1 + \Psi'(s)^\beta) s \right) (s\Psi'(s)^{\frac{1}{1-\chi}})^\alpha \Psi'(s)^{(k-1)\alpha+k-\frac{\alpha}{1-\chi}} \end{aligned}$$

and

$$(2.27) \quad \nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) s^{\gamma-1} \geq o(|p|^k) \quad \text{where } |p| = \Psi'(s).$$

Next substeps are devoted to prove that we can fulfill the two above inequalities by choosing A_2 large enough. It then leads to a contradiction which implies that the maximum M is

nonnegative concluding that v is Lipschitz continuous with constant $A_1 A_2$ and ending the proof of Theorem 1.2.

Step 2.1. Choosing A_2 such that (2.26) holds true. From Step 1, we know that v is χ -Hölder continuous for any $\chi \in (\frac{k-2}{k-1}, 1)$ with a constant $K = K_\chi$ which is independent of ϵ . We then have $s\Psi'(s) \leq \Psi(s) < K_\chi s^\chi$, hence

$$(2.28) \quad s\Psi'(s)^{\frac{1}{1-\chi}} \leq K_\chi^{\frac{1}{1-\chi}}.$$

Moreover

$$(2.29) \quad c_\gamma A_1 A_2 := \frac{\log 3 - 1}{3} \gamma A_1 A_2 \leq A_1 A_2 \left(1 - \frac{1 + \gamma}{3^\gamma}\right) = \Psi'(r) \leq \Psi'(s) \leq \Psi'(0) = A_1 A_2.$$

It follows that (2.26) holds provided

$$\nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) s^{\gamma-1} \geq \omega \left((1 + \Psi'(s)^\beta) s \right) K_\chi^{\frac{\alpha}{1-\chi}} c_\gamma^{-|(k-1)\alpha+k-\frac{\alpha}{1-\chi}|} (A_1 A_2)^{(k-1)\alpha+k-\frac{\alpha}{1-\chi}}.$$

Recalling that $1/s \geq 1/r > A_2$, the above inequality is true if

$$\nu A_1 A_2^2 \gamma (1 + \gamma) \geq \omega \left((1 + \Psi'(s)^\beta) s \right) K_\chi^{\frac{\alpha}{1-\chi}} c_\gamma^{-|(k-1)\alpha+k-\frac{\alpha}{1-\chi}|} (A_1 A_2)^{(k-1)\alpha+k-\frac{\alpha}{1-\chi}}.$$

First of all, we have $\beta < k - 1 < \frac{1}{1-\chi}$, so, by (2.26), $\omega \left((1 + \Psi'(s)^\beta) s \right)$ is small for small s . Therefore, to fulfill (2.26), it is enough to fix χ close enough to 1 such that

$$(2.30) \quad (k-1)\alpha + k - \frac{\alpha}{1-\chi} < 2 \quad \Leftrightarrow \quad \chi > 1 - \frac{\alpha}{(k-1)\alpha + k - 2}$$

and to take A_2 large enough.

Step 2.2. Choosing A_2 such that (2.27) holds true. We need to choose A_2 such that

$$\begin{aligned} \nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) &\geq s^{1-\gamma} o(|p|^k) \\ &= \frac{o(|p|^k)}{\Psi'(s)^k} s^{1-\gamma} \Psi'(s)^k = \frac{o(|p|^k)}{\Psi'(s)^k} (s\Psi'(s)^{\frac{1}{1-\chi}})^{1-\gamma} \Psi'(s)^{k-\frac{1-\gamma}{1-\chi}}. \end{aligned}$$

Using (2.28) and (2.29) again, we see that the above inequality is true provided

$$\nu A_1 A_2^{1+\gamma} \gamma (1 + \gamma) \geq \frac{o(|p|^k)}{\Psi'(s)^k} K_\chi^{\frac{1-\gamma}{1-\chi}} (A_1 A_2)^{k-\frac{1-\gamma}{1-\chi}}.$$

We fix $\chi \in (\frac{k-2}{k-1}, 1)$ close enough to 1 such that (2.30) holds and $k - \frac{1-\gamma}{1-\chi} < 1 + \gamma$. Noticing that $\Psi'(s) = |p| \rightarrow +\infty$ when $A_2 \rightarrow +\infty$, the previous inequality holds when A_2 is big enough. Therefore (2.27) holds. The proof of the theorem is complete. \square

2.5. Extensions. As said in the introduction, all proofs still hold true when when $\sigma \in C^{0,\theta}(\mathbb{T}^N; \mathcal{M}_N)$, $\theta \in [\frac{1}{2}, 1]$, instead of $W^{1,\infty}(\mathbb{T}^N; \mathcal{M}_N)$. Actually, on the one side, (2.4) and (2.5) in Lemma 2.2 are modified as follows: $-N|\sigma_x|_\infty^2 |\bar{x} - \bar{y}| \Psi'(|\bar{x} - \bar{y}|)$ (respectively $-\tilde{C} \Psi'(|\bar{x} - \bar{y}|) |\bar{x} - \bar{y}|$) are replaced by $-N|\sigma|_{C^{0,\theta}}^2 |\bar{x} - \bar{y}|^{2\theta-1} \Psi'(|\bar{x} - \bar{y}|)$ (respectively $-\tilde{C} \Psi'(|\bar{x} - \bar{y}|) |\bar{x} - \bar{y}|^{2\theta-1}$ with $\tilde{C} = \tilde{C}(N, \nu, |\sigma|_\infty, |\sigma|_{C^{0,\theta}})$). On the other side, the oscillation bound, Lemma 2.1, hold when $N|x - y| |\sigma_x|_\infty^2$ is replaced by $N|\sigma|_{C^{0,1/2}}^2$ in (1.8). The computations in the proofs of Theorems 1.1 and 1.2 are adapted accordingly.

The proofs of Theorems 1.1 and 1.2 can be adapted easily to quasilinear equations with diffusion matrices of type $A(x, p) = \sigma(x, p)\sigma(x, p)^T$ under suitable growth structures in x, p of $\sigma(x, p)$.

As far as fully nonlinear equations of Bellman-Isaacs-type

$$\epsilon v^\epsilon + \sup_{a \in A} \inf_{b \in B} \{-\text{trace}(A_{ab}(x)D^2v^\epsilon) + H_{ab}(x, Dv^\epsilon)\} = 0,$$

are concerned, our results apply provided that Assumptions (1.3), (1.8)-(1.9), (1.10)-(1.11) hold with constants independent of a, b .

3. COMPARISON PRINCIPLE, EXISTENCE AND UNIQUENESS FOR THE STATIONARY EQUATION (1.1)

We prove the well-posedness of the stationary equation in a slightly more general framework, namely, we work in an open bounded subset of \mathbb{R}^N instead of \mathbb{T}^N , assuming some Dirichlet boundary conditions hold.

More precisely, we consider

$$(3.1) \quad \begin{cases} \epsilon v^\epsilon - \text{trace}(A(x)D^2v^\epsilon) + H(x, Dv^\epsilon) = 0, & x \in \Omega, \\ v^\epsilon(x) = g(x), & x \in \overline{\Omega}. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set with $\partial\Omega \in C^{1,1}$, $g \in C(\partial\Omega)$, $\epsilon > 0$ and we need to assume that $H \in C(\overline{\Omega} \times \mathbb{R}^N; \mathbb{R})$ to prove the comparison principle.

The comparison principle follows easily from the ad-hoc inequality (3.3) which follows.

