The quasiequational theory of relational lattices, in the pure lattice signature

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Abstract. The natural join and the inner union operations combine relations in a database. Tropashko and Spight realized that these two operations are the meet and join operations in a class of lattices, known by now as the relational lattices. They proposed then lattice theory as an algebraic approach, alternative to the relational algebra, to the theory of databases. Litak et al. proposed an axiomatization in the signature extending the pure lattice signature with the header constant. They argued then that the quasiequational theory of relational lattices is undecidable in this extended signature.

We refine this result by showing that the quasiequational theory of relational lattices in the pure lattice signature is undecidable as well. We obtain this result as a consequence of the following statement: it is undecidable whether a finite subdirectly-irreducible lattice can be embedded into a relational lattice. Our proof of this statement is a reduction from a similar problem for relation algebras and from the coverability problem of a frame by a universal product frame. As corollaries, we also obtain the following results: the quasiequational theory of relational lattices has no finite base; there is a quasiequation which holds in all the finite lattices but fails in an infinite relational lattice.

1. Introduction

The natural join and the inner union operations combine relations (i.e. tables) of a database. Most of today's web programs query their databases making repeated use of the natural join, while the inner union is a variant of another well known operation, the union of tables. Tropashko and Spight realized [21, 20] that these two operations are the meet and join operations in a class of lattices, known by now as the class of relational lattices. They proposed then lattice theory as an algebraic approach, alternative to Codd's relational algebra [3], to the theory of databases.

An important first attempt to axiomatize these lattices is due to Litak, Mikulás, and Hidders [12]. The authors propose an axiomatization, comprising equations and quasiequations, in a signature that extends the pure lattice signature with a constant, the header constant. A main result of that paper is that the quasiequational theory of relational lattices is undecidable in this extended signature. Their proof mimics Maddux' proof that the equational theory of cylindric algebras of dimension $n \geq 3$ is undecidable [13].

In [19] we have investigated equational axiomatizations for relational lattices using as tool the duality theory for finite lattices developed in [18]. A conceptual contribution from [19] is to make explicit the similarity between the developing theory of relational lattices and the well established theory of combination of modal logics, see e.g. [10]. This was achieved on the syntactic side, but also on the semantic side, by identifying some key properties of the structures dual to the finite atomistic lattices in the variety generated by the relational lattices, see [19, Theorem 7]. These properties make the dual structures into frames for commutator multimodal logics in a natural way.

In this paper we fully exploit this similarity to transfer results from the theory of multidimensional modal logics to lattice theory. Our main result is a refinement of the undecidability theorem of [12]. We prove that *the quasiequational theory of relational lattices in the pure lattice signature is* also *undecidable*. We obtain this result as a consequence of the following Theorem: *it is undecidable whether a finite subdirectly irreducible lattice can be embedded into a relational lattice*. We prove this theorem by reducing to it the coverability problem of a frame by a universal $\mathrm{S5}^3$-product frame, a problem shown to be undecidable in [9]. As stated there, the coverability problem is—in light of standard duality theory—a direct reformulation of the representability problem of finite simple relation algebras, problem shown to be undecidable by Hirsch and Hodkinson [8].

The proof method we rely on allows to establish two other results. Firstly, we argue that *the quasiequational theory of relational lattices has no finite base*. Then we argue that there is a quasiequation that holds in all the finite relational lattices, but fails in an infinite one. For the latter result, we rely on the work by Hirsch, Hodkinson, and Kurucz [9] who constructed a finite 3-multimodal frame which has no finite p-morphism from a finite universal $\mathrm{S5}^3$-product frame, but has a p-morphism from an infinite one. On the methodological side, we wish to point out our use of generalized ultrametric spaces to tackle these problems. A key idea in the proof of the main result is the characterization of universal $\mathrm{S5}^A$-product frames as pairwise complete generalized ultrametric spaces with distance valued in the Boolean algebra $P(A)$, a characterization that holds when A is finite.

The paper is structured as follows. We recall in Section 2 few definitions and facts on frames and lattices. Relational lattices are introduced in Section 3. In Section 4 we give an outline of the proof of our main technical result—the undecidability of embeddability of a finite subdirectly-irreducible lattice into a relational lattice—and derive from it the other results. In Section 5 we show how to construct a lattice from a frame and use functoriality of this construction to argue that such lattice embeds into a relational lattice whenever the frame is a p-morphic image of a universal product frame. The proof of the converse statement is carried out in Section 8. The technical tools needed to prove the converse are developed Sections 6 and 7. The theory of generalized ultrametric spaces over a powerset Boolean algebra and the aforementioned characterization of $\mathrm{S5}^A$-product frames as pairwise complete spaces over $P(A)$ appear in Section 6. In Section 7 we study embeddings of finite subdirectly-irreducible lattices into relational lattices and prove that we can assume that these embeddings preserve bounds. This task is needed so to exclude the constants $\bot$ and $\top$ (denoting the bounds) from the signature of lattice theory.

2. Frames and lattices

Frames. Let A be a set of actions. An *A-multimodal frame* (briefly, an *A-frame* or a *frame*) is a structure $\mathfrak{F} = \langle X_{\mathfrak{F}}, \{R_a \mid a \in A\}\rangle$ where, for each $a \in A$, R_a is a binary relation on $X_{\mathfrak{F}}$. We say that an A-frame is S4 if each R_a is reflexive and transitive. If $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are two A-frames, then a p-morphism from $\mathfrak{F}_0$ to $\mathfrak{F}_1$ is a function $\psi : X_{\mathfrak{F}_0} \longrightarrow X_{\mathfrak{F}_1}$ such that, for each $a \in A$,

- if xR_ay, then $\psi(x)R_a\psi(y)$,

- if $\psi(x) R_a z$, then $x R_a y$ for some y with $\psi(y) = z$.

Let us mention that A-multimodal frames and p-morphisms form a category.

A frame $\mathfrak{F}$ is said to be *rooted* (or *initial*, see [17]) if there is $f_0 \in X_{\mathfrak{F}}$ such that every other $f \in X_{\mathfrak{F}}$ is reachable from f_0. We say that an A-frame $\mathfrak{F}$ is *full* if, for each $a \in A$, there exists $f, g \in X_{\mathfrak{F}}$ such that $f \neq g$ and $f R_a g$. If $G = (V, D)$ is a directed graph, then we shall say that G is rooted if it is rooted as a unimodal frame.

A particular class of frames we shall deal with are the *universal* $\mathrm{S5}^A$*-product frames*. These are the frames $\mathfrak{U}$ with $X_{\mathfrak{U}} = \prod_{a \in A} X_a$ and $x R_a y$ if and only if $x_i = y_i$ for each $i \neq a$, where $x := \langle x_i \mid i \in A \rangle$ and $y := \langle y_i \mid i \in A \rangle$.

Let $\alpha \subseteq A$, $\mathfrak{F}$ be an A-frame, $x, y \in X_{\mathfrak{F}}$. An *$\alpha$-path* from x to y is a sequence $x = x_0 R_{a_0} x_1 \ldots x_{k-1} R_{a_{k-1}} x_k = y$ with $\{a_0, \ldots, a_{k-1}\} \subseteq \alpha$. We use then the notation $x \xrightarrow{\alpha} y$ to mean that there is an α-path from x to y. Notice that if $\mathfrak{F}$ is an S4 A-frame, then $x \xrightarrow{\{a\}} y$ if and only if $x R_a y$.

Orders and lattices. We assume some basic knowledge of order and lattice theory as presented in standard monographs [4, 6]. Most of the tools we use in this paper originate from the monograph [5] and have been further developed in [18].

A *lattice* is a poset L such that every finite non-empty subset $X \subseteq L$ admits a smallest upper bound $\bigvee X$ and a greatest lower bound $\bigwedge X$. A lattice can also be understood as a structure $\mathfrak{A}$ for the functional signature $(\vee, \wedge)$, such that the interpretations of these two binary function symbols both give $\mathfrak{A}$ the structure of an idempotent commutative semigroup, the two semigroup structures being tied up by the absorption laws $x \wedge (y \vee x) = x$ and $x \vee (y \wedge x) = x$. Once a lattice is presented as such structure, the order is recovered by stating that $x \leq y$ holds if and only if $x \wedge y = x$.

A lattice L is *complete* if any subset $X \subseteq L$ admits a smallest upper bound $\bigvee X$. It can be shown that this condition implies that any subset $X \subseteq L$ admits a greatest lower bound $\bigwedge X$. A lattice is *bounded* if it has a least element $\bot$ and a greatest element $\top$. A complete lattice (in particular, a finite lattice) is bounded, since $\bigvee \emptyset$ and $\bigwedge \emptyset$ are, respectively, the least and greatest elements of the lattice.

If P and Q are partially ordered sets, then a function $f : P \longrightarrow Q$ is *order-preserving* (or *monotone*) if $p \leq p'$ implies $f(p) \leq f(p')$. If L and M are lattices, then a function $f : L \longrightarrow M$ is a *lattice morphism* if it preserves the lattice operations $\vee$ and $\wedge$. A lattice morphism is always order-preserving. A lattice morphism $f : L \longrightarrow M$ between bounded lattices L and M is *bound-preserving* if $f(\bot) = \bot$ and $f(\top) = \top$. A function $g : Q \longrightarrow P$ is said to be *left adjoint* to an order-preserving $f : P \longrightarrow Q$ if $g(q) \leq p$ holds if and only if $q \leq f(p)$ holds; such a left adjoint, when it exists, is unique. If L is finite, M is bounded, and $f : L \longrightarrow M$ is a bound-preserving lattice morphism, then a left adjoint to f always exists; such a left adjoint preserves the $\bot$ and $\vee$ operations.

A *Moore family on a set U* is a collection $\mathcal{F}$ of subsets of U which is closed under arbitrary intersections. Given a Moore family $\mathcal{F}$ on U, the correspondence sending $Z \subseteq U$ to $\overline{Z} := \bigcap\{Y \in \mathcal{F} \mid Z \subseteq Y\}$ is a closure operator on U, that is, an order-preserving inflationary and idempotent endofunction of $P(U)$. The subsets in $\mathcal{F}$, called the *closed sets*, are exactly the fixpoints of this closure operator. We can give $\mathcal{F}$ a lattice structure by defining

$$\bigwedge X := \bigcap X\,, \qquad\qquad \bigvee X := \overline{\bigcup X}\,. \tag{1}$$

Let L be a complete lattice. An element $j \in L$ is *completely join-irreducible* if $j = \bigvee X$ implies $j \in X$, for each $X \subseteq L$; the set of completely join-irreducible elements of L is denoted here $\mathcal{J}(L)$. A complete lattice is *spatial* if every element is the join of the completely join-irreducible elements below it. An element $j \in \mathcal{J}(L)$ is said to be *join-prime* if $j \leq \bigvee X$ implies $j \leq x$ for some $x \in X$, for each finite subset X of L. If x is not join-prime, then we say that x is *non-join-prime*. An *atom* of a lattice L is an element of L such that $\perp$ is the only element strictly below it. A spatial lattice is *atomistic* if every element of $\mathcal{J}(L)$ is an atom.

For $j \in \mathcal{J}(L)$, a *join-cover* of j is a subset $X \subseteq L$ such that $j \leq \bigvee X$. For $X, Y \subseteq L$, we say that X *refines* Y, and write $X \ll Y$, if for all $x \in X$ there exists $y \in Y$ such that $x \leq y$. A join-cover X of j is said to be *minimal* if $j \leq \bigvee Y$ and $Y \ll X$ implies $X \subseteq Y$; we write $j \lhd_{\mathtt{m}} X$ if X is a minimal join-cover of j. In a spatial lattice, if $j \lhd_{\mathtt{m}} X$, then $X \subseteq \mathcal{J}(L)$. If $j \lhd_{\mathtt{m}} X$, then we say that X is a *non-trivial* minimal join-cover of j if $X \neq \{j\}$. Some auhtors use the word *perfect* for a lattice which is both spatial and dually spatial. We need here something different:

Definition 1. A complete lattice is *pluperfect* if it is spatial and for each $j \in \mathcal{J}(L)$ and $X \subseteq L$, if $j \leq \bigvee X$, then $Y \ll X$ for some Y such that $j \lhd_{\mathtt{m}} Y$. The *OD-graph* of a pluperfect lattice L is the structure $\langle \mathcal{J}(L), \leq, \lhd_{\mathtt{m}} \rangle$.

That is, in a pluperfect lattice every cover refines to a minimal one. [1] Notice that every finite lattice is pluperfect. If L is a pluperfect lattice, then we say that $X \subseteq \mathcal{J}(L)$ is *closed* if it is a downset and $j \lhd_{\mathtt{m}} C \subseteq X$ implies $j \in X$. Closed subsets of $\mathcal{J}(L)$ form a Moore family. The interest of considering pluperfect lattices stems from the following representation theorem stated in [15] for finite lattices; its generalization to pluperfect lattices is straightforward.

