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# Revisiting $SU(N)$ integrals

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## Abstract

In this note, I revisit integrals over  $SU(N)$  of the form  $\int DU U_{i_1 j_1} \cdots U_{i_p j_p} U_{k_1 l_1}^\dagger \cdots U_{k_n l_n}^\dagger$ . While the case  $p = n$  is well known, it seems that explicit expressions for  $p = n + N$  had not appeared in the literature. Similarities and differences, in particular in the large  $N$  limit, between the two cases are discussed.

# 1 Introduction and results

In this note, we consider the  $SU(N)$  integrals

$$\int DU U_{i_1 j_1} \cdots U_{i_p j_p} U_{k_1 l_1}^\dagger \cdots U_{k_n l_n}^\dagger, \quad (1)$$

with  $DU$  the normalized Haar measure, or their generating functions

$$Z_{p,n}(J, K) = \int DU (\text{tr}(KU))^p (\text{tr}(JU^\dagger))^n, \quad (2)$$

where  $J$  and  $K$  are arbitrary  $N \times N$  matrices.

Such integrals, mainly over the group  $U(N)$ , have been the object of numerous publications in the past, in the context of lattice gauge theories [1, 2, 3, 4], or in the large  $N$  limit [5, 6], or for their connections with combinatorics [7, 8]. Integrals over  $SU(N)$ , however, seem to have received less attention, see, however, [2, 9]. Recent work by Rossi and Veneziano [10] has prompted this new investigation.

By  $\mathbb{Z}_N$  invariance, it is clear that the above integrals vanish if

$$p - n \neq 0 \pmod{N}. \quad (3)$$

From a representation theoretic point of view, the number of independent terms in (2), (i.e., of independent tensors with the right symmetries in (1)), is given by the number of invariants in  $(N)^{\otimes p} \otimes (\bar{N})^{\otimes n}$ , where  $(N)$  and  $(\bar{N})$  denotes the fundamental  $N$ -dimensional representation of  $SU(N)$  and its complex conjugate.

$Z_{n,n}(J, K)$  is a well known function of (traces of powers of)  $JK$ , (“Weingarten’s function” [1]), at least for  $n < N$ , and one may even collect all  $Z_{n,n}$ ’s into

$$Z_W(J, K) = \sum_{n \geq 0} \frac{\beta^{2n}}{(n!)^2} Z_{n,n} = \int DU e^{\beta \text{tr}(UK + U^\dagger J)}. \quad (4)$$

For the convenience of the reader, a certain number of known results on these integrals and their generating function are recalled in the Appendix. This case  $n = p$  in eqs. (1), (2) will be referred to as “the ordinary case”.

Consider now  $Z_{n+N,n}$ . This kind of integral appears in the strong coupling calculation of the “book observable” of ref [10]. First, for  $n = 0$ ,  $\int DU U_{i_1 j_1} \cdots U_{i_N j_N}$  is given by the only invariant in  $\otimes^N(N)$ , namely by the totally antisymmetric tensor product, (i.e., the determinant of  $U$ , equal to 1 in  $SU(N)$ ). Hence

$$\int DU U_{i_1 j_1} \cdots U_{i_N j_N} = A \epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N}$$

with a constant  $A$  determined by contraction with  $\epsilon_{i_1 \cdots i_N}$  and use of  $\epsilon_{i_1 \cdots i_N} U_{i_1 j_1} \cdots U_{i_N j_N} = \epsilon_{j_1 \cdots j_N} \det U$ , hence  $\int DU \det U = \int DU = 1 = A \epsilon_{j_1 \cdots j_N} \epsilon_{j_1 \cdots j_N} = AN!$ , and  $A = 1/N!$ . Thus

$$\int DU U_{i_1 j_1} \cdots U_{i_N j_N} = \frac{1}{N!} \epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N}. \quad (5)$$