Proposition 3.1. *Assume (1.3) and either (1.8)-(1.9) or (1.10)-(1.11) hold, where the torus \mathbb{T}^N is replaced by Ω . Let $u \in USC(\overline{\Omega})$ be a subsolution and $v \in LSC(\overline{\Omega})$ be a supersolution of (3.1) such that $u \leq g \leq v$ on $\partial\Omega$ and*

$$(3.2) \quad d := \sup_{x \in \overline{\Omega}} (u - v) > 0.$$

Then, there exists a constant C such that

$$(3.3) \quad u(x) - v(y) \leq d + C|x - y| \text{ for all } x, y \in \overline{\Omega}.$$

The proof of the proposition follows the same ideas of the proof of Theorem 1.2. We only sketch the minor changes between two proofs.

Proof. We make the proof under assumptions (1.10)-(1.11), the another case being simpler. With the assumption $\partial\Omega \in C^{1,1}$ and (1.10), the result of [10] gives

$$(3.4) \quad u \text{ is } \frac{k-2}{k-1}\text{-H\"older continuous in } \overline{\Omega} \text{ with a constant } K_0,$$

where k is given by the assumption (1.10).

Since u, v are bounded, we can set

$$(3.5) \quad \mathcal{U} = \|u\|_\infty + \|v\|_\infty.$$

By the upper semi-continuity of u and the compactness of $\partial\Omega$, there exists $r > 0$ such that

$$\begin{aligned} u(x) - u(y) &\leq d \text{ for all } y \in \partial\Omega, x \in \Omega \text{ and } |x - y| \leq r, \\ v(x) - v(y) &\leq d \text{ for all } x \in \partial\Omega, y \in \Omega \text{ and } |y - x| \leq r. \end{aligned}$$

Hence, using $u \leq v$ on $\partial\Omega$, there exists $r > 0$ such that

$$u(x) - v(y) \leq d \text{ if } |y - x| \leq r \text{ and either } x \in \partial\Omega \text{ or } y \in \partial\Omega.$$

This implies that for $C = \frac{d}{r}$, we have

$$(3.6) \quad u(x) - v(y) \leq d + C|x - y| \text{ if either } x \in \partial\Omega \text{ or } y \in \partial\Omega.$$

Step 1. Now, we prove that for any $\chi \in (\frac{k-2}{k-1}, 1)$, there exists a constant K such that

$$(3.7) \quad \max_{x,y \in \bar{\Omega}} \{u(x) - v(y) - d - K|x - y|^\chi\} \leq 0,$$

where $K > 0$ depends only on C, α, β given by the hypothesis (1.11) and will be precised later.

We argue by contradiction assuming that the maximum is positive for any $K > 0$. It is therefore achieved at (\bar{x}, \bar{y}) with $\bar{x} \neq \bar{y}$. Denote $s := |\bar{x} - \bar{y}|$. We have

$$(3.8) \quad Ks^\chi < u(\bar{x}) - v(\bar{y}) - d.$$

It follows from (3.5) that s tends to zero as $K \rightarrow +\infty$. Thanks to (3.6), we then infer that necessarily $\bar{x}, \bar{y} \in \Omega$ for K big enough. Therefore, for K big enough, we can write the viscosity inequalities for u at \bar{x} and v at \bar{y} .

From this point, the next arguments follow exactly the same ones of Part 1 and 2 in the proof of Theorem 1.2. The only minor difference is the way we get (2.23). From (3.8) and (3.4), setting $\Psi(t) = Kt^\chi$, we obtain

$$s\Psi'(s) \leq Ks^\chi < u(\bar{x}) - u(\bar{y}) \leq K_0s^{\frac{k-2}{k-1}},$$

hence

$$s(\Psi'(s))^{k-1} \leq K_0^{k-1} \text{ and } \frac{1}{s} \geq \left(\frac{K}{K_0}\right)^{\frac{1}{\chi - \frac{k-2}{k-1}}}.$$

which is exactly the estimation (2.23) as desired.

Step 2. Proof of (3.3). Consider the function $\Psi(s) = A_1[A_2s - (A_2s)^{1+\gamma}]$ defined as in (2.7) with $r, A_1, A_2 > 0$ satisfying (2.8). Consider

$$\max_{x,y \in \bar{\Omega}} \{u(x) - v(y) - d - \Psi(|x - y|)\}.$$

If the maximum is negative, (3.3) holds with $C = A_1A_2$. From now, we argue by contradiction assuming that the maximum is positive and achieved at (\bar{x}, \bar{y}) . With the choice of r in (2.8), we have $0 < s := |\bar{x} - \bar{y}| < r$. Using the same arguments as in the beginning of Step 1, up to take A_2 big enough, we can assume that $\bar{x}, \bar{y} \in \Omega$ and therefore we can write the viscosity inequalities for u at \bar{x} and v at \bar{y} .

The next arguments follow exactly the same ones of Part 3 in the proof of Theorem 1.2. The only minor difference is the way we get (2.28). Fix any $\chi \in (\frac{k-2}{k-1}, 1)$. From Step 1, we obtain

$$d + \Psi(s) < u(\bar{x}) - v(\bar{y}) \leq d + Ks^\chi.$$

We then have $s\Psi'(s) \leq \Psi(s) < Ks^\chi$, hence

$$s(\Psi'(s))^{\frac{1}{1-\chi}} \leq K^{\frac{1}{1-\chi}}.$$

This is exactly Estimate (2.28) as we want. Having on hands (2.28) we repeat readily the arguments of Part 3 in the proof of Theorem 1.2 to conclude. \square

We now prove the comparison principle

Theorem 3.2. *Assume (1.3), H is continuous and either (1.8)-(1.9) or (1.10)-(1.11) hold, where the torus \mathbb{T}^N is replaced by Ω . Let $u \in USC(\overline{\Omega})$ be a subsolution and $v \in LSC(\overline{\Omega})$ be a supersolution of (3.1) such that $u \leq g \leq v$ on $\partial\Omega$. Then*

$$u(x) \leq v(x) \quad \text{for all } x \in \overline{\Omega}.$$

Notice that we assume that the Dirichlet boundary conditions hold in the classical viscosity sense on $\partial\Omega$. This is a little bit restrictive especially when working with superquadratic Hamiltonians since it is known that loss of boundary conditions may happen, see [9] for instance. But it is enough for our purpose here since we work in the periodic setting without boundary condition.

Proof. The proof of this result is followed quite easily from the estimate (3.3). Define d as in (3.2). We assume that $d > 0$ and try to get a contradiction. Since $u \leq g \leq v$ on $\partial\Omega$, any $z \in \overline{\Omega}$ such that $d = u(z) - v(z)$ lies in Ω . The maximum

$$M_\eta = \max_{x, y \in \overline{\Omega}} \left\{ u(x) - v(y) - \frac{|x - y|^2}{2\eta^2} - d \right\} \geq 0$$

is achieved at $(x_\eta, y_\eta) \in \overline{\Omega} \times \overline{\Omega}$. If there is a sequence $\eta \rightarrow 0$ such that $x_\eta, y_\eta \rightarrow \bar{x} \in \partial\Omega$, then

$$M_\eta = u(x_\eta) - v(y_\eta) - \frac{|x_\eta - y_\eta|^2}{2\eta^2} - d \rightarrow u(\bar{x}) - v(\bar{x}) - d < 0,$$

which is a contradiction.

