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EMBEDDABILITY INTO RELATIONAL LATTICES IS UNDECIDABLE

LUIGI SANTOCANALE

ABSTRACT. The natural join and the inner union operations combine relations of a database. Tropashko and Spight realized that these two operations are the meet and join operations in a class of lattices, known by now as the relational lattices. They proposed then lattice theory as an algebraic approach to the theory of databases alternative to the relational algebra. Litak et al. proposed an axiomatization of relational lattices over the signature that extends the pure lattice signature with a constant and argued that the quasiequational theory of relational lattices over this extended signature is undecidable.

We prove in this paper that embeddability is undecidable for relational lattices. More precisely, it is undecidable whether a finite subdirectly-irreducible lattice can be embedded into a relational lattice. Our proof is a reduction from the coverability problem of a multimodal frame by a universal product frame and, indirectly, from the representability problem for relation algebras.

As corollaries we obtain the following results: the quasiequational theory of relational lattices over the pure lattice signature is undecidable and has no finite base; there is a quasiequation over the pure lattice signature which holds in all the finite relational lattices but fails in an infinite relational lattice.

1. INTRODUCTION

The natural join and the inner union operations combine relations (i.e. tables) of a database. Most of today's web programs query their databases making repeated use of the natural join and of the union, of which the inner union is a mathematically well behaved variant. Tropashko and Spight realized [26, 25] that these two operations are the meet and join operations in a class of lattices, known by now as the class of relational lattices. They proposed then lattice theory as an algebraic approach, alternative to Codd's relational algebra [4], to the theory of databases.

An important first attempt to axiomatize these lattices is due to Litak, Mikuláš, and Hidders [16]. These authors propose an axiomatization, comprising equations and quasiequations, in a signature that extends the pure lattice signature with a constant, the header constant. A main result of that paper is that the quasiequational theory of relational lattices is undecidable in this extended signature. Their proof mimics Maddux's proof that the equational theory of cylindric algebras of dimension $n \geq 3$ is undecidable [17].

We have investigated in [23] equational axiomatizations for relational lattices using as tool the duality theory for finite lattices developed in [22]. A conceptual contribution from [23] is to make explicit the similarity between the developing theory of relational lattices and the well established theory of combination of modal logics, see e.g. [14]. This was achieved on the syntactic side, but also on the semantic side, by identifying some key properties of the structures dual to the finite atomistic lattices in the variety generated by the relational lattices, see [23, Theorem 7]. These

properties make the dual structures into frames for commutator multimodal logics in a natural way.

In this paper we exploit this similarity to transfer results from the theory of multidimensional modal logics to lattice theory. Our main result is that *it is undecidable whether a finite subdirectly irreducible lattice can be embedded into a relational lattice*. We prove this statement by reducing to it the coverability problem of a frame by a universal $S5^3$ -product frame, a problem shown to be undecidable in [12]. As stated there, the coverability problem is—in light of standard duality theory—a direct reformulation of the representability problem of finite simple relation algebras, problem shown to be undecidable by Hirsch and Hodkinson [11].

Our main result and its proof allow us to derive further consequences. Firstly, we refine the undecidability theorem of [16] and prove that *the quasiequational theory of relational lattices in the pure lattice signature is undecidable* as well and *has no finite base*. Then we argue that *there is a quasiequation that holds in all the finite relational lattices, but fails in an infinite one*. For the latter result, we rely on the work by Hirsch, Hodkinson, and Kurucz [12] who constructed a finite 3-multimodal frame which has no finite p -morphism from a finite universal $S5^3$ -product frame, but has a p -morphism from an infinite one. On the methodological side, we wish to point out our use of generalized ultrametric spaces to tackle these problems. A key idea in the proof of the main result is the characterization of universal $S5^A$ -product frames as pairwise complete generalized ultrametric spaces with distance valued in the Boolean algebra $P(A)$, a characterization that holds when A is finite.

The paper is structured as follows. We recall in Section 2 few definitions and facts on frames and lattices. Relational lattices are introduced in Section 3. In Section 4 we give an outline of the proof of our main technical result—the undecidability of embeddability of a finite subdirectly-irreducible lattice into a relational lattice—and derive from it the other results. In Section 5 we show how to construct a lattice from a frame and use functoriality of this construction to argue that such lattice embeds into a relational lattice whenever the frame is a p -morphic image of a universal product frame. The proof of the converse statement is carried out in Section 8. The technical tools needed to prove the converse are developed Sections 6 and 7. The theory of generalized ultrametric spaces over a powerset Boolean algebra and the aforementioned characterization of $S5^A$ -product frames as pairwise complete spaces over $P(A)$ appear in Section 6. In Section 7 we study embeddings of finite subdirectly-irreducible lattices into relational lattices and prove that we can assume that these embeddings preserve bounds. This task is needed so to exclude the constants \perp and \top (denoting the bounds) from the signature of lattice theory.

2. FRAMES AND LATTICES

Frames. Let A be a set of actions. An A -multimodal frame (briefly, an A -frame or a *frame*) is a structure $\mathfrak{F} = \langle X_{\mathfrak{F}}, \{R_a \mid a \in A\} \rangle$ where, for each $a \in A$, R_a is a binary relation on $X_{\mathfrak{F}}$. We say that an A -frame is $S4$ if each R_a is reflexive and transitive. If \mathfrak{F}_0 and \mathfrak{F}_1 are two A -frames, then a p -morphism from \mathfrak{F}_0 to \mathfrak{F}_1 is a function $\psi : X_{\mathfrak{F}_0} \rightarrow X_{\mathfrak{F}_1}$ such that, for each $a \in A$,

- if xR_ay , then $\psi(x)R_a\psi(y)$,
- if $\psi(x)R_az$, then xR_ay for some y with $\psi(y) = z$.

Let us mention that A -multimodal frames and p -morphisms form a category.

A frame \mathfrak{F} is said to be *rooted* (or *initial*, see [21]) if there is $f_0 \in X_{\mathfrak{F}}$ such that every other $f \in X_{\mathfrak{F}}$ is reachable from f_0 . We say that an A -frame \mathfrak{F} is *full* if, for each $a \in A$, there exists $f, g \in X_{\mathfrak{F}}$ such that $f \neq g$ and fR_ag . If $G = (V, D)$ is a directed graph, then we shall say that G is rooted if it is rooted as a unimodal frame.

A particular class of frames we shall deal with are the *universal S5^A-product frames*. These are the frames \mathfrak{U} with $X_{\mathfrak{U}} = \prod_{a \in A} X_a$ and xR_ay if and only if $x_i = y_i$ for each $i \neq a$, where $x := \langle x_i \mid i \in A \rangle$ and $y := \langle y_i \mid i \in A \rangle$.

Let $\alpha \subseteq A$, \mathfrak{F} be an A -frame, $x, y \in X_{\mathfrak{F}}$. An α -*path* from x to y is a sequence $x = x_0R_{a_0}x_1 \dots x_{k-1}R_{a_{k-1}}x_k = y$ with $\{a_0, \dots, a_{k-1}\} \subseteq \alpha$. We use then the notation $x \xrightarrow{\alpha} y$ to mean that there is an α -path from x to y . Notice that if \mathfrak{F} is an S4 A -frame, then $x \xrightarrow{\{a\}} y$ if and only if xR_ay .

Orders and lattices. We assume some basic knowledge of order and lattice theory as presented in standard monographs [5, 9]. Most of the tools we use in this paper originate from the monograph [8] and have been further developed in [22].

A *lattice* is a poset L such that every finite non-empty subset $X \subseteq L$ admits a smallest upper bound $\bigvee X$ and a greatest lower bound $\bigwedge X$. A lattice can also be understood as a structure \mathfrak{A} for the functional signature (\vee, \wedge) , such that the interpretations of these two binary function symbols both give \mathfrak{A} the structure of an idempotent commutative semigroup, the two semigroup structures being connected by the absorption laws $x \wedge (y \vee x) = x$ and $x \vee (y \wedge x) = x$. Once a lattice is presented as such structure, the order is recovered by stating that $x \leq y$ holds if and only if $x \wedge y = x$.

A lattice L is *complete* if any subset $X \subseteq L$ admits a smallest upper bound $\bigvee X$. It can be shown that this condition implies that any subset $X \subseteq L$ admits a greatest lower bound $\bigwedge X$. A lattice is *bounded* if it has a least element \perp and a greatest element \top . A complete lattice (in particular, a finite lattice) is bounded, since $\bigvee \emptyset$ and $\bigwedge \emptyset$ are, respectively, the least and greatest elements of the lattice.

If P and Q are partially ordered sets, then a function $f : P \rightarrow Q$ is *order-preserving* (or *monotone*) if $p \leq p'$ implies $f(p) \leq f(p')$. If L and M are lattices, then a function $f : L \rightarrow M$ is a *lattice morphism* if it preserves the lattice operations \vee and \wedge . A lattice morphism is always order-preserving. A lattice morphism $f : L \rightarrow M$ between bounded lattices L and M is *bound-preserving* if $f(\perp) = \perp$ and $f(\top) = \top$. A function $g : Q \rightarrow P$ is said to be *left adjoint* to an order-preserving $f : P \rightarrow Q$ if $g(q) \leq p$ holds if and only if $q \leq f(p)$ holds; such a left adjoint, when it exists, is unique. If L is finite, M is bounded, and $f : L \rightarrow M$ is a bound-preserving lattice morphism, then a left adjoint to f always exists and preserves the constant \perp and the operation \vee .

A *Moore family* on a set U is a collection \mathcal{F} of subsets of U which is closed under arbitrary intersections. Given a Moore family \mathcal{F} on U , the correspondence sending $Z \subseteq U$ to $\overline{Z} := \bigcap \{Y \in \mathcal{F} \mid Z \subseteq Y\}$ is a closure operator on U , that is, an order-preserving inflationary and idempotent endofunction of $P(U)$. The subsets in \mathcal{F} , called the *closed sets*, are exactly the fixpoints of this closure operator. We can give \mathcal{F} a lattice structure by defining

$$\bigwedge X := \bigcap X, \quad \bigvee X := \overline{\bigcup X}. \quad (1)$$

Let L be a complete lattice. An element $j \in L$ is *completely join-irreducible* if $j = \bigvee X$ implies $j \in X$, for each $X \subseteq L$; the set of completely join-irreducible elements of L is denoted here $\mathcal{J}_c(L)$. A complete lattice is *spatial* if every element is the join of the completely join-irreducible elements below it. An element $j \in \mathcal{J}_c(L)$ is said to be *join-prime* if $j \leq \bigvee X$ implies $j \leq x$ for some $x \in X$, for each finite subset X of L . If x is not join-prime, then we say that x is *non-join-prime*. An *atom* of a lattice L is an element of L such that \perp is the only element strictly below it. A spatial lattice is *atomistic* if every element of $\mathcal{J}_c(L)$ is an atom.

For $j \in \mathcal{J}_c(L)$, a *join-cover* of j is a subset $X \subseteq L$ such that $j \leq \bigvee X$. For $X, Y \subseteq L$, we say that X *refines* Y , and write $X \ll Y$, if for all $x \in X$ there exists $y \in Y$ such that $x \leq y$. A join-cover X of j is said to be *minimal* if $j \leq \bigvee Y$ and $Y \ll X$ implies $X \subseteq Y$; we write $j \triangleleft_m X$ if X is a minimal join-cover of j . In a spatial lattice, if $j \triangleleft_m X$, then $X \subseteq \mathcal{J}_c(L)$. If $j \triangleleft_m X$, then we say that X is a *non-trivial* minimal join-cover of j if $X \neq \{j\}$. The word *perfect* is used in the lattice-theoretic literature with different meanings [6, 2]. We use here something different:

Definition 1. We say that a complete lattice is *pluperfect* if it is spatial and for each $j \in \mathcal{J}_c(L)$ and $X \subseteq L$, if $j \leq \bigvee X$, then $Y \ll X$ for some Y such that $j \triangleleft_m Y$. The *OD-graph* of a pluperfect lattice L is the structure $\langle \mathcal{J}_c(L), \leq, \triangleleft_m \rangle$.

That is, in a pluperfect lattice every cover refines to a minimal one. With respect to analogous definitions, such as that of a lattice with the Σ -*minimal join-cover refinement property* [27] or that of a strongly spatial lattice [24], we do not require here that the set Y in the relation $j \triangleleft_m Y$ is finite, nor that, for a given j , there are a finite number of these sets.

Notice that every finite lattice is pluperfect. If L is a pluperfect lattice, then we say that $X \subseteq \mathcal{J}_c(L)$ is *closed* if it is a downset and $j \triangleleft_m C \subseteq X$ implies $j \in X$. Closed subsets of $\mathcal{J}_c(L)$ form a Moore family. The interest of considering pluperfect lattices stems from the following representation theorem stated in [19] for finite lattices; its generalization to pluperfect lattices is straightforward.