Equivalently,

$$Z_{N,0}(J, K) = \det K. \quad (6)$$

More generally, for  $n < N$ ,  $Z_{N+n,n}(J, K)$  is the product of  $\det K$  by a polynomial of  $M := JK$ , invariant under  $M \mapsto VMV^\dagger$ , for any  $V \in SU(N)$ , and homogeneous of degree  $n$  in  $M$ , hence a polynomial that may be expanded on traces of powers of  $M$

$$Z_{N+n,n} = \int DU (\text{tr}(UK))^{N+n} (\text{tr}(U^\dagger J))^n = \det K \sum_{\alpha \vdash n} d_\alpha t_\alpha. \quad (7)$$

with a sum over partitions of  $n$  denoted  $\alpha = [1^{\alpha_1}, 2^{\alpha_2}, \dots, p^{\alpha_p}]$ ,  $\sum_q q \alpha_q = n$ , and with the notation

$$t_\alpha := \prod_q t_q^{\alpha_q}, \quad t_q := \text{tr}(JK)^q. \quad (8)$$

The coefficients  $d_\alpha$  are determined through recursion formulae resulting from a contraction of (1) with a Kronecker delta:

$$\delta_{jk} \frac{\partial^2}{\partial J_{lk} \partial K_{ji}} Z_{N+n,n} = (N+n)n \delta_{il} Z_{N+n-1,n-1}. \quad (9)$$

The first coefficients are readily determined

$$\begin{aligned} n=1 & & d_{[1]} &= 1 \\ n=2 & & d_{[2]} &= -\frac{1}{N} & d_{[1^2]} &= \frac{N+1}{N} \\ n=3 & & d_{[3]} &= \frac{4}{N(N-1)} & d_{[1,2]} &= -\frac{3(N+1)}{N(N-1)} & d_{[1^3]} &= \frac{(N+1)^2-2}{N(N-1)} \\ n=4 & & d_{[4]} &= -\frac{30}{N(N-1)(N-2)} & d_{[1,3]} &= \frac{8(2(N+1)^2-3)}{(N+1)N(N-1)(N-2)} & d_{[2^2]} &= \frac{3((N+1)^2+6)}{(N+1)N(N-1)(N-2)} \\ & & d_{[1^2,2]} &= -\frac{6((N+1)^2-4)}{N(N-1)(N-2)} & d_{[1^4]} &= \frac{(N+1)^4-8(N+1)^2+6}{(N+1)N(N-1)(N-2)} \\ & & \text{etc.} & & & & & \end{aligned} \quad (10)$$

These coefficients are in fact simply related to the analogous coefficients  $z_\alpha$  in the expansion of the ordinary generating function, see Appendix, eq. (A.3). If  $z_\alpha$ ,  $\alpha \vdash n$ , is written in the form

$$z_\alpha = \frac{P_\alpha(N)}{N^2(N^2-1)\dots(N^2-(n-1)^2)} \quad (11)$$

with  $P_\alpha(N)$  a polynomial of  $N$ , then

$$d_\alpha = \frac{P_\alpha(N+1)}{(N+1)\dots(N-(n-2))}. \quad (12)$$

This follows from the comparison between the two systems of recursion formulae, see below sect. 2, and may be verified on the first coefficients (10) and (A.5).

Note that the coefficients  $d_\alpha$  decrease more slowly than the  $z_\alpha$ , for fixed  $n$ , as  $N$  grows. This behavior has consequences on the large  $N$  limit of the generating function  $Z_D$  defined by

$$Z_D = \sum_{n \geq 0} \frac{\beta^n}{n!} Z_{N+n,n} = \int DU e^{\beta \text{tr} UK \text{tr} U^\dagger J} (\text{tr} UK)^N =: \tilde{Z}_D \det K. \quad (13)$$

Note also the different summations in (4) and (13): here  $\beta$  is a homogeneity parameter for the eigenvalues of  $M = JK$  while in (4), it is  $\beta^2$  that plays that role. In both cases, however, we take  $\beta$  of order  $N$  and set  $\beta = N\tilde{\beta}$ . Then while in the ordinary case, a non trivial limit is obtained by taking the traces  $t_n$  also of order  $N$ , resulting in an exponentiation  $Z_W = \exp N^2 W_W$ , here one finds that the traces  $t_n$  have to be kept of order 1 and then  $\tilde{Z}_D = \exp N W_D$ , with  $W_D$  a non trivial function of the  $t_n$ 's.