Therefore, $(x_\eta, y_\eta) \in \Omega \times \Omega$ for η small enough. The theory of second order viscosity solutions yields, for every $\varrho > 0$, the existence of $(p_\eta, X) \in \overline{\mathcal{J}}^{2,+} u(x_\eta)$, $(p_\eta, Y) \in \overline{\mathcal{J}}^{2,-} v(y_\eta)$ such that and the following viscosity inequalities hold

$$\begin{cases} \epsilon u(x_\eta) - \text{trace}(A(x_\eta)X) + H(x_\eta, p_\eta) \leq 0, \\ \epsilon v(y_\eta) - \text{trace}(A(y_\eta)Y) + H(y_\eta, p_\eta) \geq 0. \end{cases}$$

Thanks to Proposition 3.1, we have

$$\frac{|x_\eta - y_\eta|^2}{2\eta^2} + d \leq u(x_\eta) - v(y_\eta) \leq d + C|x_\eta - y_\eta|.$$

Thus

$$\frac{p_\eta}{2} = \frac{|x_\eta - y_\eta|}{2\eta^2} \leq K.$$

This implies that p_η is bounded independently of η . Subtracting the viscosity inequalities and using (2.4), we get $\epsilon d \leq H(y_\eta, p_\eta) - H(x_\eta, p_\eta) + O(\eta) + O(\varrho)$, which leads to a contradiction when $\varrho \rightarrow 0$, $\eta \rightarrow 0$, thanks to the uniform continuity of H on compact subsets. \square

As a consequence of the previous results, we obtain the well-posedness for (1.1) in the class of Lipschitz continuous functions.

Corollary 3.3. *Assume (1.3), $H \in C(\mathbb{T}^N \times \mathbb{R}^N; \mathbb{R})$ and either (1.8)-(1.9) or (1.10)-(1.11) hold. Then, there exists a unique continuous viscosity solution v^ϵ of (1.1) which is Lipschitz continuous with a constant independent of ϵ . Moreover, if $A = \sigma\sigma^T$ and H are C^∞ , then v^ϵ is C^∞ .*

Proof. Thanks to the comparison principle, Theorem 3.2, we can construct a unique continuous viscosity solution to (1.1) with Perron's method. To apply this method, it is enough to build some sub and supersolution to (1.1) which is easily done by considering $v^\pm(x) = \pm \frac{1}{\epsilon} |H(\cdot, 0)|_\infty$. The Lipschitz regularity of the solution is then obtained from Theorems 1.1 and 1.2. When A and H are C^α in x , the $C^{2,\alpha}$ regularity of v^ϵ is a consequence of the Lipschitz bounds and the classical elliptic regularity theory [15, Theorems 6.13 and 6.14]. \square

4. APPLICATIONS

4.1. Ergodic problem. As a first application of Theorems 1.1 and 1.2, we prove that we can solve the ergodic problem associated with (1.1), namely, there exist $v^0 \in W^{1,\infty}(\mathbb{T}^N)$ and a unique constant $c \in \mathbb{R}$ solution to

$$(4.1) \quad -\text{trace}(A(x)D^2v^0) + H(x, Dv^0) = c, \quad x \in \mathbb{T}^N.$$

Theorem 4.1. *Assume (1.3), $H \in C(\mathbb{T}^N \times \mathbb{R}^N; \mathbb{R})$ and either (1.8)-(1.9) or (1.10)-(1.11) hold. Then, there exists $(c, v^0) \in \mathbb{R} \times W^{1,\infty}(\mathbb{T}^N)$ solution to (4.1) and c is unique. If we assume moreover that $H(x, \cdot)$ is locally Lipschitz continuous then v^0 is unique up to additive constants.*

Proof. Having on hands Theorems 1.1 and 1.2, the result is an easy application of the method of [20] and the strong maximum principle. We only give a sketch of proof. Let v^ϵ be the Lipschitz continuous solution of (1.1) given by Corollary 3.3. Since $|\epsilon v^\epsilon| \leq |H(\cdot, 0)|_\infty$ and $|Dv^\epsilon|_\infty \leq K$, the sequences ϵv^ϵ and $v^\epsilon - v^\epsilon(0)$ are bounded and equicontinuous in $C(\mathbb{T}^N)$ for all $\epsilon > 0$. By Ascoli-Arzelà Theorem, they converge, up to subsequences to $-c \in \mathbb{R}$ and $v^0 \in W^{1,\infty}(\mathbb{T}^N)$ respectively. By stability, (c, v^0) is a solution of (4.1). To prove the uniqueness part of the theorem, assume we have two solutions (c_1, v_1) and (c_2, v_2) of (4.1). Then $\tilde{u}_1(x, t) := v_1(x) - c_1 t - (|v_1|_\infty + |v_2|_\infty)$ and $\tilde{u}_2(x, t) := v_2(x) - c_2 t$ are respectively subsolution and supersolution of the associated evolution problem (1.2) with

initial datas $\tilde{u}_1(x, 0) \leq \tilde{u}_2(x, 0)$. Since both \tilde{u}_1 and \tilde{u}_2 are Lipschitz continuous, we have a straightforward comparison principle for the evolution problem which yields $\tilde{u}_1(x, t) \leq \tilde{u}_2(x, t)$ for all $(x, t) \in \mathbb{T}^N \times [0, +\infty)$. Sending $t \rightarrow +\infty$, we infer $c_1 \geq c_2$ and exchanging the role of the two solutions, we conclude $c_1 = c_2$. It is then easy to prove, using the Lipschitz continuity of v_1, v_2 and H with respect to the gradient that $v = v_1 - v_2$ is a subsolution of $-\text{trace}(A(x)D^2v) - C|Dv| \leq 0$ in \mathbb{T}^N for some constant $C > 0$. By the strong maximum principle ([13]), $v_1 - v_2$ is constant. \square

4.2. The parabolic equation. In this section, we prove the well-posedness and time-independent gradient bounds for the nonlinear parabolic problem (1.2) both under the assumptions (1.8)-(1.9) and (1.10)-(1.11).

Theorem 4.2. *Assume (1.3) and that $H \in C(\mathbb{T}^N \times \mathbb{R}^N; \mathbb{R})$ satisfies either (1.8)-(1.9) or (1.10)-(1.11). For any initial data $u_0 \in C^2(\mathbb{T}^N)$, there exists a unique continuous viscosity solution u to (1.2) such that, for all $x, y \in \mathbb{T}^N$, $s, t \in [0, +\infty)$,*

$$(4.2) \quad |u(x, t) - u(y, s)| \leq K|x - y| + \Lambda|t - s| \quad \text{with } K, \Lambda \text{ independent of time.}$$

If, in addition A, H and u_0 are C^∞ , then $u \in C^\infty(\mathbb{T}^N \times [0, +\infty))$.

To prove the theorem, we adapt the proofs of Theorems 1.1 and 1.2. The proof under the set of assumptions (1.10)-(1.11) is more delicate since the proof of Theorem 1.2 requires first to construct a solution to (1.2) which is $\frac{k-2}{k-1}$ -Hölder continuous. Due to the lack of comparison principle for (1.2) in our case and since the Hölder regularity result of [10] does not apply directly to evolution equations, the task is difficult. We need to extend the result of [10] for subsolutions of (1.2) which are Lipschitz continuous in time (see Lemma 4.3) and to construct an approximate solution of (1.2) which is indeed Lipschitz continuous in time.

Proof of Theorem 4.2.

Step 1. Proof when (1.8)-(1.9) hold. We truncate the Hamiltonian H by defining

$$(4.3) \quad H_n(x, p) = \begin{cases} H(x, p) & x \in \mathbb{T}^N, |p| \leq n, \\ H(x, n\frac{p}{|p|}) & x \in \mathbb{T}^N, |p| > n. \end{cases}$$

Notice that, on the one side, for $n \geq L$, H_n satisfies (1.8). On the other side, for all n , H_n satisfies (1.9) with the same constant C as for H . Moreover H_n converges locally uniformly to H as $n \rightarrow +\infty$.

By construction, $H_n \in BUC(\mathbb{T}^N \times \mathbb{R}^N; \mathbb{R})$. It follows that the comparison principle holds for (1.2) where H is replaced by H_n . Since $H_n(x, Du_0(x)) = H(x, Du_0(x))$, for n large enough,

$$(4.4) \quad u^\pm(x, t) = u_0(x) \pm |H(\cdot, Du_0) - \text{trace}(AD^2u_0)|_\infty t$$

are respectively super and subsolutions of (1.2) with H_n , and Perron's method yields a unique continuous viscosity solution u_n of this latter equation.

