LQ-based event-triggered controller co-design for saturated linear systems

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Abstract

This paper deals with the simultaneous design of a state feedback law and an event-triggering condition ensuring local exponential stability and LQ performance in the presence of plant input saturation and of a communication channel between the controller output and the saturated plant input. To this aim, we adopt Lyapunov-based techniques in a hybrid framework. The design of the event-triggered control is based on two conditions: one to solve the event-triggered control co-design for LQ stabilization; the second one to adjust the co-design among all possible solutions of the first condition thanks to a tunable parameter. The proposed Lyapunov formulation yields an event-triggered algorithm to update the saturated plant input based on conditions involving the closed-loop state, while an estimate of the domain of attraction is provided. Furthermore, a trade-off is highlighted relating the optimality level, the size of the estimate of the basin of attraction and the reduction of the amount of transmissions.

Key words: Event-triggered control, input saturation, linear quadratic performance.

1 Introduction

In recent years, the study of sampled-data systems has provided several techniques for dealing with linear or nonlinear systems. On the one hand, a large attention has been paid to robust stability with respect to aperiodic samplings (see, for example, [8, 15, 22] and references therein), where variations on the sampling intervals are seen as a disturbance to the periodic case. Then, analysis of such systems using the discrete-time approach [15, 9], the input delay approach [10, 27], and the impulsive systems approach [21] have emerged. On the other hand, an alternative and interesting vision of sampled-data systems has been proposed in [4, 6], which suggests to adapt the sampling sequence to certain events related to the state evolution (see for example [5, 13, 16, 19, 31, 41]), in particular for control implementation as in [35, 39]. In contrast to [12, 20] where self-triggered controllers are considered, the plant evolves in continuous-time and is in closed loop with a controller providing a discrete-time input held during an asynchronous sampling period. The design of an event-triggered algorithm can be first rewritten as the stability study of a system with mixed continuous/discrete dynamics (also called hybrid dynamical system), as considered e.g. in [11, 24, 25] in a different context.

In the event-triggered control framework, two approaches can be handled. The first one is concerned with the situation where the controller is given a priori and only the event-triggered rule has to be designed. This approach corresponds to an emulation problem and is addressed in numerous works (see, for example, [14], [37], [23], [2] and references therein). The second approach refers to the situation where the design of both the controller and the event-triggering condition

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has to be performed, simultaneously, which is called a co-design problem and is addressed only in a few papers. Usually, the synthesis of the control parameters is indeed performed a priori using a fully continuous-time approach [5, 28, 30, 31] or a fully discrete-time approach [13]. Achieved performances are affected by the choice of the continuous-time or the discrete-time synthesis approach. This problem is for example addressed in [1, 3] including some optimality criteria in the synthesis phase. Another important feature when dealing with the stability/performance analysis of control design problems resides in the presence of actuators' limitations. It is now well known that the presence of saturated actuators may cause loss of performance, even unstable behavior (see, for example, [32] and references therein). To the best knowledge of the authors, a few existing results deal with event-based control for saturated plants, some notable exceptions being [17, 18, 36, 38], where specific controller-plant structures are considered.

In the current paper, we pursue the goal of characterizing a desirable behavior of the closed-loop system in terms of a control Lyapunov function, similarly to [28, 29, 30, 34]. Hence, extending the results developed in [28, 29, 30], we use a hybrid framework and Lyapunov theory to define the update policy of event-triggered control algorithms for linear plants subject to input saturation. We propose a simultaneous design of the state feedback law and the event-triggering conditions, ensuring local exponential stability and LQ performance in the presence of plant input saturation and of a communication channel between the controller output and the saturated plant input. To this aim, we adopt Lyapunov-based techniques in a hybrid framework. The originality of this paper with respect to [17, 18, 28, 30, 36, 38] relies on the design of the event-triggered control, which is composed by two conditions. The first one aims at providing a solution to the event-triggered control co-design for LQ stabilization. The second one refines and constrains the co-design among all possible solutions of the first condition thanks to a tunable parameter. The proposed Lyapunov formulation yields an event-triggered algorithm to update the saturated plant input based on conditions involving the closed-loop state, while an estimate of the domain of attraction is provided. More specifically, a trade-off is highlighted relating the optimality level, the size of the estimate of the basin of attraction and the reduction of the amount of transmissions.

The paper is organized as follows. Section 2 describes the problem we intend to solve and presents the adopted hybrid framework. Section 3 is dedicated to the co-design of the event-triggered control. Section 4 proposes an illustrative example allowing us to point out the trade-off between the guaranteed stability, the upper bound on the quadratic cost, and the number of updates performed in simulation tests. The proof of the main result is presented in Section 5. Finally, Section 6 ends the paper with concluding remarks.

Notation. $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ denote respectively the sets of positive integers, positive reals, n-dimensional vectors and $n\times m$ matrices. $|\cdot|$ stands for the Euclidean norm. Given a compact set $\mathcal{A}$, $|x|_{\mathcal{A}} = \min\{|x-y|,\ y \in \mathcal{A}\}$ indicates the distance of the vector x from the set $\mathcal{A}$. The superscript '$\top$' stands for matrix transposition. For any matrix A in $\mathbb{R}^{n\times n}$, we denote $\mathrm{He}(A) = A + A^\top$. For a partitioned matrix, the symbol $*$ stands for symmetric blocks. I and 0 represent the identity and the zero matrices of appropriate dimensions. For any matrix P in $\mathbb{R}^{n\times n}$, $P > 0$ means that P is symmetric positive definite. Given a symmetric positive definite matrix P in $\mathbb{R}^{n\times n}$, $\mathcal{E}(P)$ denotes $\{x \in \mathbb{R}^n :\ x^\top P x \le 1\}$.