Theorem 2. *Let L be a pluperfect lattice and let $\mathsf{L}(\mathcal{J}(L), \leq, \lhd_{\mathtt{m}})$ be the lattice of closed subsets of $\mathcal{J}(L)$. The mapping $l \mapsto \{j \in \mathcal{J}(L) \mid j \leq l\}$ is a lattice isomorphism from L to $\mathsf{L}(\mathcal{J}(L), \leq, \lhd_{\mathtt{m}})$.*

Proof. Let $f(l) := \{j \in \mathcal{J}(L) \mid j \leq l\}$. Clearly $f(l)$ is a downset, let us verify that it is closed as well: if $j \lhd_{\mathtt{m}} C \subseteq f(l)$, then $C \ll l$ and $j \leq \bigvee C \leq l$, so $j \in f(l)$.

Observe now that f is order-preserving; to see that f is an order isomorphism we argue that $\bigvee f(l) = l$ and $f(\bigvee X) = X$, when X is closed subset of $\mathcal{J}(L)$.

If $j \leq \bigvee f(l)$, then $j \lhd_{\mathtt{m}} C \ll f(l)$; since $f(l)$ is a downset, $C \subseteq f(l)$ follows and therefore $j \in f(l)$, since $f(l)$ is closed; that is, we have $j \leq l$. By spatiality, we have therefore that $\bigvee f(l) \leq l$; equality follows since clearly $l \leq \bigvee f(l)$. For the second relation, if $j \in X$, then $j \leq \bigvee X$ and $j \in f(\bigvee X)$, so $X \subseteq f(\bigvee X)$. Conversely, if $j \in f(\bigvee X)$, then $j \leq \bigvee X$ and $j \lhd_{\mathtt{m}} C \ll X$. Since X is a downset, then $C \subseteq X$ and since X is closed, then $j \in X$. Thus $f(\bigvee X) \subseteq X$ and equality holds. $\square$

It was shown in [18] how to extend this representation theorem to a duality between the category of finite lattices and the category of OD-graphs.

For a lattice L, a *principal ideal* of L is a subset of the form $\downarrow l := \{x \in L \mid x \leq l\}$.

[1] With respect to analogous definitions, such as that of a lattice with the Σ-*minimal join-cover refinement property*, see [22], we do not require here that the set Y in the relation $j \lhd_{\mathtt{m}} Y$ is finite, nor that, for a given j, there are a finite number of these sets.

Lemma 3. *If L is a pluperfect lattice, then every principal ideal $\downarrow l$, $l \in L$, is also pluperfect. We have $\mathcal{J}(\downarrow l) = \mathcal{J}(L) \cap \downarrow l$ and, for $\{j\} \cup C \subseteq \mathcal{J}(\downarrow l)$, the relation $j \lhd_{\mathtt{m}} C$ holds in $\downarrow l$ if and only if it holds in L.*

Proof. Each element of $\mathcal{J}(L) \cap \downarrow l$ is completely join-irreducible in $\downarrow l$. If $x \leq l$, then $x = \bigvee J$ with $J \subseteq \mathcal{J}(L)$ and clearly $J \subseteq \downarrow l$. Therefore $\downarrow l$ is spatial with $\mathcal{J}(\downarrow l) = \mathcal{J}(L) \cap \downarrow l$.

Suppose now that $\{j\} \cup X \subseteq \downarrow l$ and $j \leq \bigvee X$. If the relations $j \lhd_{\mathtt{m}} C$ and $C \ll X$ hold in L, then $C \subseteq \downarrow l$, so they hold in $\downarrow l$ as well. In particular, this shows that $\downarrow l$ is pluperfect. □

Let L be a pluperfect lattice. A subset $A \subseteq \mathcal{J}(L)$ is *D-closed* if $j \in A$ and $j \lhd_{\mathtt{m}} C$ implies $C \subseteq A$. Given a D-closed subset $A \subseteq \mathcal{J}(L)$, let L_A be the closure of A under possibly infinite joins so, in particular, L_A is a sub-join-semilattice of L. As L_A has infinite joins, it has also infinite meets. Let us define then $\pi_A : L \longrightarrow L_A$ by $\pi_A(l) := \bigvee\{x \in L_A \mid x \leq l\}$. The following Lemma generalizes to pluperfect lattices well known facts about finite lattices, see e.g. [5, Lemma 2.33].

Lemma 4. *$\pi_A : L \longrightarrow L_A$ is a surjective lattice homomorphism. Moreover, L_A is a pluperfect lattice whose OD-graph is the restriction to A of the OD-graph of L.*

Proof. L_A is subset of L closed under arbitrary joins and therefore $\pi_A : L \longrightarrow L_A$, defined by $\pi_A(l) := \bigvee\{x \in L_A \mid x \leq l\}$, is a surjective map which preserves arbitrary meets (since meets are computed in L_A via this map, e.g. $x \wedge_{L_A} y = \pi_A(x \wedge_L y)$).

Let us show that π_A preserves arbitrary joins as well. To this end, observe first that $\pi_A(l) = \bigvee\{j \in A \mid j \leq l\}$. Since π_A is order-preserving, we only need to show that $\pi_A(\bigvee X) \leq \bigvee \pi_A(X)$. Let therefore $j \in A$ with $j \leq \bigvee X$, so $j \lhd_{\mathtt{m}} C$ with $C \ll X$. Since $j \in A$ and A is D-closed, we have $C \subseteq A$, whence $C = \pi_A(C) \ll \pi_A(X)$. It follows that $j \leq \bigvee C \leq \bigvee \pi_A(X)$.

The set $A \subseteq \mathcal{J}(L)$ generates L_A under arbitrary joins and, moreover, each element of A is completely join-irreducible in L_A, since L_A is a sub-join-semilattice of L; thus L_A is spatial and $\mathcal{J}(L_A) = A$. It is easily verified that, for each $j \in A$, each minimal join-cover of j L is also a minimal join-cover of j in L_A. □

Lemma 5. *If $\{A_i \mid i \in I\}$ is a collection of D-closed subsets such that $\bigcup A_i = \mathcal{J}(L)$, then $\langle \pi_{A_i} \mid i \in I\rangle : L \longrightarrow \prod L_{A_i}$ is a subdirect decomposition of L.*

Proof. If $l \not\leq l'$, then, by spatiality, there is $j \in \mathcal{J}(L)$ such that $j \leq l$ but $j \not\leq l'$. Let $i \in I$ such that $j \in A_i$: then $j \leq \pi_{A_i}(l)$ but $j \not\leq \pi_{A_i}(l')$. It follows that $\langle \pi_{A_i} \mid i \in I\rangle$ is an injective map. □

3. The relational lattices $\mathsf{R}(D, A)$

Throughout this paper we shall use the notation Y^X for the set of functions of domain Y and codomain X, for X and Y any two sets.

Let A be a collection of attributes (or column names) and let D be a set of cell values. A *relation* on A and D is a pair (α, T) where $\alpha \subseteq A$ and $T \subseteq D^{\alpha}$. Elements of the relational lattice $\mathsf{R}(D, A)$ [2] are relations on A and D. Informally, a relation

[2]In [12] such a lattice is called *full* relational lattice. The wording "class of relational lattices" is used there for the class of lattices that have an embedding into some lattice of the form $\mathsf{R}(D, A)$.

(α, T) represents a table of a relational database, with α being the header, i.e. the collection of names of columns, while T is the collection of rows.

Before we define the natural join, the inner union operations, and the order on $\mathsf{R}(D, A)$, let us recall a few key operations. If $\alpha \subseteq \beta \subseteq A$ and $f \in D^\beta$, then we shall use $f_{\restriction_\alpha} \in D^\alpha$ for the restriction of f to α; if $T \subseteq D^\beta$, then $T\Restriction_\alpha$ shall denote projection to α, that is, the direct image of T along restriction, $T\Restriction_\alpha := \{f_{\restriction_\alpha} \mid f \in T\}$; if $T \subseteq D^\alpha$, then $i_\beta(T)$ shall denote cylindrification to β, that is, the inverse image of restriction, $i_\beta(T) := \{f \in D^\beta \mid f_{\restriction\alpha} \in T\}$. Recall that i_β is right adjoint to $\Restriction_\alpha$. With this in mind, the natural join and the inner union of relations are respectively described by the following formulas:

$$
\begin{aligned}
(\alpha_1, T_1) \wedge (\alpha_2, T_2) &:= (\alpha_1 \cup \alpha_2, T) \\
&\quad \text{where } T = \{f \mid f_{\restriction_{\alpha_i}} \in T_i, i = 1, 2\} \\
&\qquad\qquad = i_{\alpha_1 \cup \alpha_2}(T_1) \cap i_{\alpha_1 \cup \alpha_2}(T_2)\,, \\
(\alpha_1, T_1) \vee (\alpha_2, T_2) &:= (\alpha_1 \cap \alpha_2, T) \\
&\quad \text{where } T = \{f \mid \exists i \in \{1, 2\}, \exists g \in T_i \text{ s.t. } g_{\restriction_{\alpha_1 \cap \alpha_2}} = f\} \\
&\qquad\qquad = T_1\Restriction_{\alpha_1 \cap \alpha_2} \cup T_2\Restriction_{\alpha_1 \cap \alpha_2}\ .
\end{aligned}
$$

The order is then given by

$$(\alpha_1, T_1) \leq (\alpha_2, T_2) \quad \text{iff} \quad \alpha_2 \subseteq \alpha_1 \text{ and } T_1\Restriction_{\alpha_2} \subseteq T_2\,.$$

A convenient way of describing these lattices was introduced in [12, Lemma 2.1]. The authors argued that the relational lattices $\mathsf{R}(D, A)$ are isomorphic to the lattices of closed subsets of $A \cup D^A$, where $Z \subseteq A \cup D^A$ is said to be closed if it is a fixed-point of the closure operator $\overline{(\,-\,)}$ defined as

$$\overline{Z} := Z \cup \{f \in D^A \mid A \setminus Z \subseteq Eq(f, g),\ \text{for some } g \in Z\}\,,$$

where in the formula above $Eq(f, g)$ is the equalizer of f and g. Letting

$$\delta(f, g) := \{x \in A \mid f(x) \neq g(x)\}\,,$$

the above definition of the closure operator is obviously equivalent to the following one:

$$\overline{Z} := \alpha \cup \{f \in D^A \mid \delta(f, g) \subseteq \alpha,\ \text{for some } g \in (Z \cap D^A)\},\ \text{with } \alpha = Z \cap A.$$

From now on, we shall rely on this representation of relational lattices. Relational lattices are atomistic pluperfect lattices. The completely join-irreducible elements of $\mathsf{R}(D, A)$ are the singletons $\{a\}$ and $\{f\}$, for $a \in A$ and $f \in D^A$, see [12]. By an abuse of notation we shall write x for the singleton $\{x\}$, for $x \in A \cup D^A$. Under this convention, we have therefore $\mathcal{J}(\mathsf{R}(D, A)) = A \cup D^A$. Every $a \in A$ is join-prime, while the minimal join-covers are of the form

$$f \lhd_{\mathtt{m}} \delta(f, g) \cup \{g\}$$

for each $f, g \in D^A$, see [19].

We shall use the following Lemma in a few key occasions.

Lemma 6. *Let L be a finite atomistic lattice in the variety generated by the class of relational lattices. If $\{j\} \cup X \subseteq \mathcal{J}(L)$, $j \leq \bigvee X$, and all the elements of X join-prime, then j is join-prime.*

The Lemma—which is an immediate consequence of Theorem 7 in [19]—asserts that a join-cover of an element $j \in \mathcal{J}(L)$ which is not join-prime cannot be made of join-prime elements only.

4. Overview and statement of the results

For an arbitrary frame $\mathfrak{F}$, we shall construct in Section 5 a lattice $\mathsf{L}(\mathfrak{F})$; if $\mathfrak{F}$ is rooted and full, then $\mathsf{L}(\mathfrak{F})$ is a subdirectly irreducible lattice, see Proposition 22. The key Theorem leading to the undecidability results is the following one.

Theorem 7. *Let A be a finite set and let $\mathfrak{F}$ be an* S4 *finite rooted full A-frame. There is a surjective p-morphism from a universal* $\mathrm{S5}^A$*-product frame $\mathfrak{U}$ to $\mathfrak{F}$ if and only if $\mathsf{L}(\mathfrak{F})$ embeds into some relational lattice $\mathsf{R}(D,B)$.*

Proof outline. The construction L defined in Section 5 is shown to extend to a contravariant functor, so if $\mathfrak{U}$ is a universal $\mathrm{S5}^A$-product frame and $\psi : \mathfrak{U} \longrightarrow \mathfrak{F}$ is a surjective p-morphism, then we have an embedding $\mathsf{L}(\psi)$ of $L(\mathfrak{F})$ into $\mathsf{L}(\mathfrak{U})$. We can assume that all the components of $\mathfrak{U}$ are equal, i.e. that the underlying set of $\mathfrak{U}$ is of the form $\prod_{a \in A} X$; if this is the case, then $\mathsf{L}(\mathfrak{U})$ is isomorphic to the relational lattice $\mathsf{R}(X,A)$.