By Theorem 4.1, there exists a solution $(c_n, v_n) \in \mathbb{R} \times W^{1,\infty}(\mathbb{T}^N)$ of (4.1) where H is replaced by H_n . Notice that, since H_n satisfies (1.8)-(1.9) with constants independent

of n for $n > L$, both $|v_n|_\infty$ and $|Dv_n|_\infty$ are bounded independently of n . Choosing \mathcal{A} independent of n such that $\mathcal{A} \geq |v_n|_\infty + |u_0|_\infty$, the functions $(x, t) \mapsto v_n(x) - c_n t \pm \mathcal{A}$ are respectively a viscosity super and subsolutions of (1.2) with H_n . By comparison with u_n we get

$$v_n(x) - c_n t - \mathcal{A} \leq u_n(x, t) \leq v_n(x) - c_n t + \mathcal{A} \quad \text{for all } x \in \mathbb{T}^N, t \in [0, +\infty).$$

It follows that

$$\text{osc}(u_n(\cdot, t)) \leq |Dv_n|_\infty \text{diam}(\mathbb{T}^N) + 2\mathcal{A} \leq C$$

with C independent of n, t .

It is now possible to mimic the proof of Theorem 1.1 for u_n .

We begin by proving that u_n is γ -Hölder continuous with a constant independent of t, n for some $\gamma \in (0, 1)$. For any $\eta > 0$, consider

$$M_\eta := \max_{x, y \in \mathbb{T}^N, t > 0} \{u_n(x, t) - u_n(y, t) - \Psi(|x - y|) - \eta t\},$$

where $\Psi(s) = Ks^\gamma$, $0 < \gamma < 1$. If the maximum is nonpositive for some $K > 1$ and all $\eta > 0$, then we are done. Otherwise, for all $K > 1$, there exists $\eta > 0$ such that the maximum is positive. It is achieved at some $(\bar{x}, \bar{y}, \bar{t})$ with $\bar{x} \neq \bar{y}$.

If $\bar{t} = 0$, then, using that $|\bar{x} - \bar{y}| \leq \sqrt{N}$, we have

$$M_\eta \leq u_0(\bar{x}) - u_0(\bar{y}) - K|\bar{x} - \bar{y}|^\gamma \leq C_0|\bar{x} - \bar{y}| - K|\bar{x} - \bar{y}|^\gamma \leq 0$$

for $K > C_0\sqrt{N}^{1-\gamma}$, where C_0 is the Lipschitz constant of u_0 .

It follows that, for K big enough, the maximum is achieved at $\bar{t} > 0$ and we can write the viscosity inequalities for u_n using the parabolic version of Ishii's Lemma [12, Theorem 8.3]. Using Lemma 2.2 in this context, we get

$$\eta - 4\nu\Psi''(|\bar{x} - \bar{y}|) - \tilde{C}\Psi'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}| + H(\bar{x}, \Psi'(|\bar{x} - \bar{y}|)q) - H(\bar{y}, \Psi'(|\bar{x} - \bar{y}|)q) < 0.$$

We then obtain a contradiction in the above inequality repeating readily the proof of Step 1 of Theorem 1.1 with $\mathcal{O} := \sup_{t>0} \text{osc}(u_n(\cdot, t))$.

With the same adaptations as above in this parabolic context, we can reproduce the rest of the proof of Theorem 1.1. We conclude that u_n is Lipschitz continuous in space with a constant independent of t, n since we used (1.8)-(1.9) with constants independent of n and since $\text{osc}(u_n(\cdot, t))$ is bounded independently of t, n .

By Ascoli-Arzelà Theorem, up to extract subsequences, u_n converge locally uniformly in $\mathbb{T}^N \times [0, +\infty)$ as $n \rightarrow +\infty$ to a function u which is still Lipschitz continuous in space with a constant independent of t . By stability, u is a solution to (1.2).

The proof of the Lipschitz continuity of u in time requires u_0 to be C^2 and can be done exactly as in the second case below.

Step 2. Proof when (1.10)-(1.11) hold. We consider, for $q, n \geq 1$, the approximate problem

$$(4.5) \quad \begin{cases} u_t - \text{trace}(A(x)D^2u) + \frac{1}{q}|Du|^M + H_n(x, Du) = 0, & (x, t) \in \mathbb{T}^N \times (0, +\infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{T}^N, \end{cases}$$

where $M > 2$ and H_n is defined in (4.3).

We have a comparison principle for (4.5) since $H_n \in BUC(\mathbb{T}^N \times \mathbb{R}^N)$ and $\frac{1}{q}|p|^M$ is a nonlinearity which is independent of x ; when subtracting the viscosity inequality, this term disappears since we are in \mathbb{T}^N and there is no need to add a localization term in the test-function to achieve the maximum. Moreover, since (4.4) are still super and subsolutions of (4.5), by means of Perron's method, we can build a continuous viscosity solution u^{qn} of the problem (4.5).

The next lemma extends the result of [10] for USC subsolutions of parabolic equations with coercive Hamiltonian satisfying (1.10). The proof is postponed at the end of the section.

Lemma 4.3. *Assume that (1.10) holds. Let $U \in USC(\mathbb{T}^N \times [0, +\infty))$ be a subsolution of (1.2) which is bounded and Lipschitz continuous in time with constants independent of t . Then, there exists $\tilde{C} > 0$ which depends on k, A, Λ (appearing in (1.10) and (4.7)) but not on t such that*

$$|U(x, t) - U(y, t)| \leq \tilde{C}|x - y|^{\frac{k-2}{k-1}} \quad x \in \mathbb{T}^N, t \geq 0.$$

We are going to prove that u^{qn} satisfies the assumptions of Lemma 4.3. We first claim that there exists a constant c^{qn} bounded with respect to n such that $u^{qn} + c^{qn}t$ is bounded in $\mathbb{T}^N \times [0, +\infty)$ by a constant depending on q but not on n . The equation

$$(4.6) \quad \epsilon v - \text{trace}(A(x)D^2v) + \frac{1}{q}|Dv|^M + H_n(x, Dv) = 0, \quad x \in \mathbb{T}^N,$$

satisfies Assumptions (1.10)-(1.11) of Theorem 1.2 with $k = M$ and a constant C depending on q but not on n . By Theorem 4.1, there exists a solution $(c^{qn}, v^{qn}) \in \mathbb{R} \times W^{1,\infty}(\mathbb{T}^N)$ of the associated ergodic problem. By the maximum principle, $|\epsilon v| \leq |H_n(\cdot, 0)|_\infty \leq |H(\cdot, 0)|_\infty$ so c^{qn} is bounded independently of q, n . Moreover, since the constants in the assumptions in Theorem 1.2 may be taken independent of n , v^{qn} is bounded and Lipschitz continuous with constants independent on n . Noticing that $\tilde{v}^{qn}(x, t) = v^{qn}(x) - c^{qn}t \pm A_q$ are respectively viscosity super and subsolutions of (4.5) when $A_q \geq |v^{qn}|_\infty + |u_0|_\infty$ (A_q may be chosen independent of n). By comparison with u^{qn} we get

$$v^{qn}(x) - c^{qn}t - A_q \leq u^{qn}(x, t) \leq v^{qn}(x) - c^{qn}t + A_q \quad \text{for all } x \in \mathbb{T}^N, t \in [0, +\infty)$$

and the claim is proved.