Indeed one finds for the first terms

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \frac{Z_D}{\det K} = \tilde{\beta} t_1 + \frac{\tilde{\beta}^2}{2} (t_1^2 - t_2) + \frac{\tilde{\beta}^3}{3} (t_1^3 - 3t_1 t_2 + 2t_3) + \frac{\tilde{\beta}^4}{4} (t_1^4 - 6t_2 t_1^2 + 2t_2^2 + 8t_3 t_1 - 5t_4) + \dots \quad (14)$$

which is just the beginning of a simple formula in the large  $N$  limit:

$$W_D := \lim_{N \rightarrow \infty} \frac{1}{N} \log \frac{Z_D}{\det K} = \sum_{n \geq 1} \tilde{\beta}^n \sum_{\alpha \vdash n} (-1)^{n-\sum \alpha_q} \frac{(n-1)!}{(n-\sum \alpha_q+1)!} \prod_p \frac{(\text{Cat}(p-1) t_p)^{\alpha_p}}{\alpha_p!} \quad (15)$$

in terms of the Catalan numbers  $\text{Cat}(m) = \frac{(2m)!}{m!(m+1)!}$ . This will be proved below in sect. 3.

## 2 The recursion relations

Let us return to (9) and look at the action of the differential operator  $\frac{\partial}{\partial K_{ji}}$  on a typical term  $t_\alpha \det K$  of (7). Making use of the identities

$$\frac{\partial}{\partial K_{ji}} \det[K] = (\text{Cof}K)_{ji} \quad (16)$$

$$(\text{Cof}K)_{ji}(KX)_{jl} = (\det K)X_{il} \quad (17)$$

one may write for each term in the expansion of  $Z_{N+n,n}$

$$\begin{aligned} & \frac{\partial^2}{\partial K_{ji} \partial J_{lj}} (t_\alpha \det K) = \frac{\partial}{\partial K_{ji}} \sum_{q=1}^p q \alpha_q (K(JK)^{q-1})_{jl} t_{\widehat{\alpha}_{(q)}} \det K \\ &= \sum_{q=1}^p q \alpha_q \left( (N+1)(JK)_{il}^{q-1} + \sum_{s=1}^{q-1} t_s (JK)_{il}^{q-s-1} \right) t_{\widehat{\alpha}_{(q)}} \det K \\ &+ \sum_{q=1}^p q^2 \alpha_q (\alpha_q - 1) (JK)_{il}^{2q-1} t_{\widehat{\alpha}_{(q)}} \det K + 2 \sum_{1 \leq q < r \leq p} q r \alpha_q \alpha_r (JK)_{il}^{q+r-1} t_{\widetilde{\alpha}_{(q,r)}} \det K \end{aligned} \quad (18)$$

with  $\widehat{\alpha}_{(q)} := (1^{\alpha_1} \dots q^{\alpha_q-1} \dots p^{\alpha_p})$ ,  $\widehat{\alpha}_{(q)} := (1^{\alpha_1} \dots q^{\alpha_q-2} \dots p^{\alpha_p})$ , and  $\widetilde{\alpha}_{(q,r)} := (1^{\alpha_1} \dots q^{\alpha_q-1} \dots r^{\alpha_r-1} \dots p^{\alpha_p})$ .

In this way, (9) yields an (overdetermined) system of relations between the coefficients  $d_\alpha$  at ranks  $n$  and  $n-1$ , namely

$$\begin{aligned} & \sum_{\alpha \vdash n} d_\alpha \left\{ \sum_{q=1}^p q \alpha_q \left( (N+1)(JK)_{il}^{q-1} + \sum_{s=1}^{q-1} t_s (JK)_{il}^{q-s-1} \right) t_{\widehat{\alpha}_{(q)}} \det(K) \right. \\ &+ \left. \sum_{q=1}^p q^2 \alpha_q (\alpha_q - 1) (JK)_{il}^{2q-1} t_{\widehat{\alpha}_{(q)}} \det(K) + 2 \sum_{1 \leq q < r \leq p} q r \alpha_q \alpha_r (JK)_{il}^{q+r-1} t_{\widetilde{\alpha}_{(q,r)}} \det(K) \right\} \\ &= n(N+n) \delta_{il} \sum_{\alpha' \vdash n-1} d_{\alpha'} t_{\alpha'} \det(K). \end{aligned} \quad (19)$$