The converse direction, developed from Section 6 up to Section 8, is subtler. Considering that $\mathsf{L}(\mathfrak{F})$ is subdirectly-irreducible, we argue in Section 7 that if $\psi : \mathsf{L}(\mathfrak{F}) \longrightarrow \mathsf{R}(D,B)$ is a lattice embedding, then we can suppose it preserves bounds; in this case ψ has a surjective left adjoint $\mu : \mathsf{R}(D,B) \longrightarrow \mathsf{L}(\mathfrak{F})$. Let us notice that there is no general reason for ψ to be the image by L of a p-morphism. Said otherwise, the functor L is not full and, in particular, the image of an atom by μ might not be an atom. The following considerations, mostly developed in Section 8, make it possible to extract a p-morphism from the left adjoint μ. Since both $\mathsf{L}(\mathfrak{F})$ and $\mathsf{R}(D,B)$ are generated (under possibly infinite joins) by their atoms, each atom $x \in \mathsf{L}(\mathfrak{F})$ has a preimage $y \in \mathsf{R}(D,B)$ which is an atom. The set F_0 of non-join-prime atoms of $\mathsf{R}(D,B)$ such that $\mu(f)$ is a non-join-prime atom of $\mathsf{L}(\mathfrak{F})$ is endowed with a $P(A)$-valued distance δ. The pair (F_0,δ) is shown to be a pairwise complete ultrametric space over $P(A)$. Section 6 recalls and develops few observations on ultrametric spaces valued on powerset algebras. The key ones are Theorems 25 and 35, stating that—when A is finite—pairwise complete ultrametric spaces over $P(A)$ and universal $\mathrm{S5}^A$-product frames are essentially the same objects. The restriction of μ to F_0 yields then a surjective p-morphism from F_0, considered as a universal $\mathrm{S5}^A$-product frame, to $\mathfrak{F}$. □

It was shown in [9] that the following problem is undecidable: given a finite 3-frame $\mathfrak{F}$, does there exists a surjective p-morphism from a universal $\mathrm{S5}^3$-product frame $\mathfrak{U}$ to $\mathfrak{F}$? In the introduction we referred to this problem as the coverability problem of a 3-frame by a universal $\mathrm{S5}^3$-product frame. The problem was shown to be undecidable by means of a reduction from the representabity problem of finite simple relation algebras, shown to be undecidable in [8]. We need to strengthen the undecisability result of [9] with few additional observations, as stated in the following Proposition.

Proposition 8. *It is undecidable whether, given an a finite set A with* $\operatorname{card} A \geq 3$ *and an* S4 *finite rooted full A-frame $\mathfrak{F}$, there is a surjective p-morphism from a universal* $\mathrm{S5}^A$*-product $\mathfrak{U}$ to $\mathfrak{F}$.*

Proof. Throughout this proof we assume a minimum knowledge of the theory of relation algebras, see e.g. [14].

The Proposition actually holds if we restrict to the case when card $A = 3$. Given a finite simple relation algebra $\mathfrak{A}$, the authors of [9] construct a 3-multimodal frame $\mathfrak{F}_{\mathfrak{A},3}$ such that $\mathfrak{A}$ is representable if and only if $\mathfrak{F}_{\mathfrak{A},3}$ is a p-morphic image of some universal $S5^3$-product frame. The frame $\mathfrak{F}_{\mathfrak{A},3}$ is S4 and rooted [9, Claim 8]. We claim that $\mathfrak{F}_{\mathfrak{A},3}$ is also full, unless $\mathfrak{A}$ is the two elements Boolean algebra. To this goal, let us recall first that an element of $\mathfrak{F}_{\mathfrak{A},3}$ is a triple (t_0, t_1, t_2) of atoms of $\mathfrak{A}$ such that $t_2^{\smile} \leq t_0; t_1$; moreover, if t, t' are two such triples and $i \in \{0, 1, 2\}$, then tR_it' if and only if t and t' coincide in the i-th coordinate. If a is an atom of $\mathfrak{A}$, then $a \leq e_l; a$ and $a \leq a; e_r$ for two atoms e_l, e_r below the multiplicative unit of $\mathfrak{A}$. Therefore, the triples $t := (e_l, a, a^{\smile})$ and $t' = (a, e_r, a^{\smile})$ are elements of $\mathfrak{F}_{\mathfrak{A},3}$ and tR_2t'. If, for each atom a, these triples are equal, then every atom of $\mathfrak{A}$ is below the multiplicative unit, which therefore concides with the top element $\top$; since $\mathfrak{A}$ is simple, then relation $\top = \top; x; \top$ holds for each $x \neq \bot$. It follows that $x = \top; x; \top = \top$, for each $x \neq \bot$, so $\mathfrak{A}$ is the two elements Boolean algebra. Thus, if $\mathfrak{A}$ has more than two elements, then $t \neq t'$ and tR_2t' for some $t, t' \in \mathfrak{F}_{\mathfrak{A},3}$. Using the cycle law of relation algebras, one also gets pairs of distinct elements of $\mathfrak{F}_{\mathfrak{A},3}$, call them u, u' and w, w', such that uR_0u' and wR_1w'.

Therefore, if we could decide whether there is a p-morphism from some universal $S5^3$-frame to a given S4 finite rooted full frame $\mathfrak{F}$, then we could also decide whether a finite simple relation algebra $\mathfrak{A}$ is representable, by answering positively if $\mathfrak{A}$ has exactly two elements and, otherwise, by answering the existence problem of a p-morphism to $\mathfrak{F}_{\mathfrak{A},3}$. $\square$

Combining Theorem 7 with Proposition 8, we derive the following undecidability result.

Theorem 9. *It is not decidable whether a finite subdirectly irreducible atomistic lattice embeds into a relational lattice.*

Let us remark that Theorem 9 partly answers Problem 7.1 in [12].

In [12] the authors proved that the quasiequational theory of relational lattices (i.e. the set of all definite Horn sentences valid in relational lattices) in the signature $(\wedge, \vee, H)$ is undecidable. Here H is the header constant, which is interpreted in a relational lattice $\mathsf{R}(D, A)$ as the closed subset A of $A \cup D^A$. Problem 4.10 in [12] asks whether the quasiequational theory of relational lattices in the restricted signature $(\wedge, \vee)$ of pure lattice theory is undecidable as well. The following result answers this question.

Theorem 10. *The quasiequational theory of relational lattices in the pure lattice signature is undecidable.*

It is a general fact that if the embeddability problem of finite subdirectly-irreducible algebras in a class $\mathcal{K}$ is undecidable, then the quasiequational theory of $\mathcal{K}$ is undecidable as well. For completeness, we add here the proof of this fact.

Proof. Given a finite subdirectly-irreducible algebra A with least non trivial congruence $\theta(\hat{a}, \bar{a})$, we construct a quasiequation ϕ_A with the following property: for any other algebra (in the same signature) K, $K \not\models \phi_A$ if and only if A has an embedding into K.

The construction is as follows. Let $X_A = \{x_a \mid a \in A\}$ be a set of variables in bijection with the elements of A. For each function symbol f in the signature Ω, let $T_{A,f}$ be its table, that is the formula

$$T_{A,f} = \bigwedge_{(a_1,\ldots,a_{ar(f)})\in A^{ar(f)}} f(x_{a_1},\ldots,x_{ar(f)}) = x_{f(a_1,\ldots,a_{ar(f)})}.$$

We let ϕ_A be the universal closure of $\bigwedge_{f\in\Omega} T_{A,f} \Rightarrow x_{\hat{a}} = x_{\bar{a}}$. We prove next that an algebra K sastifies ϕ_A if and only if there is no embedding of A into K.

If $K \models \phi_A$ and $\psi : A \longrightarrow K$, then $v(x_a) = \psi(a)$ is a valuation such that $K, v \models \bigwedge_{f\in\Omega} T_{A,f}$, so $\psi(\hat{a}) = v(x_{\hat{a}}) = v(x_{\bar{a}}) = \psi(\bar{a})$ and ψ is not injective.

Conversely, suppose $K \not\models \phi_A$ and let v be a valuation such that $K, v \models \bigwedge_{f\in\Omega} T_{A,f}$ and $K, v \not\models x_{\hat{a}} = x_{\bar{a}}$. Define $\psi : A \longrightarrow K$ as $\psi(a) = v(x_a)$, then ψ is a morphism, since $K, v \models T_{A,f}$ for each $f \in \Omega$. Let $Ker_\psi = \{(a, a') \mid \psi(a) = \psi(a')\}$ so, supposing that ψ is not injective, Ker_ψ is a non-trivial congruence. Then $(\hat{a}, \bar{a}) \in \theta(\hat{a}, \bar{a}) \subseteq Ker_\psi$, so $v(x_{\hat{a}}) = \psi(\hat{a}) = \psi(\bar{a}) = v(x_{\bar{a}})$, a contradiction. We have therefore $Ker_\psi = \{(a, a) \mid a \in A\}$, which shows that ψ is injective.

Let now $\mathcal{K}$ be a class of algebras in the same signature. We have then

$\mathcal{K} \not\models \phi_A$ iff $K \not\models \phi_A$ for some $K \in \mathcal{K}$

iff there is an embedding of A into K, for some for some $K \in \mathcal{K}$.

Thus, if the embeddability problem of finite subdirectly-irreducible algebras into some algebra in $\mathcal{K}$ is undecidable, then the quasiequational theory of $\mathcal{K}$ is undecidable as well. □

Following [9], let us add some further observations on the quasiequational theory of relational lattices.

Lemma 11. *The class of lattices that have an embedding into a relational lattice is closed under ultraproducts.*

Proof. Let us say that a sublattice L of a lattice $\mathsf{R}(D, A)$ is H-closed if the subset A belongs to L. Let $\mathcal{R}$ denote the closure under isomorphisms of the class of H-closed sublattices of some $\mathsf{R}(D, A)$. It is proved in [12, Corollary 4.2] that $\mathcal{R}$ is closed under ultraproducts. It immediately follows from this result that the class of lattices that have an embeding into some relational lattice is closed under ultraproducts, as follows. Let $\{L_i \longrightarrow \mathsf{R}(D_i, A_i) \mid i \in I\}$ be a family of lattice embeddings and let $\mathcal{F}$ be an ultrafilter over I. The ultraproduct constructions on $\{L_i \mid i \in I\}$ and $\{\mathsf{R}(D_i, A_i) \mid i \in I\}$ yield a lattice embedding $\prod_{\mathcal{F}} L_i \longrightarrow \prod_{\mathcal{F}} \mathsf{R}(D_i, A_i)$. Clearly, each $\mathsf{R}(D_i, A_i)$ belongs to $\mathcal{R}$, whence the ultraproduct $\prod_{\mathcal{F}} \mathsf{R}(D_i, A_i)$ belongs to $\mathcal{R}$ as well: thus $\prod_{\mathcal{F}} \mathsf{R}(D_i, A_i)$ embeds into some $\mathsf{R}(D, A)$, and so does $\prod_{\mathcal{F}} L_i$. □

Theorem 12. *The quasiequational theory of relational lattices is not finitely axiomatizable.*

Proof. A known result in universal algebra—see e.g. [2, Theorem 2.25]—states that a subdirectly-irreducible algebra satisfies all the quasiequations satisfied by a class of algebras if and only if it embeds in an ultraproduct of algebras in this class. Lemma 11 implies that the class of lattices that have an embedding into an ultraproduct of relational lattices and the class of lattices that have an embedding into some relational lattices are the same. Therefore a subdirectly-irreducible lattice L embeds in a relational lattice if and only if it satisfies all the quasiequations satisfied by the relational lattices. If this collection of quasiequations was a logical consequence of a finite set of quasiequations, then we could decide whether a finite subdirectly-irreducible L satisfies all these quasiequations, by verifying whether L satisfies the finite set of quasiequations. In this way, we could also decide whether such an L embeds into some relational lattice. □

Finally, the following Theorem, showing that the quasiequational theory of the finite relational lattices is stronger than the quasiequational theory of all the relational lattices, partly answers Problem 3.6 in [12].

Theorem 13. *There is a quasiequation which holds in all the finite relational lattices which, however, fails into an infinite relational lattice.*

Proof. In the first appendix of [9] an S4 finite rooted full 3-frame $\mathfrak{F}$ is constructed that has no surjective p-morphism from a finite universal $S5^3$-product frame, but has such a p-morphism from an infinite one.

Since $\mathsf{L}(\mathfrak{F})$ is finite whenever $\mathfrak{F}$ is finite, we obtain by using Theorem 7 a subdirectly-irreducible finite lattice L which embeds into an infinite relational lattice, but has no embedding into a finite one.

Let ϕ_L be the quasiequation as in the proof of Theorem 10. We have therefore that, for any lattice K, $K \models \phi_L$ if and only if L does not embed into K.

Correspondingly, any finite relational lattice satisfies ϕ_L and, on the other hand, $K \not\models \phi_L$ if K is the infinite lattice into which L embeds. □

5. The lattice of a multimodal frame

We assume throughout this Section that A is a finite set of actions. Given an A-frame $\mathfrak{F} = \langle X_{\mathfrak{F}}, \{R_a \mid a \in A\}\rangle$, we construct a lattice as follows. For $\alpha \subseteq A$, we say that $Y \subseteq X_{\mathfrak{F}}$ is *α-closed* if $x \in Y$, whenever there is a α-path from x to some $y \in Y$. We say that a subset $Z \subseteq A \cup X_{\mathfrak{F}}$ is closed if $Z \cap X_{\mathfrak{F}}$ is $Z \cap A$-closed.