We then claim that u^{qn} is Lipschitz continuous in time, i.e., there exists $\Lambda > 0$ independent of t, q, n such that

$$(4.7) \quad |u^{qn}(x, t) - u^{qn}(x, s)| \leq \Lambda|x - y| \quad x \in \mathbb{T}^N, s, t \geq 0.$$

The proof is classical and relies on the comparison principle together with the fact that $u_0 \in C^2(\mathbb{T}^N)$. We only give a sketch of proof. Since A and the Hamiltonian in (4.5) do not depend on t , for all $h > 0$, $u^{qn}(\cdot, \cdot + h)$ is solution to (4.5) with initial data $u^{qn}(\cdot, h)$. By comparison, we obtain

$$(4.8) \quad u^{qn}(x, t + h) - u^{qn}(x, t) \leq \sup_{y \in \mathbb{T}^N} (u^{qn}(y, h) - u_0(y))^+ \quad x \in \mathbb{T}^N, t \geq 0.$$

Setting

$$\Lambda := |\text{trace}(AD^2u_0)|_\infty + |Du_0|_\infty^M + |H(\cdot, 0)|_\infty$$

(notice that Λ does not depend neither on q nor n), we have that $u_0(x) \pm \Lambda t$ are respectively super and subsolutions of (4.5). By comparison, it follows $|u^{qn}(x, t) - u_0(x)| \leq \Lambda t$. Using this inequality in (4.8), we obtain (4.7).

Therefore, we can apply Lemma 4.3 to $U(x, t) = u^{qn}(x, t) + c^{qn}t$ which is Lipschitz continuous in time with a constant independent of t, q, n since c^{qn} is bounded independently on q, n . We obtain that $u^{qn}(x, t) + c^{qn}t$ and so u^{qn} is $\frac{M-2}{M-1}$ -Hölder continuous in space with a constant depending on q (but not on n, t). By Ascoli-Arzela Theorem, u^{qn} converges, up to subsequences, locally uniformly in $\mathbb{T}^N \times [0, +\infty)$ as $n \rightarrow +\infty$ to a function u^q which still satisfies (4.7) (with the same constant Λ). Moreover, by stability, u^q is solution to (4.5) with H_n replaced by H .

Arguing as above on (4.6) where H_n is replaced by H , we can construct a solution (c^q, v^q) to the ergodic problem associated to (4.6) with H . Using that

$$(4.9) \quad \frac{1}{q}|p|^M + H(x, p) \geq \frac{1}{C}|p|^k - C$$

this time and that (1.11) holds for datas independent of q , we can prove that c^q is bounded and v^q is bounded and Lipschitz continuous with constants independent of q . By comparison, $u^q + c^qt$ is bounded independently of q, t .

Applying again Lemma 4.3 to $u^q + c^qt$ but using (4.9), we obtain that u^q is $\frac{k-2}{k-1}$ -Hölder continuous with a constant independent of q now. Thanks again to Ascoli-Arzela Theorem, we can send $q \rightarrow +\infty$ to obtain, up to subsequences, a solution u of (1.2) which is still $\frac{k-2}{k-1}$ -Hölder continuous with a constant independent of t .

We are not in position to mimic the proof of Theorem 1.2 for this solution u , which is done easily adapting the proof in the time-dependent case.

In conclusion, we built a Lipschitz continuous (in space and time) solution to (1.2) with constants independent of t .

Step 3. Uniqueness in the class of continuous functions and upper regularity. Even if a strong comparison principle between semicontinuous viscosity sub and supersolutions does not necessarily hold for (1.2) under our assumptions, it is easy to see that a comparison principle holds if either the subsolution or the supersolution is Lipschitz continuous. It allows to compare any continuous viscosity solution of (1.2) with u .

The regularity of u when the data $u_0 \in C^{2,\alpha}$ and H is C^α in x -variable is a consequence of the Lipschitz bounds and the classical parabolic regularity theory, see [17] for instance.

The proof of the theorem is complete. \square

Remark 4.4. When $\sigma \in C^{0,1/2}(\mathbb{T}^N; \mathcal{M}_N)$ instead of $W^{1,\infty}(\mathbb{T}^N; \mathcal{M}_N)$, we need to regularize also σ into a Lipschitz continuous matrix to build a continuous solution. The estimates on the approximate solutions are not affected by this regularization thanks to the results of Sections 2 and 3 and the result of [10], which hold for $\sigma \in C^{0,1/2}(\mathbb{T}^N; \mathcal{M}_N)$.

Proof of Lemma 4.3. To prove the lemma, it is sufficient to prove that there exists $C > 0$ such that, for every $t > 0$,

$$(4.10) \quad -\text{trace}(A(x)D^2U(x, t)) + H(x, DU(x, t)) \leq C \quad \text{for } x \in \mathbb{T}^N \text{ in the viscosity sense.}$$

Indeed, once (4.10) is established, we can repeat readily the proof of [10, Theorem 2.7].

Fix $t > 0$ and suppose that $x_0 \in \mathbb{T}^N$ is a strict maximum point of $x \mapsto U(x, t) - \varphi(x)$ in \mathbb{T}^N , where $\varphi \in C^2(\mathbb{T}^N)$. The supremum

$$\sup_{x, y \in \mathbb{T}^N, t \geq 0} \left\{ U(x, s) - \varphi(x) - \frac{(t-s)^2}{\eta^2} \right\}$$

is achieved at (\bar{x}, \bar{s}) and, since U is bounded, $\frac{(t-\bar{s})^2}{\eta^2} \rightarrow 0$ and $\bar{x} \rightarrow x_0$ as $\eta \rightarrow 0$. Writing that (\bar{x}, \bar{s}) is a maximum point we have

$$U(\bar{x}, t) - \varphi(\bar{x}) \leq U(\bar{x}, \bar{s}) - \varphi(\bar{x}) - \frac{(t-\bar{s})^2}{\eta^2}.$$

Using the Lipschitz continuity with respect to time of U (let us say with constant Λ independent of t), we obtain

$$(4.11) \quad \frac{|t-\bar{s}|}{\eta^2} \leq \Lambda.$$

Since U is a viscosity subsolution of (1.2), we get

$$\frac{\bar{s}-t}{\eta^2} - \text{trace}(A(\bar{x})D^2\varphi(\bar{x})) + H(\bar{x}, D\varphi(\bar{x})) \leq 0.$$

Taking into account (4.11) and letting $\eta \rightarrow 0$, we infer

$$-\text{trace}(A(x_0)D^2\varphi(x_0)) + H(x_0, D\varphi(x_0)) \leq \Lambda,$$

which proves (4.10). \square

We end this section with a general bound for the oscillation of continuous solutions to (1.2) when the comparison result holds. It is the analogous of Lemma 2.1 in the parabolic setting and is a result interesting by itself. We give below as an easy application the convergence of $u(x, t)/t$ towards a constant.

Lemma 4.5. *Suppose that comparison principle holds for (1.2). Let $u_0 \in C^2(\mathbb{T}^N)$ and assume that H, A, u_0 satisfy*

$$(4.12) \quad \left\{ \begin{array}{l} \text{there exists } L > 1 \text{ such that for all } x, y \in \mathbb{T}^N, \text{ if } |p|=L, \text{ then} \\ H(x, p) \geq |p| \left[H(y, \frac{p}{|p|}) + |H(\cdot, Du_0) - \text{trace}(AD^2u_0)|_\infty + N|x-y|\sigma_x|_\infty^2 \right]. \end{array} \right.$$

Then, the unique continuous solution u of (1.2) satisfies

$$(4.13) \quad u(x, t) - u(y_t, t) \leq L|x - y_t|, \quad \text{for all } t \geq 0, x \in \mathbb{T}^N$$

and y_t such that $u(y_t, t) = \min_{x \in \mathbb{T}^N} u(x, t)$.

Notice that (4.12) is a parabolic version of (1.8) which holds as soon as H is superlinear.

Remark 4.6. Assuming that the comparison principle holds is a bit restrictive in this context but we do not succeed to skip it.