Compare these equations with those satisfied by the coefficients  $z_\alpha$  in the similar expansion of  $Z_{n,n}(J, K)$  in the ordinary case, see (A.4). Their structure is the same, except for changes on the terms of (19) marked in red. In the linear system on the  $d_\alpha$ , the parameter  $N$  in the lhs of (A.4) has been changed into  $N+1$  while the right hand side is multiplied by  $(N+n)$ . As a result the solutions of the  $d_\alpha$  linear system are obtained from those of the  $z_\alpha$  one by

$$\text{for a given } n \quad d_\alpha = (N+n)(N+n-1) \dots (N+1) \left( z_\alpha |_{N \rightarrow N+1} \right).$$

If  $z_\alpha$  is written as in (11), it follows that  $d_\alpha$  has the form (12), qed.

In particular, this relation implies that  $z_\alpha$  and  $d_\alpha$  have the same overall sign, namely  $(-1)^{\#\text{cy}(\alpha)+n}$ .

## 3 Exponentiation

The differential equation (9) carries over to the generating function  $Z_D$  of (13) in the form

$$\delta_{jk} \frac{\partial^2}{\partial J_{lk} \partial K_{ji}} Z_D = \delta_{il} \left( (N+1)\beta + \beta^2 \frac{\partial}{\partial \beta} \right) Z_D. \quad (20)$$

We write  $Z_D = \det K \tilde{Z}_D$ , in which  $\tilde{Z}_D$  is a function of  $M := JK$  invariant under  $M \rightarrow VMV^\dagger$  for any  $V \in \text{SU}(N)$ , and we rewrite the differential equation as

$$\frac{\partial^2 \tilde{Z}_D}{\partial M_{lk} \partial M_{ji}} M_{jk} + (N+1) \frac{\partial \tilde{Z}_D}{\partial M_{li}} = \delta_{il} \left( (N+1)\beta + \beta^2 \frac{\partial}{\partial \beta} \right) \tilde{Z}_D. \quad (21)$$

This may be reexpressed as a differential equation wrt to the eigenvalues  $\lambda_i$  of  $M$  (generically  $M$  is diagonalizable). This is a standard procedure [5] with the result that for any  $i$ ,  $1 \leq i \leq N$

$$\lambda_i \frac{\partial^2 \tilde{Z}_D}{\partial \lambda_i^2} + (N+1) \frac{\partial \tilde{Z}_D}{\partial \lambda_i} + \sum_{j \neq i} \lambda_j \frac{\frac{\partial \tilde{Z}_D}{\partial \lambda_i} - \frac{\partial \tilde{Z}_D}{\partial \lambda_j}}{\lambda_i - \lambda_j} = \left( (N+1)\beta + \beta^2 \frac{\partial}{\partial \beta} \right) \tilde{Z}_D. \quad (22)$$

Finally we write  $\tilde{Z}_D = \exp NW_D$ , which results in

$$\lambda_i \left( N \frac{\partial^2 \tilde{W}_D}{\partial \lambda_i^2} + N^2 \left( \frac{\partial W_D}{\partial \lambda_i} \right)^2 \right) + (N+1)N \frac{\partial W_D}{\partial \lambda_i} + N \sum_{j \neq i} \lambda_j \frac{\frac{\partial W_D}{\partial \lambda_i} - \frac{\partial W_D}{\partial \lambda_j}}{\lambda_i - \lambda_j} = (N+1)\beta + N\beta^2 \frac{\partial W_D}{\partial \beta}. \quad (23)$$

In the large  $N$  limit, we rescale  $\beta = N\tilde{\beta}$ , keeping all  $t_n = \sum_i \lambda_i^n$  of order 1, and after dropping the subdominant terms, we get for  $w_i := \frac{\partial W_D}{\partial \lambda_i}$  the equation

$$\lambda_i w_i^2 + w_i = \tilde{\beta} + \tilde{\beta}^2 \frac{\partial W_D}{\partial \tilde{\beta}}. \quad (24)$$