Theorem 2. Let L be a pluperfect lattice and let $\mathbb{L}(\mathcal{J}_c(L), \leq, \triangleleft_m)$ be the lattice of closed subsets of $\mathcal{J}_c(L)$. The mapping $l \mapsto \{j \in \mathcal{J}_c(L) \mid j \leq l\}$ is a lattice isomorphism from L to $\mathbb{L}(\mathcal{J}_c(L), \leq, \triangleleft_m)$.

Proof. Let $f(l) := \{j \in \mathcal{J}_c(L) \mid j \leq l\}$. Clearly $f(l)$ is a downset, let us verify that it is closed as well: if $j \triangleleft_m C \subseteq f(l)$, then $C \ll l$ and $j \leq \bigvee C \leq l$, so $j \in f(l)$.

Observe now that f is order-preserving; to see that f is an order isomorphism we argue that $\bigvee f(l) = l$ and $f(\bigvee X) = X$, when X is closed subset of $\mathcal{J}_c(L)$.

If $j \leq \bigvee f(l)$, then $j \triangleleft_m C \ll f(l)$; since $f(l)$ is a downset, $C \subseteq f(l)$ follows and therefore $j \in f(l)$, since $f(l)$ is closed; that is, we have $j \leq l$. By spatiality, we have therefore that $\bigvee f(l) \leq l$; equality follows since clearly $l \leq \bigvee f(l)$. For the second relation, if $j \in X$, then $j \leq \bigvee X$ and $j \in f(\bigvee X)$, so $X \subseteq f(\bigvee X)$. Conversely, if $j \in f(\bigvee X)$, then $j \leq \bigvee X$ and $j \triangleleft_m C \ll X$. Since X is a downset, then $C \subseteq X$ and since X is closed, then $j \in X$. Thus $f(\bigvee X) \subseteq X$ and equality holds. \square

It was shown in [22] how to extend this representation theorem to a duality between the category of finite lattices and the category of OD-graphs. We develop next some observations about pluperfect lattices, that generalize well known facts on finite lattices. We shall need these observations mainly in the course of Section 7.

For a lattice L , a *principal ideal* of L is a subset of the form $\downarrow l := \{x \in L \mid x \leq l\}$.

Lemma 3. *If L is a pluperfect lattice, then every principal ideal $\downarrow l$, $l \in L$, is also pluperfect. We have $\mathcal{J}_c(\downarrow l) = \mathcal{J}_c(L) \cap \downarrow l$ and, for $\{j\} \cup C \subseteq \mathcal{J}_c(\downarrow l)$, the relation $j \triangleleft_m C$ holds in $\downarrow l$ if and only if it holds in L .*

Proof. Each element of $\mathcal{J}_c(L) \cap \downarrow l$ is completely join-irreducible in $\downarrow l$. If $x \leq l$, then $x = \bigvee J$ with $J \subseteq \mathcal{J}_c(L)$ and clearly $J \subseteq \downarrow l$. Therefore $\downarrow l$ is spatial with $\mathcal{J}_c(\downarrow l) = \mathcal{J}_c(L) \cap \downarrow l$.

Suppose now that $\{j\} \cup X \subseteq \downarrow l$ and $j \leq \bigvee X$. If the relations $j \triangleleft_m C$ and $C \ll X$ hold in L , then $C \subseteq \downarrow l$, so they hold in $\downarrow l$ as well. In particular, this shows that $\downarrow l$ is pluperfect. \square

Let L be a pluperfect lattice. A subset $A \subseteq \mathcal{J}_c(L)$ is *D-closed* if $j \in A$ and $j \triangleleft_m C$ implies $C \subseteq A$. Given a *D-closed* subset $A \subseteq \mathcal{J}_c(L)$, let L_A be the closure of A under possibly infinite joins so, in particular, L_A is a sub-join-semilattice of L . As L_A has infinite joins, it has also infinite meets. Let us define then $\pi_A : L \rightarrow L_A$ by $\pi_A(l) := \bigvee \{x \in L_A \mid x \leq l\}$. The following Lemma generalizes to pluperfect lattices well known facts about finite lattices, see e.g. [8, Lemma 2.33].

Lemma 4. *$\pi_A : L \rightarrow L_A$ is a surjective lattice homomorphism. Moreover, L_A is a pluperfect lattice whose OD-graph is the restriction to A of the OD-graph of L .*

Proof. L_A is subset of L closed under arbitrary joins and therefore $\pi_A : L \rightarrow L_A$, defined by $\pi_A(l) := \bigvee \{x \in L_A \mid x \leq l\}$, is a surjective map which preserves arbitrary meets (since meets are computed in L_A via this map, e.g. $x \wedge_{L_A} y = \pi_A(x \wedge_L y)$).

Let us show that π_A preserves arbitrary joins as well. To this end, observe first that $\pi_A(l) = \bigvee \{j \in A \mid j \leq l\}$. Since π_A is order-preserving, we only need to show that $\pi_A(\bigvee X) \leq \bigvee \pi_A(X)$. Let therefore $j \in A$ with $j \leq \bigvee X$, so $j \triangleleft_m C$ with $C \ll X$. Since $j \in A$ and A is *D-closed*, we have $C \subseteq A$, whence $C = \pi_A(C) \ll \pi_A(X)$. It follows that $j \leq \bigvee C \leq \bigvee \pi_A(X)$.

The set $A \subseteq \mathcal{J}_c(L)$ generates L_A under arbitrary joins and, moreover, each element of A is completely join-irreducible in L_A , since L_A is a sub-join-semilattice of L ; thus L_A is spatial and $\mathcal{J}_c(L_A) = A$. It is easily verified that, for each $j \in A$, each minimal join-cover of j in L is also a minimal join-cover of j in L_A . \square

Lemma 5. *If $\{A_i \mid i \in I\}$ is a collection of *D-closed* subsets such that $\bigcup A_i = \mathcal{J}_c(L)$, then $\langle \pi_{A_i} \mid i \in I \rangle : L \rightarrow \prod L_{A_i}$ is a lattice embedding, that is, a subdirect decomposition of L .*

Proof. If $l \not\leq l'$, then, by spatiality, there is $j \in \mathcal{J}_c(L)$ such that $j \leq l$ but $j \not\leq l'$. Let $i \in I$ such that $j \in A_i$: then $j \leq \pi_{A_i}(l)$ but $j \not\leq \pi_{A_i}(l')$. It follows that $\langle \pi_{A_i} \mid i \in I \rangle$ is an injective map. \square

3. THE RELATIONAL LATTICES $R(D, A)$

Throughout this paper we shall use the notation Y^X for the set of functions of domain Y and codomain X , for X and Y any two sets.

Let A be a collection of attributes (or column names) and let D be a set of cell values. A *relation* on A and D is a pair (α, T) where $\alpha \subseteq A$ and $T \subseteq D^\alpha$. We shall use $R(D, A)$ to denote the set of relations on A and D . Informally, a relation

(α, T) represents a table of a relational database, with α being the header, i.e. the collection of names of columns, while T is the collection of rows.

Before we define the natural join, the inner union operations, and the order on $R(D, A)$, let us recall some key operations. If $\alpha \subseteq \beta \subseteq A$ and $f \in D^\beta$, then we shall use $f|_\alpha \in D^\alpha$ for the restriction of f to α ; if $T \subseteq D^\beta$, then $T\upharpoonright_\alpha$ shall denote projection to α , that is, the direct image of T along restriction, $T\upharpoonright_\alpha := \{f|_\alpha \mid f \in T\}$; if $T \subseteq D^\alpha$, then $i_\beta(T)$ shall denote cylindrification to β , that is, the inverse image of restriction, $i_\beta(T) := \{f \in D^\beta \mid f|_\alpha \in T\}$. Recall that i_β is right adjoint to \upharpoonright_α . With this in mind, the natural join \bowtie and the inner union \oplus of relations are respectively described by the following formulas:

$$\begin{aligned} (\alpha_1, T_1) \bowtie (\alpha_2, T_2) &:= (\alpha_1 \cup \alpha_2, T) \\ &\text{where } T = \{f \mid f|_{\alpha_i} \in T_i, i = 1, 2\} \\ &= i_{\alpha_1 \cup \alpha_2}(T_1) \cap i_{\alpha_1 \cup \alpha_2}(T_2), \\ (\alpha_1, T_1) \oplus (\alpha_2, T_2) &:= (\alpha_1 \cap \alpha_2, T) \\ &\text{where } T = \{f \mid \exists i \in \{1, 2\}, \exists g \in T_i \text{ s.t. } g|_{\alpha_1 \cap \alpha_2} = f\} \\ &= T_1\upharpoonright_{\alpha_1 \cap \alpha_2} \cup T_2\upharpoonright_{\alpha_1 \cap \alpha_2}. \end{aligned}$$

The set $R(D, A)$ is then ordered as follows:

$$(\alpha_1, T_1) \leq (\alpha_2, T_2) \quad \text{iff} \quad \alpha_2 \subseteq \alpha_1 \text{ and } T_1\upharpoonright_{\alpha_2} \subseteq T_2.$$

Proposition 6 (Tropashko [26]). *The poset $(R(D, A), \leq)$ is a lattice, with \bowtie as the meet operation and \oplus as the join operation.*

We shall therefore use $R(D, A)$ to denote such a lattice and call it a *relational lattice*. Let us remark however that in [16] such a lattice is called *full relational lattice* and that the wording “class of relational lattices” is used there for the class of lattices that have an embedding into some lattice of the form $R(D, A)$. As our concerns are lattice theoretical, we shall avoid to use the symbols \bowtie and \oplus , and prefer instead the usual meet and join symbols \wedge and \vee .

A convenient way of describing these lattices was introduced in [16, Lemma 2.1]. The authors argued that the relational lattices $R(D, A)$ are isomorphic to the lattices of closed subsets of $A \cup D^A$, where $Z \subseteq A \cup D^A$ is said to be closed if it is a fixed-point of the closure operator $\overline{(-)}$ defined as

$$\overline{Z} := Z \cup \{f \in D^A \mid A \setminus Z \subseteq Eq(f, g), \text{ for some } g \in Z\},$$

where in the formula above $Eq(f, g)$ is the equalizer of f and g . Letting

$$\delta(f, g) := \{x \in A \mid f(x) \neq g(x)\},$$

the above definition of the closure operator is obviously equivalent to the following one:

$$\overline{Z} := \alpha \cup \{f \in D^A \mid \delta(f, g) \subseteq \alpha, \text{ for some } g \in (Z \cap D^A)\}, \text{ with } \alpha = Z \cap A.$$

From now on, we rely on this representation of relational lattices. Relational lattices are atomistic pluperfect lattices. The completely join-irreducible elements of $R(D, A)$ are the singletons $\{a\}$ and $\{f\}$, for $a \in A$ and $f \in D^A$, see [16]. By an abuse of notation we shall write x for the singleton $\{x\}$, for $x \in A \cup D^A$. Under this

convention, we have therefore $\mathcal{J}_c(\mathbf{R}(D, A)) = A \cup D^A$. Every $a \in A$ is join-prime, while the minimal join-covers are of the form

$$f \triangleleft_m \delta(f, g) \cup \{g\}$$

for each $f, g \in D^A$, see [23]. The only non-trivial result from [23] that we use later (for Theorem 43 and Lemma 50) is the following:

Lemma 7. *Let L be a finite atomistic lattice in the variety generated by the class of relational lattices. If $\{j\} \cup X \subseteq \mathcal{J}_c(L)$, $j \leq \bigvee X$, and all the elements of X are join-prime, then j is join-prime.*

The Lemma—which is an immediate consequence of Theorem 7 in [23]—asserts that a join-cover of an element $j \in \mathcal{J}_c(L)$ which is not join-prime cannot be made of join-prime elements only.

4. OVERVIEW AND STATEMENT OF THE RESULTS

For an arbitrary frame \mathfrak{F} , we construct in Section 5 a lattice $\mathbf{L}(\mathfrak{F})$; if \mathfrak{F} is rooted and full, then $\mathbf{L}(\mathfrak{F})$ is a subdirectly irreducible lattice, see Proposition 23. The key Theorem leading to the undecidability results is the following one.

Theorem 8. *Let A be a finite set and let \mathfrak{F} be an S4 finite rooted full A -frame. There is a surjective p -morphism from a universal $\mathbf{S5}^A$ -product frame \mathfrak{U} to \mathfrak{F} if and only if $\mathbf{L}(\mathfrak{F})$ embeds into some relational lattice $\mathbf{R}(D, B)$.*

outline. The construction \mathbf{L} defined in Section 5 extends to a contravariant functor, so if \mathfrak{U} is a universal $\mathbf{S5}^A$ -product frame and $\psi : \mathfrak{U} \rightarrow \mathfrak{F}$ is a surjective p -morphism, then we have an embedding $\mathbf{L}(\psi)$ of $\mathbf{L}(\mathfrak{F})$ into $\mathbf{L}(\mathfrak{U})$. We can assume that all the components of \mathfrak{U} are equal, i.e. that the underlying set of \mathfrak{U} is of the form $\prod_{a \in A} X$; if this is the case, then $\mathbf{L}(\mathfrak{U})$ is isomorphic to the relational lattice $\mathbf{R}(X, A)$.