Proof of Lemma 4.5. Setting

$$(4.14) \quad \mathcal{A} := |H(\cdot, Du_0) - \text{trace}(AD^2u_0)|_\infty,$$

we have that $u_0(x) \pm \mathcal{A}t$ are respectively super and subsolutions of (1.2). By comparison, it follows $|u(x, t) - u_0(x)| \leq \mathcal{A}t$. By comparison again, we get

$$(4.15) \quad |u(x, t + s) - u(x, t)| \leq \mathcal{A}s.$$

Fix $T > 0$. We define

$$M = \max_{x, y \in \mathbb{T}^N, t \in [0, T]} \{u(x, t) - Lu(y, t) + (L - 1) \min_{x \in \mathbb{T}^N} u(x, t) - L|x - y|\},$$

where the constant L is the one in (4.12). If $M \leq 0$, then (4.13) is straightforward. Otherwise, $M \geq L\delta > 0$ for $\delta > 0$ enough small.

Thanks to (4.15), we can approximate $\phi(t) := \min_{x \in \mathbb{T}^N} u(x, t)$ from below over the compact interval $[0, T]$ by a sequence of smooth functions $\phi_n(t)$ whose lipschitz norm is bounded by \mathcal{A} given by (4.14). Up to choosing n big enough, we may assume $0 \leq \phi - \phi_n \leq \delta$. For $n \in \mathbb{N}$, we consider

$$M_n = \max_{x, y \in \mathbb{T}^N, t \in [0, T]} \{u(x, t) - Lu(y, t) + (L - 1)\phi_n(t) - L|x - y|\}.$$

It is clear that $M_n \geq \delta > 0$. The above positive maximum is achieved at (x_n, y_n, t_n) with $x_n \neq y_n$. Unless $u(x_n, t_n) - Lu(x_n, t_n) + (L - 1)\phi_n(t_n) \geq \delta$, which is impossible since $\phi_n(t) \leq \phi(t) = \min_{x \in \mathbb{T}^N} u(x, t)$. Moreover, by replacing L with $\max\{L, \|Du_0\|_\infty\}$ if necessary, we can see easily that $t_n > 0$. The claim is proved and the maximum in M_n is achieved at a differentiable point of the test-function.

The theory of second order viscosity solutions [12, Theorem 8.3] yields, for every $\varrho > 0$, the existence of $(a, p, X) \in \bar{J}^{2,+} u(x_n, t_n)$ and $(b/L, p/L, Y/L) \in \bar{J}^{2,-} u(y_n, t_n)$, with $p = L \frac{x_n - y_n}{|x_n - y_n|}$, $a - b = -(L - 1)\phi'(t_n)$, such that

$$\begin{cases} a - \text{trace}(A(x_n)X) + H(x_n, p) \leq 0, \\ \frac{b}{L} - \text{trace}(A(y_n)\frac{Y}{L}) + H(y_n, \frac{p}{L}) \geq 0. \end{cases}$$

It follows

$$-(L - 1)\phi'_n(t_n) - \text{trace}(A(x_n)X - A(y_n)Y) + H(x_n, p) - LH(y_n, \frac{p}{L}) \leq 0.$$

Using Lemma 2.2, we have

$$-\text{trace}(A(x_n)X - A(y_n)Y) \geq -LN|x_n - y_n|\sigma_x|_\infty^2 + O(\varrho)$$

Finally, we obtain

$$H(x_n, p) - L \left[H(y_n, \frac{p}{L}) + N|x_n - y_n|\sigma_x|_\infty^2 \right] + O(\varrho) \leq (L - 1)\phi'_n(t_n) < L\mathcal{A}.$$

Letting $\varrho \rightarrow 0$ and applying (4.12) yields a contradiction. \square

We end this section by an application of the oscillation bound.

Proposition 4.7. *Assume (4.12) and suppose that a comparison principle for (1.2) holds. For every $u_0 \in C(\mathbb{T}^N)$, there exists $c \in \mathbb{R}$ such that the unique solution u of (1.2) satisfies*

$$\lim_{t \rightarrow \infty} \frac{u(x, t)}{t} = -c \text{ uniformly with respect to } x \in \mathbb{T}^N.$$

For related results in the case of Bellman equations, see [2, 1].

Sketch of proof of Proposition 4.7. Without loss of generality, we assume that $u_0 \in C^2(\mathbb{T}^N)$. The general case where $u_0 \in C(\mathbb{T}^N)$ can be handled using an approximation of u_0 in the class of C^2 functions and the comparison principle.

Set $m(t) = \min_{\mathbb{T}^N} u(\cdot, t)$. Since $(x, t) \mapsto u_0(x) - \mathcal{A}t$, where \mathcal{A} is given by (4.14), is a subsolution of (1.2), we have $m(t) \geq -C(1+t)$. Moreover, an easy application of the comparison principle yields that m is subadditive, namely $m(t+s) \leq m(t) + m(s)$ for all $t, s \geq 0$. By the subadditive theorem, there exists $c \in \mathbb{R}$ such that $m(t)/t \rightarrow -c$ as $t \rightarrow +\infty$. By Lemma 4.5, $0 \leq u(x, t) - m(t) \leq L \text{diam}(\mathbb{T}^N)$. This implies the uniform convergence of $u(\cdot, t)/t$ to $-c$. \square

4.3. Large time behavior of solutions of nonlinear strictly parabolic equations.

In this section, we use the uniform gradient bound proved in Theorems 1.1 and 1.2 to study the large time behavior of the solution of (1.2).

The first results on the large time behavior of solutions for second order parabolic equations were established in Barles-Souganidis [8]. They prove the uniform gradient bounds (1.4) and (1.5) for (1.1) and (1.2) in two cases. The first one is for Hamiltonians with a sublinear growth with respect to the gradient. A typical example is

$$(4.16) \quad H(x, p) = \langle b(x), p \rangle + \ell(x), \quad b \in C(\mathbb{T}^N; \mathbb{R}^N), \ell \in C(\mathbb{T}^N).$$

The second case is for *superlinear Hamiltonians*. The precise assumptions ([8, (H2)]) are more involved and require both local Lipschitz regularity properties and convexity-type assumptions on H . These assumptions are designed to allow the use of weak Bernstein-type arguments ([5]). The typical example is with a superlinear growth with respect to the gradient

$$(4.17) \quad H(x, p) = a(x)|p|^{1+\alpha} + \ell(x), \quad \alpha > 0, a, \ell \in W^{1,\infty}(\mathbb{T}^N) \text{ and } a > 0.$$

The proof of the large time behavior of the solution of (1.2) is then a consequence of the strong maximum principle (we give a sketch of proof below).

On the one hand, our results generalizes the assumptions on sublinear Hamiltonians made in [8]. More importantly, our results allow to deal with a class of superlinear Hamiltonians which is very different with the superlinear case of [8].

Theorem 4.8. *(Large time behavior) Assume that either the assumptions of Theorem 1.1 or the assumptions of Theorem 1.2 hold. Moreover, suppose that H is continuous and locally Lipschitz with respect to p . Then, there exists a unique $c \in \mathbb{R}$ such that, for all $u_0 \in C(\mathbb{T}^N)$, the solution u of (1.2) satisfies*

$$(4.18) \quad u(x, t) + ct \rightarrow v^0(x) \text{ uniformly as } t \rightarrow +\infty,$$

where (c, v^0) is a solution of (4.1).

Sketch of proof of Theorem 4.8. First of all, it is enough to assume that $u_0 \in C^2(\mathbb{T}^N)$. The general case where $u_0 \in C(\mathbb{T}^N)$ can be handled using an approximation of u_0 in the class of C^2 functions and the comparison principle.