(Note that this is in contrast with the ‘‘ordinary case’’ where the term  $\sum_{j \neq i} \dots$  contributes in the large  $N$  limit [5].) Now  $\tilde{\beta}$  is just an homogeneity variable of the  $\lambda$ 's, and we may thus substitute  $\tilde{\beta} \frac{\partial W_D}{\partial \tilde{\beta}} = \sum_j \lambda_j \frac{\partial W_D}{\partial \lambda_j} = \sum_j \lambda_j w_j$ . The equation finally reduces to a system of *algebraic* equations for the  $w_i$ 's

$$\lambda_i w_i^2 + w_i = \tilde{\beta} \left( 1 + \sum_j \lambda_j w_j \right). \quad (25)$$

Assuming  $w := \sum_i \lambda_i w_i$  known, one finds

$$\begin{aligned} \lambda_i w_i^2 + w_i &= \tilde{\beta}(1+w) \\ w_i &= \frac{-1 + \sqrt{1 + 4\lambda_i \tilde{\beta}(1+w)}}{2\lambda_i} \end{aligned} \quad (26)$$

and  $1+w$  is thus the root of

$$1+w = 1 - \frac{N}{2} + \sum_i \frac{\sqrt{1 + 4\lambda_i \tilde{\beta}(1+w)}}{2}. \quad (27)$$

Equation (26) should be compared with that of the generating function of Catalan numbers  $\text{Cat}(m) = \frac{(2m)!}{m!(m+1)!}$ , namely  $C(t) = \sum_{n \geq 0} \text{Cat}(n)t^n$ ,  $tC^2(t) - C(t) + 1 = 0$ . We find that  $w_i = \tilde{\beta}(1+w)C(-\tilde{\beta}(1+w)\lambda_i)$ , hence

$$w_i = \tilde{\beta}(1+w) \sum_{n \geq 0} \text{Cat}(n) (-\tilde{\beta}(1+w)\lambda_i)^n \quad (28)$$

and

$$w = \sum_i \lambda_i w_i = - \sum_{n \geq 1} \text{Cat}(n-1) (-\tilde{\beta}(1+w))^n t_n. \quad (29)$$

Let  $f_0 = 1$ ,  $f_n := (-1)^{n-1} \text{Cat}(n-1) t_n$  for  $n \geq 1$ ,  $y := \tilde{\beta}(1+w)$ , then (29) reads  $y = \tilde{\beta} \sum_{n \geq 0} f_n y^n$ , whose solution is given by Lagrange formula

$$\begin{aligned} y = \tilde{\beta}(1+w) &= \sum_{n \geq 0} \frac{\tilde{\beta}^{n+1}}{(n+1)!} a_z^n \left( \sum_{m \geq 0} f_m z^m \right)^{n+1} \Big|_{z=0} \\ &= \tilde{\beta} + \sum_{n \geq 1} \frac{\tilde{\beta}^{n+1}}{(n+1)!} \sum_{\alpha \vdash n} \frac{(n+1)!}{(n+1 - \sum_q \alpha_q)! \prod_q \alpha_q!} n! f_\alpha \\ w = \sum_{n \geq 1} m_n \tilde{\beta}^n &= \sum_{n \geq 1} \tilde{\beta}^n \sum_{\alpha \vdash n} \frac{n!}{(n+1 - \sum_q \alpha_q)! \prod_q \alpha_q!} f_\alpha \end{aligned}$$

where the multinomial coefficient appears naturally in the expansion of the  $n+1$ -th power, and the  $n!$  results from the  $n$ -th derivative of  $z^n$ . Upon integration we get

$$W_D = \sum_{n \geq 1} \tilde{\beta}^n \sum_{\alpha \vdash n} (-1)^{n - \sum_q \alpha_q} \frac{(n-1)!}{(n+1 - \sum_q \alpha_q)! \prod_q \alpha_q!} \prod_p (\text{Cat}(p-1) t_p)^{\alpha_p}$$

which establishes (15).

Note the similarity of this calculation with the relation between ordinary moments  $m_n$  and planar cumulants  $f_n$  of a given distribution. The combinatorial or diagrammatical interpretation of (15) remains to be found.

