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# Information uncertainty related to marked random times and optimal investment

Ying Jiao\*    Idris Kharroubi†

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## Abstract

We study an optimal investment problem under default risk where related information such as loss or recovery at default is considered as an exogenous random mark added at default time. Two types of agents who have different levels of information are considered. We first make precise the insider's information flow by using the theory of enlargement of filtrations and then obtain explicit logarithmic utility maximization results to compare optimal wealth for the insider and the ordinary agent.

MSC: 60G20, 91G40, 93E20

Keywords : information uncertainty, marked random times, enlargement of filtrations, utility maximization.

## 1 Introduction

The optimization problem in presence of uncertainty on a random time is an important subject in finance and insurance, notably for risk and asset management when it concerns a default event or a catastrophe occurrence. Another related source of risk is the information associated to the random time concerning resulting payments, the price impact, the loss given default or the recovery rate etc. Measuring these random quantities is in general difficult since the relevant information on the underlying firm is often not accessible to investors on the market. For example, in the credit risk analysis, modelling the recovery rate is a subtle task (see e.g. Duffie and Singleton [12, Section 6], Bakshi et al. [4] and Guo et al. [16]). In this paper, we study the optimal investment problem with a random time and consider the information revealed at the random time as an exogenous factor of risk. We suppose that all investors on the market can observe the arrival of the random time such as the occurrence of a default event. However,

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for the associated information such as the recovery rate, there are two types of investors: the first one is an informed insider and the second one is an ordinary investor. For example, the insider has private information on the loss or recovery value of a distressed firm at the default time and the ordinary investor has to wait for the legitimate procedure to be finished to know the result. Both investors aim at maximizing the expected utility on the terminal wealth and each of them will determine the investment strategy based on the corresponding information set. Following Amendinger et al. [2, 3], we will compare the optimization results and deduce the additional gain of the insider.

Let the financial market be described by a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  equipped with a reference filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  which satisfies the usual conditions. In the literature, the theory of enlargements of filtrations provides essential tools for the modelling of different information flows. In general, the observation of a random time, in particular a default time, is modelled by the progressive enlargement of filtration, as proposed by Elliott et al. [13] and Bielecki and Rutkowski [5]. The knowledge of insider information is usually studied by using the initial enlargement of filtration as in [2, 3] and Grorud and Pontier [15]. In this paper, we suppose that the filtration  $\mathbb{F}$  represents the market information known by all investors including the default information. Let  $\tau$  be an  $\mathbb{F}$ -stopping time which represents the default time. The information flow associated to  $\tau$  is modelled by a random variable  $G$  on  $(\Omega, \mathcal{A})$  valued in a measurable space  $(E, \mathcal{E})$ . In the classic setting of insider information,  $G$  is added to  $\mathbb{F}$  at the initial time  $t = 0$ , while in our model, the information is added punctually at the random time  $\tau$ . Therefore, we need to specify the corresponding filtration which is a mixture of the initial and the progressive enlargements. Let the insider's filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  be a punctual enlargement of  $\mathbb{F}$  by adding the information of  $G$  at the random time  $\tau$ . In other words,  $\mathbb{G}$  is the smallest filtration which contains  $\mathbb{F}$  and such that the random variable  $G$  is  $\mathcal{G}_\tau$ -measurable. We shall make precise the adapted and predictable processes in the filtration  $\mathbb{G}$  in order to describe investment strategy and wealth processes. As usual, we suppose the density hypothesis of Jacod [17] that the  $\mathbb{F}$ -conditional law of  $G$  admits a density with respect to its probability law. By adapting arguments in Föllmer and Imkeller [14] and in [15], we deduce the insider martingale measure  $\mathbb{Q}$  which plays an important role in the study of (semi)martingale processes in the filtration  $\mathbb{G}$ . We give the decomposition formula of an  $\mathbb{F}$ -martingale as a semimartingale in  $\mathbb{G}$ , which gives a positive answer to the Jacod's (H')-hypothesis and allows us to characterize the  $\mathbb{G}$ -portfolio wealth processes as in [3].

In the optimization problem with random default times, it is often supposed that the random time satisfies the intensity hypothesis (e.g. Lim and Quenez [24] and Kharroubi et al. [23]) or the density hypothesis (e.g. Blanchet-Scalliet et al. [6], Jeanblanc et al. [19] and Jiao et al. [21]), so that it is a totally inaccessible stopping time in the market filtration. In particular, in [21], we consider marked random times where the random mark represents the loss at default and we suppose that the couple of default time and mark admits a conditional density. In this current paper, the random time  $\tau$  we consider does not necessarily satisfy the intensity nor the density hypothesis: it is a general stopping time in  $\mathbb{F}$  and may also contain a predictable part. We obtain the optimal strategy and wealth for the two types of investors with a logarithmic utility function

and deduce the additional gain due to the extra information. As a concrete case, we consider a hybrid default model similar as in Campi et al. [8] where the filtration  $\mathbb{F}$  is generated by a Brownian motion and a Poisson process, and the default time is the minimum of two random times: the first hitting time of a Brownian diffusion and the first jump time of the Poisson process and we compute the additional expected logarithmic utility wealth.

The rest of the paper is organized as following. We model in Section 2 the filtration which represents the default time together with the random mark and we study its theoretical properties. Section 3 focuses on the logarithmic utility optimization problem for the insider and compares the result with the case for ordinary investor. In Section 4, we present the optimization results for an explicit hybrid default model. Section 5 concludes the paper.

## 2 Model framework

In this section, we present our model setup. In particular, we study the enlarged filtration including the random mark which is a mixture of the initial and the progressive enlargements of filtrations.

### 2.1 The enlarged filtration and martingale processes

Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space equipped with a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  which satisfies the usual conditions and  $\tau$  be an  $\mathbb{F}$ -stopping time. Let  $G$  be a random variable valued in a measurable space  $(E, \mathcal{E})$  and  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  be the smallest filtration containing  $\mathbb{F}$  such that  $G$  is  $\mathcal{G}_\tau$ -measurable. By definition, one has

$$\forall t \in \mathbb{R}_+, \quad \mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(\{A \cap \{\tau \leq s\} \mid A \in \sigma(G), s \leq t\}\right). \quad (2.1)$$

In particular, similar as in Jeulin [20] (see also Callegaro et al. [7]), a stochastic process  $Z$  is  $\mathbb{G}$ -adapted if and only if it can be written in the form

$$Z_t = \mathbb{1}_{\{\tau > t\}} Y_t + \mathbb{1}_{\{\tau \leq t\}} Y_t(G), \quad t \geq 0 \quad (2.2)$$

where  $Y$  is an  $\mathbb{F}$ -adapted process and  $Y(\cdot)$  is an  $\mathbb{F} \otimes \mathcal{E}$ -adapted process on  $\Omega \times E$ , where  $\mathbb{F} \otimes \mathcal{E}$  denotes the filtration  $(\mathcal{F}_t \otimes \mathcal{E})_{t \geq 0}$ . The following proposition characterizes the  $\mathbb{G}$ -predictable processes. The proof combines the techniques in those of [20] Lemma 3.13 and 4.4 and is postponed in Appendix.