Set $m(t) = \max_{x \in \mathbb{T}^N} (u(x, t) + ct - v^0(x))$. By the comparison principle, m is nonincreasing and, since it is bounded from below, $m(t) \rightarrow \ell$ as $t \rightarrow \infty$. From Theorem 4.2, $\{u(\cdot, t) + ct, t > 0\}$ is relatively compact in $W^{1,\infty}(\mathbb{T}^N)$. So we can extract a sequence, $t_j \rightarrow +\infty$ such that $u(\cdot, t_j) + ct_j \rightarrow \bar{u} \in W^{1,\infty}(\mathbb{T}^N)$. Applying the comparison principle for (1.2) in $W^{1,\infty}(\mathbb{T}^N \times [0, +\infty))$, we obtain, for every $x \in \mathbb{T}^N$, $t \geq 0$, $p \in \mathbb{N}$,

$$|u(x, t + t_j) + ct_j - u(x, t + t_p) - ct_p| \leq \max_{y \in \mathbb{T}^N} |u(y, t_j) + ct_j - u(y, t_p) - ct_p|,$$

which proves that $(u(\cdot, \cdot + t_j) + c(\cdot + t_j))_j$ is a Cauchy sequence in $C(\mathbb{T}^N \times [0, +\infty))$. We call u_∞ its limit. Notice, on one hand, that $|Du_\infty(\cdot, t)|_\infty \leq K$ for all t and, on the other hand, that $u_\infty - ct$ is solution of (1.2) with initial data \bar{u} by stability.

Passing to the limit with respect to j in $m(t + t_j)$ we obtain

$$(4.19) \quad \ell = \max_x (u_\infty(x, t) - v^0(x)) \quad \text{for any } t > 0.$$

Since u_∞ is solution of (1.2) with c in the right-hand side and v^0 is solution of (4.1), thanks to the Lipschitz continuity of u_∞, v^0 with respect to x and H with respect to the gradient, we obtain that there exists $C > 0$ such that $w = u_\infty - v^0$ is subsolution of $w_t - \text{trace}(A(x)D^2w) - C|Dw| \leq 0$ in $\mathbb{T}^N \times [0, +\infty)$. Using (4.19) and the strong maximum principle ([13]), we infer $u_\infty(x, t) - v^0(x) = \ell$ for every $(x, t) \in \mathbb{T}^N \times [0, +\infty)$. Noticing that $\ell + v^0(x)$ does not depend on the choice of subsequences, we obtain $u(x, t) + ct - \ell - v^0(x) \rightarrow 0$ uniformly in x as $t \rightarrow \infty$. \square

4.4. Existence result of Hölder continuous solutions for equations without comparison principle. Usually, existence results for Equations like (1.1) or (1.2) are consequence of a strong comparison principle as Theorem 3.2 together with Perron's method or using the value function of an optimal control problem when H is convex. In this section, we use Theorem 3.2 and the result of [10] to build Hölder continuous solutions under assumptions which are too weak to expect any comparison principle.

Theorem 4.9. *Assume $A \geq 0$, H is continuous and satisfies*

$$(4.20) \quad \frac{|p|^m}{C} - C \leq H(x, p) \leq C(|p|^M + 1), \quad x \in \mathbb{T}^N, p \in \mathbb{R}^N, 2 < m \leq M.$$

Then there exists a viscosity solution v^ϵ of (1.1) which is $\frac{m-2}{m-1}$ -Hölder continuous solution and, for every $u_0 \in C^2(\mathbb{T}^N)$, a viscosity solution u of (1.2) which is $\frac{m-2}{m-1}$ -Hölder continuous in space and Lipschitz continuous in t .

Proof. The proof follows the approach used in Step 2 of the proof of Theorem 4.2.

Step 1. Existence for the stationary problem (1.1). Equation (1.1) with H replaced by $H_q(x, p) = \frac{|p|^{M+1}}{q} + H(x, p)$ and A replaced by $A + \frac{1}{q}I$ satisfies the conditions of Theorem 3.2,

hence we have the strong comparison principle for this new equation. Therefore, we can apply Perron's method to obtain the existence of a continuous solution v_q^ϵ . From [10], v_q^ϵ is $\frac{m-2}{m-1}$ -Hölder continuous. Using Ascoli-Arzelà Theorem and stability when $q \rightarrow +\infty$, we obtain the existence of a viscosity solution v^ϵ which is $\frac{m-2}{m-1}$ -Hölder continuous (with a constant independent of ϵ).

Step 2. Existence of Hölder continuous solutions to the ergodic problem. We can reproduce the beginning of the proof of Theorem 4.1 with v^ϵ : the sequences ϵv^ϵ and $v^\epsilon - v^\epsilon(0)$ are still equicontinuous and therefore, we can build a solution $(c, v^0) \in \mathbb{R} \times C^{0, \frac{m-2}{m-1}}(\mathbb{T}^N)$ to (4.1).

Step 3. Existence for the parabolic problem. We now consider (4.5). This equation satisfies a strong comparison principle. We can follow readily the proof of Step 2 of Theorem 4.2 up to obtain a Hölder continuous solution u^q . Notice it is possible to build a solution to (4.1) as explained in Step 2 above. The comparison of u^q with $v^q - c^q t \pm C$ where C is a big constant is not anymore straightforward as in the proof of Theorem 4.2 since v^q is only Hölder continuous and not Lipschitz continuous. To continue, we need to adapt the proof of Theorem 3.2 to the parabolic case which can be done easily since u^q, v^q are $\frac{m-2}{m-1}$ -Hölder continuous in space. It is then possible to send a subsequence $q \rightarrow +\infty$ to obtain a Hölder continuous (in space) solution u to (1.2) as desired. \square

5. APPENDIX

Proof of Lemma 2.1. For simplicity, we skip the ϵ superscript in v^ϵ . The constant L which appears below is the one of (1.8). Consider

$$M = \max_{x, y \in \mathbb{T}^N} \{v(x) - Lv(y) + (L-1) \min v - L|x-y|\}.$$

We are done if $M \leq 0$. Otherwise, the above positive maximum is achieved at (\bar{x}, \bar{y}) with $\bar{x} \neq \bar{y}$. Notice that the continuity of v is crucial at this step. The theory of second order viscosity solutions yields, for every $\varrho > 0$, the existence of $(p, X) \in \overline{J}^{2,+} v(\bar{x})$ and $(p/L, Y/L) \in \overline{J}^{2,-} v(\bar{y})$, $p = L \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}$, such that

$$\begin{cases} \epsilon v(\bar{x}) - \text{trace}(A(\bar{x})X) + H(\bar{x}, p) \leq 0, \\ \epsilon v(\bar{y}) - \text{trace}(A(\bar{y})\frac{Y}{L}) + H(\bar{y}, \frac{p}{L}) \geq 0. \end{cases}$$

Using Lemma 2.2, we have

$$-\text{trace}(A(\bar{x})X - A(\bar{y})Y) \geq -LN|x-y||\sigma_x|_\infty^2 + O(\varrho)$$

It follows

$$\epsilon(v(\bar{x}) - Lv(\bar{y})) - \text{trace}(A(\bar{x})X - A(\bar{y})Y) + H(\bar{x}, p) - LH(\bar{y}, \frac{p}{L}) \leq 0.$$

Recall that $\epsilon \min v \leq |H(\cdot, 0)|_\infty$, then

$$\epsilon(v(\bar{x}) - Lv(\bar{y})) > -(L-1)\epsilon \min v \geq -L|H(\cdot, 0)|_\infty.$$

Finally, we obtain

$$H(\bar{x}, p) - L \left[H(\bar{y}, \frac{p}{L}) + |H(\cdot, 0)|_\infty + N|x - y||\sigma_x|_\infty^2 \right] < 0.$$

Applying (1.8) yields a contradiction. \square

5.1. Proof of Lemma 2.2. For simplicity, we skip the ϵ superscript in v^ϵ . The theory of second order viscosity solutions yields (see [12, Theorem 3.2] for instance), for every $\varrho > 0$, the existence of $(p, X) \in \bar{\mathcal{J}}^{2,+} v(\bar{x})$, $(p, Y) \in \bar{\mathcal{J}}^{2,-} v(\bar{y})$ such that (2.1), (2.2), (2.3) hold.