The converse direction, developed from Section 6 up to Section 8, is subtler. Considering that $\mathbf{L}(\mathfrak{F})$ is subdirectly-irreducible, we argue in Section 7 that if $\psi : \mathbf{L}(\mathfrak{F}) \rightarrow \mathbf{R}(D, B)$ is a lattice embedding, then we can suppose it preserves bounds; in this case ψ has a surjective left adjoint $\mu : \mathbf{R}(D, B) \rightarrow \mathbf{L}(\mathfrak{F})$. Let us notice that there is no general reason for ψ to be the image by \mathbf{L} of a p -morphism. Said otherwise, the functor \mathbf{L} is not full and, in particular, the image of an atom by μ might not be an atom. The following considerations, mostly developed in Section 8, make it possible to extract a p -morphism from the left adjoint μ . Since both $\mathbf{L}(\mathfrak{F})$ and $\mathbf{R}(D, B)$ are generated (under possibly infinite joins) by their atoms, each atom $x \in \mathbf{L}(\mathfrak{F})$ has a preimage $y \in \mathbf{R}(D, B)$ which is an atom. The set F_0 of non-join-prime atoms of $\mathbf{R}(D, B)$ such that $\mu(f)$ is a non-join-prime atom of $\mathbf{L}(\mathfrak{F})$ is endowed with a $P(A)$ -valued distance δ . The pair (F_0, δ) is shown to be a pairwise complete ultrametric space over $P(A)$. Section 6 recalls and develops some observations on ultrametric spaces valued on powerset algebras. The key ones are Theorems 27 and 39, stating that—when A is finite—pairwise complete ultrametric spaces over $P(A)$ and universal $\mathbf{S5}^A$ -product frames are essentially the same objects. The restriction of μ to F_0 yields then a surjective p -morphism from F_0 , considered as a universal $\mathbf{S5}^A$ -product frame, to \mathfrak{F} . \square

The following problem was shown to be undecidable in [12]: given a finite 3-frame \mathfrak{F} , does there exist a surjective p -morphism from a universal $\mathbf{S5}^3$ -product frame \mathfrak{U} to \mathfrak{F} ? In the introduction we referred to this problem as the coverability

problem of a 3-frame by a universal $S5^3$ -product frame. The problem was shown to be undecidable by means of a reduction from the representability problem of finite simple relation algebras, shown to be undecidable in [11]. We need to strengthen the undecidability result of [12] with some additional observations—rootedness and fullness—as stated in the following Proposition.

Proposition 9. *It is undecidable whether, given a finite set A with $\text{card } A \geq 3$ and an $S4$ finite rooted full A -frame \mathfrak{F} , there is a surjective p -morphism from a universal $S5^A$ -product \mathfrak{U} to \mathfrak{F} .*

Proof. Throughout this proof we assume a minimum knowledge of the theory of relation algebras, see e.g. [18].

The Proposition actually holds if we restrict to the case when $\text{card } A = 3$. Given a finite simple relation algebra \mathfrak{A} , the authors of [12] construct a 3-multimodal frame $\mathfrak{F}_{\mathfrak{A},3}$ such that \mathfrak{A} is representable if and only if $\mathfrak{F}_{\mathfrak{A},3}$ is a p -morphic image of some universal $S5^3$ -product frame. The frame $\mathfrak{F}_{\mathfrak{A},3}$ is $S4$ and rooted [12, Claim 8]. We claim that $\mathfrak{F}_{\mathfrak{A},3}$ is also full, unless \mathfrak{A} is the two elements Boolean algebra. To prove this claim, let us recall first that an element of $\mathfrak{F}_{\mathfrak{A},3}$ is a triple (t_0, t_1, t_2) of atoms of \mathfrak{A} such that $t_2 \leq t_0; t_1$; moreover, if t, t' are two such triples and $i \in \{0, 1, 2\}$, then $tR_i t'$ if and only if t and t' coincide in the i -th coordinate. If a is an atom of \mathfrak{A} , then $a \leq e_l; a$ and $a \leq a; e_r$ for two atoms e_l, e_r below the multiplicative unit of \mathfrak{A} . Therefore, the triples $t := (e_l, a, a^-)$ and $t' := (a, e_r, a^-)$ are elements of $\mathfrak{F}_{\mathfrak{A},3}$ and $tR_2 t'$. If, for each atom a , these triples are equal, then every atom of \mathfrak{A} is below the multiplicative unit, which therefore coincides with the top element \top ; since \mathfrak{A} is simple, then relation $\top = \top; x; \top$ holds for each $x \neq \perp$. It follows that $x = \top; x; \top = \top$, for each $x \neq \perp$, so \mathfrak{A} is the two elements Boolean algebra. Thus, if \mathfrak{A} has more than two elements, then $t \neq t'$ and $tR_2 t'$ for some $t, t' \in \mathfrak{F}_{\mathfrak{A},3}$. Using the cycle law of relation algebras, one also gets pairs of distinct elements of $\mathfrak{F}_{\mathfrak{A},3}$, call them u, u' and w, w' , such that $uR_0 u'$ and $wR_1 w'$.

Therefore, if we could decide whether there is a p -morphism from some universal $S5^3$ -frame to a given $S4$ finite rooted full frame \mathfrak{F} , then we could also decide whether a finite simple relation algebra \mathfrak{A} is representable, by answering positively if \mathfrak{A} has exactly two elements and, otherwise, by answering the existence problem of a p -morphism to $\mathfrak{F}_{\mathfrak{A},3}$. \square

Combining Theorem 8 with Proposition 9, we derive the following undecidability result.

Theorem 10. *It is not decidable whether a finite subdirectly irreducible atomistic lattice embeds into a relational lattice.*

Let us remark that Theorem 10 partly answers Problem 7.1 in [16].

In [16] the authors proved that the quasiequational theory of relational lattices (i.e. the set of all definite Horn sentences valid in relational lattices) in the signature (\wedge, \vee, H) is undecidable. Here H is the header constant, which is interpreted in a relational lattice $R(D, A)$ as the closed subset A of $A \cup D^A$. Problem 4.10 in [16] asks whether the quasiequational theory of relational lattices in the restricted signature (\wedge, \vee) of pure lattice theory is undecidable as well. We positively answer this question.

Theorem 11. *The quasiequational theory of relational lattices in the pure lattice signature is undecidable.*

It is a general fact that if the embeddability problem of finite subdirectly-irreducible algebras in a class \mathcal{K} is undecidable, then the quasiequational theory of \mathcal{K} is undecidable as well. We thank one of the authors of [13] for pointing out to us how this can be derived from Evans' work [7] as well as the connection to their work. We add here the proof of this fact, since we shall need it later in the proof of Theorem 14.

Proof of Theorem 11. Given a finite subdirectly-irreducible algebra A with least non trivial congruence $\theta(\hat{a}, \bar{a})$, we construct a quasiequation ϕ_A with the following property: for any other algebra (in the same signature) K , $K \models \phi_A$ if and only if A has an embedding into K .

The construction is as follows. Let $X_A = \{x_a \mid a \in A\}$ be a set of variables in bijection with the elements of A . For each function symbol f in the signature Ω , let $T_{A,f}$ be its table, that is the formula

$$T_{A,f} = \bigwedge_{(a_1, \dots, a_{ar(f)}) \in A^{ar(f)}} f(x_{a_1}, \dots, x_{a_{ar(f)}}) = x_{f(a_1, \dots, a_{ar(f)})}.$$

We let ϕ_A be the universal closure of $\bigwedge_{f \in \Omega} T_{A,f} \Rightarrow x_{\hat{a}} = x_{\bar{a}}$. We prove next that an algebra K satisfies ϕ_A if and only if there is no embedding of A into K .

If $K \models \phi_A$ and $\psi : A \rightarrow K$, then $v(x_a) = \psi(a)$ is a valuation such that $K, v \models \bigwedge_{f \in \Omega} T_{A,f}$, so $\psi(\hat{a}) = v(x_{\hat{a}}) = v(x_{\bar{a}}) = \psi(\bar{a})$ and ψ is not injective.

Conversely, suppose $K \not\models \phi_A$ and let v be a valuation such that $K, v \models \bigwedge_{f \in \Omega} T_{A,f}$ and $K, v \not\models x_{\hat{a}} = x_{\bar{a}}$. Define $\psi : A \rightarrow K$ as $\psi(a) = v(x_a)$, then ψ is a morphism, since $K, v \models T_{A,f}$ for each $f \in \Omega$. Let $\text{Ker}_\psi = \{(a, a') \mid \psi(a) = \psi(a')\}$ so, supposing that ψ is not injective, Ker_ψ is a non-trivial congruence. Then $(\hat{a}, \bar{a}) \in \theta(\hat{a}, \bar{a}) \subseteq \text{Ker}_\psi$, so $v(x_{\hat{a}}) = \psi(\hat{a}) = \psi(\bar{a}) = v(x_{\bar{a}})$, a contradiction. We have therefore $\text{Ker}_\psi = \{(a, a) \mid a \in A\}$, which shows that ψ is injective.

Let now \mathcal{K} be a class of algebras in the same signature. We have then

$$\begin{aligned} \mathcal{K} \models \phi_A &\text{ iff } K \not\models \phi_A \text{ for some } K \in \mathcal{K} \\ &\text{ iff there is an embedding of } A \text{ into } K, \text{ for some } K \in \mathcal{K}. \end{aligned}$$

Thus, if the embeddability problem of finite subdirectly-irreducible algebras into some algebra in \mathcal{K} is undecidable, then the quasiequational theory of \mathcal{K} is undecidable as well. \square

Following [12], let us add some further observations on the quasiequational theory of relational lattices.

Lemma 12. *The class of lattices that have an embedding into a relational lattice is closed under ultraproducts.*

Proof. Let us say that a sublattice L of a lattice $R(D, A)$ is H -closed if the subset A belongs to L . Let \mathcal{R} denote the closure under isomorphisms of the class of H -closed sublattices of some $R(D, A)$. It is proved in [16, Corollary 4.2] that \mathcal{R} is closed under ultraproducts. It immediately follows from this result that the class of lattices that have an embedding into some relational lattice is closed under ultraproducts, as follows. Let $\{L_i \rightarrow R(D_i, A_i) \mid i \in I\}$ be a family of lattice embeddings and let \mathcal{F} be an ultrafilter over I . The ultraproduct constructions on $\{L_i \mid i \in I\}$ and $\{R(D_i, A_i) \mid i \in I\}$ yield a lattice embedding $\prod_{\mathcal{F}} L_i \rightarrow \prod_{\mathcal{F}} R(D_i, A_i)$. Clearly, each $R(D_i, A_i)$ belongs to \mathcal{R} , whence the ultraproduct $\prod_{\mathcal{F}} R(D_i, A_i)$ belongs to \mathcal{R} as well: thus $\prod_{\mathcal{F}} R(D_i, A_i)$ embeds into some $R(D, A)$, and so does $\prod_{\mathcal{F}} L_i$. \square

Theorem 13. *The quasiequational theory of relational lattices is not finitely axiomatizable.*

Proof. A known result in universal algebra—see e.g. [3, Theorem 2.25]—states that a subdirectly-irreducible algebra satisfies all the quasiequations satisfied by a class of algebras if and only if it embeds in an ultraproduct of algebras in this class. Lemma 12 implies that the class of lattices that have an embedding into an ultraproduct of relational lattices and the class of lattices that have an embedding into some relational lattices are the same. Therefore a subdirectly-irreducible lattice L embeds in a relational lattice if and only if it satisfies all the quasiequations satisfied by the relational lattices. If this collection of quasiequations was a logical consequence of a finite set of quasiequations, then we could decide whether a finite subdirectly-irreducible L satisfies all these quasiequations, by verifying whether L satisfies the finite set of quasiequations. In this way, we could also decide whether such an L embeds into some relational lattice. \square

Finally, the following Theorem, showing that the quasiequational theory of the finite relational lattices is stronger than the quasiequational theory of all the relational lattices, partly answers Problem 3.6 in [16].

Theorem 14. *There is a quasiequation which holds in all the finite relational lattices which, however, fails into an infinite relational lattice.*

Proof. In the first appendix of [12] an S4 finite rooted full 3-frame \mathfrak{F} is constructed that has no surjective p -morphism from a finite universal $S5^3$ -product frame, but has such a p -morphism from an infinite one.

Since $L(\mathfrak{F})$ is finite whenever \mathfrak{F} is finite, we obtain by using Theorem 8 a subdirectly-irreducible finite lattice L which embeds into an infinite relational lattice, but has no embedding into a finite one.