Lemma 14. *The collection of closed subsets of $A \cup X$ is a Moore family.*

The Lemma, whose proof is straightforward, allows us to define the lattice of an A-frame.

Definition 15. The lattice $\mathsf{L}(\mathfrak{F})$ is the lattice of closed subsets of $A \cup X_{\mathfrak{F}}$.

The lattice operations on $\mathsf{L}(\mathfrak{F})$ are defined as in the display (1). In order to master the formula for the join, we need a more explicit description of the closure operator associated to this Moore family. If $\alpha \subseteq A$ and $Y \subseteq X_{\mathfrak{F}}$, define

$$\overline{Y}^{\alpha} := \{x \in X_{\mathfrak{F}} \mid \exists y \in Y, x \xrightarrow{\alpha} y\}\,.$$

Lemma 16. *For $Z \subseteq A \cup X_{\mathfrak{F}}$, we have*

$$\overline{Z} = \alpha \cup \overline{Z \cap X_{\mathfrak{F}}}^{\alpha}\,, \quad \textit{where } \alpha = Z \cap A. \tag{2}$$

In particular, for $x \in X_{\mathfrak{F}}$, $x \in \overline{Z}$ if and only if there exists $y \in Z \cap X_{\mathfrak{F}}$ and an α-path from x to y, with $\alpha = Z \cap A$.

The above formula (2) allows to make $\mathsf{L}(-)$ into a contravariant functor from the category of frames to the category of lattices. Namely, for a p-morphism $\psi : \mathfrak{F}_0 \longrightarrow \mathfrak{F}_1$ and any $Z \subseteq A \cup X_{\mathfrak{F}_1}$, define

$$\mathsf{L}(\psi)(Z) := (Z \cap A) \cup \psi^{-1}(Z \cap X_{\mathfrak{F}_1})\,.$$

Let $\psi^A : A \cup X_{\mathfrak{F}_0} \longrightarrow A \cup X_{\mathfrak{F}_1}$ be the function such that $\psi^A(a) = a$, for each $a \in A$, and $\psi^A(x) = \psi(x)$, for each $x \in X_{\mathfrak{F}_0}$. Notice that $\mathsf{L}(\psi)$ is the inverse image of ψ^A, so in particular $\mathsf{L}(\psi)$ commutes with intersections and unions.

Proposition 17. *$\mathsf{L}(\psi)$ sends closed subsets of $A \cup X_{\mathfrak{F}_1}$ to closed subsets of $A \cup X_{\mathfrak{F}_0}$. Its restriction to $\mathsf{L}(\mathfrak{F}_1)$ yields a bound-preserving lattice morphism $L(\psi) : \mathsf{L}(\mathfrak{F}_1) \longrightarrow \mathsf{L}(\mathfrak{F}_0)$.*

Proof. The key observation is that, for each $\alpha \subseteq A$ and each $Y \subseteq X_{\mathfrak{F}_1}$, we have $\psi^{-1}(\overline{Y}^{\alpha}) = \overline{\psi^{-1}(Y)}^{\alpha}$:

$$\begin{aligned}
\psi^{-1}(\overline{Y}^{\alpha}) &= \psi^{-1}(\{x \in X_{\mathfrak{F}_1} \mid \exists y \in Y,\, x \xrightarrow{\alpha} y\}) \\
&= \{x \in X_{\mathfrak{F}_0} \mid \exists y \in Y,\, \psi(x) \xrightarrow{\alpha} y\} \\
&= \{x \in X_{\mathfrak{F}_0} \mid \exists z \in \psi^{-1}(Y),\, x \xrightarrow{\alpha} z\} \qquad \text{since } \psi \text{ is a } p\text{-morphism,} \\
&= \overline{\psi^{-1}(Y)}^{\alpha}\,.
\end{aligned}$$

This implies that, for $Z \subseteq A \cup X_{\mathfrak{F}_1}$, we have

$$\mathsf{L}(\psi)(\overline{Z}) = \overline{\mathsf{L}(\psi)(Z)}\,.$$

In particular, if $Z \subseteq A \cup X_{\mathfrak{F}_1}$ is closed, then

$$\mathsf{L}(\psi)(Z) = \mathsf{L}(\psi)(\overline{Z}) = \overline{\mathsf{L}(\psi)(Z)}$$

so $\mathsf{L}(\psi)$ sends closed subsets to closed subsets. $\mathsf{L}(\psi)$ preserves all meets, since it commutes with intersections. Moreover

$$\mathsf{L}(\psi)(\bigvee_{i \in I} Z_i) = \mathsf{L}(\psi)(\overline{\bigcup_{i \in I} Z_i}) = \overline{\bigcup_{i \in I} \mathsf{L}(\psi)(Z_i)} = \bigvee_{i \in I} \mathsf{L}(\psi)(Z_i)\,,$$

so $\mathsf{L}(\psi)$ is a lattice morphism. □

As $\mathsf{L}(\psi)$ is the restriction of the inverse image of ψ^A defined above, it immediately follows that L is a contravariant functor from the category of A-frames to the category of lattices.

Lemma 18. *If $\psi : \mathfrak{F}_0 \longrightarrow \mathfrak{F}_1$ is surjective, then $\mathsf{L}(\psi)$ is injective.*

Proof. If ψ is surjective, then ψ^A is also surjective. As $\mathsf{L}(\psi)$ is the inverse image of ψ^A, then $\mathsf{L}(\psi)$ is injective. □

We are ready to state the main result of this Section.

Theorem 19. *If there exists a p-morphism from a universal $\mathrm{S5}^A$-product frame $\mathfrak{U}$ to an A-frame $\mathfrak{F}$, then $\mathsf{L}(\mathfrak{F})$ embeds into a relational lattice.*

Proof. We say that $\mathfrak{U}$ is uniform on X if all the components of $\mathfrak{U}$ are equal to X. Spelled out, this means that $X_{\mathfrak{U}} = \prod_{a \in A} X$. Let $\psi : \mathfrak{U} \longrightarrow \mathfrak{F}$ be a p-morphism as in the statement of the Theorem. W.l.o.g. we can assume that $\mathfrak{U}$ is uniform on some set X. If this is not the case, then we choose $a_0 \in A$ such that X_{a_0} has maximum cardinality and surjective mappings $p_a : X_{a_0} \longrightarrow X_a$, for each $a \in A$. The product frame $\mathfrak{U}'$ on $\prod_{a \in A} X_{a_0}$ is uniform and $\prod_{a \in A} p_a : \mathfrak{U}' \longrightarrow \mathfrak{U}$ is a surjective p-morphism. By pre-composing ψ with this p-morphism, we obtain a surjective p-morphism from the uniform $\mathfrak{U}'$ to $\mathfrak{F}$. Now, if $\mathfrak{U}$ is uniform on X, then

$\mathsf{L}(\mathfrak{U})$ is equal to the relational lattice $\mathsf{R}(X, A)$. Then, by functoriality of L, we have a lattice morphism

$$L(\psi) : \mathsf{L}(\mathfrak{F}) \longrightarrow \mathsf{L}(\mathfrak{U}) = \mathsf{R}(X, A)\,.$$

By Lemma 18 $\mathsf{L}(\psi)$ is an embedding. □

Some properties of the lattices $\mathsf{L}(\mathfrak{F})$.

Proposition 20. *The completely join-irreducible elements of $\mathsf{L}(\mathfrak{F})$ are the singletons, so $\mathsf{L}(\mathfrak{F})$ is an atomistic lattice.*

Proof. Each singleton set is closed. It immediately follows that the join-irreducible elements of $\mathsf{L}(\mathfrak{F})$ are the singletons and clearly they are atoms. □

Identifying the singletons of $P(A \cup X_{\mathfrak{F}})$ with their elements, we can write

$$\mathcal{J}(\mathsf{L}(\mathfrak{F})) = A \cup X_{\mathfrak{F}}\,.$$

To state the next Proposition, let us say that an α-path from x to y is *minimal* if there is no β-path from x to y, for each proper subset β of α.

Proposition 21. *$\mathsf{L}(\mathfrak{F})$ is a pluperfect lattice. Each element of A is join-prime, while the minimal join-covers of $x \in X_{\mathfrak{F}}$ are of the form $x \lhd_{\mathtt{m}} \alpha \cup \{y\}$, for a minimal α-path from x to y.*

Proof. We observe that, for $Z \subseteq A \cup X_{\mathfrak{F}}$, the relation

$$x \in \bigvee Z = \overline{Z}$$

holds if and only if either (i) $x \in A \cap Z$, or (ii) $x \in X_{\mathfrak{F}}$ and $x \xrightarrow{\alpha} y$ for some $y \in X_{\mathfrak{F}} \cap Z$ and some $\alpha \subseteq A \cap Z$. Thus, in particular, each element of A is join-prime. If $x \in X_{\mathfrak{F}}$, $Z \subseteq A \cup X_{\mathfrak{F}}$ and $x \leq \bigvee Z$, then we can find $y \in X_{\mathfrak{F}} \cap Z$ and an α-path from x to y with $\alpha \subseteq A \cap Z$. Clearly, we can assume $x \xrightarrow{\alpha} y$ is minimal, so

$$x \in \overline{\alpha \cup \{y\}} = \bigvee \alpha \vee y\,,$$

with $\alpha \cup \{y\} \subseteq Z$. This proves that every cover of x refines to a cover of the form $x \leq \bigvee \alpha \vee y$ with $x \xrightarrow{\alpha} y$ minimal. □

Notice that if an α-path is minimal, then α is necessarily finite. Therefore $\mathsf{L}(\mathfrak{F})$ is actually a lattice with the Σ-weak minimal join-cover refinement property as defined in [22], where Σ is here the set of completely join-irreducible elements of the lattice.

Before stating the next Proposition, let us recall from [5, Corollary 2.37], see also [18, Section 5.2], that a finite lattice L is subdirectly-irreducible if and only if the directed graph $(\mathcal{J}(L), D)$ is rooted. Here D is the *join-dependency relation* on the join-irreducible elements of L, defined as follows:

$$jDk \text{ iff } j \neq k \text{ and, for some } p \in L,\ j \leq p \vee k \text{ and } j \not\leq p \vee k_*\,,$$

where k_* denotes the unique lower cover of $k \in \mathcal{J}(L)$. It can be shown that jDk if and only if $k \neq j$ and $k \in C$ for some subset $C \subseteq \mathcal{J}(L)$ such that $j \lhd_{\mathtt{m}} C$, see e.g. [5, Lemma 2.31]. If a lattice is atomistic, then $k_* = \bot$ for each $k \in \mathcal{J}(L)$, and therefore jDk if and only if $j \neq k$ and $j \leq p \vee k$ for some $p \in L$ with $j \not\leq p$.

Proposition 22. *If a finite A-frame $\mathfrak{F}$ is rooted and full, then $\mathsf{L}(\mathfrak{F})$ is a subdirectly-irreducible lattice.*

Proof. We argue that the digraph $(\mathcal{J}(\mathsf{L}(\mathfrak{F})), D)$ is rooted. Observe that $x \in \overline{\{a, y\}} = a \vee y$ whenever $xR_a y$. This implies that xDy and xDa when $x, y \in X_{\mathfrak{F}}$, $a \in A$, $x \neq y$ and $xR_a y$. The fact that of $(\mathcal{J}(\mathsf{L}(\mathfrak{F})), D)$ is rooted follows now from $\mathfrak{F}$ being rooted and full. □

6. Some theory of generalized ultrametric spaces

Generalized ultrametric spaces over a Boolean algebra $P(A)$ turn out to be a useful tool for studying relational lattices [12, 19] as well as universal product frames from multidimensional modal logic [10]. Metrics are well known tools from graph theory, see e.g. [7]. Generalized ultrametric spaces over a Boolean algebra $P(A)$ were introduced in [16] to study equivalence relations. The main results of this Section are Theorem 25 and Proposition 35 which together instantiate the claim that, when A is finite, universal $\mathrm{S5}^A$-product frames are pairwise complete ultrametric spaces valued in the Boolean algebra $P(A)$.

Some of the observations we shall develop are not strictly necessary to prove the undecidability result, which is the main result of this paper; namely, we can always suppose that the set A is finite. Nonetheless we include these observations since they are part of a coherent set of results and, as far as we are aware of, they are original.

Definition 23. An *ultrametric space over* $P(A)$ (briefly, a *space*) is a pair (X, δ), with $\delta : X \times X \longrightarrow P(A)$ such that, for every $f, g, h \in X$,

$$\delta(f, f) \subseteq \emptyset\,, \qquad\qquad \delta(f, g) \subseteq \delta(f, h) \cup \delta(h, g)\,.$$

That is, we have defined an ultrametric space over $P(A)$ as a category (with a small set of objects) enriched over $(P(A)^{op}, \emptyset, \cup)$, see [11]. We shall assume in this paper that such a space (X, δ) is also *reduced* and *symmetric*, that is, that the following two properties hold for every $f, g \in X$:

$$\delta(f, g) = \emptyset \text{ implies } f = g, \qquad\qquad \delta(f, g) = \delta(g, f)\,.$$

Under these hypothesis, it is easily seen that if A is empty or a singleton, then the categories of spaces over $P(A)$ are trivial. Thus, we shall assume here that A has at least two elements.