2 Problem statement and sampled-data architectures

Consider a linear plant with a saturated input

$$\begin{aligned} \dot{x} &= Ax + Bs, \\ s &= \mathrm{sat}(u), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $s \in \mathbb{R}^m$ and $u \in \mathbb{R}^m$ are the state variable, the input vector and the control law. $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant and given matrices such that pair (A, B) is stabilizable. The function $\mathrm{sat}(\cdot)$ in (1) is a decentralized symmetric saturation with saturation bounds $u_0 = \begin{bmatrix} u_{01} & \cdots & u_{0m} \end{bmatrix}^\top$, namely $s = \mathrm{sat}(u)$ corresponds to enforcing $s_i = \max(-u_{0i}, \min(u_{0i}, u_i))$, where s_i and u_i denote the i-th components of s and u, respectively, for all $i = 1, \ldots, m$.

In this paper we address the problem of event-triggered implementation of a static state-feedback stabilizing law for plant (1), given by the following equation

$$u = Kx. \tag{2}$$

The gain $K \in \mathbb{R}^{m\times n}$ is a matrix to be designed that should ensure local asymptotic stability of the zero equilibrium of the arising closed-loop system (1), (2) with a guaranteed basin of attraction containing the ball $\mathcal{B}(\alpha) := \{x \in \mathbb{R}^n :\ |x| \le \alpha\}$ where $\alpha \in \mathbb{R}$ is a design parameter. Moreover, we require some optimality guarantee in the sense that for any initial condition $x(0) \in \mathcal{B}(\alpha)$, the corresponding (unique) solution to the closed-loop system (1), (2) is required to satisfy an LQ type of bound. More specifically, we provide an upper bound for the following integral cost:

$$J(x(\cdot), u(\cdot)) = \int_0^\infty \begin{bmatrix} x(t) \\ s(t) \end{bmatrix}^\top Q \begin{bmatrix} x(t) \\ s(t) \end{bmatrix} \mathrm{d}t, \tag{3}$$

where $Q \in \mathbb{R}^{(n+m)\times(n+m)}$ is a symmetric positive definite matrix.

Sampled and hold devices could be used to obtain a sampled-data implementation of the feedback law (2) for plant (1). In the most common case of zero-order-hold, this corresponds to breaking the continuous-time closed loop given by $s(t) = \mathrm{sat}(u(t))$, for all $t \geq 0$, where $u(t) = Kx(t)$, and converting this into a zero order hold $\dot{s} = 0$ combined with the update rule $s^+ = \mathrm{sat}(u)$ for s, which should be performed at suitable times according to the specific sampled-data architecture. As mentioned in the introduction, this architecture could selected as *event-triggered sampling*. We explain this architectures below and use the hybrid formalism of [11, 24, 26]. *Event-triggered sampling* corresponds to performing the update rule $s^+ = \mathrm{sat}(Kx)$ whenever the augmented state (x, s) belongs to suitable sets that should be designed in such a way to guarantee asymptotic stability. In this case, the sampled-data system can be written as

$$\begin{cases} \dot{x} = Ax + Bs, \\ \dot{s} = 0, \end{cases} \quad (x,s) \in \mathcal{F}_E, \qquad \begin{cases} x^+ = x, \\ s^+ = \mathrm{sat}(Kx), \end{cases} \quad (x,s) \in \mathcal{J}_E, \tag{4}$$

where $s \in \mathbb{R}^m$ represents the held value of the control input and $\mathcal{F}_E$ and $\mathcal{J}_E$ are two subsets of $\mathbb{R}^n \times \mathbb{R}^m$ indicating where the solution is allowed to flow and/or to jump. Sets $\mathcal{F}_E$ and $\mathcal{J}_E$ are respectively called flow set and jump set and are the available degrees of freedom in the design of the event-triggered algorithm. This hybrid dynamical model allows representing in an efficient manner the sampled-data nature of the system since a jump corresponds to an update of the control input. In this paper, we address the following problem:

Problem 1 *Given plant (1), a scalar $\alpha > 0$ and the linear state feedback law (2), determine gain K and an event-triggered sampled-data implementation of the state feedback law guaranteeing local exponential stability of a suitable attractor (containing the origin in the projection to the plant state) for the sampled-data system, with a basin of attraction containing $\mathcal{B}(\alpha)$ and an upper bound on the performance index (3).*

Remark 1 The attractor is precisely defined in Theorem 1 below. In particular, when dealing with asymptotic stability, the convergence has to be understood in terms of the distance from the attractor and not in terms of the usual Euclidean norm (see [11, Definition 3.5] for the distance definition and the associated stability notions). ◦

3 Co-design result

3.1 Main theorem

The main result of this paper for LQ stabilization and the associated optimization problem is stated in the following theorem, whose proof is given in Section 5. Special care must be taken when considering the LQ cost (3), due to the hybrid time domain of solutions. A possible way to hybridize cost (3) with selection (2), when considering a hybrid solution (x, s) to (4), is the following one:

$$J(x(\cdot,\cdot), s(\cdot,\cdot)) = \sum_{j \in \mathrm{dom}_j(x)} \int_{t_j}^{t_{j+1}} \psi(x(t,j), s(t,j)) \mathrm{d}t. \tag{6}$$

where $\mathrm{dom}_j(x)$ is the set of j such that x is defined at (t, j) for at least one t and t_j, $j \in \mathrm{dom}_j(x)$ are the jump times associated to the solution (except for $t_0 = 0$). Furthermore, $\psi(x, s) = \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]^\top Q \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]$, where we recall that $Q = Q^\top > 0$. For simplicity, throughout the paper we use J to denote $J(x(\cdot,\cdot), s(\cdot,\cdot))$ without indicating its dependence on the solution under consideration.