**Proposition 2.1** *Let  $\mathcal{P}(\mathbb{F})$  be the predictable  $\sigma$ -algebra of the filtration  $\mathbb{F}$ . A  $\mathbb{G}$ -adapted process  $Z$  is  $\mathbb{G}$ -predictable if and only if it can be written in the form*

$$Z_t = \mathbb{1}_{\{\tau \geq t\}} Y_t + \mathbb{1}_{\{\tau < t\}} Y_t(G), \quad t > 0, \quad (2.3)$$

where  $Y$  is an  $\mathbb{F}$ -predictable process and  $Y(\cdot)$  is a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ -measurable function.

We study the martingale processes in the filtrations  $\mathbb{F}$  and  $\mathbb{G}$ . One basic martingale in  $\mathbb{F}$  is related to the random time  $\tau$ . Let  $D = (\mathbb{1}_{\{\tau \leq t\}}, t \geq 0)$  be the indicator process of the  $\mathbb{F}$ -stopping time  $\tau$ . Recall that the  $\mathbb{F}$ -compensator process  $\Lambda$  of  $\tau$  is the  $\mathbb{F}$ -predictable increasing process  $\Lambda$  such that  $N := D - \Lambda$  is an  $\mathbb{F}$ -martingale. In particular, if  $\tau$  is a predictable  $\mathbb{F}$ -stopping time, then  $\Lambda$  coincides with  $D$ .

To study  $\mathbb{G}$ -martingales, we assume the following hypothesis for the random variable  $G$  with respect to the filtration  $\mathbb{F}$  (c.f. [15] in the initial enlargement setting, see also [17] for comparison).

**Assumption 2.2** *For any  $t \geq 0$ , the  $\mathcal{F}_t$ -conditional law of  $G$  is equivalent to the probability law  $\eta$  of  $G$ , i.e.,  $\mathbb{P}(G \in \cdot | \mathcal{F}_t) \sim \eta(\cdot)$ , a.s.. Moreover, we denote by  $p_t(\cdot)$  the conditional density*

$$\mathbb{P}(G \in dx | \mathcal{F}_t) = p_t(x)\eta(dx), \quad a.s.. \quad (2.4)$$

As pointed out in [17, Lemma 1.8], we can choose a version of the conditional probability density  $p(\cdot)$ , such that  $p_t(\cdot)$  is  $\mathcal{F}_t \otimes \mathcal{E}$ -measurable for any  $t \geq 0$  and that  $(p_t(x), t \geq 0)$  is a positive càdlàg  $(\mathbb{F}, \mathbb{P})$ -martingale for any  $x \in E$ . In the following we will fix such a version of the conditional density.

**Remark 2.3** We assume the hypothesis of Jacod which is widely adopted in the study of initial and progressive enlargements of filtrations. Compared to the standard initial enlargement of  $\mathbb{F}$  by  $G$ , the information of the random variable  $G$  is added at a random time  $\tau$  but not at the initial time; compared to the progressive enlargement, the random variable added here is the associated information  $G$  instead of the random time  $\tau$ . In particular, the behavior of  $\mathbb{G}$ -martingales is quite different from the classic settings, and worth to be examined in detail.

Similar as in [14] and [15], we introduce the insider martingale measure  $\mathbb{Q}$  which will be useful in the sequel.

**Proposition 2.4** *There exists a unique probability measure  $\mathbb{Q}$  on  $\mathcal{F}_\infty \vee \sigma(G)$  which verifies the following conditions:*

- (1) *the probability measures  $\mathbb{Q}$  and  $\mathbb{P}$  are equivalent;*
- (2)  *$\mathbb{Q}$  identifies with  $\mathbb{P}$  on  $\mathbb{F}$  and on  $\sigma(G)$ ;*
- (3)  *$G$  is independent of  $\mathbb{F}$  under the probability  $\mathbb{Q}$ .*

Moreover, the Radon-Nikodym density of  $\mathbb{Q}$  with respect to  $\mathbb{P}$  on  $\mathcal{G}_t$  is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{G}_t} = \mathbb{1}_{\{\tau > t\}} + \mathbb{1}_{\{\tau \leq t\}} p_t(G)^{-1} \quad (2.5)$$

**Proof.** Let  $\mathbb{Q}$  be defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t \vee \sigma(G)} = p_t(G)^{-1}, \quad t \geq 0.$$

Since  $(\mathcal{F}_t \vee \sigma(G))_{t \geq 0}$  is the initial enlargement of  $\mathbb{F}$  by  $G$ , we obtain by [14,15] that  $\mathbb{Q}$  is the unique equivalent probability measure on  $\mathcal{F}_\infty \vee \sigma(G)$  which satisfies the conditions (1)–(3). Moreover, the Radon-Nikodym density  $d\mathbb{Q}/d\mathbb{P}$  on  $\mathcal{G}_t$  is given by

$$\mathbb{E}^{\mathbb{P}}[p_t(G)^{-1} | \mathcal{G}_t] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > t\}} p_t(G)^{-1} | \mathcal{G}_t] + \mathbb{1}_{\{\tau \leq t\}} p_t(G)^{-1}.$$

Let  $Z_t$  be a bounded  $\mathcal{G}_t$ -measurable random variable. By the decomposed form (2.2) we obtain that  $\mathbb{1}_{\{\tau > t\}} Z_t$  is  $\mathcal{F}_t$ -measurable. Hence

$$\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > t\}} p_t(G)^{-1} Z_t] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > t\}} Z_t \mathbb{E}^{\mathbb{P}}[p_t(G)^{-1} | \mathcal{F}_t]],$$

which leads to

$$\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > t\}} p_t(G)^{-1} | \mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}}[p_t(G)^{-1} | \mathcal{F}_t] = \mathbb{1}_{\{\tau > t\}} \quad \text{a.s.}$$

Hence we obtain (2.5).  $\square$

The following proposition shows that the filtration  $\mathbb{G}$  also satisfies the usual conditions under the  $\mathbb{F}$ -density hypothesis on the random variable  $G$ . The idea follows [1, Proposition 3.3].