A *morphism* of spaces[3] $\psi : (X, \delta_X) \longrightarrow (Y, d_Y)$ is a function $\psi : X \longrightarrow Y$ such that $\delta_Y(\psi(f), \psi(g)) \leq \delta_X(f, g)$, for each $f, g \in X$. If $\delta_Y(\psi(f), \psi(g)) = \delta_X(f, g)$, for each $f, g \in X$, then ψ is said to be an *isometry.* For (X, δ) a space over $P(A)$, $f \in X$ and $\alpha \subseteq A$, the ball centered in f of radius α is defined as usual: $B(f, \alpha) := \{g \in X \mid \delta(f, g) \subseteq \alpha\}$. In [1] a space (X, δ) is said to be *pairwise complete* if, for each $f, g \in X$ and $\alpha, \beta \subseteq A$,

$$B(f, \alpha \cup \beta) = B(g, \alpha \cup \beta) \text{ implies } B(f, \alpha) \cap B(g, \beta) \neq \emptyset\,.$$

This property is easily seen to be equivalent to:

$$\delta(f, g) \subseteq \alpha \cup \beta \text{ implies } \delta(f, h) \subseteq \alpha \text{ and } \delta(h, g) \subseteq \beta\,, \quad \text{for some } h \in X.$$

Recall also from [1] that a space is said to be *spherically complete* if the intersection $\bigcap_{i \in I} B(f_i, \alpha_i)$ of every chain $\{B(f_i, \alpha_i) \mid i \in I\}$ of balls is non-empty. It was shown in [16] that, when A is finite, every space over $P(A)$ is spherically complete. Let us

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[3]As $P(A)$ is not totally ordered, we avoid calling a morphism "*non expanding map*" as it is often done in the literature.

recall that pairwise and spherically complete spaces are characterized as injective objects in the category of spaces [1].

If (X, δ_X) is a space and $Y \subseteq X$, then the restriction of δ_X to Y induces a space (Y, δ_X); we say then that (Y, δ_X) is a *subspace* of X. Notice that the inclusion of Y into X yields an isometry of spaces.

Our main example of space over $P(A)$ is (D^A, δ), with D^A the set of functions from A to D and the distance defined by

$$\delta(f, g) := \{a \in A \mid f(a) \neq g(a)\}\,. \tag{3}$$

A second example is a slight generalization of the previous one. Given a surjective function $\pi : E \longrightarrow A$, let $\mathsf{Sec}(\pi)$ denote the set of all sections of π, that is the functions $f : A \longrightarrow E$ such that $\pi \circ f = id_A$; the formula in (3) also defines a distance on $\mathsf{Sec}(\pi)$. Clearly, (D^A, δ) and $(\mathsf{Sec}(\pi), \delta)$ are pairwise complete and $(\mathsf{Sec}(\pi), \delta)$ is an induced subspace of (E^A, δ). Considering the first projection $\pi_1 : A \times D \longrightarrow A$, we can see that (D^A, δ) is isomorphic to $(\mathsf{Sec}(\pi_1), \delta)$. By identifying $f \in \mathsf{Sec}(\pi)$ with a vector $\langle f_a \in \pi^{-1}(a) \mid a \in A\rangle$, we see that

$$\mathsf{Sec}(\pi) = \prod_{a \in A} E_a\,, \quad \text{where } E_a := \pi^{-1}(a). \tag{4}$$

That is, the underlying set of a space $(\mathsf{Sec}(\pi), \delta)$ is that of a universal $\mathrm{S5}^A$-product frame.

Proposition 24. *Every space of the form* $(\mathsf{Sec}(\pi), \delta)$ *is spherically complete.*

Proof. Let $\mathcal{C} := \{B(f_i, \alpha_i) \mid i \in I\}$ be a chain of balls. For each $a \in A$ pick $*_a \in D_a$ and define f as follows:

$$f(a) = \begin{cases} f_i(a)\,, & \text{if } a \notin \alpha_i \text{ for some } i \in I, \\ *_a\,, & \text{otherwise.} \end{cases}$$

Let us show that f is well defined. Namely, suppose that $a \notin \alpha_i$ and $a \notin \alpha_j$. Since $\mathcal{C}$ is a chain, we can suppose, without loss of generality, that $B(f_i, \alpha_i) \subseteq B(f_j, \alpha_j)$ so $\delta(f_i, f_j) \subseteq \alpha_j$. Since $a \notin \alpha_j$ it follows that $f_i(a) = f_j(a)$.

Let now $i \in I$ be arbitrary; if $a \notin \alpha_i$, then $f(a) = f_i(a)$, so $\delta(f, f_i) \subseteq \alpha_i$ and $f \in B(f_i, \alpha_i)$. It follows that $f \in \bigcap_{i \in I} B(f, \alpha_i)$. □

Theorem 25. *Every space* (X, δ) *over* $P(A)$ *has an isometry into some* $(\mathsf{Sec}(\pi), \delta)$*. If A is pairwise and spherically complete, then this isometry is an isomorphism.*

Proof. For each $a \in A$, let $D_a = \{B(f, A \backslash \{a\}) \mid f \in X\}$. That is, D_a is the quotient of X by the equivalence relation identifying f and g when $\delta(f, g) \subseteq A \setminus \{a\}$. Let $\pi : \sum_{a \in A} D_a \longrightarrow A$ be the obvious projection.

We associate to $f \in X$ the vector $\psi(f) = \langle B(f, A \setminus \{a\}) \mid a \in A\rangle$. Let us argue that the correpondence ψ is an isometry:

$$\begin{aligned} a \notin \delta(\psi(f), \psi(g)) &\text{ iff } B(f, A \setminus \{a\}) = B(g, A \setminus \{a\}) \\ &\text{ iff } \delta(f, g) \subseteq A \setminus \{a\} \text{ iff } a \notin \delta(f, g)\,, \end{aligned}$$

thus $\delta(\psi(f), \psi(g)) = \delta(f, g)$. In particular, when the space is reduced (i.e. $\delta(f, g) = \emptyset$ implies $f = g$), ψ is an injective map.

Next, we suppose that (X, δ) is pairwise and spherically complete and argue that ψ is surjective. To this goal, we fix a well-ordering on A, say $A = \{a_\lambda \mid \lambda < \tau\}$.

For $\lambda < \tau$, let us also set $A_{\lambda^-} := \{a_\beta \in A \mid \beta \leq \lambda\}$ and $A_{\lambda^+} := A \setminus A_{\lambda^-} = \{a_\beta \in A \mid \beta > \lambda\}$.

Let $v := \langle B(f_\lambda, A \setminus \{a_\lambda\}) \mid \lambda < \tau \rangle \in \mathsf{Sec}(\pi)$; we need to construct a preimage of v by ψ. To this end, we construct, by induction on $\lambda < \tau$, a family $\{g_\lambda \in X \mid \lambda < \tau\}$ such that $g_\lambda \in B(g_\beta, A_{\beta^+})$ for $\beta \leq \lambda$ and $\delta(g_\lambda, f_\lambda) \subseteq A \setminus \{a_\lambda\}$. Let $\lambda < \tau$ be an ordinal and suppose that we have defined g_β with these properties for each $\beta < \lambda$. As $\{B(g_\beta, A_{\beta^+}) \mid \beta < \lambda\}$ is a chain, we can pick $g \in \bigcap_{\beta<\lambda} B(g_\beta, A_{\beta^+})$. Notice that if $\lambda = \gamma + 1$ is a successor cardinal, then we can simply pick g_γ.

We use pairwise completeness to define g_λ as some h with $\delta(g, h) \subseteq \{a_\lambda\}$ and $\delta(h, f_\lambda) \subseteq \delta(g, f_\lambda) \setminus \{a_\lambda\}$ (if $a_\lambda \notin \delta(g, f_\lambda)$, then we can take $h = g$). Clearly, $\delta(g_\lambda, g_\lambda) = \emptyset \subseteq A_{\lambda^+}$ and, for $\beta < \lambda$, we have $\delta(g_\lambda, g_\beta) \subseteq \delta(g_\lambda, g) \cup \delta(g, g_\beta) = \{a_\lambda\} \cup \delta(g, g_\beta) \subseteq \{a_\lambda\} \cup A_{\beta^+} \subseteq A_{\beta^+}$.

Let now $g \in \bigcap_{\lambda<\tau} B(g_\beta, A_{\beta^+})$. If $\lambda < \tau$, then

$$\delta(g, f_\lambda) \subseteq \delta(g, g_\lambda) \cup \delta(g_\lambda, f_\lambda) \subseteq A_{\lambda^+} \cup (A \setminus \{a_\lambda\}) \subseteq A \setminus \{a_\lambda\}\,.$$

This shows that $B(g, A \setminus \{a_\lambda\}) = B(f_\lambda, A \setminus \{a_\lambda\})$ or, stated otherwise, $\psi(g)_\lambda = v_\lambda$, for each $\lambda < \tau$, so g is a preimage of v. □

In the following, let (X, δ_X) be a fixed pairwise complete space; our next goal is to devise criteria to recognize pairwise complete subspaces of (X, δ_X). More precisely, we shall be interested in particular subspaces for which the inclusion is continuous. To this goal, for a subspace Y of X, let us define

$$\nu_Y(f) := \bigcap \{\delta(f, g) \mid g \in Y\}\,.$$

We say that a subspace Y of X is *continuous* if, for each $f \in X$, $\nu_Y(f) = \emptyset$ implies $f \in Y$.

Lemma 26. *A continuous subspace Y of X is itself pairwise complete.*

Proof. Let $f, g \in Y$ with $\delta(f, g) \subseteq \alpha \cup \beta$. Let $h \in X$ be such that $\delta(f, h) \subseteq \alpha \setminus \beta$ and $\delta(h, g) \subseteq \beta$; then $\bigcap_{k\in Y} \delta(h, k) \subseteq \delta(h, f) \cap \delta(h, g) \subseteq \alpha \setminus \beta \cap \beta = \emptyset$, so $h \in Y$ since Y is continuous. □

We give next a partial converse of the statement of Lemma 26: Corollary 28 shall clarify that if A is finite, then continuous subspace and pairwise complete subspace are equivalent conditions.

Lemma 27. *If Y is pairwise complete subspace of X, then, for each $f \in X$, the set $\alpha \subseteq A$ for which $B(f, \alpha) \cap Y \neq \emptyset$ is closed under finite intersections. Consequently, if Y is spherically complete, then, for each $f \in X$, there exists $g \in Y$ with $\delta(f, g) = \nu_Y(f)$.*

Proof. Let $\mathcal{A}$ be the set of all $\alpha \subseteq A$ such that $B(f, \alpha) \cap Y \neq \emptyset$. For each $\alpha \in \mathcal{A}$, choose $t_\alpha \in B(f, \alpha) \cap Y$.

Observe that, for $\alpha, \alpha' \in \mathcal{A}$, $\delta(t_\alpha, t_{\alpha'}) \subseteq \delta(t_\alpha, f) \cup \delta(f, t_{\alpha'}) \subseteq \alpha \cup \alpha'$. That is, the function t sending α to t_α is a γ-Cauchy function to Y as defined in [1, Definition 2.8], where $\gamma = \bigcap \mathcal{A}$.

Observe next that $\delta(t_\alpha, t_{\alpha'}) \subseteq \alpha \cup \alpha'$ so, by pairwise completeness of Y, $\delta(t_\alpha, h) \subseteq \alpha$ and $\delta(h, t_{\alpha'}) \subseteq \alpha'$ for some $h \in Y$. It follows that $\delta(f, h) \subseteq \alpha \cap \alpha'$, showing that $\alpha \cap \alpha' \in \mathcal{A}$. In particular, t is a γ-Cauchy net, as defined in [1, Definition 2.9].

If we assume that Y is spherically complete, then we can use Proposition 2.16 in [1] to deduce that, for some $g \in Y$, $\delta(g, t_\alpha) \subseteq \alpha$, for each $\alpha \in \mathcal{A}$. For such a

$g \in Y$, we have $\delta(f,g) \subseteq \delta(f,t_\alpha) \cup \delta(t_\alpha,g) \subseteq \alpha$, showing that $\delta(f,g) \subseteq \nu_Y(f)$. As $g \in Y$, we also have $\nu_Y(f) \subseteq \delta(f,g)$ and $\nu_Y(f) = \delta(f,g)$. □

Corollary 28. *If Y is a pairwise complete and spherically complete subspace of X, then Y is a continuous subspace of X. In particular, if A is finite, a subspace of X is continuous if and only if it is pairwise complete.*

Proof. Let $f \in X$ be such that $\nu_Y(f) = \bigcap_{g \in Y} \delta(f,g) = \emptyset$. By Lemma 27, let $h \in Y$ such that $\nu_Y(f) = \delta(f,h)$, so $\delta(f,h) = \emptyset$ and $f = h \in Y$.