Theorem 1 *Given $Q = Q^\top > 0$ in $\mathbb{R}^{(n+m)\times(n+m)}$, assume that there exist matrices $W = W^\top > 0$ in $\mathbb{R}^{n\times n}$, Y, X in $\mathbb{R}^{m\times n}$, a diagonal positive definite matrix $S > 0$ in $\mathbb{R}^{m\times m}$ and positive scalar tuning parameters T, α, μ satisfying the following linear matrix inequalities:*

$$\begin{bmatrix} W & X_i \\ X_i^\top & u_{0i}^2 \end{bmatrix} \geq 0 \;\forall i = 1, ..., m, \qquad \begin{bmatrix} I & \alpha I \\ \alpha I & W \end{bmatrix} \geq 0, \tag{7}$$

$$\Psi < 0, \tag{8}$$

$$\Phi(T) < 0 \tag{9}$$

where X_i denotes the i-th row of matrix X, Ψ and $\Phi(T)$ are given in (5), $C = [I\ 0]$, $M(T) = e^{\left(\left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right] T\right)}$ and $\bar{Q}(T) = \int_0^T M^\top(\tau) Q M(\tau) \mathrm{d}\tau$.

Then, consider the corresponding values $P = W^{-1}$, $K = YW^{-1}$, α, μ. Given any scalar $\hat{\mu} \geq \mu$, define the following flow and jump sets for (4):

$$\mathcal{F}_E = \left\{ (x,s) \in \mathbb{R}^n \times \mathcal{U}_0 : \begin{bmatrix} x \\ s \end{bmatrix}^\top \Pi_{\hat{\mu}} \begin{bmatrix} x \\ s \end{bmatrix} \leq 0 \right\}, \tag{10a}$$

$$\mathcal{J}_E = \left\{ (x,s) \in \mathbb{R}^n \times \mathcal{U}_0 : \begin{bmatrix} x \\ s \end{bmatrix}^\top \Pi_{\hat{\mu}} \begin{bmatrix} x \\ s \end{bmatrix} \geq 0 \right\}, \tag{10b}$$

$$\Pi_{\hat{\mu}} = \begin{bmatrix} PA + A^\top P & PB \\ B^\top P & 0 \end{bmatrix} + Q/\hat{\mu}. \tag{10c}$$

Denoting by $\mathcal{U}_0 = \{u \in \mathbb{R}^m : |\operatorname{diag}(u_0)^{-1} u|_\infty \leq 1\}$ the range of the saturation function, the event-triggered closed-loop system (4), (10) is such that the set

$$\mathcal{A} = \{0\} \times \mathcal{U}_0, \tag{11}$$

$$\Psi = \begin{bmatrix} \begin{bmatrix} \mathrm{He}(AW+BY) & * \\ (Y+X)-SB^\top & -2S \end{bmatrix} & * \\ \begin{bmatrix} W & 0 \\ Y & -S \end{bmatrix} & -\mu Q^{-1} \end{bmatrix}, \qquad \Phi(T) = \begin{bmatrix} \begin{bmatrix} -W & * \\ T(Y+X) & -2TS \end{bmatrix} & * & * \\ \begin{bmatrix} W & 0 \\ Y & -S \end{bmatrix} & -\mu \bar{Q}^{-1}(T) & * \\ CM(T) \begin{bmatrix} W & 0 \\ Y & -S \end{bmatrix} & 0 & -W \end{bmatrix}. \tag{5}$$

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is locally exponentially stable with basin of attraction including the set $\mathcal{E}(P) \times \mathcal{U}_0$, where $\mathcal{E}(P) = \{x \in \mathbb{R}^n : x^\top P x \leq 1\}$.

Moreover, for each initial condition in $\mathcal{E}(P) \times \mathcal{U}_0$, there exists at least one solution having unbounded time domain in the ordinary time direction t and there is a minimum dwell time for all solutions starting in $\mathcal{E}(P) \times \mathcal{U}_0$, except for those reaching the attractor in finite time. Finally, for each $x(0,0) \in \mathcal{E}(P) \supset \mathcal{B}(\alpha)$, the LQ cost given by (6) satisfies $J \leq \hat{\mu} x(0,0)^\top P x(0,0) \leq \hat{\mu}$.

Theorem 1 provides an appealing co-design tool. Indeed, by solving the matrix inequalities (7)–(9), it is possible to obtain both the feedback gain K and the parameter P of the event-triggering conditions in (10). Note also that the inequalities are nonlinear but become linear after fixing the scalar T. It is therefore interesting to study the feasibility properties of the arising LMIs obtained for different values of T. When finding a solution to (7)–(9) it is important to keep in mind the several performance criterias that can be optimized. One of them is the size of the guaranteed basin of attraction (well quantified by scalar α), another one is the level of LQ optimality (characterized by μ), and a last one is the size of the parameter T, whose role is connected to the expected average sampling rate of the event triggered implementation, as explained in the next discussions. In our simulations we will adopt the optimality goal of maximizing α. Additional details and discussions are given in Section 4.

Remark 2 As compared to the event-triggered control proposed in [17] for plants with saturated inputs, Theorem 1 guarantees local exponential stability of the attractor while the results proposed in [17] only assess the convergence of the closed-loop trajectories to a bounded set around the origin. ◦

In the reminder of this section, we will provide several motivations and explanations about Theorem 1. This discussion deals with the following issues

- Existence of the tuning parameter T such that the conditions of Theorem 1 are feasible;
- Motivations for each condition of Theorem 1;
- Influence of the tuning parameters T, α, μ;
- Existence of Zeno solutions in the attractor.

3.2 *Existence of the design parameter T*

The next feasibility result establishes the existence of a tuning parameters T, for which solutions to the LMI problem (7), (8) exist. Its proof is given in Section 5.