Let us prove (2.4) and (2.5). From (2.1), for every $\zeta, \xi \in \mathbb{R}^N$, we have

$$\langle X\zeta, \zeta \rangle - \langle Y\xi, \xi \rangle \leq \Psi' \langle \zeta - \xi, B(\zeta - \xi) \rangle + \Psi'' \langle \zeta - \xi, (q \otimes q)(\zeta - \xi) \rangle + O(\varrho).$$

We estimate $\text{trace}(A(\bar{x})X)$ and $\text{trace}(A(\bar{y})Y)$ using two orthonormal bases (e_1, \dots, e_N) and $(\tilde{e}_1, \dots, \tilde{e}_N)$ in the following way:

$$\begin{aligned} T := \text{trace}(A(\bar{x})X - A(\bar{y})Y) &= \sum_{i=1}^N \langle X\sigma(\bar{x})e_i, \sigma(\bar{x})e_i \rangle - \langle Y\sigma(\bar{y})\tilde{e}_i, \sigma(\bar{y})\tilde{e}_i \rangle \\ &\leq \sum_{i=1}^N \Psi' \langle \zeta_i, B\zeta_i \rangle + \Psi'' \langle \zeta_i, (q \otimes q)\zeta_i \rangle + O(\varrho) \\ (5.1) \qquad \qquad \qquad &\leq \Psi'' \langle \zeta_1, (q \otimes q)\zeta_1 \rangle + \sum_{i=1}^N \Psi' \langle \zeta_i, B\zeta_i \rangle + O(\varrho), \end{aligned}$$

where we set $\zeta_i = \sigma(\bar{x})e_i - \sigma(\bar{y})\tilde{e}_i$ and noticing that $\Psi'' \langle \zeta_i, (q \otimes q)\zeta_i \rangle = \Psi'' \langle \zeta_i, q \rangle^2 \leq 0$ since Ψ is concave.

We now build a suitable base to prove (2.4) and another one to prove (2.5).

In the case of (2.4) where σ could be degenerate, we choose any orthonormal basis such that $e_i = \tilde{e}_i$. It follows

$$\begin{aligned} T &\leq \sum_{i=1}^N \Psi' \langle (\sigma(\bar{x}) - \sigma(\bar{y}))e_i, B(\sigma(\bar{x}) - \sigma(\bar{y}))e_i \rangle + O(\varrho) \\ &\leq \Psi' N |\sigma(\bar{x}) - \sigma(\bar{y})|^2 |B| + O(\varrho) \\ &\leq \Psi' N |\sigma_x|_\infty^2 |\bar{x} - \bar{y}| + O(\varrho) \end{aligned}$$

since $|B| \leq 1/|\bar{x} - \bar{y}|$. Thus (2.4) holds.

When (1.3) holds, i.e., $A(x) \geq \nu I$ for every x , the matrix $\sigma(x)$ is invertible and we can set

$$e_1 = \frac{\sigma(\bar{x})^{-1}q}{|\sigma(\bar{x})^{-1}q|}, \quad \tilde{e}_1 = -\frac{\sigma(\bar{y})^{-1}q}{|\sigma(\bar{y})^{-1}q|}, \quad \text{where } q \text{ is given by (2.2).}$$

If e_1 and \tilde{e}_1 are collinear, then we complete the basis with orthogonal unit vectors $e_i = \tilde{e}_i \in e_1^\perp$, $2 \leq i \leq N$. Otherwise, in the plane $\text{span}\{e_1, \tilde{e}_1\}$, we consider a rotation \mathcal{R} of angle $\frac{\pi}{2}$ and define

$$e_2 = \mathcal{R}e_1, \quad \tilde{e}_2 = -\mathcal{R}\tilde{e}_1.$$

Finally, noticing that $\text{span}\{e_1, e_2\}^\perp = \text{span}\{\tilde{e}_1, \tilde{e}_2\}^\perp$, we can complete the orthonormal basis with unit vectors $e_i = \tilde{e}_i \in \text{span}\{e_1, e_2\}^\perp$, $3 \leq i \leq N$.

From (1.3), we have

$$(5.2) \quad \nu \leq \frac{1}{|\sigma(x)^{-1}q|^2} \leq |\sigma|_\infty^2.$$

It follows

$$\langle \zeta_1, (q \otimes q)\zeta_1 \rangle = \left(\frac{1}{|\sigma(\bar{x})^{-1}q|} + \frac{1}{|\sigma(\bar{y})^{-1}q|} \right)^2 \geq 4\nu.$$

From (2.2), we deduce $Bq = 0$. Therefore

$$\langle \zeta_1, B\zeta_1 \rangle = 0.$$

For $3 \leq i \leq N$, we have

$$\langle \zeta_i, B\zeta_i \rangle = \langle (\sigma(\bar{x}) - \sigma(\bar{y}))e_i, B(\sigma(\bar{x}) - \sigma(\bar{y}))e_i \rangle \leq |\sigma_x|_\infty^2 |\bar{x} - \bar{y}|.$$

Now, we estimate ζ_2

$$|\zeta_2| = |(\sigma(\bar{x}) - \sigma(\bar{y}))\mathcal{R}e_1 + \sigma(\bar{y})\mathcal{R}(e_1 + \tilde{e}_1)| \leq |\sigma_x|_\infty |\bar{x} - \bar{y}| + |\sigma|_\infty |e_1 + \tilde{e}_1|.$$

It remains to estimate

$$\begin{aligned} |e_1 + \tilde{e}_1| &\leq \frac{1}{|\sigma(\bar{x})^{-1}q|} |\sigma(\bar{x})^{-1}q - \sigma(\bar{y})^{-1}q| + |\sigma(\bar{y})^{-1}q| \left| \frac{1}{|\sigma(\bar{x})^{-1}q|} - \frac{1}{|\sigma(\bar{y})^{-1}q|} \right| \\ &\leq \frac{2|\sigma|_\infty |\sigma_x|_\infty}{\nu} |\bar{x} - \bar{y}|, \end{aligned}$$

from (5.2) and $|(\sigma^{-1})_x|_\infty \leq |\sigma_x|_\infty / \nu$.

From (5.1), we finally obtain $T \leq 4\nu\Psi'' + \tilde{C}\Psi'|\bar{x} - \bar{y}| + O(\varrho)$ where

$$(5.3) \quad \tilde{C} = \tilde{C}(N, \nu, |\sigma|_\infty, |\sigma_x|_\infty) := |\sigma_x|_\infty^2 (N - 2 + (1 + \frac{2|\sigma|_\infty^2}{\nu})^2).$$

This completes the proof of (2.5).

We finally prove (2.6). Writing the viscosity inequality for the subsolution v of (1.1) at \bar{x} and the supersolution v at \bar{y} , we get

$$\begin{cases} \epsilon v(\bar{x}) - \text{trace}(A(\bar{x})X) + H(\bar{x}, p) \leq 0, \\ \epsilon v(\bar{y}) - \text{trace}(A(\bar{y})Y) + H(\bar{y}, p) \geq 0. \end{cases}$$

Since the maximum is supposed to be positive and $\Psi \geq 0$, we have $v(\bar{x}) > v(\bar{y})$ and obtain

$$-\text{trace}(A(\bar{x})X - A(\bar{y})Y) + H(\bar{x}, p) - H(\bar{y}, p) < 0.$$

Estimate (2.6) follows from a straightforward application of (2.5).

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