**Proposition 2.5** *Under Assumption 2.2, the enlarged filtration  $\mathbb{G}$  is right continuous.*

**Proof.** The statement does not involve the underlying probability measure. Hence we may assume without loss of generality (by Proposition 2.4) that  $G$  is independent of  $\mathbb{F}$  under the probability  $\mathbb{P}$ . Let  $t \geq 0$  and  $\varepsilon > 0$ . Let  $X_{t+\varepsilon}$  be a bounded  $\mathcal{G}_{t+\varepsilon}$ -measurable random variable. We write it in the form

$$X_{t+\varepsilon} = Y_{t+\varepsilon} \mathbb{1}_{\{\tau > t+\varepsilon\}} + Y_{t+\varepsilon}(G) \mathbb{1}_{\{\tau \leq t+\varepsilon\}},$$

where  $Y_{t+\varepsilon}$  and  $Y_{t+\varepsilon}(\cdot)$  are respectively bounded  $\mathcal{F}_{t+\varepsilon}$ -measurable and  $\mathcal{F}_{t+\varepsilon} \otimes \mathcal{E}$ -measurable functions. Then for  $\delta \in (0, \varepsilon)$ , by the independence between  $G$  and  $\mathbb{F}$  one has

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[X_{t+\varepsilon} | \mathcal{G}_{t+\delta}] &= \mathbb{1}_{\{\tau > t+\delta\}} \mathbb{E}^{\mathbb{P}}[Y_{t+\varepsilon} \mathbb{1}_{\{\tau > t+\varepsilon\}} + Y_{t+\varepsilon}(G) \mathbb{1}_{\{t+\delta < \tau \leq t+\varepsilon\}} | \mathcal{F}_{t+\delta}] \\ &\quad + \mathbb{1}_{\{\tau \leq t+\delta\}} \mathbb{E}^{\mathbb{P}}[Y_{t+\varepsilon}(x) | \mathcal{F}_{t+\delta}]_{x=G} \\ &= \mathbb{1}_{\{\tau > t+\delta\}} \left( \mathbb{E}^{\mathbb{P}}[Y_{t+\varepsilon} \mathbb{1}_{\{\tau > t+\varepsilon\}} | \mathcal{F}_{t+\delta}] + \int_E \mathbb{E}^{\mathbb{P}}[Y_{t+\varepsilon}(x) \mathbb{1}_{\{t+\delta < \tau \leq t+\varepsilon\}} | \mathcal{F}_{t+\delta}] \eta(dx) \right) \\ &\quad + \mathbb{1}_{\{\tau \leq t+\delta\}} \mathbb{E}^{\mathbb{P}}[Y_{t+\varepsilon}(x) | \mathcal{F}_{t+\delta}]_{x=G}, \end{aligned}$$

where  $\eta$  is the probability law of  $G$ . Since the filtration  $\mathbb{F}$  satisfies the usual conditions, any  $\mathbb{F}$ -martingale admits a càdlàg version. Therefore, by taking a suitable version of the expectations  $\mathbb{E}^{\mathbb{P}}[X_{t+\varepsilon} | \mathcal{F}_{t+\delta}]$ , we have

$$\begin{aligned} \lim_{\delta \rightarrow 0} \mathbb{E}^{\mathbb{P}}[X_{t+\varepsilon} | \mathcal{G}_{t+\delta}] &= \mathbb{1}_{\{\tau > t\}} \left( \mathbb{E}^{\mathbb{P}}[Y_{t+\varepsilon} \mathbb{1}_{\{\tau > t+\varepsilon\}} | \mathcal{F}_t] + \int_E \mathbb{E}^{\mathbb{P}}[Y_{t+\varepsilon}(x) \mathbb{1}_{\{t < \tau \leq t+\varepsilon\}} | \mathcal{F}_t] \eta(dx) \right) \\ &\quad + \mathbb{1}_{\{\tau \leq t\}} \mathbb{E}^{\mathbb{P}}[Y_{t+\varepsilon}(x) | \mathcal{F}_t]_{x=G} = \mathbb{E}^{\mathbb{P}}[X_{t+\varepsilon} | \mathcal{G}_t]. \end{aligned}$$

In particular, if  $X$  is a bounded  $\mathcal{G}_{t+} := \bigcap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon}$ -measurable random variable, then one has  $\mathbb{E}^{\mathbb{P}}[X | \mathcal{G}_t] = X$  almost surely. Hence  $\mathcal{G}_{t+} = \mathcal{G}_t$ .  $\square$

Under the probability measure  $\mathbb{Q}$ , the random variable  $G$  is independent of  $\mathbb{F}$ . This observation leads to the following characterization of  $(\mathbb{G}, \mathbb{Q})$ -(local)-martingales. In the particular case where  $\tau = 0$ , we recover the classic result on initial enlargement of filtrations.

**Proposition 2.6** *Let  $Z = (\mathbb{1}_{\{\tau>t\}}Y_t + \mathbb{1}_{\{\tau\leq t\}}Y_t(G), t \geq 0)$  be a  $\mathbb{G}$ -adapted process. We assume that*

(1)  $Y(\cdot)$  is an  $\mathbb{F} \otimes \mathcal{E}$ -adapted process such that  $Y(x)$  is an  $(\mathbb{F}, \mathbb{P})$ -square-integrable martingale for any  $x \in E$  (resp. an  $(\mathbb{F}, \mathbb{P})$ -locally square-integrable martingale with a common localizing stopping time sequence independent of  $x$ ),

(2) the process

$$\tilde{Y}_t := \mathbb{1}_{\{\tau>t\}}Y_t + \int_E \left( \int_{]0,t]} Y_{u-}(x) d\Lambda_u + \langle N, Y(x) \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx), \quad t \geq 0$$

is well defined and is an  $(\mathbb{F}, \mathbb{P})$ -martingale (resp. an  $(\mathbb{F}, \mathbb{P})$ -local martingale).

Then the process  $Z$  is a  $(\mathbb{G}, \mathbb{Q})$ -martingale (resp. a  $(\mathbb{G}, \mathbb{Q})$ -local martingale).

**Proof.** We can reduce the local martingale case to the martingale case by taking a sequence of  $\mathbb{F}$ -stopping times which localizes the processes appearing in the conditions (1) and (2). Therefore, we only treat the martingale case. Note that since  $N$  and  $Y(x)$  are square integrable (c.f. [11, Chapitre VII (15.1)] for the square integrability of  $N$ ),  $NY(x) - \langle N, Y(x) \rangle^{\mathbb{F}, \mathbb{P}}$  is an  $(\mathbb{F}, \mathbb{P})$ -martingale by [18, Chapter I, Theorem 4.2].