Proposition 1 *For any control system* (1)-(2) *and for any matrix Q in* (3)*, if there exist matrices $W = W^\top > 0 \in \mathbb{R}^{n\times n}$, Y, X in $\mathbb{R}^{m\times n}$, a diagonal positive definite matrix $S > 0$ in $\mathbb{R}^{m\times m}$ and positive scalars α, μ, which are solutions to* (7) *and* (8)*, then there exists a sufficiently small tuning parameter T such that conditions* (7)*,* (8) *and* (9) *are also satisfied.*

Proposition 1 states that, if one can find a solution to the continuous-time LMI problem (7) and (8), then there exists a sufficiently small T such that both the continuous- and the discrete-time LMI problems are solvable at the same time. Proposition 1 also ensures that for sufficiently small T the same performance level as in the continuous-time design is obtained. Then, if wanting to solve the conditions of Theorem 1, the feasibility of (7), (8) and (9) should be first evaluated for small values of T and then, T should be gradually increased until the problem becomes infeasible.

3.3 *Discussion on the conditions of Theorem 1*

Considering the Lyapunov function $V(x) = x^\top P x$, with $P = W^{-1}$, the following statements hold

- if the LMIs (7) and (8) are satisfied, then, for any $x \in \mathcal{E}(P)$, the solutions to the system with a continuous-time implementation of the control law $s = Kx$ satisfies the Lyapunov condition

$$\dot{V}(x) + \left[\begin{smallmatrix} x \\ Kx \end{smallmatrix}\right]^\top Q \left[\begin{smallmatrix} x \\ Kx \end{smallmatrix}\right] < 0; \tag{12}$$

- if the LMI (7) and (9) are satisfied, then, for any $x \in \mathcal{E}(P)$, the solutions to the system with a periodic sampled-data implementation of the control law, satisfies the Lyapunov condition for a T periodic sampled-data implementation,

$$\Delta_T V(x) + \left[\begin{smallmatrix} x \\ Kx \end{smallmatrix}\right]^\top \bar{Q}(T) \left[\begin{smallmatrix} x \\ Kx \end{smallmatrix}\right] < 0,$$

where $\Delta_T V(x) = V(CM(T)x) - V(x)$, matrices C and $M(T)$ are defined in the statement of Theorem 1.

The proofs of the two facts above can be found in [30], [40, Section 3.5] and [32, Proposition 3.1], for the continuous-time case and in [32, Proposition 3.35] for the case of periodic sampled-data implementation. These proofs use the decentralized deadzone function $dz(u) = u - \mathrm{sat}(u)$ and the associated sector condition exposed in [32, Lemma 1.6].

The design of the control gain K and of the jump and flow sets via the matrix P is based on both conditions $\Psi < 0$ in (8) and $\Phi(T) < 0$ in (9), for a given parameter T. Differently from [30], the addition of condition (9) adjusts the choice of the control parameter K and the Lyapunov matrix P to fit in a better manner the sampled-data implementation. As a consequence, the event-triggered algorithm requires less updates of the control input, as we will see in Section 4.

3.4 *Influence of the tuning parameters*

On the one hand, the jump and flow sets in (10) can be suitably modified by selecting the desired LQ performance level $\hat{\mu}$. In particular, note that for $\hat{\mu} = \mu$ one recovers the same LQ performance as the continuous-time feedback guaranteeing (12). However, one may increase $\hat{\mu}$ and trade in some performance because this leads to strictly larger flow sets and strictly smaller jump sets. Furthermore, it is expected that enlarging the flow set and reducing the jump set leads to less jumps in solutions starting from the same initial conditions. It yields smaller average sampling rate, which is desirable from an event-triggered viewpoint. While this observation is only qualitative, its advantages are readily illustrated through the numerical results of Section 4.

On the other hand, the shape of the jump and flow sets in (10) heavily relies on the properties of the Lyapunov function $V(x) = x^\top Px$ that are established in Theorem 1. For example, an advantageous feature of these sets is given by the fact that whenever the continuous-time feedback of Theorem 1 would lead to a control input that remains saturated for some time interval (this is the case, for example, during the initial transient of a trajectory that starts far from the attractor), the event-triggered solution (4), (10) does not experience any jump. This fact can be seen by noticing that the flow set $\mathcal{F}_E$ in (10a) is defined as the set where the Lyapunov-like function $V(x) = x^\top Px$ experiences a suitable decrease, as established in Theorem 1. Since for all such responses the plant input remains constant also for the continuous-time solution, then the event-triggered solution remains in the flow set without triggering any sampling. This aspect is well illustrated by some of the simulations reported in Section 4.

3.5 *Existence of Zeno phenomenon*

The hybrid system exhibits Zeno solutions in the attractor (even though it also admits solutions that never jump and forever flow). Actually, both jump and flow sets are closed, which causes a nonempty intersection (including the attractor) that could be avoided by picking a jump set that is not closed. Selecting closed jump and flow sets however ensures well-posedness of the hybrid dynamics, as illustrated in [11, Ch. 4 and 6], which, among other things, enables the use of La Salle's invariance principle, and implies robustness of asymptotic stability. Closed flow and jump sets also allow capturing, in the set of the (non-necessarily unique) solutions to the well-posed dynamics, any possible limiting solution produced by vanishing perturbations affecting the nominal dynamics. This type of "rich" behavior within the attractor is a common feature in event-triggered designs and has been already observed in [23, §IV.B and IV.C]. If wanting to enforce some kind of semiglobal practical dwell-time property, one may consider the possibility of modifying the triggering laws as suggested in [23, §IV.C] to obtain practical stability results.