Let ϕ_L be the quasiequation as in the proof of Theorem 11. We have therefore that, for any lattice K , $K \models \phi_L$ if and only if L does not embed into K .

Correspondingly, any finite relational lattice satisfies ϕ_L and, on the other hand, $K \not\models \phi_L$ if K is the infinite lattice into which L embeds. \square

5. THE LATTICE OF A MULTIMODAL FRAME

We assume throughout this Section that A is a finite set of actions. Given an A -frame $\mathfrak{F} = \langle X_{\mathfrak{F}}, \{R_a \mid a \in A\} \rangle$, we construct a lattice as follows. For $\alpha \subseteq A$, we say that $Y \subseteq X_{\mathfrak{F}}$ is α -closed if $x \in Y$, whenever there is a α -path from x to some $y \in Y$. We say that a subset $Z \subseteq A \cup X_{\mathfrak{F}}$ is *closed* if $Z \cap X_{\mathfrak{F}}$ is $Z \cap A$ -closed.

Lemma 15. *The collection of closed subsets of $A \cup X$ is a Moore family.*

The Lemma, whose proof is straightforward, allows us to define the lattice of an A -frame.

Definition 16. The lattice $L(\mathfrak{F})$ is the lattice of closed subsets of $A \cup X_{\mathfrak{F}}$.

The lattice operations on $L(\mathfrak{F})$ are defined as in the display (1). In order to master the formula for the join, we need a more explicit description of the closure operator associated to this Moore family. If $\alpha \subseteq A$ and $Y \subseteq X_{\mathfrak{F}}$, define

$$\overline{Y}^{\alpha} := \{x \in X_{\mathfrak{F}} \mid \exists y \in Y, x \xrightarrow{\alpha} y\}.$$

Lemma 17. *For $Z \subseteq A \cup X_{\mathfrak{F}}$, we have*

$$\overline{Z} = \alpha \cup \overline{Z \cap X_{\mathfrak{F}}}^{\alpha}, \quad \text{where } \alpha = Z \cap A. \quad (2)$$

In particular, for $x \in X_{\mathfrak{F}}$, $x \in \overline{Z}$ if and only if there exists $y \in Z \cap X_{\mathfrak{F}}$ and an α -path from x to y , with $\alpha = Z \cap A$.

The above formula (2) allows to make $L(-)$ into a contravariant functor from the category of frames to the category of lattices. Namely, for a p -morphism $\psi : \mathfrak{F}_0 \rightarrow \mathfrak{F}_1$ and any $Z \subseteq A \cup X_{\mathfrak{F}_1}$, define

$$L(\psi)(Z) := (Z \cap A) \cup \psi^{-1}(Z \cap X_{\mathfrak{F}_1}).$$

Let $\psi^A : A \cup X_{\mathfrak{F}_0} \rightarrow A \cup X_{\mathfrak{F}_1}$ be the function such that $\psi^A(a) = a$, for each $a \in A$, and $\psi^A(x) = \psi(x)$, for each $x \in X_{\mathfrak{F}_0}$. Notice that $L(\psi)$ is the inverse image of ψ^A , so in particular $L(\psi)$ commutes with intersections and unions.

Proposition 18. *$L(\psi)$ sends closed subsets of $A \cup X_{\mathfrak{F}_1}$ to closed subsets of $A \cup X_{\mathfrak{F}_0}$. Its restriction to $L(\mathfrak{F}_1)$ yields a bound-preserving lattice morphism $L(\psi) : L(\mathfrak{F}_1) \rightarrow L(\mathfrak{F}_0)$.*

Proof. The key observation is that, for each $\alpha \subseteq A$ and each $Y \subseteq X_{\mathfrak{F}_1}$, we have $\psi^{-1}(\overline{Y}^{\alpha}) = \overline{\psi^{-1}(Y)}^{\alpha}$:

$$\begin{aligned} \psi^{-1}(\overline{Y}^{\alpha}) &= \psi^{-1}(\{x \in X_{\mathfrak{F}_1} \mid \exists y \in Y, x \xrightarrow{\alpha} y\}) \\ &= \{x \in X_{\mathfrak{F}_0} \mid \exists y \in Y, \psi(x) \xrightarrow{\alpha} y\} \\ &= \{x \in X_{\mathfrak{F}_0} \mid \exists z \in \psi^{-1}(Y), x \xrightarrow{\alpha} z\} \quad \text{since } \psi \text{ is a } p\text{-morphism,} \\ &= \overline{\psi^{-1}(Y)}^{\alpha}. \end{aligned}$$

This implies that, for $Z \subseteq A \cup X_{\mathfrak{F}_1}$, we have

$$L(\psi)(\overline{Z}) = \overline{L(\psi)(Z)}.$$

In particular, if $Z \subseteq A \cup X_{\mathfrak{F}_1}$ is closed, then

$$L(\psi)(Z) = L(\psi)(\overline{Z}) = \overline{L(\psi)(Z)}$$

so $L(\psi)$ sends closed subsets to closed subsets. $L(\psi)$ preserves all meets, since it commutes with intersections. Moreover

$$L(\psi)\left(\bigvee_{i \in I} Z_i\right) = L(\psi)\left(\overline{\bigcup_{i \in I} Z_i}\right) = \overline{\bigcup_{i \in I} L(\psi)(Z_i)} = \bigvee_{i \in I} L(\psi)(Z_i),$$

so $L(\psi)$ is a lattice morphism. \square

As $L(\psi)$ is the restriction of the inverse image of ψ^A defined above, it immediately follows that L is a contravariant functor from the category of A -frames to the category of lattices.

Lemma 19. *If $\psi : \mathfrak{F}_0 \rightarrow \mathfrak{F}_1$ is surjective, then $L(\psi)$ is injective.*

Proof. If ψ is surjective, then ψ^A is also surjective. As $L(\psi)$ is the inverse image of ψ^A , then $L(\psi)$ is injective. \square

We are ready to state the main result of this Section.

Theorem 20. *If there exists a surjective p -morphism from a universal $S5^A$ -product frame \mathfrak{U} to an A -frame \mathfrak{F} , then $L(\mathfrak{F})$ embeds into a relational lattice.*

Proof. We say that \mathfrak{U} is uniform on X if all the components of \mathfrak{U} are equal to X . Spelled out, this means that $X_{\mathfrak{U}} = \prod_{a \in A} X$. Let $\psi : \mathfrak{U} \rightarrow \mathfrak{F}$ be a p -morphism as in the statement of the Theorem. W.l.o.g. we can assume that \mathfrak{U} is uniform on some set X . If this is not the case, then we choose $a_0 \in A$ such that X_{a_0} has maximum cardinality and surjective mappings $p_a : X_{a_0} \rightarrow X_a$, for each $a \in A$. The product frame \mathfrak{U}' on $\prod_{a \in A} X_{a_0}$ is uniform and $\prod_{a \in A} p_a : \mathfrak{U}' \rightarrow \mathfrak{U}$ is a surjective p -morphism. By pre-composing ψ with this p -morphism, we obtain a surjective p -morphism from the uniform \mathfrak{U}' to \mathfrak{F} . Now, if \mathfrak{U} is uniform on X , then $L(\mathfrak{U})$ is equal to the relational lattice $R(X, A)$. Then, by functoriality of L , we have a lattice morphism

$$L(\psi) : L(\mathfrak{F}) \rightarrow L(\mathfrak{U}) = R(X, A).$$

By Lemma 19 $L(\psi)$ is an embedding. \square

Some properties of the lattices $L(\mathfrak{F})$.

Proposition 21. *The completely join-irreducible elements of $L(\mathfrak{F})$ are the singletons, so $L(\mathfrak{F})$ is an atomistic lattice.*

Proof. Each singleton set is closed. It immediately follows that the join-irreducible elements of $L(\mathfrak{F})$ are the singletons and clearly they are atoms. \square

Identifying the singletons of $P(A \cup X_{\mathfrak{F}})$ with their elements, we can write

$$\mathcal{J}_c(L(\mathfrak{F})) = A \cup X_{\mathfrak{F}}.$$

To state the next Proposition, let us say that an α -path from x to y is *minimal* if there is no β -path from x to y , for each proper subset β of α .

Proposition 22. *$L(\mathfrak{F})$ is a pluperfect lattice. Each element of A is join-prime, while the minimal join-covers of $x \in X_{\mathfrak{F}}$ are of the form $x \triangleleft_m \alpha \cup \{y\}$, for a minimal α -path from x to y .*

Proof. We observe that, for $Z \subseteq A \cup X_{\mathfrak{F}}$, the relation

$$x \in \bigvee Z = \overline{Z}$$

holds if and only if either (i) $x \in A \cap Z$, or (ii) $x \in X_{\mathfrak{F}}$ and $x \xrightarrow{\alpha} y$ for some $y \in X_{\mathfrak{F}} \cap Z$ and some $\alpha \subseteq A \cap Z$. Thus, in particular, each element of A is join-prime. If $x \in X_{\mathfrak{F}}$, $Z \subseteq A \cup X_{\mathfrak{F}}$ and $x \leq \bigvee Z$, then we can find $y \in X_{\mathfrak{F}} \cap Z$ and an α -path from x to y with $\alpha \subseteq A \cap Z$. Clearly, we can assume $x \xrightarrow{\alpha} y$ is minimal, so

$$x \in \overline{\alpha \cup \{y\}} = \bigvee \alpha \vee y,$$

with $\alpha \cup \{y\} \subseteq Z$. This proves that every cover of x refines to a cover of the form $x \leq \bigvee \alpha \vee y$ with $x \xrightarrow{\alpha} y$ minimal. \square

Notice that if an α -path is minimal, then α is necessarily finite. Therefore $L(\mathfrak{F})$ is actually a lattice with the Σ -weak minimal join-cover refinement property as defined in [27], where Σ is here the set of completely join-irreducible elements of the lattice.

Before stating the next Proposition, let us recall from [8, Corollary 2.37], see also [22, Section 5.2], that a finite lattice L is subdirectly-irreducible if and only if

the directed graph $(\mathcal{J}_c(L), D)$ is rooted. Here D is the *join-dependency relation* on the join-irreducible elements of L , defined as follows:

$$jDk \text{ iff } j \neq k \text{ and, for some } p \in L, j \leq p \vee k \text{ and } j \neq p \vee k_*,$$

where k_* denotes the unique lower cover of $k \in \mathcal{J}_c(L)$. It can be shown that jDk if and only if $k \neq j$ and $k \in C$ for some subset $C \subseteq \mathcal{J}_c(L)$ such that $j \triangleleft_m C$, see e.g. [8, Lemma 2.31]. If a lattice is atomistic, then $k_* = \perp$ for each $k \in \mathcal{J}_c(L)$, and therefore jDk if and only if $j \neq k$ and $j \leq p \vee k$ for some $p \in L$ with $j \not\leq p$.

Proposition 23. *If a finite A -frame \mathfrak{F} is rooted and full, then $\mathsf{L}(\mathfrak{F})$ is a subdirectly-irreducible lattice.*

Proof. We argue that the digraph $(\mathcal{J}_c(\mathsf{L}(\mathfrak{F})), D)$ is rooted. Observe that $x \in \overline{\{a, y\}} = a \vee y$ whenever $xR_a y$. This implies that xDy and xDa when $x, y \in X_{\mathfrak{F}}$, $a \in A$, $x \neq y$ and $xR_a y$. The fact that $(\mathcal{J}_c(\mathsf{L}(\mathfrak{F})), D)$ is rooted follows now from \mathfrak{F} being rooted and full. \square

6. SOME THEORY OF GENERALIZED ULTRAMETRIC SPACES

Generalized ultrametric spaces over a Boolean algebra $P(A)$ turn out to be a useful tool for relational lattices [16, 23]—as well as, we claim here, for universal product frames from multidimensional modal logic [14]. The use of metrics is well known in graph theory, where universal product frames are known as Hamming graphs, see e.g. [10]. Generalized ultrametric spaces over a Boolean algebra $P(A)$ were introduced in [20] to study equivalence relations. The main results of this Section are Theorem 27 and Proposition 39 which together substantiate the claim that when A is finite, universal $\mathsf{S5}^A$ -product frames are pairwise complete ultrametric spaces valued in the Boolean algebra $P(A)$. It is this abstract point of view that shall allow us to construct a universal product frame given a lattice embedding $\mathsf{L}(\mathfrak{F}) \longrightarrow \mathsf{R}(D, A)$.

Some of the observations we shall develop are not strictly necessary to prove the undecidability result, which is the main result of this paper; namely, we can always suppose that the set A is finite. Nonetheless we include these observations since they are part of a coherent set of results and, as far as we are aware of, they are original.