For  $t \geq s \geq 0$ , one has

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[Z_t | \mathcal{G}_s] &= \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{\tau>t\}}Y_t | \mathcal{G}_s] + \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{\tau\leq t\}}Y_t(G) | \mathcal{G}_s] \\ &= \mathbb{1}_{\{\tau>s\}} \left( \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{\tau>t\}}Y_t | \mathcal{F}_s] + \int_E \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{s<\tau\leq t\}}Y_t(x) | \mathcal{F}_s] \eta(dx) \right) + \mathbb{1}_{\{\tau\leq s\}} \mathbb{E}^{\mathbb{Q}}[Y_t(x) | \mathcal{F}_s]_{x=G} \\ &= \mathbb{1}_{\{\tau>s\}} \left( \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau>t\}}Y_t | \mathcal{F}_s] + \int_E \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{s<\tau\leq t\}}Y_t(x) | \mathcal{F}_s] \eta(dx) \right) + \mathbb{1}_{\{\tau\leq s\}} \mathbb{E}^{\mathbb{P}}[Y_t(x) | \mathcal{F}_s]_{x=G} \\ &= \mathbb{1}_{\{\tau>s\}} \left( \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau>t\}}Y_t | \mathcal{F}_s] + \int_E \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{s<\tau\leq t\}}Y_t(x) | \mathcal{F}_s] \eta(dx) \right) + \mathbb{1}_{\{\tau\leq s\}} Y_s(G) \end{aligned} \tag{2.6}$$

where the second equality comes from the fact that  $G$  is independent of  $\mathbb{F}$  under the probability  $\mathbb{Q}$  and that  $\eta$  coincides with the  $\mathbb{Q}$ -probability law of  $\mathbb{G}$ , and the third equality comes from the fact that the probability measures  $\mathbb{P}$  and  $\mathbb{Q}$  coincide on the filtration  $\mathbb{F}$ .

Since  $Y(x)$  is an  $(\mathbb{F}, \mathbb{P})$ -martingale, one has

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{s<\tau\leq t\}}Y_t(x) | \mathcal{F}_s] &= \mathbb{E}^{\mathbb{P}}[D_t Y_t(x) | \mathcal{F}_s] - D_s Y_s(x) \\ &= \mathbb{E}^{\mathbb{P}}[N_t Y_t(x) - N_s Y_s(x) | \mathcal{F}_s] + \mathbb{E}^{\mathbb{P}}[\Lambda_t Y_t(x) - \Lambda_s Y_s(x) | \mathcal{F}_s] \\ &= \mathbb{E}^{\mathbb{P}}[\langle N, Y(x) \rangle_t^{\mathbb{F}, \mathbb{P}} - \langle N, Y(x) \rangle_s^{\mathbb{F}, \mathbb{P}} | \mathcal{F}_s] + \mathbb{E}^{\mathbb{P}}[\Lambda_t Y_t(x) - \Lambda_s Y_s(x) | \mathcal{F}_s], \end{aligned} \tag{2.7}$$

where the last equality comes from the fact that  $NY(x) - \langle N, Y(x) \rangle^{\mathbb{F}, \mathbb{P}}$  is an  $(\mathbb{F}, \mathbb{P})$ -martingale. Moreover, since  $Y(x)$  is an  $(\mathbb{F}, \mathbb{P})$ -martingale, its predictable projection is  $Y_-(x)$  (see [18, Chapter I, Corollary 2.31]), and hence

$$\mathbb{E}^{\mathbb{P}}[\Lambda_t Y_t(x) - \Lambda_s Y_s(x) | \mathcal{F}_s] = \mathbb{E} \left[ \int_{(s,t]} Y_{u-}(x) d\Lambda_u \middle| \mathcal{F}_s \right] \quad (2.8)$$

since  $\Lambda$  is an integrable increasing process which is  $\mathbb{F}$ -predictable (see [11, VI.61]). Therefore, by (2.6) we obtain

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}}[Z_t | \mathcal{G}_s] - Z_s \\ &= \mathbb{1}_{\{\tau > s\}} \left( \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > t\}} Y_t - \mathbb{1}_{\{\tau > s\}} Y_s | \mathcal{F}_s] + \int_E \mathbb{E}^{\mathbb{P}} \left[ \langle N, Y(x) \rangle_t^{\mathbb{F}, \mathbb{P}} - \langle N, Y(x) \rangle_s^{\mathbb{F}, \mathbb{P}} + \int_{(s,t]} Y_{u-}(x) d\Lambda_u \middle| \mathcal{F}_s \right] \eta(dx) \right) \\ &= \mathbb{1}_{\{\tau > s\}} \mathbb{E}^{\mathbb{P}}[\tilde{Y}_t - \tilde{Y}_s | \mathcal{F}_s] = 0. \end{aligned}$$

The proposition is thus proved.  $\square$

**Corollary 2.7** *Let  $Z = (\mathbb{1}_{\{\tau > t\}} Y_t + \mathbb{1}_{\{\tau \leq t\}} Y_t(G), t \geq 0)$  be a  $\mathbb{G}$ -adapted process. Then  $Z$  is a  $(\mathbb{G}, \mathbb{P})$ -martingale (resp. local  $(\mathbb{G}, \mathbb{P})$ -martingale) if the following conditions are fulfilled:*

- (1) *for any  $x \in E$ ,  $(Y_t(x)p_t(x), t \geq 0)$  is an  $(\mathbb{F}, \mathbb{P})$ -square integrable martingale (resp. a  $(\mathbb{F}, \mathbb{P})$ -locally square integrable martingale with a common localizing stopping time sequence);*
- (2) *the process*

$$\mathbb{1}_{\{\tau > t\}} Y_t + \int_E \left( \int_{]0,t]} Y_{u-}(x) p_{u-}(x) d\Lambda_u + \langle N, Y(x)p(x) \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx), \quad t \geq 0$$

*is a  $(\mathbb{F}, \mathbb{P})$ -martingale (resp. a local  $(\mathbb{F}, \mathbb{P})$ -martingale).*

**Proof.** By Proposition 2.4,  $Z$  is a  $(\mathbb{G}, \mathbb{P})$ -(local)-martingale if and only if the process  $Z(\mathbb{1}_{]0,\tau[} + \mathbb{1}_{[\tau,+\infty[} p(G))$  is a  $(\mathbb{G}, \mathbb{Q})$ -(local)-martingale. Therefore the assertion results from Proposition 2.6.  $\square$

**Proposition 2.8** *Let  $Z$  be a  $(\mathbb{G}, \mathbb{P})$ -martingale on  $[0, T]$  such that the process  $\mathbb{1}_{[\tau, +\infty[} Zp(G)$  is bounded. Then there exists an  $\mathbb{F}$ -adapted process  $Y$  and an  $\mathbb{F} \otimes \mathcal{E}$ -adapted process  $Y(\cdot)$  such that  $Z_t = \mathbb{1}_{\{\tau > t\}} Y_t + \mathbb{1}_{\{\tau \leq t\}} Y_t(G)$  and that the following conditions are fulfilled:*