4 Simulation example

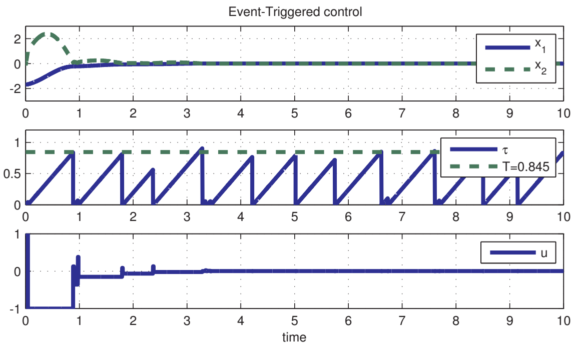
Consider plant (1) as the mass-spring system with destabilizing friction studied in [40, Example 7.2.6], corresponding to

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -f/m \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \qquad (13)$$

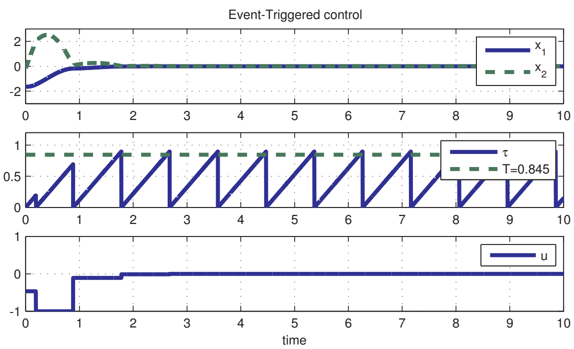
where $k = 1$, $m = 0.1$, $f = -0.01$ and $u_0 = 1$. The cost function (3) is selected with $Q = \left[\begin{smallmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0.5 \end{smallmatrix}\right]$. The plant is exponentially unstable. Therefore it is only possible to design a state feedback gain K that locally exponentially stabilizes the origin.

We aim at presenting first the influence of parameter T on the size of the estimated domain of attraction. Second we show the effects of parameter $\hat{\mu}$ on the number of control updates required by the event-triggered controller. Finally, we present a comparison between our event-triggered control and a periodic sampled-data implementation.

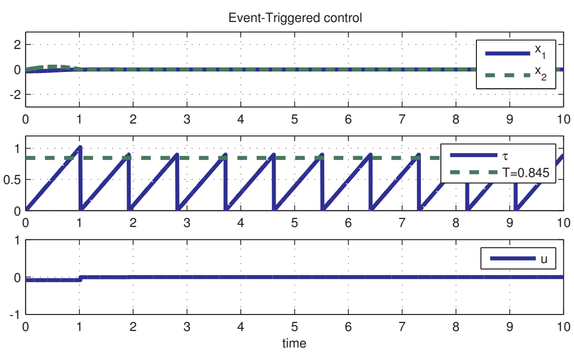
Influence of T on the event-triggered algorithm: Table 1 summarizes the result of the LMI optimization



(a) $T = 0$: Number of control updates: 37.



(b) $T = 0.2$: Number of control updates: 13



(c) $T = 0.845$: Number of control updates: 11.

Fig. 1. Simulation results representing the state x (top), the timer τ (middle), and the control inputs u (bottom), the three associated gains being reported in Table 1.

problem

$$\max_{W,X,Y,S,\alpha} \alpha, \text{ subject to (7), (8) and (9),} \qquad (14)$$

with $\mu = 5$ for several values of T. This optimization problem aims at maximizing the size of the estimate of the basin of attraction. Note that selecting $T = 0$ refers to optimization problem (14) only considering the continuous-time LMI constraints (7) and (8) as presented in [30]. Table 1 reports the Lyapunov matrix $P = W^{-1}$, the controller gain K, the size of the estimate of the domain of attraction represented by α and the average number N of updates resulting from 80 simulations

T	P	K	α	N
0	$\begin{bmatrix} 0.3481 & 0.0275 \\ 0.0275 & 0.0661 \end{bmatrix}$	$\begin{bmatrix} -3.7412 & -8.7110 \end{bmatrix}$	1.688	49.6
0.1	$\begin{bmatrix} 0.3491 & 0.4517 \\ 0.4517 & 0.0669 \end{bmatrix}$	$\begin{bmatrix} 0.1855 & -0.5161 \end{bmatrix}$	1.685	30.7
0.2	$\begin{bmatrix} 0.3624 & 0.0328 \\ 0.0328 & 0.0699 \end{bmatrix}$	$\begin{bmatrix} 0.3437 & -0.5477 \end{bmatrix}$	1.652	13.0
0.4	$\begin{bmatrix} 0.6487 & 0.0685 \\ 0.0685 & 0.1168 \end{bmatrix}$	$\begin{bmatrix} 0.3966 & -0.2457 \end{bmatrix}$	1.233	13.2
0.6	$\begin{bmatrix} 0.9203 & 0.1364 \\ 0.1364 & 0.2090 \end{bmatrix}$	$\begin{bmatrix} 0.4874 & -0.1373 \end{bmatrix}$	1.028	12.8
0.84	$\begin{bmatrix} 41.614 & 4.503 \\ 4.503 & 10.136 \end{bmatrix}$	$\begin{bmatrix} 0.5576 & -0.0588 \end{bmatrix}$	0.153	11.8

Table 1
Solutions to optimization problem (14) under the LMI constraints (7), (8) and (9), and average number over 80 different initial conditions of control updates in simulations of $10s$ with the corresponding event-triggered algorithm of Theorem 1.

of $10s$, with initial conditions x_0 satisfying $x_0^\top P x_0 = 1$ and with $\hat{\mu} = \mu = 5$. Table 1 shows that increasing parameter T in the LMI formulation of Theorem 1 leads to increasingly smaller values of α (namely the size of the estimate of the domain of attraction). Increasing T leads to increasingly smaller values of α (next to last column of Table 1). On the other hand, the benefits of introducing the sampled-data criteria appear when implementing the event-triggered algorithm, since increasing T leads to a notable reduction of the number of updates during the considered simulations (last column of Table 1). The maximal value of T such that the conditions of Theorem 1 are satisfied is 0.845. Indeed, we can see on Table 1 that the size α of the estimate of the domain of attraction becomes small.