Definition 24. An *ultrametric space* over $P(A)$ (briefly, a *space*) is a pair (X, δ) , with $\delta : X \times X \longrightarrow P(A)$ such that, for every $f, g, h \in X$,

$$\delta(f, f) \subseteq \emptyset, \quad \delta(f, g) \subseteq \delta(f, h) \cup \delta(h, g).$$

That is, we have defined an ultrametric space over $P(A)$ as a category (with a small set of objects) enriched over $(P(A)^{op}, \emptyset, \cup)$, see [15]. We shall assume in this paper that such a space (X, δ) is also *reduced* and *symmetric*, that is, that the following two properties hold for every $f, g \in X$:

$$\delta(f, g) = \emptyset \text{ implies } f = g, \quad \delta(f, g) = \delta(g, f).$$

Under these hypothesis, it is easily seen that if A is empty or a singleton, then the categories of spaces over $P(A)$ are trivial. Thus, we shall assume here that A has at least two elements.

A *morphism* of spaces¹ $\psi : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is a function $\psi : X \longrightarrow Y$ such that $\delta_Y(\psi(f), \psi(g)) \leq \delta_X(f, g)$, for each $f, g \in X$. If $\delta_Y(\psi(f), \psi(g)) = \delta_X(f, g)$, for each $f, g \in X$, then ψ is said to be an *isometry*. For (X, δ) a space over $P(A)$, $f \in X$ and $\alpha \subseteq A$, the ball centered in f of radius α is defined as usual: $B(f, \alpha) := \{g \in X \mid \delta(f, g) \subseteq \alpha\}$. In [1] a space (X, δ) is said to be *pairwise complete* if, for each $f, g \in X$ and $\alpha, \beta \subseteq A$,

$$B(f, \alpha \cup \beta) = B(g, \alpha \cup \beta) \text{ implies } B(f, \alpha) \cap B(g, \beta) \neq \emptyset.$$

This property is easily seen to be equivalent to:

$$\delta(f, g) \subseteq \alpha \cup \beta \text{ implies } \delta(f, h) \subseteq \alpha \text{ and } \delta(h, g) \subseteq \beta, \text{ for some } h \in X.$$

Recall also from [1] that a space is said to be *spherically complete* if the intersection $\bigcap_{i \in I} B(f_i, \alpha_i)$ of every chain $\{B(f_i, \alpha_i) \mid i \in I\}$ of balls is non-empty. It was shown in [20] that, when A is finite, every space over $P(A)$ is spherically complete. To stress the importance of these two conditions, pairwise completeness and spherical completeness, let us recall the following result from [1].

Proposition 25. *In the category of spaces, the injective objects are the spaces that are both pairwise complete and spherically complete.*

If (X, δ_X) is a space and $Y \subseteq X$, then the restriction of δ_X to Y induces a space (Y, δ_X) ; we say then that (Y, δ_X) is a *subspace* of X . Notice that the inclusion of Y into X yields an isometry of spaces.

Our main example of space over $P(A)$ is (D^A, δ) , with D^A the set of functions from A to D and the distance defined by

$$\delta(f, g) := \{a \in A \mid f(a) \neq g(a)\}. \quad (3)$$

A second example is a slight generalization of the previous one. Given a surjective function $\pi : E \longrightarrow A$, let $\text{Sec}(\pi)$ denote the set of all sections of π , that is the functions $f : A \longrightarrow E$ such that $\pi \circ f = \text{id}_A$; the formula in (3) also defines a distance on $\text{Sec}(\pi)$. Clearly, $(\text{Sec}(\pi), \delta)$ is a subspace of (E^A, δ) and, considering the first projection $\pi_1 : A \times D \longrightarrow A$, we can see that (D^A, δ) is isomorphic to the space $(\text{Sec}(\pi_1), \delta)$. By identifying $f \in \text{Sec}(\pi)$ with the vector $\langle f_a \in \pi^{-1}(a) \mid a \in A \rangle$, we see that

$$\text{Sec}(\pi) = \prod_{a \in A} E_a, \quad \text{where } E_a := \pi^{-1}(a). \quad (4)$$

That is, the underlying set of a space $(\text{Sec}(\pi), \delta)$ is that of a universal $S5^A$ -product frame. Our next observations are meant to identify the particular role of the universal $S5^A$ -product frames as spaces.

Proposition 26. *Every space of the form $(\text{Sec}(\pi), \delta)$ is both pairwise complete and spherically complete.*

Proof. It is immediate to verify that $(\text{Sec}(\pi), \delta)$ is pairwise complete. For spherical completeness, let $\mathcal{C} := \{B(f_i, \alpha_i) \mid i \in I\}$ be a chain of balls. For each $a \in A$ pick $*_a \in D_a$ and define f as follows:

$$f(a) = \begin{cases} f_i(a), & \text{if } a \notin \alpha_i \text{ for some } i \in I, \\ *_a, & \text{otherwise.} \end{cases}$$

¹As $P(A)$ is not totally ordered, we avoid calling a morphism “non expanding map” as it is often done in the literature.

Let us show that f is well defined. Namely, suppose that $a \notin \alpha_i$ and $a \notin \alpha_j$. Since \mathcal{C} is a chain, we can suppose, without loss of generality, that $B(f_i, \alpha_i) \subseteq B(f_j, \alpha_j)$ so $\delta(f_i, f_j) \subseteq \alpha_j$. Since $a \notin \alpha_j$ it follows that $f_i(a) = f_j(a)$.

Let now $i \in I$ be arbitrary; if $a \notin \alpha_i$, then $f(a) = f_i(a)$, so $\delta(f, f_i) \subseteq \alpha_i$ and $f \in B(f_i, \alpha_i)$. It follows that $f \in \bigcap_{i \in I} B(f, \alpha_i)$. \square

Theorem 27. *Every space (X, δ) over $P(A)$ has an isometry into some space of the form $(\text{Sec}(\pi), \delta)$. If (X, δ) is pairwise and spherically complete, then this isometry is an isomorphism.*

Proof. For each $a \in A$, let $D_a = \{B(f, A \setminus \{a\}) \mid f \in X\}$. That is, D_a is the quotient of X by the equivalence relation identifying f and g when $\delta(f, g) \subseteq A \setminus \{a\}$. Let $\pi : \sum_{a \in A} D_a \rightarrow A$ be the obvious projection.

We associate to $f \in X$ the vector $\psi(f) = \langle B(f, A \setminus \{a\}) \mid a \in A \rangle$. Let us argue that the correspondence ψ is an isometry:

$$\begin{aligned} a \notin \delta(\psi(f), \psi(g)) &\text{ iff } B(f, A \setminus \{a\}) = B(g, A \setminus \{a\}) \\ &\text{ iff } \delta(f, g) \subseteq A \setminus \{a\} \text{ iff } a \notin \delta(f, g), \end{aligned}$$

thus $\delta(\psi(f), \psi(g)) = \delta(f, g)$. In particular, when the space is reduced (i.e. $\delta(f, g) = \emptyset$ implies $f = g$), ψ is an injective map.

Next, we suppose that (X, δ) is pairwise and spherically complete and argue that ψ is surjective. To this goal, we fix a well-ordering on A , say $A = \{a_\lambda \mid \lambda < \tau\}$. For $\lambda < \tau$, let us also set $A_{\lambda-} := \{a_\beta \in A \mid \beta \leq \lambda\}$ and $A_{\lambda+} := A \setminus A_{\lambda-} = \{a_\beta \in A \mid \beta > \lambda\}$.

Let $v := \langle B(f_\lambda, A \setminus \{a_\lambda\}) \mid \lambda < \tau \rangle \in \text{Sec}(\pi)$; we need to construct a preimage of v by ψ . To this end, we construct, by induction on $\lambda < \tau$, a family $\{g_\lambda \in X \mid \lambda < \tau\}$ such that $g_\lambda \in B(g_\beta, A_{\beta+})$ for $\beta \leq \lambda$ and $\delta(g_\lambda, f_\lambda) \subseteq A \setminus \{a_\lambda\}$. Let $\lambda < \tau$ be an ordinal and suppose that we have defined g_β with these properties for each $\beta < \lambda$. As $\{B(g_\beta, A_{\beta+}) \mid \beta < \lambda\}$ is a chain, we can pick $g \in \bigcap_{\beta < \lambda} B(g_\beta, A_{\beta+})$. Notice that if $\lambda = \gamma + 1$ is a successor cardinal, then we can simply pick g_γ .

We use pairwise completeness to define g_λ as some h with $\delta(g, h) \subseteq \{a_\lambda\}$ and $\delta(h, f_\lambda) \subseteq \delta(g, f_\lambda) \setminus \{a_\lambda\}$ (if $a_\lambda \notin \delta(g, f_\lambda)$, then we can take $h = g$). Clearly, $\delta(g_\lambda, g_\lambda) = \emptyset \subseteq A_{\lambda+}$ and, for $\beta < \lambda$, we have $\delta(g_\lambda, g_\beta) \subseteq \delta(g_\lambda, g) \cup \delta(g, g_\beta) = \{a_\lambda\} \cup \delta(g, g_\beta) \subseteq \{a_\lambda\} \cup A_{\beta+} \subseteq A_{\beta+}$.

Let now $g \in \bigcap_{\lambda < \tau} B(g_\beta, A_{\beta+})$. If $\lambda < \tau$, then

$$\delta(g, f_\lambda) \subseteq \delta(g, g_\lambda) \cup \delta(g_\lambda, f_\lambda) \subseteq A_{\lambda+} \cup (A \setminus \{a_\lambda\}) \subseteq A \setminus \{a_\lambda\}.$$

This shows that $B(g, A \setminus \{a_\lambda\}) = B(f_\lambda, A \setminus \{a_\lambda\})$ or, stated otherwise, $\psi(g)_\lambda = v_\lambda$, for each $\lambda < \tau$, so g is a preimage of v . \square

By the last two Propositions and the last Theorem, we obtain:

Corollary 28. *Universal product frames are, up to isomorphism, the injective objects in the category of spaces.*

Continuous subspaces and completeness. In the following, let (X, δ_X) be a fixed pairwise complete space. Our next goal is to devise criteria to recognize pairwise complete subspaces of (X, δ_X) . To this end, let us introduce the notion of continuous subspaces of (X, δ_X) as follows: for a subspace Y of X , we define

$$\nu_Y(f) := \bigcap \{\delta(f, g) \mid g \in Y\},$$

and say that a subspace Y of X is *continuous* if, for each $f \in X$, $\nu_Y(f) = \emptyset$ implies $f \in Y$.

Lemma 29. *A continuous subspace Y of X is pairwise complete.*

Proof. Let $f, g \in Y$ with $\delta(f, g) \subseteq \alpha \cup \beta$. Let $h \in X$ be such that $\delta(f, h) \subseteq \alpha \setminus \beta$ and $\delta(h, g) \subseteq \beta$; then $\bigcap_{k \in Y} \delta(h, k) \subseteq \delta(h, f) \cap \delta(h, g) \subseteq \alpha \setminus \beta \cap \beta = \emptyset$, so $h \in Y$ since Y is continuous. \square

Lemma 30. *If X is also spherically complete, then a continuous subspace Y of X is pairwise and spherically complete.*

Proof. By Theorem 27, we can suppose that X is of the form $\prod_{a \in A} E_a$. A subset Y of X is then continuous if $f \in Y$, whenever for all $a \in A$ there exists $g \in Y$ with $f(a) = g(a)$. Let us put $Y_a := \{g(a) \mid g \in Y\}$, so $Y \subseteq \prod_{a \in A} Y_a$. If $f \in \prod_{a \in A} Y_a$, then for each $a \in A$ there is some $g \in Y$ such that $f(a) = g(a)$ and therefore $f \in Y$. We have therefore $Y = \prod_{a \in A} Y_a$, so Y is pairwise and spherically complete. \square

Lemma 31. *If Y is pairwise complete subspace of X , then, for each $f \in X$, the set $\{\alpha \subseteq A \mid B(f, \alpha) \cap Y \neq \emptyset\}$ is closed under finite intersections. Consequently, if Y is spherically complete, then, for each $f \in X$, there exists $g \in Y$ with $\delta(f, g) = \nu_Y(f)$.*

Proof. Let \mathcal{A} be the set of all $\alpha \subseteq A$ such that $B(f, \alpha) \cap Y \neq \emptyset$. For each $\alpha \in \mathcal{A}$, choose $t_\alpha \in B(f, \alpha) \cap Y$.

Observe that, for $\alpha, \alpha' \in \mathcal{A}$, $\delta(t_\alpha, t_{\alpha'}) \subseteq \delta(t_\alpha, f) \cup \delta(f, t_{\alpha'}) \subseteq \alpha \cup \alpha'$. That is, the function t sending α to t_α is a γ -Cauchy function to Y as defined in [1, Definition 2.8], where $\gamma = \bigcap \mathcal{A}$.