In particular, when A is finite, every subspace of X is spherically complete, so in this case pairwise completeness implies spherically completeness. □

A function $v : X \longrightarrow P(A)$ is said to be a *module*[4] if

$$v(f) \subseteq \delta(f,g) \cup v(g)\,.$$

We let $Modules(X)$ be the set of all modules v; we order this set by letting $v \leq w$ if and only if $w(f) \subseteq v(f)$, for each $f \in Y$ (that is, we take the reverse pointwise order). Let use $Sub(X)$ for the set of subspaces of X, ordered by inclusion—thus $Sub(X)$ is the usual power set of X.

Lemma 29. *For each subspace Y, ν_Y is a module. Moreover, the correspondence $\nu : Sub(X) \longrightarrow Modules(X)$ is order-preserving.*

Proof. If $Y = \emptyset$, then $\nu_Y(f) = A$, for each $f \in X$. Thus we can suppose that $Y \neq \emptyset$. Let $a \in \nu_Y(f)$. This means that $a \in \delta(f,h)$ whenever $h \in Y$. Suppose now that $a \notin \delta(f,g)$. Pick now $k \in Y$; from $a \in \delta(f,k) \subseteq \delta(f,g) \cup \delta(g,k)$, we have $a \in \delta(g,k)$. As k was aribitrary, $a \in \nu_Y(g)$.

For monotonicity, notice that, by definition, if $Z \subseteq Y \subseteq X$, then $\nu_Y(f) \subseteq \nu_Z(f)$ for each $f \in Y$, thus $\nu_Z \leq \nu_Y$. □

Next, given a module v, let us define

$$\mathsf{S}_v := \{x \in X \mid v(x) = \emptyset\}\,.$$

It is easily seen that $\mathsf{S} : Modules(X) \longrightarrow Sub(X)$, sending v to S_v, is also order-preserving.

Lemma 30. *The map ν is left adjoint to S.*

Proof. As both maps are order-preserving, we shall show that the usual unit and counit laws hold. If $f \in Y$, then $\nu_Y(f) \subseteq \delta(f,f) = \emptyset$; thus $Y \subseteq \mathsf{S}_{\nu_Y}$. Let us argue for the counit law. For each $f \in X$ and $g \in \mathsf{S}_v$—i.e. when $v(g) = \emptyset$—we have $v(f) \subseteq \delta(f,g) \cup v(g) = \delta(f,g)$. It follows that $v(f) \subseteq \nu_{\mathsf{S}_v}(f)$, for each $f \in X$. This means that $\nu_{\mathsf{S}_v} \leq v$ in $Modules(X)$. □

Lemma 31. *For each module v, S_v is a continuous subspace of X.*

Proof. We already observed that $v(f) \subseteq \nu_{\mathsf{S}_v}(f)$, that is, $v(f) \subseteq \bigcap_{g \in \mathsf{S}_v} \delta(f,g)$. If the latter expression is equal to the emptyset, then $v(f) = \emptyset$, whence $f \in \mathsf{S}_v$. This shows that S_v is a continuous subspace of X. □

Proposition 32. *A subspace Y of X is continuous if and only if $\mathsf{S}_{\nu_Y} = Y$. Thus, for any $Y \subseteq X$, $\mathsf{S}_{\nu_Y} \subseteq X$ is the least continuous subspace of X containing Y.*

[4] Module is a standard naming for a space morphism (thus, an enriched functor) from (X,δ) to the space $(P(A),\Delta)$, where Δ is the symmetric difference, see for example `https://ncatlab.org/nlab/show/bimodule`.

Proof. Observe first that if $Y = \mathsf{S}_{\nu_Y}$, then Y is continuous by Lemma 31.

Conversely, let us suppose that Y is continuous. By adjointness, $Y \subseteq \mathsf{S}_{\nu_Y}$ holds, so we argue for the reverse inclusion. If $f \in \mathsf{S}_{\nu_Y}$, then $\emptyset = \nu_Y(f) = \bigcap_{g \in Y} \delta(f,g)$, so $f \in Y$. Therefore $\mathsf{S}_{\nu_Y} \subseteq Y$.

The last statement follows from the characterization of the continuous subspaces of X as the fixed-points of the closure operator $\mathsf{S}_{\nu(-)}$. ☐

Proposition 33. *A module v is such that $\nu_{\mathsf{S}_v} = v$ if and only if either $v(f) = A$, for each $f \in X$, or $v(f) = \emptyset$, for some $f \in X$.*

Proof. If $v(f) = A$ for each $f \in X$, then $S_v = \emptyset$ (since $A \neq \emptyset$). It follows that $\nu_{\mathsf{S}_v}(f) = A$ for each $f \in X$ and $\nu_{\mathsf{S}_v} = v$.

Suppose now that $v(g) = \emptyset$ for some $g \in X$. By adjointness, we have $v(f) \subseteq \nu_{\mathsf{S}_v}(f)$, for all $f \in X$; thus we need to argue for the opposite inclusion. Fix $f \in X$; we exhibit next $h \in X$ such that $v(h) = \emptyset$ and $\delta(f,h) \subseteq v(f)$. It shall follow that $\nu_{\mathsf{S}_v}(f) = \bigcap_{v(h)=\emptyset} \delta(f,h) \subseteq v(f)$. Since $v(f) \subseteq \delta(f,g) \cup v(g) = \delta(f,g)$, we can write $\delta(f,g) = v(f) \cup (\delta(f,g) \setminus v(f))$. We use now pairwise completeness to pick $h \in X$ such $\delta(f,h) \subseteq v(f)$ and $\delta(h,g) \subseteq \delta(f,g) \setminus v(f)$. Then $v(h) \subseteq \delta(h,f) \cup v(f) \subseteq v(f)$, and $v(h) \subseteq \delta(h,g) \cup v(g) = \delta(h,g) \subseteq \delta(f,g) \setminus v(f)$. It follows that $v(h) \subseteq v(f) \cap A \setminus v(f)$, whence $v(h) = \emptyset$.

For the converse direction, suppose that $v(f) = \nu_{\mathsf{S}_v}(f)$ for each $f \in X$. If $v(f) \neq \emptyset$, for each $f \in Y$, then $v(f) = \nu_{\mathsf{S}_v}(f) = A$, for each $f \in X$. Otherwise, $v(f) = \emptyset$, for some $f \in X$. ☐

Remark 34. Proposition 32, characterizing continuous subspaces as closed subsets of a closure operator, suggests that pairwise complete spaces might have some algebraic nature as well. This is actually the case. It is easily verified that a space (X, δ) is pairwise complete if and only if, *for each α, β such that* $\alpha \cap \beta = \emptyset$, and for each $f, g \in X$, $\delta(f,g) \subseteq \alpha \cup \beta$ implies $\delta(f,h) \subseteq \alpha$ and $\delta(h,g) \subseteq \beta$ for some $h \in X$. We observe that *such an h is unique*. Suppose that $\alpha \cap \beta = \emptyset$ and let h_i, $i = 1,2$ with $\delta(f,h_i) \subseteq \alpha$ and $\delta(h_i,g) \subseteq \beta$. Then $\delta(h_1,h_2) \subseteq \delta(h_1,f) \cup \delta(f,h_2) \subseteq \alpha$ and similarly $\delta(h_1,h_2) \subseteq \beta$. It follows that $\delta(h_1,h_2) \subseteq \alpha \cap \beta = \emptyset$ and $h_1 = h_2$.

Pairwise complete spaces and universal product frames. We already observed—see equation (4)—that the underlying set of a space of the form $(\mathsf{Sec}(\pi), \delta)$ with $\pi : E \longrightarrow A$ is that of a universal $\mathrm{S5}^A$-product frame. Something more is true: we can define the transition relations of the universal $\mathrm{S5}^A$-product frame by means of the metric. Indeed, for each $a \in A$, we have

$$f R_a g \text{ iff } \delta(f,g) \subseteq \{a\}\,.$$

On the other hand, if A is finite, then the metric is completely determined from the transition relation of the frame, using the notion of α-path introduced in Section 2, as follows:

$$\delta(f,g) = \bigcap \{\alpha \subseteq A \mid \text{there exists an } \alpha\text{-path from } f \text{ to } g\ \}\,.$$

We cast our observations in a Proposition:

Proposition 35. *If A is finite, then there is a bijective correspondence bewtween spaces over $P(A)$ of the form $(\mathsf{Sec}(\pi), \delta)$ and universal $\mathrm{S5}^A$-product frames.*

Pairwise complete spaces and lattices. We can generalize the construction of the relational lattice $\mathsf{R}(D, A)$ starting from an arbitrary space (X, δ). We say that a subset $Z \subseteq A \cup X$ is closed if $x \in Z$ whenever $\delta(x, y) \subseteq Z \cap A$ and $y \in Z$. The set of closed subsets of $A \cup X$ is then a Moore family.

Definition 36. The lattice $\mathcal{L}(X, \delta)$ is the lattice of closed subsets of $A \cup X$.

Obviously, $\mathcal{L}(D^A, \delta)$ is the relational lattice $\mathsf{R}(D, A)$. The lattice $\mathcal{L}(X, \delta)$ can be shown to be pluperfect when (X, δ) is reduced and symmetric. Yet, for the sake of the undecidability result, we shall only need that $\mathcal{L}(X, \delta)$ is an atomistic pluperfect lattice when A is finite and (X, δ) is pairwise complete.

Proposition 37. *If A is finite and (X, δ) is pairwise complete, then $\mathcal{L}(X, \delta)$ is isomorphic to the lattice $\mathsf{L}(\mathfrak{U})$ for some universal* $\mathrm{S5}^A$*-product frame* $\mathfrak{U}$.

Proof. By Theorem 25 the space (X, δ) is isomorphic to the space $(\mathsf{Sec}(\pi), \delta)$, for some surjective $\pi : E \longrightarrow A$. The construction $\mathcal{L}$ clearly sends isomorphic spaces to isomorphic to isomorphic lattices. Therefore, we assume that $(X, \delta) = (\mathsf{Sec}(\pi), \delta)$ and prove that $\mathcal{L}(X, \delta) = \mathsf{L}(\mathfrak{U})$. We have $X = \prod_{a \in A} E_a$ with $E_a = \pi^{-1}(a)$, while $\delta(f, g) = \{a \in A \mid f(a) \neq g(a)\}$. It is easily verified that $\delta(f, g) \subseteq \alpha$ if and only if there is an α-path from f to g in the universal $\mathrm{S5}^A$ product frame $\mathfrak{U}$ on $\prod_{a \in A} E_a$. Therefore, the two Moore families, $\mathcal{L}(X, \delta)$ and $\mathsf{L}(\mathfrak{U})$, are the same. □

From the above theorem and from the preliminary investigation of the structure of the lattices $\mathsf{L}(\mathfrak{F})$ in Section 5, we can infer the following statement.

Corollary 38. *If A is finite and (X, δ) is pairwise complete, then $\mathcal{L}(X, \delta)$ is an atomistic pluperfect lattice, where the set of join-irreducible elements can be identified with $A \cup X$, every element $a \in A$ is join-prime, and minimal join-covers of $f \in X$ are of the form*

$$f \lhd_{\mathtt{m}} \delta(f, g) \cup \{g\},$$

for each $g \in X$.

Proof. The statement follows from Propositon 21 and from the observation that an α-path from f to g is minimal if and only if $\alpha = \delta(f, g)$. □

Let us remark that the above statement holds even when A is not finite or when (X, δ) is not pairwise complete. In particular, if A is infinite and the universal $\mathrm{S5}^A$-product frame $\mathfrak{U}$ has $(\mathsf{Sec}(\pi), \delta)$ as underlying space, then $\mathsf{L}(\mathfrak{U})$ and $\mathcal{L}(\mathsf{Sec}(\pi), \delta)$ need not to be equal. For instance, if $\delta(f, g)$ is an infinite set, then $\delta(f, g) \cup \{g\}$ is an infinite minimal join-cover of f, while we observed before that any minimal join-cover in $\mathsf{L}(\mathfrak{U})$ is finite.

7. Principal ideals and filters in relational lattices

The purpose of this Section is to prove the following statement.

Theorem 39. *If L is a finite subdirectly-irreducible atomistic lattice which has a lattice embedding into some relational lattice $\mathsf{R}(D, A)$, then there exists an embeddings of L into some other relational lattice $\mathsf{R}(D, B)$ which moreover preserves $\bot$ and $\top$.*

The theorem is an immediate consequence of Propositions 40 and 42 that follows. These propositions mainly deal with the structure of principal ideals and filters in a relational lattice $\mathsf{R}(D, A)$, namely the sublattices of the form $\downarrow Z := \{W \in \mathsf{R}(D, A) \mid W \subseteq Z\}$ and $\uparrow Z := \{W \in \mathsf{R}(D, A) \mid Z \subseteq W\}$.

In the proofs of these propositions we use the isomorphism between $P(A \sqcup D^A)$ and $P(A) \times P(D^A)$ to represent the lattice $\mathsf{R}(D, A)$ as the set of pairs (α, Y) with $\alpha \subseteq A$ and $Y \subseteq D^A$ α-closed.