- (1) *for any  $x \in E$ ,  $(Y_t(x)p_t(x), t \geq 0)$  is a bounded  $(\mathbb{F}, \mathbb{P})$ -martingale;*
- (2) *the process*

$$\mathbb{1}_{\{\tau > t\}} Y_t + \int_E \left( \int_{]0,t]} Y_{u-}(x) p_{u-}(x) d\Lambda_u + \langle N, Y(x)p(x) \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx), \quad t \geq 0$$

*is well defined and is an  $(\mathbb{F}, \mathbb{P})$ -martingale.*



**Proof.** Since  $Z_T$  is a  $\mathcal{G}_T$ -measurable random variable, we can write it in the form

$$Z_T = \mathbb{1}_{\{\tau > T\}} Y_T + \mathbb{1}_{\{\tau \leq T\}} Y_T(G), \quad (2.9)$$

where  $Y_T$  is an  $\mathcal{F}_T$ -measurable random variable, and  $Y_T(\cdot)$  is an  $\mathcal{F}_T \otimes \mathcal{E}$ -measurable function such that  $Y_T(\cdot)p_T(\cdot)$  is bounded. Similarly to [17, Lemma 1.8], we can construct an  $\mathbb{F} \otimes \mathcal{E}$ -adapted process  $Y(\cdot)$  on  $[0, T]$  such that  $Y(x)p(x)$  is a càdlàg  $(\mathbb{F}, \mathbb{P})$ -martingale for any  $x \in E$ . In particular, for  $t \in [0, T]$  one has

$$Y_t(x) = \mathbb{E}^{\mathbb{P}} \left[ \frac{Y_T(x)p_T(x)}{p_t(x)} \middle| \mathcal{F}_t \right]. \quad (2.10)$$

We then let, for  $t \in [0, T]$

$$\tilde{Y}_t := \mathbb{E}^{\mathbb{P}} \left[ Y_T \mathbb{1}_{\{\tau > T\}} + \int_E \left( \int_{]0, T]} Y_{u-}(x)p_{u-}(x) d\Lambda_u + \langle N, Y(x)p(x) \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx) \middle| \mathcal{F}_t \right]. \quad (2.11)$$

Then  $\tilde{Y}$  is an  $(\mathbb{F}, \mathbb{P})$ -martingale. For any  $t \in [0, T]$ , we let  $Y_t$  be an  $\mathcal{F}_t$ -measurable random variable such that

$$\mathbb{1}_{\{\tau > t\}} Y_t = \tilde{Y}_t - \int_E \left( \int_{]0, t]} Y_{u-}(x)p_{u-}(x) d\Lambda_u + \langle N, Y(x)p(x) \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx).$$

This is always possible since

$$\begin{aligned} & \mathbb{1}_{\{\tau \leq t\}} \left( \tilde{Y}_t - \int_E \left( \int_{]0, t]} Y_{u-}(x)p_{u-}(x) d\Lambda_u + \langle N, Y(x)p(x) \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx) \right) \\ &= \mathbb{1}_{\{\tau \leq t\}} \mathbb{E}^{\mathbb{P}} \left[ \int_E \left( \int_{]t, T]} Y_{u-}(x)p_{u-}(x) d\Lambda_u + d\langle N, Y(x)p(x) \rangle_u^{\mathbb{F}, \mathbb{P}} \right) \eta(dx) \middle| \mathcal{F}_t \right] \\ &= \mathbb{1}_{\{\tau \leq t\}} \int_E \mathbb{E}^{\mathbb{P}} [\mathbb{1}_{\{t < \tau \leq T\}} Y_T(x)p_T(x) | \mathcal{F}_t] \eta(dx) = 0, \end{aligned}$$

where the second equality is obtained by an argument similar to (2.7) and (2.8). We finally show that  $Z_t = \mathbb{1}_{\{\tau > t\}} Y_t + \mathbb{1}_{\{\tau \leq t\}} Y_t(G)$   $\mathbb{P}$ -a.s. for any  $t \in [0, T]$ . Note that we already have  $Z_T = \mathbb{1}_{\{\tau > T\}} Y_T + \mathbb{1}_{\{\tau \leq T\}} Y_T(G)$ . Therefore it remains to prove that the  $\mathbb{G}$ -adapted process  $(\mathbb{1}_{\{\tau > t\}} Y_t + \mathbb{1}_{\{\tau \leq t\}} Y_t(G))_{t \in [0, T]}$  is an  $(\mathbb{G}, \mathbb{P})$ -martingale. This follows from the construction of the processes  $Y$ ,  $Y(\cdot)$  and Corollary 2.7.  $\square$

**Remark 2.9** (1) We observe from the proof of the previous proposition that, if  $Z$  is a  $(\mathbb{G}, \mathbb{P})$ -martingale on  $[0, T]$  (without boundedness hypothesis) such that  $Z_T$  can be written into the form (2.9) with  $Y_T(x)p_T(x) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$  for any  $x \in E$ , then we can construct the  $\mathbb{F} \otimes \mathcal{E}$ -adapted process  $Y(\cdot)$  by using the relation (2.10). Note that for any  $x \in E$ , the process  $Y(x)p(x)$  is a square-integrable  $(\mathbb{F}, \mathbb{P})$ -martingale. Therefore, the result of Proposition 2.8 remains true provided that the conditional expectation in (2.11) is well defined.

(2) Let  $Z$  be a  $(\mathbb{G}, \mathbb{P})$ -martingale on  $[0, T]$ . In general, the decomposition of  $Z$  into the form  $Z = \mathbb{1}_{]0, \tau[} Y + \mathbb{1}_{[\tau, +\infty[} Y(G)$  with  $Y$  being  $\mathbb{F}$ -adapted and  $Y(\cdot)$  being  $\mathbb{F} \otimes \mathcal{E}$ -adapted is not

unique. Namely, there may exist an  $\mathbb{F}$ -adapted process  $\tilde{Y}$  and an  $\mathbb{F} \otimes \mathcal{E}$ -adapted process  $\tilde{Y}(\cdot)$  such that  $\tilde{Y}$  is not a version of  $Y$ ,  $\tilde{Y}(\cdot)$  is not a version of  $Y(\cdot)$ , but we still have  $Z = \mathbb{1}_{[0, \tau[} \tilde{Y} + \mathbb{1}_{[\tau, +\infty[} \tilde{Y}(G)$ . Moreover, although the proof of Proposition 2.8 provides an explicit way to construct the decomposition of the  $(\mathbb{G}, \mathbb{P})$ -martingale  $Z$  which satisfies the two conditions, in general such decomposition is not unique neither.

- (3) Concerning the local martingale analogue of Proposition 2.8, the main difficulty is that a local  $(\mathbb{G}, \mathbb{P})$ -martingale need not be localized by a sequence of  $\mathbb{F}$ -stopping times. To solve this problem, it is crucial to understand the  $\mathbb{G}$ -stopping times and their relation with  $\mathbb{F}$ -stopping times.