Observe next that $\delta(t_\alpha, t_{\alpha'}) \subseteq \alpha \cup \alpha'$ so, by pairwise completeness of Y , $\delta(t_\alpha, h) \subseteq \alpha$ and $\delta(h, t_{\alpha'}) \subseteq \alpha'$ for some $h \in Y$. It follows that $\delta(f, h) \subseteq \alpha \cap \alpha'$, showing that $\alpha \cap \alpha' \in \mathcal{A}$. In particular, t is a γ -Cauchy net, as defined in [1, Definition 2.9].

If we assume that Y is spherically complete, then we can use Proposition 2.16 in [1] to deduce that, for some $g \in Y$, $\delta(g, t_\alpha) \subseteq \alpha$, for each $\alpha \in \mathcal{A}$. For such a $g \in Y$, we have $\delta(f, g) \subseteq \delta(f, t_\alpha) \cup \delta(t_\alpha, g) \subseteq \alpha$, showing that $\delta(f, g) \subseteq \nu_Y(f)$. As $g \in Y$, we also have $\nu_Y(f) \subseteq \delta(f, g)$ and $\nu_Y(f) = \delta(f, g)$. \square

Corollary 32. *If Y is a pairwise complete and spherically complete subspace of X , then Y is a continuous subspace of X .*

Proof. Let $f \in X$ be such that $\nu_Y(f) = \bigcap_{g \in Y} \delta(f, g) = \emptyset$. By Lemma 31, let $h \in Y$ such that $\nu_Y(f) = \delta(f, h)$, so $\delta(f, h) = \emptyset$ and $f = h \in Y$. \square

Corollary 33. *If X is also spherically complete, then a subspace of X is continuous if and only if it is pairwise and spherically complete.*

Let us remark that when A is finite, then (X, δ_X) and all its subspaces are spherically complete, so in this case continuity and pairwise completeness are equivalent conditions.

Modules. We say that a function $v : X \longrightarrow P(A)$ is a *module* if

$$v(f) \subseteq \delta(f, g) \cup v(g).$$

In (enriched) category theory “module” is a standard naming for an enriched functor from an enriched category to the base category enriched on itself. Here a module

can be seen as a space morphism from (X, δ) to the space $(P(A), \Delta)$, where Δ is the symmetric difference.

We let $Modules(X)$ be the set of all modules v ; we order this set by letting $v \leq w$ if and only if $w(f) \subseteq v(f)$, for each $f \in Y$ (that is, we take the reverse pointwise order). Let us use $Sub(X)$ for the set of subspaces of X , ordered by inclusion—thus $Sub(X)$ is the usual power set of X . Given a module v , let us define

$$S_v := \{x \in X \mid v(x) = \emptyset\}.$$

It is easily seen that $S : Modules(X) \rightarrow Sub(X)$, sending v to S_v , is also order-preserving.

Lemma 34. *The map ν is left adjoint to S .*

Proof. As both maps are order-preserving, we shall show that the usual unit and counit laws hold. If $f \in Y$, then $\nu_Y(f) \subseteq \delta(f, f) = \emptyset$; thus $Y \subseteq S_{\nu_Y}$. Let us argue for the counit law. For each $f \in X$ and $g \in S_v$ —i.e. when $v(g) = \emptyset$ —we have $v(f) \subseteq \delta(f, g) \cup v(g) = \delta(f, g)$. It follows that $v(f) \subseteq \nu_{S_v}(f)$, for each $f \in X$. This means that $\nu_{S_v} \leq v$ in $Modules(X)$. \square

Lemma 35. *For each module v , S_v is a continuous subspace of X .*

Proof. We already observed that $v(f) \subseteq \nu_{S_v}(f)$, that is, $v(f) \subseteq \bigcap_{g \in S_v} \delta(f, g)$. If the latter expression is equal to the emptyset, then $v(f) = \emptyset$, whence $f \in S_v$. This shows that S_v is a continuous subspace of X . \square

Proposition 36. *A subspace Y of X is continuous if and only if $S_{\nu_Y} = Y$. Thus, for any $Y \subseteq X$, $S_{\nu_Y} \subseteq X$ is the least continuous subspace of X containing Y .*

Proof. Observe first that if $Y = S_{\nu_Y}$, then Y is continuous by Lemma 35.

Conversely, let us suppose that Y is continuous. By adjointness, $Y \subseteq S_{\nu_Y}$ holds, so we argue for the reverse inclusion. If $f \in S_{\nu_Y}$, then $\emptyset = \nu_Y(f) = \bigcap_{g \in Y} \delta(f, g)$, so $f \in Y$. Therefore $S_{\nu_Y} \subseteq Y$.

The last statement follows from the characterization of the continuous subspaces of X as the fixed-points of the closure operator $S_{\nu(-)}$. \square

Proposition 37. *A module v is such that $\nu_{S_v} = v$ if and only if either $v(f) = A$, for each $f \in X$, or $v(f) = \emptyset$, for some $f \in X$.*

Proof. If $v(f) = A$ for each $f \in X$, then $S_v = \emptyset$ (since we are assuming that $A \neq \emptyset$). It follows that $\nu_{S_v}(f) = A$ for each $f \in X$ and $\nu_{S_v} = v$.

Suppose now that $v(g) = \emptyset$ for some $g \in X$. By adjointness, we have $v(f) \subseteq \nu_{S_v}(f)$, for all $f \in X$; thus we need to argue for the opposite inclusion. Fix $f \in X$; we exhibit next $h \in X$ such that $v(h) = \emptyset$ and $\delta(f, h) \subseteq v(f)$. It shall follow that $\nu_{S_v}(f) = \bigcap_{v(h)=\emptyset} \delta(f, h) \subseteq v(f)$. Since $v(f) \subseteq \delta(f, g) \cup v(g) = \delta(f, g)$, we can write $\delta(f, g) = v(f) \cup (\delta(f, g) \setminus v(f))$. We use now pairwise completeness to pick $h \in X$ such $\delta(f, h) \subseteq v(f)$ and $\delta(h, g) \subseteq \delta(f, g) \setminus v(f)$. Then $v(h) \subseteq \delta(h, f) \cup v(f) \subseteq v(f)$, and $v(h) \subseteq \delta(h, g) \cup v(g) = \delta(h, g) \subseteq \delta(f, g) \setminus v(f)$. It follows that $v(h) \subseteq v(f) \cap A \setminus v(f)$, whence $v(h) = \emptyset$.

For the converse direction, suppose that $v(f) = \nu_{S_v}(f)$ for each $f \in X$. If $v(f) \neq \emptyset$, for each $f \in Y$, then $v(f) = \nu_{S_v}(f) = A$, for each $f \in X$. Otherwise, $v(f) = \emptyset$, for some $f \in X$. \square

Remark 38. Proposition 36, characterizing continuous subspaces as closed subsets of a closure operator, suggests that pairwise complete spaces might have some algebraic nature as well. This is actually the case. It is easily verified that a space (X, δ) is pairwise complete if and only if, for each α, β such that $\alpha \cap \beta = \emptyset$, and for each $f, g \in X$, $\delta(f, g) \subseteq \alpha \cup \beta$ implies $\delta(f, h) \subseteq \alpha$ and $\delta(h, g) \subseteq \beta$ for some $h \in X$. We observe that such an h is unique. Suppose that $\alpha \cap \beta = \emptyset$ and let $h_i, i = 1, 2$ with $\delta(f, h_i) \subseteq \alpha$ and $\delta(h_i, g) \subseteq \beta$. Then $\delta(h_1, h_2) \subseteq \delta(h_1, f) \cup \delta(f, h_2) \subseteq \alpha$ and similarly $\delta(h_1, h_2) \subseteq \beta$. It follows that $\delta(h_1, h_2) \subseteq \alpha \cap \beta = \emptyset$ and $h_1 = h_2$.

Pairwise complete spaces and universal product frames. We already observed, in the display (4), that the underlying set of a space of the form $(\text{Sec}(\pi), \delta)$ with $\pi : E \rightarrow A$ is that of a universal $S5^A$ -product frame. Something more is true: we can define the transition relations of the universal $S5^A$ -product frame by means of the metric. Indeed, for each $a \in A$, we have

$$fR_ag \text{ iff } \delta(f, g) \subseteq \{a\}.$$

On the other hand, if A is finite, then the metric is completely determined from the transition relation of the frame, using the notion of α -path introduced in Section 2, as follows:

$$\delta(f, g) = \bigcap \{ \alpha \subseteq A \mid \text{there exists an } \alpha\text{-path from } f \text{ to } g \}.$$

We cast our observations in a Proposition:

Proposition 39. *If A is finite, then there is a bijective correspondence between spaces over $P(A)$ of the form $(\text{Sec}(\pi), \delta)$ and universal $S5^A$ -product frames.*

Pairwise complete spaces and lattices. We can generalize the construction of the relational lattice $R(D, A)$ starting from an arbitrary space (X, δ) . We say that a subset $Z \subseteq A \cup X$ is closed if $x \in Z$ whenever $\delta(x, y) \subseteq Z \cap A$ and $y \in Z$. The set of closed subsets of $A \cup X$ is then a Moore family.

Definition 40. The lattice $\mathcal{L}(X, \delta)$ is the lattice of closed subsets of $A \cup X$.

Obviously, $\mathcal{L}(D^A, \delta)$ is the relational lattice $R(D, A)$. The lattice $\mathcal{L}(X, \delta)$ can be shown to be pluperfect when (X, δ) is reduced and symmetric. Yet, for the sake of the undecidability result, we shall only need that $\mathcal{L}(X, \delta)$ is an atomistic pluperfect lattice when A is finite and (X, δ) is pairwise complete.

Proposition 41. *If A is finite and (X, δ) is pairwise complete, then $\mathcal{L}(X, \delta)$ is isomorphic to the lattice $L(\mathfrak{U})$ for some universal $S5^A$ -product frame \mathfrak{U} .*

Proof. By Theorem 27 the space (X, δ) is isomorphic to the space $(\text{Sec}(\pi), \delta)$, for some surjective $\pi : E \rightarrow A$. The construction \mathcal{L} clearly sends isomorphic spaces to isomorphic lattices. Therefore, we assume that $(X, \delta) = (\text{Sec}(\pi), \delta)$ and prove that $\mathcal{L}(X, \delta) = L(\mathfrak{U})$. We have $X = \prod_{a \in A} E_a$ with $E_a = \pi^{-1}(a)$, while $\delta(f, g) = \{a \in A \mid f(a) \neq g(a)\}$. It is easily verified that $\delta(f, g) \subseteq \alpha$ if and only if there is an α -path from f to g in the universal $S5^A$ product frame \mathfrak{U} on $\prod_{a \in A} E_a$. Therefore, the two Moore families, $\mathcal{L}(X, \delta)$ and $L(\mathfrak{U})$, are the same. \square

From the above theorem and from the preliminary investigation of the structure of the lattices $L(\mathfrak{F})$ in Section 5, we can infer the following statement.

Corollary 42. *If A is finite and (X, δ) is pairwise complete, then $\mathcal{L}(X, \delta)$ is an atomistic pluperfect lattice, where the set of join-irreducible elements can be identified with $A \cup X$, every element $a \in A$ is join-prime, and minimal join-covers of $f \in X$ are of the form*

$$f \triangleleft_m \delta(f, g) \cup \{g\},$$

for each $g \in X$.

Proof. The statement follows from Proposition 22 and from the observation that an α -path from f to g is minimal if and only if $\alpha = \delta(f, g)$. \square

Let us remark that the above statement holds even when A is not finite or when (X, δ) is not pairwise complete. In particular, if A is infinite and the universal $S5^A$ -product frame \mathfrak{U} has $(\text{Sec}(\pi), \delta)$ as underlying space, then the lattice $L(\mathfrak{U})$ defined in Section 5 and the lattice $\mathcal{L}(\text{Sec}(\pi), \delta)$ defined here need not to be equal. For instance, if $\delta(f, g)$ is an infinite set, then $\delta(f, g) \cup \{g\}$ is an infinite minimal join-cover of f , while we observed before that any minimal join-cover in $L(\mathfrak{U})$ is finite.

7. PRINCIPAL IDEALS AND FILTERS IN RELATIONAL LATTICES

The purpose of this Section is to prove the following statement.

Theorem 43. *If L is a finite subdirectly-irreducible atomistic lattice which has a lattice embedding into some relational lattice $R(D, A)$, then there exists an embeddings of L into some other relational lattice $R(D, B)$ which moreover preserves \perp and \top .*

The theorem is an immediate consequence of Propositions 44 and 46 that follows. These propositions mainly deal with the structure of principal ideals and filters in a relational lattice $R(D, A)$, namely the sublattices of the form $\downarrow Z := \{W \in R(D, A) \mid W \subseteq Z\}$ and $\uparrow Z := \{W \in R(D, A) \mid Z \subseteq W\}$.

In the proofs of these propositions we use the isomorphism between $P(A \sqcup D^A)$ and $P(A) \times P(D^A)$ to represent the lattice $R(D, A)$ as the set of pairs (α, Y) with $\alpha \subseteq A$ and $Y \subseteq D^A$ α -closed.