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## Appendix

In this Appendix, we recall a certain number of results on the Weingarten’s functions  $Z_{n,n}(J, K)$  and their generating function  $Z_W(J, K)$  of (4). The  $Z_{n,n}$  are well known functions of (traces of powers of)  $JK$ , at least for  $n < N$ , see also [2, 3, 4]. For  $n \geq N$ , the traces are no longer independent, or equivalently the tensors  $\prod \delta_{i\pi(l)} \delta_{j\rho(k)}$ ,  $\pi, \rho \in S_n$ , are no longer independent, and we have to deal with a “pseudo-inverse” of the Gram matrix, see [7, 8].

Consider the integrals on  $U(N)$  (or  $SU(N)$ , this is irrelevant here, assuming  $n < N$ ) [1, 4, 7]

$$\begin{aligned} \int DU U_{i_1 j_1} \cdots U_{i_n j_n} U_{k_1 \ell_1}^\dagger \cdots U_{k_n \ell_n}^\dagger &= \sum_{\tau, \sigma \in S_n} C([\sigma]) \prod_{a=1}^n \delta_{i_a \ell_{\tau(a)}} \delta_{j_a k_{\sigma(a)}} \\ &= \sum_{\tau, \sigma \in S_n} \sum_{\substack{Y \text{ Young diagr.} \\ |Y|=n}} \frac{(\chi^{(\lambda)}(1))^2 \chi^{(\lambda)}([\sigma])}{n!^2 s_\lambda(I)} \prod_{q=1}^p \delta_{i_q \ell_{\tau(q)}} \delta_{j_q k_{\sigma(q)}} \end{aligned} \quad (\text{A.1})$$

where  $\chi^{(\lambda)}([\sigma])$  is the character of the symmetric group  $S_n$  associated with the Young diagram  $Y$ , (a function of the class  $[\sigma]$  of  $\sigma$ ); thus  $\chi^{(\lambda)}(1)$  is the dimension of that representation;  $s_\lambda(X)$  is the character of the linear group  $GL(N)$  associated with Young diagram  $Y$ , that is a Schur function when expressed in terms of the eigenvalues of  $X$ ;  $s_\lambda(I)$  is thus the dimension of that representation. Finally the coefficient  $C([\sigma])$  will be determined below.

Alternatively, in terms of generating functions with sources  $J$  and  $K$

$$Z_{n,n}(J, K) = \int DU (\text{tr } KU)^n (\text{tr } JU^\dagger)^n = \sum_{\alpha \vdash n} n! |\alpha| C([\alpha]) t_\alpha \quad (\text{A.2})$$

where  $|\alpha|$  is the cardinal of class  $[\alpha]$  in  $S_n$ , thus  $|\alpha| = \frac{n!}{\prod_p p^{\alpha_p} \alpha_p!}$ . In (A.2), the factor  $n!$  comes from the sum over  $\tau$  and the factor  $|\alpha|$  from that over the elements  $\sigma \in [\alpha]$ . Thus

$$Z_W(J, K) := \int DU \exp[\beta \text{tr}(KU + JU^\dagger)] = \sum_{n=0}^{\infty} \frac{\beta^{2n}}{n!} \sum_{\alpha \vdash n} z_\alpha t_\alpha \quad (\text{A.3})$$

with  $z_\alpha = |\alpha| C([\alpha])$ .

By the same argument as in sect. 2, the  $z_\alpha$  satisfy the linear system of recursion relations

$$\begin{aligned} \sum_{\alpha \vdash n} z_\alpha \left\{ \sum_{q=1}^p q \alpha_q \left( N(JK)_{il}^{q-1} + \sum_{s=1}^{q-1} t_s (JK)_{il}^{q-s-1} \right) t_{\widehat{\alpha}_{(q)}} \right. \\ \left. + \sum_{q=1}^p q^2 \alpha_q (\alpha_q - 1) (JK)_{il}^{2q-1} t_{\widehat{\alpha}_{(q)}} + 2 \sum_{1 \leq q < r \leq p} q r \alpha_q \alpha_r (JK)_{il}^{q+r-1} t_{\widehat{\alpha}_{(q,r)}} \right\} \\ = n \delta_{il} \sum_{\alpha' \vdash n-1} z_{\alpha'} t_{\alpha'} . \end{aligned} \quad (\text{A.4})$$