Proposition 40. *If L is a subdirectly-irreducible lattice which has an embedding $i : L \longrightarrow \mathsf{R}(D, A)$, then there is a subset $\alpha \subseteq A$ and an embedding $j : L \longrightarrow \mathsf{R}(D, \alpha)$ that preserves $\top$.*

Proof. Suppose $i(\top) \neq (A, D^A)$, say $i(\top) = (\alpha, Y)$, with Y α-closed. Call M the ideal $\downarrow(\alpha, Y)$, so L embeds into M while preserving $\top$. Let us study the structure of M. This lattice is clearly atomistic and pluperfect by Lemma 3. Its set of atoms is $\alpha \cup Y$, while the minimal join-covers are of the form $f \lhd_{\mathtt{m}} \delta(f, g) \cup \{g\}$ whenever $f, g \in Y$ and $\delta(f, g) \subseteq \alpha$.

Notice now that if $f \in Y$, then the ball $B(f, \alpha)$ is contained in Y, since Y is α-closed. This implies that $A_f := \alpha \cup B(f, \alpha)$ is a D-closed subset of $\mathcal{J}(M)$ and, M_{A_f} defined in Lemma 4, is a lattice quotient of M. We notice that the OD-graph of M_{A_f} is isomorphic to the one of $\mathsf{R}(D, \alpha)$, so M_{A_f} itself is isomorphic to $\mathsf{R}(D, \alpha)$.

Since moreover $\bigcup_{f \in Y} \alpha \cup B(f, \alpha) = \alpha \cup Y$, then $\langle \pi_{A_f} \mid f \in Y \rangle : M \longrightarrow \prod_{f \in Y} M_{A_f}$ is, by Lemma 5, a subdirect decomposition of M. Therefore L embeds into $\prod_{f \in Y} M_{A_f}$ and since L is subdirectly-irreducible, it embeds into some M_{A_f} and such embedding preserves $\top$. Since M_{A_f} is isomorphic to $\mathsf{R}(D, \alpha)$, we conclude that L embeds into $\mathsf{R}(D, \alpha)$ while preserving $\top$. $\square$

For $B \subseteq A$, let us define $\psi_{A,B} : P(A) \times P(D^A) \longrightarrow P(B) \times P(D^B)$ by the following formula:

$$\psi_{A,B}(\alpha, X) := (\alpha \cap B, X{\upharpoonright\!\!\upharpoonright}_B)\,.$$

Lemma 41. *The map $\psi_{A,B}$ restricts to an order-preserving map from $\mathsf{R}(D, A)$ to $\mathsf{R}(D, B)$. Its further restriction to the filter $\uparrow(B^{\mathsf{c}}, \emptyset) \subseteq \mathsf{R}(D, A)$ yields an isomorphism with $\mathsf{R}(D, B)$.*

Proof. We suppose that X is α-closed and argue $X{\upharpoonright\!\!\upharpoonright}_B$ is $\alpha \cap B$-closed. If $g \in X$, $f \in D^B$, and $\delta_{D^B}(f, g_{\upharpoonright B}) \subseteq \alpha \cap B$, then we can extend f to $f' \in D^A$, so $f'_{\upharpoonright B} = f$ and $f'(x) = g(x)$ for all $x \in B^{\mathsf{c}}$. It follows that $\delta_{D^A}(f', g) = \delta_{D^B}(f, g_{\upharpoonright B}) \subseteq \alpha \cap B$, so $f' \in X$ since X is α-closed. Then $f = f'_{\upharpoonright B} \in X{\upharpoonright\!\!\upharpoonright}_B$.

We argue similarly that if $(\beta, Y) \in \mathsf{R}(D, B)$, then $(\beta \cup B^{\mathsf{c}}, i_A(Y))$ belongs to $\mathsf{R}(D, A)$, namely that $i_A(Y)$ is $\beta \cup B^{\mathsf{c}}$-closed when Y is β-closed. Let $f \in D^A$ and $g \in i_A(Y)$ be such that $\delta(f, g) \subseteq \beta \cup B^{\mathsf{c}}$. Now $\delta(f_{\upharpoonright B}, g_{\upharpoonright B}) \subseteq (\beta \cup B^{\mathsf{c}}) \cap B = \beta$, and since $g_{\upharpoonright B} \in Y$ and Y is β-closed, we have $f_{\upharpoonright B} \in Y$, that is $f \in i_A(Y)$.

Observe moreover that $(\alpha \cap B, X{\upharpoonright\!\!\upharpoonright}_B) \subseteq (\beta, Y)$ holds if and only if $(\alpha, X) \subseteq (\beta \cup B^{\mathsf{c}}, i_A(Y))$, so the two maps are adjoints to each other, in particular they are monotonic.

Next $\psi_{A,B}(\beta \cup B^{\mathsf{c}}, i_A(Y)) = ((\beta \cup B^{\mathsf{c}}) \cap B, i_A(Y){\upharpoonright\!\!\upharpoonright}_B) = (\beta, Y)$, thus $\psi_{A,B}$ is surjective. Finally, let us argue that $\psi_{A,B}$ is injective if restricted to $\uparrow(B^{\mathsf{c}}, \emptyset)$. Let $(\alpha, X), (\alpha', X') \in \mathsf{R}(D, A)$ with $B^{\mathsf{c}} \subseteq \alpha \cap \alpha'$ and $\psi_{A,B}(\alpha, X) = \psi_{A,B}(\alpha', X')$. Then $\alpha \cap B = \alpha' \cap B$, $\alpha = (\alpha \cap B) \cup B^{\mathsf{c}}$, and $\alpha' = (\alpha' \cap B) \cup B^{\mathsf{c}}$, imply $\alpha = \alpha'$. Let $f \in X$, so $f_{\upharpoonright B} \in X{\upharpoonright\!\!\upharpoonright}_B = X'{\upharpoonright\!\!\upharpoonright}_B$, so there exists $f' \in X'$ with $f'_{\upharpoonright B} = f_{\upharpoonright B}$. Since X'

is B^{c} closed and $\delta(f, f') \subseteq B^{\mathsf{c}}$ we have $f \in X'$. Thus we have $X \subseteq X'$; a similar argument yields $X' \subseteq X$, so $X = X'$. ☐

Proposition 42. *If a finite subdirectly-irreducible atomistic lattice L has a $\top$-preserving lattice embedding $i : L \longrightarrow \mathsf{R}(D, A)$, then there exists an embedding $j : L \longrightarrow \mathsf{R}(D, B)$ which preserves $\top$ and $\bot$.*

Proof. By Lemma 41, if $i(\bot) = (B^{\mathsf{c}}, X)$ for some $B \subseteq A$, then $j = \psi_{A,B} \circ i : L \longrightarrow \mathsf{R}(D, B)$ is an embedding which preserves $\top$ and such that $j(\bot) = (\emptyset, Y)$ for some $Y \subseteq D^B$.

Suppose now that $Y \neq \emptyset$. Let $\mu : \mathsf{R}(D, B) \longrightarrow L$ be left adjoint to j, so μ is surjective and, moreover, each atom $a \in \mathcal{J}(L)$ has some $k \in B \cup D^B$ with $\mu(k) = a$. Notice also that $\mu(k) = \bot$ if and only if $k \in Y$, for each $k \in B \cup D^B$. Let us argue that every element of $\mathcal{J}(L)$ is join-prime. Let $a \in \mathcal{J}(L)$ and pick $k \in B \cup D^B$ such that $\mu(k) = a$. If $k \in B$, then $a = \mu(k)$ is join-prime, since μ sends a join-prime element either to a join-prime element or to $\bot$. Suppose now $k = f \in D^B$ and recall that $\mu(f) = a \neq \bot$ implies $f \notin Y$. Pick $g \in Y$, so $f \leq \bigvee \delta(f, g) \vee g$ and $a = \mu(f) \leq \bigvee \mu(\delta(f, g)) \vee \mu(g) = \bigvee \mu(\delta(f, g))$. Since μ sends join-prime elements to join-prime elements or to $\bot$, we see that a has a join-cover made up of join-prime elements only. Lemma 6 implies then that a is join-prime.

We have argued that either $Y = \emptyset$, so j preserves $\bot$; or $Y \neq \emptyset$, in which case all the elements of $\mathcal{J}(L)$ are join-prime and atoms. In the last case, however, L is a two elements Boolean algebra, since L is subdirectly-irreducible and distributive. Such an algebra can obviously be embedded into a relational lattice while preserving $\top$ and $\bot$. ☐

8. From lattice embeddings to surjective p-morphisms

The goal of this Section is to prove the following statement:

Theorem 43. *Let A be a finite set, let $\mathfrak{F}$ be a finite rooted full* S4 *A-frame. If $\mathsf{L}(\mathfrak{F})$ embeds into a relational lattice $\mathsf{R}(D, B)$, then there exists a universal* $\mathrm{S5}^A$*-product frame $\mathfrak{U}$ and a surjective p-morphism from $\mathfrak{U}$ to $\mathfrak{F}$.*

To prove the Theorem, we study bound-preserving embeddings of finite atomistic lattices into lattices of the form $\mathsf{R}(D, B)$. Let in the following

$$i : L \longrightarrow \mathsf{R}(D, B)$$

be a fixed bound-preserving lattice embedding, with L a finite atomistic lattice. Since L is finite, i has a left adjoint $\mu : \mathsf{R}(D, B) \longrightarrow L$. By abuse of notation, we shall also use the same letter μ to denote the restriction of this left adjoint to the set of completely join-irreducible elements of $\mathsf{R}(D, B)$ which, we recall, is identified with the set $B \cup D^B$.

Lemma 44. *If $b \in B$, then $\mu(b)$ is join-prime.*

Proof. Suppose $b \in B$ and $\mu(b) \leq \bigvee X$. Then $b \leq i(\bigvee X) = \bigvee i(X)$, so $b \leq i(x)$ for some $x \in X$, since b is join-prime. It follows that $\mu(b) \leq x$, for some $x \in X$. ☐

It is not in general true that μ sends join-irreducible elements to join-irreducible elements, and this is a main difficulty towards a proof of Theorem 43. Yet, the following holds:

Lemma 45. *For each $x \in \mathcal{J}(L)$ there exists $y \in B \cup D^B$ such that $\mu(y) = x$.*

Proof. Since i is an embedding, then its left adjoint μ is surjective. So if $x \in \mathcal{J}(L)$, then there exists $y \in \mathsf{R}(D, B)$ with $\mu(y) = x$. Write $y = \bigvee_{i \in I} z_i$ with each $z_i \in B \cup D^B$. Then $x = \bigvee_{i \in I} \mu(z_i)$, so $x = \mu(z_i)$ for some $i \in I$ and such a z_i is a preimage of y by μ which belongs to $B \cup D^B$. □

Lemma 46. *Let $g \in D^B$ such that $\mu(g)$ is join-reducible in L. There exists $h \in D^B$ such that $\mu(h) \in \mathcal{J}(L)$ and $\mu(g) = \bigvee \mu(\delta(g, h)) \vee \mu(h)$. If L is not a Boolean algebra, then $\mu(h)$ is non-join-prime.*

Proof. Write $\mu(g) = \bigvee \alpha$ with $\alpha \subseteq \mathcal{J}(L)$ and α minimal with these two properties. We have then $g \leq \bigvee i(\alpha)$ so $\delta(g, h) \cup \{h\} \ll i(\alpha)$ for some $h \in D^B$. We have then $\mu(\delta(g, h)) \cup \{\mu(h)\} \ll \alpha$ and this relation implies that $\mu(\delta(g, h)) \cup \{\mu(h)\} \subseteq \alpha$. Indeed, since i preserves the least element, $\mu(x) = \bot$ implies $x = \bot$. Thus every element of $\mu(\delta(g, h)) \cup \{\mu(h)\}$ is distinct from $\bot$ and below an atom in α, so it is necessarily equal to such an atom. In particular, we have $\mu(h) \in \mathcal{J}(L)$.

We also have $\bigvee \alpha \leq \mu(g) \leq \bigvee \mu(\delta(g, h)) \vee \mu(h) \leq \bigvee \alpha$, so $\mu(g) = \bigvee \mu(\delta(g, h)) \vee \mu(h)$. By minimality, it follows $\alpha = \mu(\delta(g, h)) \cup \{\mu(h)\}$.

Suppose that L is not a Boolean algebra, so we can find an atom $a \in \mathcal{J}(L)$ which is non-join-prime. Pick $f \in D^B$ such that $\mu(f) = a$. Observe that every element $\delta(f, g) \cup \delta(g, h)$ is join-prime, so every element of $\mu(\delta(f, g) \cup \delta(g, h))$ is also join-prime. If $\mu(h)$ is join-prime, then we deduce $a \leq \bigvee \mu(\delta(f, g)) \vee \bigvee \mu(\delta(g, h)) \vee \mu(h)$, so the non-join-prime a has a join-cover all made of join-prime elements. Since L is in the variety generated by the relational lattices, this contradicts Lemma 6. □

Let A be the set of atoms of L that are join-prime. While (D^B, δ) is an ultrametric space over $P(B)$, we need to transform it into an ultrametric space over $P(A)$. To this end, we define a $P(A)$-valued distance δ_A on D^B by

$$\delta_A(f, g) := \{\mu(b) \mid b \in \delta(f, g)\} .$$

Because of Lemma 44, we have $\delta_A(f, g) \subseteq A$.