## 2.2 (H')-hypothesis and semimartingale decomposition

In this subsection, we prove that under Assumption 2.2, the (H')-hypothesis (see [17]) is satisfied and we give the semimartingale decomposition of an  $\mathbb{F}$ -martingale in  $\mathbb{G}$ .

**Theorem 2.10** *We suppose that Assumption 2.2 holds. Let  $M$  be an  $(\mathbb{F}, \mathbb{P})$ -locally square integrable martingale, then it is a  $(\mathbb{G}, \mathbb{P})$ -semimartingale. Moreover, the process*

$$\tilde{M}_t = M_t - \mathbb{1}_{\{\tau \leq t\}} \int_{]0, t]} \frac{d\langle M - M^\tau, p(x) \rangle_s^{\mathbb{F}, \mathbb{P}}}{p_{s-}(x)} \Big|_{x=G}, \quad t \geq 0$$

*is a  $(\mathbb{G}, \mathbb{P})$ -local martingale, where  $M^\tau = (M_t^\tau, t \geq 0)$  is the stopped process with  $M_t^\tau = M_{t \wedge \tau}$ .*

We present two proofs of Theorem 2.10. The first one relies on the following Lemma, which computes the  $(\mathbb{G}, \mathbb{Q})$ -predictable bracket of an  $(\mathbb{F}, \mathbb{P})$ -local martingale with a general  $(\mathbb{F}, \mathbb{P})$ -local martingale. This approach is more computational, but Lemma 2.11 has its own interest, in particular for the study of  $\mathbb{G}$ -adapted processes. The second proof is more conceptual and relies on a classic result of Jacod [17] on initial enlargement of filtrations under an additional (positivity or integrability) assumption on the process  $\tilde{M}$ .

**Lemma 2.11** *Let  $Y$  be an  $\mathbb{F}$ -adapted process and  $Y(\cdot)$  be an  $\mathbb{F} \otimes \mathcal{E}$ -adapted process such that (as in Proposition 2.6)*

- (1)  $Y(x)$  is an  $(\mathbb{F}, \mathbb{P})$ -locally square integrable martingale for any  $x \in E$ ,  
(2) the process

$$H_t := \int_E \left( \int_{]0, t]} Y_{u-}(x) d\Lambda_u + \langle N, Y(x) \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx), \quad t \geq 0$$

*is well defined and of finite variation, and  $\tilde{Y} = \mathbb{1}_{[0, \tau[} Y + H$  is an  $(\mathbb{F}, \mathbb{P})$ -locally square-integrable martingale.*

Let  $Z$  be the process  $\mathbb{1}_{[0,\tau]}Y + \mathbb{1}_{[\tau,+\infty]}Y(G)$ . Then one has

$$\begin{aligned} \langle M, Z \rangle_t^{\mathbb{G},\mathbb{Q}} &= \langle M^\tau, \tilde{Y} \rangle_t^{\mathbb{F},\mathbb{P}} - \int_{]0,t]} M_{s-}^\tau dH_s + \int_E \left( \int_{]0,t]} U_{s-}(x) d\Lambda_s + \langle N, U(x) \rangle_t^{\mathbb{F},\mathbb{P}} \right) \eta(dx) \\ &\quad + \langle M - M^\tau, Y(x) \rangle^{\mathbb{F},\mathbb{P}} \Big|_{x=G}, \end{aligned} \quad (2.12)$$

where

$$U_t(x) = M_t^\tau Y_t(x) - \langle M^\tau, Y(x) \rangle_t^{\mathbb{F},\mathbb{P}} + \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau < +\infty\}} \langle M^\tau, Y(x) \rangle_\tau^{\mathbb{F},\mathbb{P}} | \mathcal{F}_t], \quad x \in E.$$

**Proof.** It follows from Proposition 2.6 that  $Z$  is a  $(\mathbb{G}, \mathbb{Q})$ -martingale. In the following, we establish the equality (2.12).

We first treat the case where the martingale  $M$  begins at  $\tau$  with  $M_\tau = 0$ , namely  $M_t \mathbb{1}_{\{\tau \geq t\}} = 0$  for any  $t \geq 0$ . Therefore  $W(x) := MY(x) - \langle M, Y(x) \rangle^{\mathbb{F},\mathbb{P}}$  is a local  $(\mathbb{F}, \mathbb{P})$ -martingale which vanishes on  $\llbracket 0, \tau \rrbracket$ . In particular one has

$$\int_{]0,t]} W_{u-}(x) d\Lambda_u = 0 \quad \text{and} \quad \langle N, W(x) \rangle^{\mathbb{P},\mathbb{F}} = 0$$

since both processes  $N$  and  $\Lambda$  are stopped at  $\tau$ . By Proposition 2.6, we obtain that the process  $W(G) = \mathbb{1}_{[\tau,+\infty]}W(G)$  is actually a local  $(\mathbb{G}, \mathbb{Q})$ -martingale. Note that

$$W(G) = MY(G) - \langle M, Y(x) \rangle^{\mathbb{F},\mathbb{P}} \Big|_{x=G},$$

and  $\langle M, Y(x) \rangle^{\mathbb{P},\mathbb{F}} \Big|_{x=G}$  is  $\mathbb{G}$ -predictable (by Proposition 2.1, we also use the fact that  $\langle M, Y(x) \rangle^{\mathbb{P},\mathbb{F}}$  vanishes on  $\llbracket 0, \tau \rrbracket$ ), therefore we obtain  $\langle M, Z \rangle^{\mathbb{G},\mathbb{Q}} = \langle M, Y(x) \rangle^{\mathbb{P},\mathbb{F}} \Big|_{x=G}$ .