Proposition 44. *If L is a subdirectly-irreducible lattice which has an embedding $i : L \longrightarrow R(D, A)$, then there is a subset $\alpha \subseteq A$ and an embedding $j : L \longrightarrow R(D, \alpha)$ that preserves \top .*

Proof. Suppose $i(\top) \neq (A, D^A)$, say $i(\top) = (\alpha, Y)$, with Y α -closed. Call M the ideal $\downarrow(\alpha, Y)$, so L embeds into M while preserving \top . Let us study the structure of M . This lattice is clearly atomistic and pluperfect by Lemma 3. Its set of atoms is $\alpha \cup Y$, while the minimal join-covers are of the form $f \triangleleft_m \delta(f, g) \cup \{g\}$ whenever $f, g \in Y$ and $\delta(f, g) \subseteq \alpha$.

Notice now that if $f \in Y$, then the ball $B(f, \alpha)$ is contained in Y , since Y is α -closed. This implies that $A_f := \alpha \cup B(f, \alpha)$ is a D -closed subset of $\mathcal{J}_c(M)$ and, M_{A_f} defined in Lemma 4, is a lattice quotient of M . We notice that the OD-graph of M_{A_f} is isomorphic to the one of $R(D, \alpha)$, so M_{A_f} itself is isomorphic to $R(D, \alpha)$.

Since moreover $\bigcup_{f \in Y} \alpha \cup B(f, \alpha) = \alpha \cup Y$, then $\langle \pi_{A_f} \mid f \in Y \rangle : M \longrightarrow \prod_{f \in Y} M_{A_f}$ is, by Lemma 5, a subdirect decomposition of M . Therefore L embeds into $\prod_{f \in Y} M_{A_f}$ and since L is subdirectly-irreducible, it embeds into some

M_{A_f} and such embedding preserves \top . Since M_{A_f} is isomorphic to $R(D, \alpha)$, we conclude that L embeds into $R(D, \alpha)$ while preserving \top . \square

For $B \subseteq A$, let us define $\psi_{A,B} : P(A) \times P(D^A) \longrightarrow P(B) \times P(D^B)$ by the following formula:

$$\psi_{A,B}(\alpha, X) := (\alpha \cap B, X \upharpoonright_B).$$

Lemma 45. *The map $\psi_{A,B}$ restricts to an order-preserving map from $R(D, A)$ to $R(D, B)$. Its further restriction to the filter $\uparrow(B^c, \emptyset) \subseteq R(D, A)$ yields an isomorphism with $R(D, B)$.*

Proof. We suppose that X is α -closed and argue $X \upharpoonright_B$ is $\alpha \cap B$ -closed. If $g \in X$, $f \in D^B$, and $\delta_{D^B}(f, g \upharpoonright_B) \subseteq \alpha \cap B$, then we can extend f to $f' \in D^A$, so $f' \upharpoonright_B = f$ and $f'(x) = g(x)$ for all $x \in B^c$. It follows that $\delta_{D^A}(f', g) = \delta_{D^B}(f, g \upharpoonright_B) \subseteq \alpha \cap B$, so $f' \in X$ since X is α -closed. Then $f = f' \upharpoonright_B \in X \upharpoonright_B$.

We argue similarly that if $(\beta, Y) \in R(D, B)$, then $(\beta \cup B^c, i_A(Y))$ belongs to $R(D, A)$, namely that $i_A(Y)$ is $\beta \cup B^c$ -closed when Y is β -closed. Let $f \in D^A$ and $g \in i_A(Y)$ be such that $\delta(f, g) \subseteq \beta \cup B^c$. Now $\delta(f \upharpoonright_B, g \upharpoonright_B) \subseteq (\beta \cup B^c) \cap B = \beta$, and since $g \upharpoonright_B \in Y$ and Y is β -closed, we have $f \upharpoonright_B \in Y$, that is $f \in i_A(Y)$.

Observe moreover that $(\alpha \cap B, X \upharpoonright_B) \subseteq (\beta, Y)$ holds if and only if $(\alpha, X) \subseteq (\beta \cup B^c, i_A(Y))$, so the two maps are adjoints to each other, in particular they are monotonic.

Next $\psi_{A,B}(\beta \cup B^c, i_A(Y)) = ((\beta \cup B^c) \cap B, i_A(Y) \upharpoonright_B) = (\beta, Y)$, thus $\psi_{A,B}$ is surjective. Finally, let us argue that $\psi_{A,B}$ is injective if restricted to $\uparrow(B^c, \emptyset)$. Let $(\alpha, X), (\alpha', X') \in R(D, A)$ with $B^c \subseteq \alpha \cap \alpha'$ and $\psi_{A,B}(\alpha, X) = \psi_{A,B}(\alpha', X')$. Then $\alpha \cap B = \alpha' \cap B$, $\alpha = (\alpha \cap B) \cup B^c$, and $\alpha' = (\alpha' \cap B) \cup B^c$, imply $\alpha = \alpha'$. Let $f \in X$, so $f \upharpoonright_B \in X \upharpoonright_B = X' \upharpoonright_B$, so there exists $f' \in X'$ with $f' \upharpoonright_B = f \upharpoonright_B$. Since X' is B^c closed and $\delta(f, f') \subseteq B^c$ we have $f \in X'$. Thus we have $X \subseteq X'$; a similar argument yields $X' \subseteq X$, so $X = X'$. \square

Proposition 46. *If a finite subdirectly-irreducible atomistic lattice L has a \top -preserving lattice embedding $i : L \longrightarrow R(D, A)$, then there exists an embedding $j : L \longrightarrow R(D, B)$ which preserves \top and \perp .*

Proof. By Lemma 45, if $i(\perp) = (B^c, X)$ for some $B \subseteq A$, then $j = \psi_{A,B} \circ i : L \longrightarrow R(D, B)$ is an embedding which preserves \top and such that $j(\perp) = (\emptyset, Y)$ for some $Y \subseteq D^B$.

Suppose now that $Y \neq \emptyset$. Let $\mu : R(D, B) \longrightarrow L$ be left adjoint to j , so μ is surjective and, moreover, each atom $a \in \mathcal{J}_c(L)$ has some $k \in B \cup D^B$ with $\mu(k) = a$. Notice also that $\mu(k) = \perp$ if and only if $k \in Y$, for each $k \in B \cup D^B$. Let us argue that every element of $\mathcal{J}_c(L)$ is join-prime. Let $a \in \mathcal{J}_c(L)$ and pick $k \in B \cup D^B$ such that $\mu(k) = a$. If $k \in B$, then $a = \mu(k)$ is join-prime, since μ sends a join-prime element either to a join-prime element or to \perp . Suppose now $k = f \in D^B$ and recall that $\mu(f) = a \neq \perp$ implies $f \notin Y$. Pick $g \in Y$, so $f \leq \bigvee \delta(f, g) \vee g$ and $a = \mu(f) \leq \bigvee \mu(\delta(f, g)) \vee \mu(g) = \bigvee \mu(\delta(f, g))$. Since μ sends join-prime elements to join-prime elements or to \perp , we see that a has a join-cover made up of join-prime elements only. Lemma 7 implies then that a is join-prime.

We have argued that either $Y = \emptyset$, so j preserves \perp ; or $Y \neq \emptyset$, in which case all the elements of $\mathcal{J}_c(L)$ are join-prime and atoms. In the last case, however, L is a two elements Boolean algebra, since L is subdirectly-irreducible and distributive.

Such an algebra can obviously be embedded into a relational lattice while preserving \top and \perp . \square

8. FROM LATTICE EMBEDDINGS TO SURJECTIVE p -MORPHISMS

We prove in this Section the converse of Theorem 20:

Theorem 47. *Let A be a finite set, let \mathfrak{F} be a finite rooted full S4 A -frame. If $\mathbf{L}(\mathfrak{F})$ embeds into a relational lattice $\mathbf{R}(D, B)$, then there exists a universal $\mathbf{S5}^A$ -product frame \mathfrak{U} and a surjective p -morphism from \mathfrak{U} to \mathfrak{F} .*

To prove the Theorem, we study bound-preserving embeddings of finite atomistic lattices into lattices of the form $\mathbf{R}(D, B)$. Let in the following

$$i : L \longrightarrow \mathbf{R}(D, B)$$

be a fixed bound-preserving lattice embedding, with L a finite atomistic lattice. Since L is finite, i has a left adjoint $\mu : \mathbf{R}(D, B) \longrightarrow L$. By abuse of notation, we shall also use the same letter μ to denote the restriction of this left adjoint to the set of completely join-irreducible elements of $\mathbf{R}(D, B)$ which, we recall, is identified with $B \cup D^B$. It is a general fact—and the main ingredient of Birkhoff's duality for finite distributive lattices—that left adjoints to bound-preserving lattice morphism preserve join-prime elements. Thus we have:

Lemma 48. *If $b \in B$, then $\mu(b)$ is join-prime.*

Proof. Suppose $b \in B$ and $\mu(b) \leq \bigvee X$. Then $b \leq i(\bigvee X) = \bigvee i(X)$, so $b \leq i(x)$ for some $x \in X$, since b is join-prime. It follows that $\mu(b) \leq x$, for some $x \in X$. \square

It is not in general true that left adjoints send join-irreducible elements to join-irreducible elements, and this is a main difficulty towards a proof of Theorem 47. Yet, the following statements hold:

Lemma 49. *For each $x \in \mathcal{J}_c(L)$ there exists $y \in B \cup D^B$ such that $\mu(y) = x$.*

Proof. Since i is an embedding, then its left adjoint μ is surjective. So if $x \in \mathcal{J}_c(L)$, then there exists $y \in \mathbf{R}(D, B)$ with $\mu(y) = x$. Write $y = \bigvee_{i \in I} z_i$ with each $z_i \in B \cup D^B$. Then $x = \bigvee_{i \in I} \mu(z_i)$, so $x = \mu(z_i)$ for some $i \in I$ and such a z_i is a preimage of y by μ which belongs to $B \cup D^B$. \square

Lemma 50. *Let $g \in D^B$ such that $\mu(g)$ is join-reducible in L . There exists $h \in D^B$ such that $\mu(h) \in \mathcal{J}_c(L)$ and $\mu(g) = \bigvee \mu(\delta(g, h)) \vee \mu(h)$; moreover, $\mu(h)$ is non-join-prime whenever L is not a Boolean algebra.*

Proof. Write $\mu(g) = \bigvee \alpha$ with $\alpha \subseteq \mathcal{J}_c(L)$ and α minimal with these two properties. We have then $g \leq \bigvee i(\alpha)$ so $\delta(g, h) \cup \{h\} \ll i(\alpha)$ for some $h \in D^B$. We have then $\mu(\delta(g, h)) \cup \{\mu(h)\} \ll \alpha$ and this relation implies that $\mu(\delta(g, h)) \cup \{\mu(h)\} \subseteq \alpha$. Indeed, since i preserves the least element, $\mu(x) = \perp$ implies $x = \perp$. Thus every element of $\mu(\delta(g, h)) \cup \{\mu(h)\}$ is distinct from \perp and below an atom in α , so it is necessarily equal to such an atom. In particular, we have $\mu(h) \in \mathcal{J}_c(L)$.

We also have $\bigvee \alpha \leq \mu(g) \leq \bigvee \mu(\delta(g, h)) \vee \mu(h) \leq \bigvee \alpha$, so $\mu(g) = \bigvee \mu(\delta(g, h)) \vee \mu(h)$. By minimality, it follows $\alpha = \mu(\delta(g, h)) \cup \{\mu(h)\}$.

Suppose that L is not a Boolean algebra, so we can find an atom $a \in \mathcal{J}_c(L)$ which is non-join-prime. Pick $f \in D^B$ such that $\mu(f) = a$. Observe that every element

$\delta(f, g) \cup \delta(g, h)$ is join-prime, so every element of $\mu(\delta(f, g) \cup \delta(g, h))$ is also join-prime. If $\mu(h)$ is join-prime, then we deduce $a \leq \bigvee \mu(\delta(f, g)) \vee \bigvee \mu(\delta(g, h)) \vee \mu(h)$, so the non-join-prime a has a join-cover all made of join-prime elements. Since L is in the variety generated by the relational lattices, this contradicts Lemma 7. \square

Let A be the set of atoms of L that are join-prime. While (D^B, δ) is a space over $P(B)$, we need to transform D^B into a space over $P(A)$. To this end, we define a $P(A)$ -valued distance δ_A on D^B by

$$\delta_A(f, g) := \{\mu(b) \mid b \in \delta(f, g)\}.$$

Because of Lemma 48, we have $\delta_A(f, g) \subseteq A$.

Proposition 51. *(D^B, δ_A) is a pairwise complete ultrametric space over $P(A)$.*

Proof. δ_A satisfies the properties defining a distance (including being reduced and symmetric), mainly because the direct image of any function (here of μ) preserves unions.