Proposition 47. *(D^B, δ_A) is a pairwise complete ultrametric space over $P(A)$.*

Proof. δ_A satisfies the properties defining a distance (including being reduced and symmetric), mainly because the direct image of any function (here of μ) preserves unions.

For pairwise completeness, observe that if $\delta_A(f, g) \subseteq \alpha_0 \cup \alpha_1$, then $\delta(f, g) \subseteq \beta_0 \cup \beta_1$, where $\beta_i := \{b \in B \mid \mu(b) \in \alpha_i\}$, $i = 0, 1$. Taking h such that $\delta(f, h) \subseteq \beta_0$ and $\delta(h, g) \subseteq \beta_1$, we obtain $\delta_A(f, h) \subseteq \alpha_0$ and $\delta_A(h, g) \subseteq \alpha_1$. □

We define next $v : D^B \longrightarrow P(A)$ by letting

$$v(f) := \{a \in A \mid a \leq \mu(f)\} .$$

Lemma 48. *$v : D^B \longrightarrow P(A)$ is a module on (D^B, δ_A). That is, the relation*

$$v(f) \subseteq \delta_A(f, g) \cup v(g) .$$

holds.

Proof. Suppose that $a \in v(f)$ and $a \notin \delta_A(f, g)$. This means that $a \leq \mu(f)$ but $b \notin \delta(f, g)$ whenever $\mu(b) = a$. Recall that if $b \in B$, then b is join-prime, whence $\mu(b)$ is join-prime as well. Thus if $a \in A$ and $a \leq \mu(b)$, then $a = \mu(b)$, since we are assuming that L is atomistic. Since $a \leq \mu(f) \leq \bigvee_{b \in \delta(f,g)} \mu(b) \vee \mu(g)$, a is join-prime, $a \leq \mu(b)$ implies $a = \mu(b)$, we necessarily have $a \leq \mu(g)$, so $a \in v(g)$. □

Lemma 49. *$v(f) = \emptyset$ if and only if $\mu(f) \in \mathcal{J}(L) \setminus A$.*

Proof. Suppose that $\mu(f) \in \mathcal{J}(L) \setminus A$. If $v(f) \neq \emptyset$, then let $a \in A$ with $a \leq \mu(f)$. Since we are assuming that $\mu(f)$ is join-irreducible and that L is atomistic, we deduce $\mu(f) = a \in A$, a contradiction.

Conversely, suppose that $v(f) = \emptyset$. This immediately gives $\mu(f) \notin A$. By the way of contradiction, suppose now that $\mu(f)$ is reducible, so use Lemma 46 to find $h \in D^B$ such that $\mu(h) \in \mathcal{J}(L)$ and $\mu(f) = \bigvee \mu(\delta(f,h)) \vee \mu(h)$. Since $\mu(h)$ is join-irreducible, then $h \neq f$ and $\delta(f,h) \neq \emptyset$. Pick $b \in \delta(f,h)$, then $\mu(b) \in A$ and $\mu(b) \leq \mu(f)$. This gives $\mu(b) \in v(f)$, so $v(f) \neq \emptyset$, a contradiction. □

Corollary 50. *The subspace*

$$F_0 := \{f \in D^B \mid \mu(f) \in \mathcal{J}(L) \setminus A\} \tag{5}$$

of D^B is pairwise complete.

Proof. By Lemma 49, $f \in F_0$ if and only if $v(f) = \emptyset$. Since v is a module, the set $\{f \in D^B \mid v(f) = \emptyset\}$ is, by Lemma 31, a pairwise complete metric space over $P(A)$. □

Proposition 51. *Let L be a finite atomistic lattice and let A be the set of its join-prime elements. If L is not a Boolean algebra and $i : L \longrightarrow \mathsf{R}(D,B)$ is a bound-preserving lattice embedding, then there exists a pairwise complete ultrametric space (F_0, δ) over $P(A)$ and a bound-preserving lattice embedding $j : L \longrightarrow \mathcal{L}(F_0, \delta_A)$ whose left adjoint ν satisfies the following condition: for each $k \in A \cup F_0$, if $k \in A$ then $\nu(k) = k$ and, otherwise, $\nu(k) \in \mathcal{J}(L) \setminus A$.*

Proof. Let (F_0, δ_A) be the pairwise complete space over $P(A)$ as defined in Corollary 50. By the definition of F_0, the restriction of μ to F_0 takes values in $\mathcal{J}(L) \setminus A$. Therefore we can define $\nu : A \cup F_0 \longrightarrow \mathcal{J}(L)$ as follows:

$$\nu(k) := \begin{cases} k\,, & k \in A, \\ \mu(k)\,, & k \in F_0. \end{cases}$$

We notice next that ν is surjective. If $x \in \mathcal{J}(L) \setminus A$, then by Lemma 45, there is $y \in B \cup D^B$ such that $\mu(y) = x$. By Lemma 44, $y \notin B$, so $y \in D^B$. Since $\mu(y) = x \in \mathcal{J}(L) \setminus A$, then y belongs to F_0.

Let $j : L \longrightarrow \mathcal{L}(F_0, \delta_A)$ be the function defined by

$$j(l) := \{x \in A \cup F_0 \mid \nu(x) \leq l\}\,.$$

Let us argue, in the order, that

(1) for each $l \in L$, $j(l)$ is a closed subset of $A \cup F_0$,
(2) j is injective,
(3) j preserves meets and (4) it preserves joins.

(1). Let $f, g \in F_0$ and suppose that $\delta_A(f,g) \cup \{g\} \subseteq j(l)$. This condition means that $\nu(\delta_A(f,g)) = \delta_A(f,g) = \mu(\delta(f,g)) \ll \{l\}$ and $\mu(g) \leq l$; it follows that $\nu(f) = \mu(f) \leq \bigvee \mu(\delta(f,g)) \vee \mu(g) \leq l$, so $f \in j(l)$.

(2). We have $j(l_0) = j(l_1)$ if and only if, for all $x \in A \cup F_0$, the condition $\nu(x) \leq l_0$ is equivalent to $\nu(x) \leq l_1$. As ν is surjective, this means that l_0 and l_1 have the same atoms below them, thus that they are equal.

(3). It is easily verified that

$$j(\top) = A \cup F_0\,, \quad \text{and} \quad j(l_0 \wedge l_1) = j(l_0) \cap j(l_1)\,.$$

In particular, j is order-preserving.

(4). Since j is order-preserving, we only need to show that $j(l_0 \vee l_1) \leq j(l_0) \vee j(l_1)$. To this end, we suppose that $x \in A \cup F_0$ is such that $x \in j(l_0 \vee l_1)$, so $\nu(x) \leq l_0 \vee l_1$. If $x \in A$, then $\nu(x) = x \leq l_0 \vee l_1$, and since x is join-prime, this gives $\nu(x) = x \leq l_i$ for some $i \in \{0, 1\}$. This immediately yields $x \in j(l_i) \leq j(l_0) \vee j(l_1)$.

Suppose now that $x = f \in F_0$ so $\mu(f) = \nu(f) \leq l_0 \vee l_1$. We have, therefore, $f \leq i(l_0) \vee i(l_1)$, so $f \lhd_{\mathtt{m}} \delta(f, g) \cup \{g\} \ll \{i(l_0), i(l_1)\}$ for some $g \in D^B$. We can use now Lemma 46 to pick $h \in D^B$ with $\mu(h) \in \mathcal{J}(L) \setminus A$ (so $h \in F_0$) and $\mu(g) = \bigvee \mu(\delta(g, h)) \vee \mu(h)$.

We have then that $\mu(\delta(f, h)) \cup \{\mu(h)\} \subseteq \mu(\delta(f, g) \cup \delta(g, h)) \cup \{\mu(h)\} \ll \{l_0, l_1\}$. This relation yields

$$\nu(\delta_A(f, h)) \cup \{\nu(h)\} \ll \{l_0, l_1\}$$

or, said otherwise,

$$\delta_A(f, h) \cup \{h\} \ll \{j(l_0), j(l_1)\}\,.$$

This implies that $f \in j(l_0) \vee j(l_1)$.

Let us argue that j preserves the least element. If $x \in j(\bot)$, then $\nu(x) \leq \bot$. We cannot have $x \in B$, so $x = f \in F_0$. Then $\mu(f) \leq \bot$ and $f \in i(\bot)$, contradicting the assumption that i preserves bounds.

Finally, let us observe that the left adjoint of j agrees, on join-irreducible elements, with ν. Indeed, for each $x \in A \cup F_0$, we have $\nu(x) \leq y$ iff $x \in j(y)$, iff $x \leq j(y)$, where we identify, as usual, a singleton with its only element. $\square$

We conclude next the proof of the main result of this Section, Theorem 43.

Proof of Theorem 43. Since $\mathfrak{F}$ is rooted and full, $\mathsf{L}(\mathfrak{F})$ is a finite atomistic subdirectly-irreducible lattice by Proposition 22. Therefore, if $i : \mathsf{L}(\mathfrak{F}) \longrightarrow \mathsf{R}(D, B)$ is a lattice embedding, then we can assume, using Theorem 39, that i preserves the bounds. Also, if $\mathsf{L}(\mathfrak{F})$ is a Boolean algebra, then it is the two elements Boolean algebra, since we are assuming that $\mathsf{L}(\mathfrak{F})$ is subdirectly-irreducible. But then, $\mathfrak{F}$ is a singleton, and the statement of the Theorem trivially holds in this case.

We can therefore assume that $\mathsf{L}(\mathfrak{f})$ is not a Boolean algebra. Let us recall that A is the set of join-prime elements of $\mathsf{L}(\mathfrak{F})$, see Proposition 21. Let (F_0, δ_A) be the pairwise complete space over $P(A)$ and let $j : \mathsf{L}(\mathfrak{F}) \longrightarrow \mathcal{L}(F_0, \delta_A)$ be the lattice morphism with the properties stated in Proposition 51; let ν be the left adjoint to j. Using Corollary 37, we can also assume that $\mathcal{L}(F_0, \delta_A) = \mathsf{L}(\mathfrak{U})$ for some universal $\mathrm{S5}^A$-product frame $\mathfrak{U}$.

To avoid confusions, we depart from now on from the convention of identifying singletons with their elements. We define $\psi : X_{\mathfrak{U}} \longrightarrow X_{\mathfrak{F}}$ by saying that $\psi(x) = y$ when $\nu(\{x\}) = \{y\}$. This is well defined since in $\mathsf{L}(\mathfrak{U})$ (respectively $\mathsf{L}(\mathfrak{F})$) the non-join-prime join-irreducible-elements are the singletons $\{x\}$ with $x \in X_{\mathfrak{F}_{\mathfrak{U}}}$ (resp. $x \in X_{\mathfrak{F}}$); moreover, we have $X_{\mathfrak{U}} = F_0$ and each singleton $\{x\}$ with $x \in F_0$ is sent by ν to a singleton $\{y\} \in \mathcal{J}(\mathsf{L}(\mathfrak{F})) \setminus \{\{a\} \mid a \in A\} = \{\{x\} \mid x \in X_{\mathfrak{F}}\}$. The function ψ is surjective since every non-join-prime atom $\{x\}$ in $\mathsf{L}(\mathfrak{F})$ has a preimage by ν an atom $\{y\}$ and such a preimage cannot be join-prime, so $y \in X_{\mathfrak{U}}$.

We are left to argue that ψ is a p-morphism. To this end, let us remark that, for each $a \in A$ and $x, y \in X_{\mathfrak{F}}$ (or $x, y \in X_{\mathfrak{U}}$), the relation $xR_a y$ holds exactly when there is an $\{a\}$-path from x to y, i.e. when $\{x\} \subseteq \overline{\{a, y\}} = \{a\} \vee \{y\}$ (we need here that $\mathfrak{F}$ and $\mathfrak{U}$ are S4 frames).

Thus, let $x, y \in X_{\mathfrak{U}}$ be such that $xR_a y$. Then $\{x\} \subseteq \{a\} \vee \{y\}$ and $\nu(\{x\}) \subseteq \nu(\{a\}) \vee \nu(\{y\}) = \{a\} \vee \nu(\{y\})$. We have therefore $\psi(x) R_a \psi(y)$. Conversely, let $x \in X_{\mathfrak{U}}$ and $z \in X_{\mathfrak{F}}$ be such that $\psi(x) R_a z$. We have therefore $\nu(\{x\}) \subseteq \{a\} \vee \{z\}$, whence, by adjointness,

$$\begin{aligned}\{x\} \subseteq j(\{a\} \vee \{z\}) &= j(\{a\}) \vee j(\{z\}) \\ &= \{a\} \vee \{y \mid \nu(\{y\}) = \{z\}\} \\ &= \overline{\{a\} \cup \{y \mid \nu(\{y\}) = \{z\}\}}\,.\end{aligned}$$

Then, by formula (2), there is some $y \in X_{\mathfrak{U}}$ with $\psi(y) = z$ and a $\{a\}$-path from x to y. But then, we also have $xR_a y$. □

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