Explicitly, the first coefficients  $z_\alpha = |\alpha|C([\alpha])$  read

$$\begin{aligned}
n = 1 & & z_{[1]} &= \frac{1}{N} \\
n = 2 & & z_{[2]} &= -\frac{1}{(N^2-1)N}, \quad z_{[1^2]} = \frac{1}{(N^2-1)} \\
n = 3 & & z_{[3]} &= \frac{4}{(N^2-4)(N^2-1)N}, \quad z_{[1,2]} = -\frac{3}{(N^2-4)(N^2-1)}, \quad z_{[1^3]} = \frac{N^2-2}{(N^2-4)(N^2-1)N} \\
n = 4 & & z_{[4]} &= -\frac{30}{(N^2-9)(N^2-4)(N^2-1)N}, \quad z_{[1,3]} = \frac{8(2N^2-3)}{(N^2-9)(N^2-4)(N^2-1)N^2} \\
& & z_{[2^2]} &= \frac{3(N^2+6)}{(N^2-9)(N^2-4)(N^2-1)N^2}, \quad z_{[1^2,2]} = -\frac{6(N^2-4)}{(N^2-9)(N^2-4)(N^2-1)N}, \\
& & z_{[1^4]} &= \frac{N^4-8N^2+6}{(N^2-9)(N^2-4)(N^2-1)N^2} \\
& & \text{etc.} &
\end{aligned} \tag{A.5}$$

The overall sign of  $z_\alpha$  is  $(-1)^{\#\text{cycles}(\alpha)-n}$  and its large  $N$  behavior  $|z_\alpha| \sim N^{-2n+\#\text{cycles}(\alpha)}$ .

To study the large  $N$  limit, we take  $\beta = N\tilde{\beta}$  and  $t_p = N\tau_p$ , with  $\tilde{\beta}$  and  $\tau_p$  of order 1. Then Brézin and Gross [5] have shown that

$$W_W(JK) := \lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_W(J, K; N\tilde{\beta}) \tag{A.6}$$

exists and satisfies the coupled equations

$$W_W = \frac{2}{N} \sum_i (\tilde{\beta}^2 \lambda_i + c)^{\frac{1}{2}} - \frac{1}{2N^2} \sum_{i,j} \log[(\tilde{\beta}^2 \lambda_i + c)^{\frac{1}{2}} + (\tilde{\beta}^2 \lambda_j + c)^{\frac{1}{2}}] - c - \frac{3}{4} \tag{A.7}$$

$$\text{with } c = \begin{cases} \frac{1}{N} \sum_i (\tilde{\beta}^2 \lambda_i + c)^{-\frac{1}{2}} & \text{for } \frac{1}{N} \sum_i (\tilde{\beta}^2 \lambda_i)^{-\frac{1}{2}} \geq 2 \quad (\text{“strong coupling”}) \\ 0 & \text{for } \frac{1}{N} \sum_i (\tilde{\beta}^2 \lambda_i)^{-\frac{1}{2}} \leq 2 \quad (\text{“weak coupling”}) \end{cases} \tag{A.8}$$

The solution has two determinations, in a strong coupling and in a weak coupling phase. Here we are concerned with the strong coupling regime in which we may expand

$$W_W = \sum_{n \geq 1} \tilde{\beta}^{2n} \sum_{\alpha \vdash n} w_\alpha \tau_\alpha \tag{A.9}$$

with the first terms

$$W_W = \tilde{\beta} \tau_1 + \frac{\tilde{\beta}^2}{2} (\tau_1^2 - \tau_2) + \frac{2\tilde{\beta}^3}{3} (2\tau_1^3 - 3\tau_1 \tau_2 + \tau_3) + \frac{\tilde{\beta}^4}{4} (24\tau_1^4 - 48\tau_1^2 \tau_2 + 9\tau_2^2 + 20\tau_1 \tau_3 - 5\tau_4) + \dots \tag{A.10}$$

and the general term given in [6]

$$w_\alpha = (-1)^n \frac{(2n-3+\sum_q \alpha_q)!}{(2n)!} \prod_q \left( -\frac{(2q)!}{(q!)^2} \right)^{\alpha_q} \frac{1}{\alpha_q!}. \tag{A.11}$$