For pairwise completeness, observe that if $\delta_A(f, g) \subseteq \alpha_0 \cup \alpha_1$, then $\delta(f, g) \subseteq \beta_0 \cup \beta_1$, where $\beta_i := \{b \in B \mid \mu(b) \in \alpha_i\}$, $i = 0, 1$. Taking h such that $\delta(f, h) \subseteq \beta_0$ and $\delta(h, g) \subseteq \beta_1$, we obtain $\delta_A(f, h) \subseteq \alpha_0$ and $\delta_A(h, g) \subseteq \alpha_1$. \square

We define next $v : D^B \rightarrow P(A)$ by letting

$$v(f) := \{a \in A \mid a \leq \mu(f)\}.$$

Lemma 52. *$v : D^B \rightarrow P(A)$ is a module on (D^B, δ_A) . That is, the relation*

$$v(f) \subseteq \delta_A(f, g) \cup v(g).$$

holds.

Proof. Suppose that $a \in v(f)$ and $a \notin \delta_A(f, g)$. This means that $a \leq \mu(f)$ but $b \notin \delta(f, g)$ whenever $\mu(b) = a$. Recall that if $b \in B$, then b is join-prime, whence $\mu(b)$ is join-prime as well. Thus if $a \in A$ and $a \leq \mu(b)$, then $a = \mu(b)$, since we are assuming that L is atomistic. Since $a \leq \mu(f) \leq \bigvee_{b \in \delta(f, g)} \mu(b) \vee \mu(g)$, a is join-prime, $a \leq \mu(b)$ implies $a = \mu(b)$, we necessarily have $a \leq \mu(g)$, so $a \in v(g)$. \square

Lemma 53. *The map $v : D^B \rightarrow P(A)$ is a module on (D^B, δ_A) . Moreover $v(f) = \emptyset$ if and only if $\mu(f) \in \mathcal{J}_c(L) \setminus A$.*

Proof. Suppose that $\mu(f) \in \mathcal{J}_c(L) \setminus A$. If $v(f) \neq \emptyset$, then let $a \in A$ with $a \leq \mu(f)$. Since we are assuming that $\mu(f)$ is join-irreducible and that L is atomistic, we deduce $\mu(f) = a \in A$, a contradiction.

Conversely, suppose that $v(f) = \emptyset$. This immediately gives $\mu(f) \notin A$. By the way of contradiction, suppose now that $\mu(f)$ is reducible, so use Lemma 50 to find $h \in D^B$ such that $\mu(h) \in \mathcal{J}_c(L)$ and $\mu(f) = \bigvee \mu(\delta(f, h)) \vee \mu(h)$. Since $\mu(h)$ is join-irreducible, then $h \neq f$ and $\delta(f, h) \neq \emptyset$. Pick $b \in \delta(f, h)$, then $\mu(b) \in A$ and $\mu(b) \leq \mu(f)$. This gives $\mu(b) \in v(f)$, so $v(f) \neq \emptyset$, a contradiction. \square

Using Lemmas 35 and 53, we derive:

Corollary 54. *The subspace*

$$F_0 := \{f \in D^B \mid \mu(f) \in \mathcal{J}_c(L) \setminus A\} \quad (5)$$

of D^B is pairwise complete.

Proof. By Lemma 53, $f \in F_0$ if and only if $v(f) = \emptyset$. Since v is a module, the set $\{f \in D^B \mid v(f) = \emptyset\}$ is, by Lemma 35, a pairwise complete metric space over $P(A)$. \square

The following Proposition, which ends the study of bound-preserving lattice embeddings into relational lattices, shows that modulo the shift of the codomain to the lattice of a universal product frame, such a lattice embedding can always be normalized, meaning that join-irreducible elements are sent to join-irreducible elements by the left adjoint.

Proposition 55. *Let L be a finite atomistic lattice and let A be the set of its join-prime elements. If L is not a Boolean algebra and $i : L \rightarrow \mathbf{R}(D, B)$ is a bound-preserving lattice embedding, then there exists a pairwise complete ultrametric space (F_0, δ) over $P(A)$ and a bound-preserving lattice embedding $j : L \rightarrow \mathcal{L}(F_0, \delta_A)$ whose left adjoint ν satisfies the following condition: for each $k \in A \cup F_0$, if $k \in A$ then $\nu(k) = k$ and, otherwise, $\nu(k) \in \mathcal{J}_c(L) \setminus A$.*

Proof. Let (F_0, δ_A) be the pairwise complete space over $P(A)$ as defined in Corollary 54. By the definition of F_0 , the restriction of μ to F_0 takes values in $\mathcal{J}_c(L) \setminus A$. Therefore we can define $\nu : A \cup F_0 \rightarrow \mathcal{J}_c(L)$ as follows:

$$\nu(k) := \begin{cases} k, & k \in A, \\ \mu(k), & k \in F_0. \end{cases}$$

We notice next that ν is surjective. If $x \in \mathcal{J}_c(L) \setminus A$, then by Lemma 49, there is $y \in B \cup D^B$ such that $\mu(y) = x$. By Lemma 48, $y \notin B$, so $y \in D^B$. Since $\mu(y) = x \in \mathcal{J}_c(L) \setminus A$, then y belongs to F_0 .

Let $j : L \rightarrow \mathcal{L}(F_0, \delta_A)$ be the function defined by

$$j(l) := \{x \in A \cup F_0 \mid \nu(x) \leq l\}.$$

Let us argue, in the order, that

- (1) for each $l \in L$, $j(l)$ is a closed subset of $A \cup F_0$,
- (2) j is injective,
- (3) j preserves meets and (4) it preserves joins.

(1). Let $f, g \in F_0$ and suppose that $\delta_A(f, g) \cup \{g\} \subseteq j(l)$. This condition means that $\nu(\delta_A(f, g)) = \delta_A(f, g) = \mu(\delta(f, g)) \ll \{l\}$ and $\mu(g) \leq l$; it follows that $\nu(f) = \mu(f) \leq \bigvee \mu(\delta(f, g)) \vee \mu(g) \leq l$, so $f \in j(l)$.

(2). We have $j(l_0) = j(l_1)$ if and only if, for all $x \in A \cup F_0$, the condition $\nu(x) \leq l_0$ is equivalent to $\nu(x) \leq l_1$. As ν is surjective, this means that l_0 and l_1 have the same atoms below them, thus that they are equal.

(3). It is easily verified that

$$j(\top) = A \cup F_0, \quad \text{and} \quad j(l_0 \wedge l_1) = j(l_0) \cap j(l_1).$$

In particular, j is order-preserving.

(4). Since j is order-preserving, we only need to show that $j(l_0 \vee l_1) \leq j(l_0) \vee j(l_1)$. To this end, we suppose that $x \in A \cup F_0$ is such that $x \in j(l_0 \vee l_1)$, so $\nu(x) \leq l_0 \vee l_1$. If $x \in A$, then $\nu(x) = x \leq l_0 \vee l_1$, and since x is join-prime, this gives $\nu(x) = x \leq l_i$ for some $i \in \{0, 1\}$. This immediately yields $x \in j(l_i) \leq j(l_0) \vee j(l_1)$.

Suppose now that $x = f \in F_0$ so $\mu(f) = \nu(f) \leq l_0 \vee l_1$. We have, therefore, $f \leq i(l_0) \vee i(l_1)$, so $f \triangleleft_m \delta(f, g) \cup \{g\} \ll \{i(l_0), i(l_1)\}$ for some $g \in D^B$. We

can use now Lemma 50 to pick $h \in D^B$ with $\mu(h) \in \mathcal{J}_c(L) \setminus A$ (so $h \in F_0$) and $\mu(g) = \bigvee \mu(\delta(g, h)) \vee \mu(h)$.

We have then that $\mu(\delta(f, h)) \cup \{\mu(h)\} \subseteq \mu(\delta(f, g) \cup \delta(g, h)) \cup \{\mu(h)\} \ll \{l_0, l_1\}$. This relation yields

$$\nu(\delta_A(f, h)) \cup \{\nu(h)\} \ll \{l_0, l_1\}$$

or, said otherwise,

$$\delta_A(f, h) \cup \{h\} \ll \{j(l_0), j(l_1)\}.$$

This implies that $f \in j(l_0) \vee j(l_1)$.

Let us argue that j preserves the least element. If $x \in j(\perp)$, then $\nu(x) \leq \perp$. We cannot have $x \in B$, so $x = f \in F_0$. Then $\mu(f) \leq \perp$ and $f \in i(\perp)$, contradicting the assumption that i preserves bounds.

Finally, let us observe that the left adjoint of j agrees, on join-irreducible elements, with ν . Indeed, for each $x \in A \cup F_0$, we have $\nu(x) \leq y$ iff $x \in j(y)$, iff $x \leq j(y)$, where we identify, as usual, a singleton with its only element. \square

We conclude next the proof of the main result of this Section, Theorem 47.

Proof of Theorem 47. Since \mathfrak{F} is rooted and full, $L(\mathfrak{F})$ is a finite atomistic subdirectly-irreducible lattice by Proposition 23. Therefore, if $i : L(\mathfrak{F}) \rightarrow R(D, B)$ is a lattice embedding, then we can assume, using Theorem 43, that i preserves the bounds. Also, if $L(\mathfrak{F})$ is a Boolean algebra, then it is the two elements Boolean algebra, since we are assuming that $L(\mathfrak{F})$ is subdirectly-irreducible. But then, \mathfrak{F} is a singleton, and the statement of the Theorem trivially holds in this case.

We can therefore assume that $L(\mathfrak{F})$ is not a Boolean algebra. Let us recall that A is the set of join-prime elements of $L(\mathfrak{F})$, see Proposition 22. According to in Proposition 55, let (F_0, δ_A) be the pairwise complete space over $P(A)$ and let $j : L(\mathfrak{F}) \rightarrow \mathcal{L}(F_0, \delta_A)$ be the lattice morphism with the properties stated there; let ν be the left adjoint to j . Using Corollary 41, we can also assume that $\mathcal{L}(F_0, \delta_A) = L(\mathfrak{U})$ for some universal $S5^A$ -product frame \mathfrak{U} .

To avoid confusions, we depart from now on from the convention of identifying singletons with their elements. We define $\psi : X_{\mathfrak{U}} \rightarrow X_{\mathfrak{F}}$ by saying that $\psi(x) = y$ when $\nu(\{x\}) = \{y\}$. This is well defined since in $L(\mathfrak{U})$ (respectively $L(\mathfrak{F})$) the non-join-prime join-irreducible-elements are the singletons $\{x\}$ with $x \in X_{\mathfrak{U}}$ (resp. $x \in X_{\mathfrak{F}}$); moreover, we have $X_{\mathfrak{U}} = F_0$ and each singleton $\{x\}$ with $x \in F_0$ is sent by ν to a singleton $\{y\} \in \mathcal{J}_c(L(\mathfrak{F})) \setminus \{\{a\} \mid a \in A\} = \{\{x\} \mid x \in X_{\mathfrak{F}}\}$. The function ψ is surjective since every non-join-prime atom $\{x\}$ in $L(\mathfrak{F})$ has a preimage by ν an atom $\{y\}$ and such a preimage cannot be join-prime, so $y \in X_{\mathfrak{U}}$.

We are left to argue that ψ is a p -morphism. To this end, let us remark that, for each $a \in A$ and $x, y \in X_{\mathfrak{F}}$ (or $x, y \in X_{\mathfrak{U}}$), the relation $xR_a y$ holds exactly when there is an $\{a\}$ -path from x to y , i.e. when $\{x\} \subseteq \overline{\{a, y\}} = \{a\} \vee \{y\}$ (we need here that \mathfrak{F} and \mathfrak{U} are $S4$ frames).

Thus, let $x, y \in X_{\mathfrak{U}}$ be such that $xR_a y$. Then $\{x\} \subseteq \{a\} \vee \{y\}$ and $\nu(\{x\}) \subseteq \nu(\{a\}) \vee \nu(\{y\}) = \{a\} \vee \nu(\{y\})$. We have therefore $\psi(x)R_a \psi(y)$. Conversely, let $x \in X_{\mathfrak{U}}$ and $z \in X_{\mathfrak{F}}$ be such that $\psi(x)R_a z$. We have therefore $\nu(\{x\}) \subseteq \{a\} \vee \{z\}$,

whence, by adjointness,

$$\begin{aligned} \{x\} &\subseteq j(\{a\} \vee \{z\}) = j(\{a\}) \vee j(\{z\}) \\ &= \{a\} \vee \{y \mid \nu(\{y\}) = \{z\}\} \\ &= \overline{\{a\} \cup \{y \mid \nu(\{y\}) = \{z\}\}}. \end{aligned}$$

But this means that there is some $y \in X_{\mathcal{U}}$ with $\psi(y) = z$ and a $\{a\}$ -path from x to y . But then, we also have $xR_a y$. \square

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