Some simulations are depicted in Figure 1 for some values of T. The evolution of the state x (top traces) shows that the performance is better when T is zero (block (a)), at the price of a larger number of control updates as visible from the evolution of a timer [1] (middle traces). Moreover, looking at the control input u (lower traces), one realizes that even if all the initial conditions are chosen at the boundary of the estimate of the domain of attraction (i.e. $x_0^\top P x_0 = 1$), the control input u obtained with $T = 0.845$ (case (c)) never reaches the saturation level. This is a consequence of the reduction of the domain of attraction when T increases, as mentioned earlier when commenting the results of Table 1. Finally, Figures 1(b) and (c) depict the fact that the sampling eventually becomes pseudoperiodic, as noticed in [7]. Note that in this pseudoperiodic regime, its pseudoperiod is slightly larger than $T = 0.845$, corresponding to the largest value of T solving the condition of Theorem 1.

Influence of $\hat{\mu}$: Figure 2 shows the average number of updates resulting from simulations of $10s$ of the event-triggered algorithm for 80 initial conditions satisfying

[1] In order to show the instants where the control law is updated, a timer has been incorporated to system (4). It consists of a variable τ, whose dynamics is governed by $\dot{\tau} = 1$ along flow and $\tau^+ = 0$ across jumps.

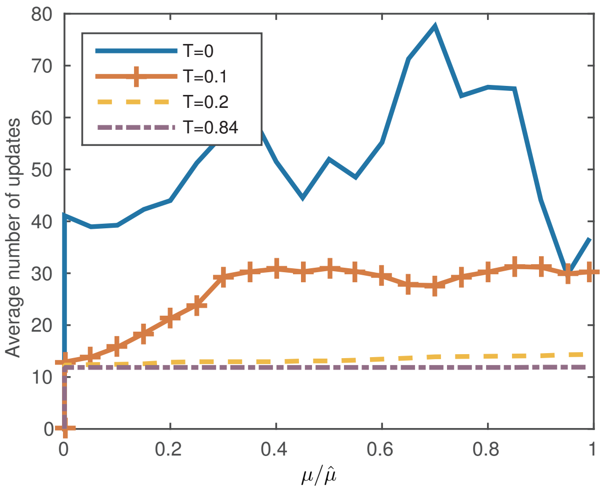


Fig. 2. Average number of updates over 80 simulations as a function of the degree of optimality $\mu/\hat{\mu}$ and for several values of the parameter T for system (13).

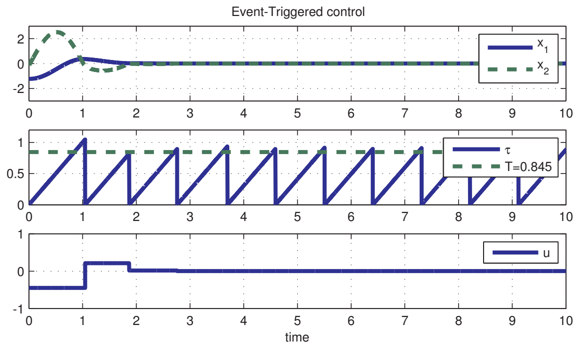
$x_0^\top P x_0 = 1$, for several values of the parameter T and several levels of (normalized) performance $\mu/\hat{\mu}$. It is shown that increasing the sampling period T reduces notably the number of updates. Moreover, for larger selections of T, the choice of the performance level $\hat{\mu}$ does not seem to significantly affect the number of updates in these simulations. Nevertheless, one can see that for large values of T, the influence of $\mu/\hat{\mu}$ becomes more regular and monotone.

Event-triggered vs. periodic samplings: As a final illustration, we compare our event-triggered algorithm with a periodic sampling implementation of period T_p. To do so, we select the maximal value of T solving the conditions of Theorem 1, i.e. $T_p = 0.845$, and we obtain a controller resulting from the LMI optimization problem

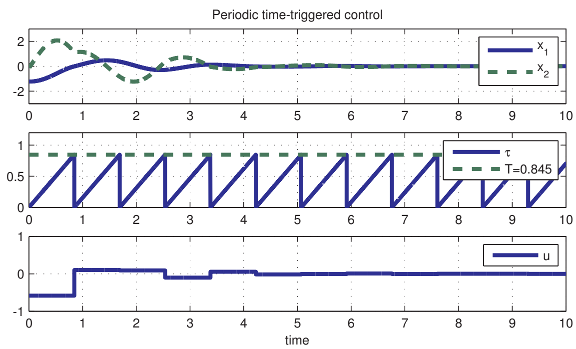
$$\max_{W,X,Y,S,\alpha} \alpha, \text{ subject to (7) and (9)}, \tag{15}$$

with $\mu = 5$, where (7) and (9) correspond to a stabilization criterion for saturated discrete-time systems. The solution of this problem gives a controller gain $K = [0.4661\ 0.0981]$ and $\alpha = 1.222$. Then, in order to provide a fair comparison, we select our event-triggered algorithm with $T = 0.4$ and $\hat{\mu} = \mu = 5$ leading to a similar size of the estimated domain of attraction as reported in Table 1.

Figure 3 shows the evolution of state x, timer τ and control input u with the same initial conditions. We see that the state of the system with the event-triggered control converges faster than the system with the periodic sampling, where oscillations notably affect the convergence. Note that it is possible to find a solution to problem (15) for larger values of T_p. This would lead to a reduction of the number of control updates but at the price of a performance degradation.