In the second step, we assume that  $M$  is stopped at  $\tau$ . In this case one has

$$\forall t \geq 0, \quad U_t(x) = M_t Y_t(x) - \langle M, Y(x) \rangle_t^{\mathbb{F},\mathbb{P}} + \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau < +\infty\}} \langle M, Y(x) \rangle_\tau^{\mathbb{P},\mathbb{F}} | \mathcal{F}_t].$$

It is a local  $(\mathbb{F}, \mathbb{P})$ -martingale. Moreover, since  $M$  is stopped at  $\tau$ , also is  $\langle M, Y(x) \rangle^{\mathbb{F},\mathbb{P}}$ . In particular, since  $\mathbb{1}_{\{\tau \leq t\}} \langle M, Y(x) \rangle_\tau^{\mathbb{F},\mathbb{P}}$  is  $\mathcal{F}_t$ -measurable, one has

$$\forall t \geq 0, \quad \mathbb{1}_{\{\tau \leq t\}} M_t Y_t(G) = \mathbb{1}_{\{\tau \leq t\}} U_t(G).$$

In addition, by definition  $\tilde{Y} = \mathbb{1}_{[0,\tau]}Y + H$ . Hence one has

$$M \mathbb{1}_{[0,\tau]}Y = M(\tilde{Y} - H) = (M\tilde{Y} - \langle M, \tilde{Y} \rangle^{\mathbb{F},\mathbb{P}}) + \langle M, \tilde{Y} \rangle^{\mathbb{F},\mathbb{P}} - M_- \cdot H - H_- \cdot M - [M, H],$$

where  $M_- \cdot H$  and  $H_- \cdot M$  denote respectively the integral processes

$$\int_0^t M_{s-} dH_s, \quad \text{and} \quad \int_0^t H_{s-} dM_s, \quad t \geq 0.$$

Since  $H$  is a predictable process of finite variation and  $M$  is an  $\mathbb{F}$ -martingale, the process  $[M, H]$  is a local  $\mathbb{F}$ -martingale (see [18] Chapter I, Proposition 4.49). In particular,

$$M \mathbb{1}_{[0,\tau]}Y - \langle M, \tilde{Y} \rangle^{\mathbb{F},\mathbb{P}} + M_- \cdot H$$

is a local  $\mathbb{F}$ -martingale. Let

$$A_t = \langle M, \tilde{Y} \rangle_t^{\mathbb{F}, \mathbb{P}} - \int_{]0, t]} M_{s-} dH_s + \int_E \left( \int_{]0, t]} U_{s-}(x) d\Lambda_s + \langle N, U(x) \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx), \quad t \geq 0.$$

This is an  $\mathbb{F}$ -predictable process, and hence is  $\mathbb{G}$ -predictable. Moreover, this process is stopped at  $\tau$ . Let  $V$  be the  $(\mathbb{F}, \mathbb{P})$ -martingale defined as

$$V_t = \mathbb{E}^{\mathbb{P}}[A_\tau \mathbb{1}_{\{\tau < +\infty\}} | \mathcal{F}_t], \quad t \geq 0. \quad (2.13)$$

Note that  $V_t \mathbb{1}_{\{\tau \leq t\}} = A_\tau \mathbb{1}_{\{\tau \leq t\}} = A_t \mathbb{1}_{\{\tau \leq t\}}$ . Hence

$$AD = VD = V_- \cdot D + D_- \cdot V + [V, D] = V_- \cdot N + V_- \cdot \Lambda + D_- \cdot V + [V, N] + [V, \Lambda],$$

where  $D = (\mathbb{1}_{\{\tau \leq t\}}, t \geq 0) = N + \Lambda$ . In particular,

$$AD - V_- \cdot \Lambda - \langle V, N \rangle^{\mathbb{F}, \mathbb{P}} = V_- \cdot N + D_- \cdot V + ([V, N] - \langle V, N \rangle^{\mathbb{P}, \mathbb{F}}) + [V, \Lambda]$$

is a local  $(\mathbb{F}, \mathbb{P})$ -martingale. Therefore, one has

$$\begin{aligned} & \mathbb{1}_{\{\tau > t\}}(M_t Y_t - A_t) + \int_E \left( \int_{]0, t]} (U_{s-}(x) - V_{s-}) d\Lambda_s + \langle N, U(x) - V \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx) \\ &= \mathbb{1}_{\{\tau > t\}} M_t Y_t - A_t + \mathbb{1}_{\{\tau \leq t\}} A_t + \int_E \left( \int_{]0, t]} U_{s-}(x) d\Lambda_s + \langle N, U(x) \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx) - (V_- \cdot \Lambda)_t - \langle V, N \rangle_t^{\mathbb{F}, \mathbb{P}} \\ &= \left( \mathbb{1}_{\{\tau > t\}} M_t Y_t - \langle M, \tilde{Y} \rangle_t^{\mathbb{F}, \mathbb{P}} + (M_- \cdot H)_t \right) + \left( \mathbb{1}_{\{\tau \leq t\}} A_t - (V_- \cdot \Lambda)_t - \langle V, N \rangle_t^{\mathbb{F}, \mathbb{P}} \right), \end{aligned}$$

which is a local  $(\mathbb{F}, \mathbb{P})$ -martingale.

We write the process  $MZ - A$  in the form

$$M_t Z_t - A_t = \mathbb{1}_{\{\tau > t\}}(Y_t M_t - A_t) + \mathbb{1}_{\{\tau \leq t\}}(U_t(G) - A_t) = \mathbb{1}_{\{\tau > t\}}(Y_t M_t - A_t) + \mathbb{1}_{\{\tau \leq t\}}(U_t(G) - V_t)$$

where the last equality comes from (2.13). We have seen that  $U(x) - V$  is a local  $(\mathbb{F}, \mathbb{P})$ -martingale for any  $x \in E$ . Hence by Proposition 2.6 we obtain that  $MZ - A$  is a local  $(\mathbb{G}, \mathbb{Q})$ -martingale.

In the final step, we consider the general case. We decompose the  $(\mathbb{F}, \mathbb{P})$ -martingale into the sum of two parts  $M^\tau$  and  $M - M^\tau$ , where  $M^\tau$  is an  $(\mathbb{F}, \mathbb{P})$ -martingale stopped at  $\tau$ , and  $M - M^\tau$  is an  $(\mathbb{F}, \mathbb{P})$ -martingale which vanishes on  $[[0, \tau]]$ . Combining the results obtained in the two previous steps, we obtain the formula (2.12).  $\square$

**Proof of Theorem 2.10.** Since  $\mathbb{P}$  and  $\mathbb{Q}$  coincide on  $\mathbb{F}$ , we obtain that  $M$  is an  $(\mathbb{F}, \mathbb{Q})$ -martingale. Moreover, since  $G$  is independent of  $\mathbb{F}$  under the probability  $\mathbb{Q}$ ,  $M$  is also a  $(\mathbb{G}, \mathbb{Q})$ -martingale.