(a) $T = 0.4$.



(b) $T_p = 0.845$.

Fig. 3. Simulation results representing x (top), the timer τ (middle), and the control inputs u (bottom) for a periodic sampled-data control with a period $T_p = 0.845$ and the event-triggered control (4) provided with $T = 0.4$.

5 Proofs of the main results

Proof of Theorem 1. Define the function ζ as

$$(x,s) \mapsto \zeta(x,s) = 2x^\top P(Ax + Bs).$$

According to (10c), consider function $x \mapsto V(x) = x^\top P x$ and along the flow dynamics of (4), we have

$$\begin{aligned} \dot{V}(x,s) &:= \langle \nabla V(x), Ax + Bs \rangle = \zeta(x.s) \\ &= \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]^\top \Pi_{\hat{\mu}} \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] - \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]^\top \tfrac{Q}{\hat{\mu}} \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]. \end{aligned} \tag{16}$$

As a consequence, and due to the definition of the flow set in (10a), which implies $\left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]^\top \Pi_{\hat{\mu}} \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] \le 0$, for all $(x,s) \in \mathcal{F}_E$, we have:

$$\begin{aligned} \dot{V}(x,s) &\le -\left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]^\top (Q/\hat{\mu}) \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] \\ &\le -\varepsilon \left|\left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]\right|^2, \quad \forall \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] \in \mathcal{F}_E, \end{aligned} \tag{17a}$$

where $\varepsilon = \lambda_{min}(Q/\hat{\mu}) > 0$. Moreover, across jumps one trivially has

$$V(x^+) - V(x) = 0, \quad \forall \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] \in \mathcal{J}_E, \tag{17b}$$

because $x^+ = x$.

To show local asymptotic stability, we focus on the set $\mathcal{E}(P) = \{x \in \mathbb{R}^n : \; x^\top P x \leq 1\}$ that satisfies $\mathcal{E}(P) \supset \mathcal{B}(\alpha)$, because of the LMI on the right side of (7). In particular, from (17) we have (strong) forward invariance of $\mathcal{E}(P) \times \mathcal{U}_0$. Let us make the following change of variables in (7): $W = P^{-1}$, $S = U^{-1}$, $Y = KP^{-1}$, $X = HP^{-1}$. Then, pre- and post-multiplying Ψ defined in (5) by diag$\Big(P,\ U,\ I,\ I\Big)$, one gets

$$\tilde{\Psi} = \begin{bmatrix} \begin{bmatrix} \mathrm{He}(P(A+BK)) & * \\ U(K+H)-B^\top P & -2U \end{bmatrix} & * \\ \begin{bmatrix} I & 0 \\ K & -I \end{bmatrix} & -\mu Q^{-1} \end{bmatrix}.$$

By applying a Schur complement to $\tilde{\Psi}$, yielding a matrix $\tilde{Y}$ whose size is the one of the upper left block of $\tilde{\Psi}$, we develop $\left[\begin{smallmatrix} x^+ \\ s^+ \end{smallmatrix}\right]^\top \tilde{Y} \left[\begin{smallmatrix} x^+ \\ s^+ \end{smallmatrix}\right]$ noticing that $x^+ = x$ and $s^+ = \mathrm{sat}(Kx) = Kx^+ - \mathrm{dz}(Kx^+)$ (where $\mathrm{dz}(w) = w - \mathrm{sat}(w)$ is the deadzone function). Then, it follows that (8) implies, for all (x, q),

$$\zeta(x^+, s^+) + 2q^\top U((K+H)x^+ - q) + \frac{1}{\mu}\psi(x^+, s^+) \leq 0, \tag{18}$$

where $q = \mathrm{dz}(Kx^+)$. Using well-known generalized sector condition approaches (see, e.g., [32, Lemma 1.6]), and exploiting the forward invariance of $\mathcal{E}(P)\times\mathcal{U}_0$, we obtain that the left constraints in (7) imply $q^\top U((K+H)x^+ - q) \geq 0$ for all $\left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] \in (\mathcal{E}(P) \times \mathcal{U}_0)$, which may be exploited together with (18) to obtain for all $\left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right] \in (\mathcal{E}(P) \times \mathcal{U}_0)$,

$$\left[\begin{smallmatrix} x^+ \\ s^+ \end{smallmatrix}\right]^\top \Pi_{\hat{\mu}} \left[\begin{smallmatrix} x^+ \\ s^+ \end{smallmatrix}\right] + \left(\frac{1}{\mu} - \frac{1}{\hat{\mu}}\right)\psi(x^+, s^+) < 0. \tag{19}$$

This, together with $\hat{\mu} \geq \mu$ and positive definiteness of ψ implies that (x^+, s^+) is in the interior of the flow set where some flow will necessarily occur, associated with a strict decrease of V as established in (17a). Summarizing, either $x = 0$ (namely $(x, s) \in \mathcal{A}$) or the solution has to flow after each jump. Then, no complete discrete solution (namely a complete solution that never flows) exists outside $\mathcal{A}$ and asymptotic stability follows from item (ii) of [11, Cor. 8.9]. Moreover, the solutions do not escape in finite time, implying forward completeness of maximal solutions thanks to [11, Prop. 6.10]. Finally, due to homogeneity in a neighborhood of the attractor, and compactness of $\mathcal{E}(P) \times \mathcal{U}_0$, exponential stability follows from [33, Prop. 1].

Let us now prove that at least one solution has unbounded domain in the ordinary time direction. First recall that the set $\mathcal{E}(P) \times \mathcal{U}_0$ is forward invariant from (17). Then notice that in a small neighborhood of the attractor we have $\mathrm{sat}(Kx) = Kx$ so that the hybrid dynamics is homogeneous. This means that there exists a small enough η such that in the set $\mathcal{E}(P/\eta) = \{x \in \mathbb{R}^n : \; x^\top P x \leq \eta\}$ all nontrivial solutions are scaled versions of the solutions starting on its boundary $\{x \in \mathbb{R}^n : \; x^\top P x = \eta\}$. Consider now any solution jumping from the compact set $(\overline{\mathcal{E}(P) \setminus \mathcal{E}(P/\eta)} \times \mathcal{U}_0) \cap \mathcal{J}_E$ and notice that from (19) that solution has to flow for some nonzero time (because it jumps to the interior of the flow set). Then all solutions jumping from this compact set will flow for some uniform minimum time τ_{min}. From homogeneity, this property is enjoyed by any nonzero solution jumping from the whole set $(\mathcal{E}(P) \times \mathcal{U}_0) \cap \mathcal{J}_E \setminus \{0\}$ which ensures a minimum dwell time for all such solutions, therefore unboundedness of their domain in the ordinary time direction. Since the attractor belongs to the flow set and it is an equilibrium for the flow dynamics, then there exists a solution with unbounded domain in the ordinary time direction also from $\{0\}$.

Let us now prove the property of the cost function. Given any solution x, for each $j \in \mathrm{dom}_j(x)$, we know that the solution flows in the open time interval (t_j, t_{j+1}) (which could be empty and where it could be that $t_{j+1} = \infty$). Then for all (t, j) in such a "flowing" ordinary time interval, from $\left[\begin{smallmatrix} x(t,j) \\ s(t,j) \end{smallmatrix}\right] \in \mathcal{F}_E$ we may rearrange (16) as:

$$\dot{V}(x(t,j), s(t,j)) + \frac{1}{\hat{\mu}}\psi(x(t,j), s(t,j)) \leq 0,$$

which can be integrated from t_j to t_{j+1} to get

$$\begin{aligned} &V(x(t_{j+1}, j)) - V(x(t_j, j)) \\ &\quad + \tfrac{1}{\hat{\mu}} \textstyle\int_{t_j}^{t_{j+1}} \psi(x(t,j), s(t,j))dt \leq 0. \end{aligned} \tag{20}$$

Since $x^+ = x$ across jumps, for all $j \in \mathrm{dom}_j(x)$ satisfying $j \geq 1$ one has $V(x(t_{j+1}, j+1)) = V(x(t_{j+1}, j))$ (see also (17b)). Then, also recalling that $t_0 = 0$, we can sum the inequalities (20) for all $j \in \mathrm{dom}(x)$ to get, also using positive definiteness of V,

$$\begin{aligned} &\frac{1}{\hat{\mu}} \sum_{j \in \mathrm{dom}_j(x)} \int_{t_j}^{t_{j+1}} \psi(x(t,j), s(t,j))dt \\ &\qquad \leq V(x(0,0)) = x(0,0)^T P x(0,0) \leq 1, \end{aligned}$$

as to be proven. $\diamondsuit$

Proof of Proposition 1. Applying twice the Schur complement to the LMI (9) leads to

$$\mathcal{I} := \begin{bmatrix} -W & * \\ T(Y+X) & -2TS \end{bmatrix}$$

$$+\begin{bmatrix} W & Y^\top \\ 0 & -S \end{bmatrix} M^\top(T) C^\top W^{-1} C M(T) \begin{bmatrix} W & 0 \\ Y & -S \end{bmatrix}$$
$$+\begin{bmatrix} W & Y^\top \\ 0 & -S \end{bmatrix} \bar{Q}(T)/\mu \begin{bmatrix} W & 0 \\ Y & -S \end{bmatrix}.$$

From the definitions of $M(T)$ and $\bar{Q}(T)$, it holds

$$\begin{aligned} CM(T)\begin{bmatrix} W & 0 \\ Y & -S \end{bmatrix} &= \begin{bmatrix} W & 0 \end{bmatrix} + T\begin{bmatrix} AW+BY & -BS \end{bmatrix} \\ &\quad + o(T) \\ \bar{Q}(T) = TQ + o(T), \end{aligned}$$

where the notation $o(T)$ represents some quantities, which are negligible for some sufficiently small positive values of T. Then, by re-injecting this expression into $\mathcal{I}$, some calculations show that

$$\begin{aligned} \mathcal{I} = T\Bigg(&\begin{bmatrix} \mathrm{He}(AW+BY) & * \\ Y+X-SB^\top & -2S \end{bmatrix} \\ &+ \begin{bmatrix} W & Y^\top \\ 0 & -S \end{bmatrix} Q/\mu \begin{bmatrix} W & 0 \\ Y & -S \end{bmatrix}\Bigg) + o(T). \end{aligned}$$

Applying the Schur complement, the continuous-time stability conditions (7), (8) are retrieved. Then, if there exist matrices W, X, Y, S solutions to (7), (8), there exists a sufficiently small positive T such that matrix $\mathcal{I}$ is negative definite, which implies that the same matrices are a solution to (9). ♢

6 Conclusion

Focusing on linear plants we proposed the simultaneous design of the state feedback law and the event-triggering conditions ensuring local exponential stability and LQ performance in the presence of plant input saturation and of a communication channel between the controller output and the saturated plant input. The design of the event-triggered control is composed by two conditions derived from Lyapunov-based techniques in a hybrid framework. The first condition provides a solution to the event-triggered control co-design for LQ stabilization. The second one constrains the co-design among all possible solutions of the first condition thanks to a tunable parameter. The proposed event-triggered algorithm consists of updating the saturated plant input based on conditions involving the closed-loop state, while an estimate of the domain of attraction is provided. Moreover, the trade-off relying the optimality level, the size of the estimate of the basin of attraction and the reduction of the amount of transmissions has been highlighted.

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