We keep the notation of the Lemma 2.11 and specify the terms in the situation of the theorem. Note that the Radon-Nikodym derivative of  $\mathbb{P}$  with respect to  $\mathbb{Q}$  on  $\mathcal{G}_t$  equals

$$Z_t := \mathbb{1}_{\{\tau > t\}} + \mathbb{1}_{\{\tau \leq t\}} p_t(G).$$

In particular, with the notation of the lemma, one has

$$Y_t = 1, \quad Y_t(x) = p_t(x), \quad t \geq 0, \quad x \in E.$$

Since  $\int_E Y_t(x) \eta(dx) = 1$ , for any  $t \geq 0$

$$\tilde{Y}_t = \mathbb{1}_{\{\tau > t\}} + \Lambda_t = 1 - N_t,$$

and

$$H_t = \int_E \left( \int_{]0,t]} Y_{u-}(x) d\Lambda_u + \langle N, Y(x) \rangle_t^{\mathbb{F}, \mathbb{P}} \right) \eta(dx) = \Lambda_t.$$

Moreover, one has

$$\int_E U_t(x) \eta(dx) = \int_E \left( M_t^\tau Y_t(x) - \langle M^\tau, Y(x) \rangle_t^{\mathbb{F}, \mathbb{P}} + \mathbb{E}^{\mathbb{P}}[\langle M^\tau, Y(x) \rangle_\tau^{\mathbb{F}, \mathbb{P}} \mathbb{1}_{\{\tau < +\infty\}} | \mathcal{F}_t] \right) \eta(dx) = M^\tau.$$

Therefore, by Lemma 2.11 one has

$$\begin{aligned} \langle M, Z \rangle^{\mathbb{G}, \mathbb{Q}} &= -\langle M^\tau, N \rangle^{\mathbb{F}, \mathbb{P}} - M_-^\tau \cdot \Lambda + M_-^\tau \cdot \Lambda + \langle M^\tau, N \rangle^{\mathbb{F}, \mathbb{P}} + \langle M - M^\tau, Y(x) \rangle^{\mathbb{F}, \mathbb{P}} \Big|_{x=G} \\ &= \langle M - M^\tau, Y(x) \rangle^{\mathbb{F}, \mathbb{P}} \Big|_{x=G}. \end{aligned}$$

Finally, since  $M$  is a  $(\mathbb{G}, \mathbb{Q})$ -local martingale, by Girsanov's theorem (cf. [18] Chapter III, Theorem 3.11), the process

$$\tilde{M}_t = M_t - \int_{]0,t]} \frac{1}{Z_{s-}} d\langle M, Z \rangle_s^{\mathbb{G}, \mathbb{Q}}, \quad t \geq 0$$

is a local  $(\mathbb{G}, \mathbb{P})$ -martingale. The theorem is thus proved.  $\square$

**Second proof of Theorem 2.10.** Let  $\mathbb{H} = \mathbb{F} \vee \sigma(G)$  be the initial enlargement of the filtration  $\mathbb{F}$  by  $\sigma(G)$ . Clearly the filtration  $\mathbb{H}$  is larger than  $\mathbb{G}$ . More precisely, the filtration  $\mathbb{G}$  coincides with  $\mathbb{F}$  before the stopping time  $\tau$ , and coincides with  $\mathbb{H}$  after the stopping time  $\tau$ . We first observe that the stopped process at  $\tau$  of an  $(\mathbb{F}, \mathbb{P})$ -martingale  $L$  is a  $(\mathbb{G}, \mathbb{P})$ -martingale. In fact, for  $t \geq s \geq 0$  one has

$$\mathbb{E}[L_t^\tau | \mathcal{G}_s] = \mathbb{1}_{\{\tau > s\}} \mathbb{E}[L_{\tau \wedge t} | \mathcal{F}_s] + \mathbb{1}_{\{\tau \leq s\}} \mathbb{E}[L_\tau | \mathcal{G}_s] = \mathbb{1}_{\{\tau > s\}} L_s + \mathbb{1}_{\{\tau \leq s\}} L_\tau = L_{\tau \wedge s}.$$

We remark that, as shown by Jeulin's formula, this result holds more generally for any enlargement  $\mathbb{G}$  which coincides with  $\mathbb{F}$  before a random time  $\tau$ .

We now consider the decomposition of  $M$  as  $M = M^\tau + (M - M^\tau)$ , where  $M^\tau$  is the stopped process of  $M$  at  $\tau$ . Since  $\mathbb{G}$  coincides with  $\mathbb{F}$  before  $\tau$ , we obtain by the above argument that  $M^\tau$  is an  $(\mathbb{G}, \mathbb{P})$ -local martingale. Consider now the process  $Y := M - M^\tau$ , which begins at  $\tau$ . It is also an  $(\mathbb{F}, \mathbb{P})$ -local martingale. By Jacod's decomposition formula (see [17, Theorem 2.1]), the process

$$\tilde{Y}_t = Y_t - \int_{]0,t]} \frac{d\langle Y, p(x) \rangle_s^{\mathbb{F}, \mathbb{P}}}{p_{s-}(x)} \Big|_{x=G}, \quad t \geq 0 \tag{2.14}$$

is an  $(\mathbb{H}, \mathbb{P})$ -local martingale. Note that the predictable quadratic variation process  $\langle Y, p(x) \rangle_s^{\mathbb{F}, \mathbb{P}}$  vanishes on  $\llbracket 0, \tau \rrbracket$  since the process  $Y$  begins at  $\tau$ . Hence

$$\int_{]0, t]} \frac{d\langle Y, p(x) \rangle_s^{\mathbb{F}, \mathbb{P}}}{p_{s-}(x)} = \mathbb{1}_{\{\tau \leq t\}} \int_{]0, t]} \frac{d\langle Y, p(x) \rangle_s^{\mathbb{F}, \mathbb{P}}}{p_{s-}(x)}.$$

This observation also shows that the process (2.14) is  $\mathbb{G}$ -adapted. Hence it is a  $(\mathbb{G}, \mathbb{P})$ -local martingale under the supplementary assumption that  $\tilde{Y}$  is positive or  $\|\tilde{Y}\|_1 < +\infty$ , by Stricker [25, Theorem 1.2], where  $\|\tilde{Y}\|_1$  is defined as the supremum of  $\|\tilde{Y}_\sigma\|_{L^1}$  with  $\sigma$  running over all finite  $\mathbb{G}$ -stopping times. Note that the condition  $\|\tilde{Y}\|_1 < +\infty$  is satisfied if and only if the process  $\tilde{Y}$  is a  $(\mathbb{G}, \mathbb{P})$ -quasimartingale (see [22]).

**Remark 2.12** *Even if the second proof of Theorem 2.10 needs the additional assumption on the positivity or integrability of the process  $\tilde{M} - M^\tau$ , it remains interesting since it allows to weaken Assumption 2.2. Indeed, to apply Jacod's decomposition formula we only need to assume that the conditional law  $\mathbb{P}(G \in \cdot | \mathcal{F}_t)$  is absolutely continuous w.r.t.  $\mathbb{P}(G \in \cdot)$ .*

### 3 Logarithmic utility maximization

In this section, we study the optimization problem for two types of investors: an insider and an ordinary agent. We consider a financial market composed by  $d$  stocks with discounted prices given by the  $d$ -dimensional process  $X = (X^1, \dots, X^d)^\top$ . This process is observed by both agents and is  $\mathbb{F}$ -adapted. We suppose that each  $X^i$ ,  $i = 1, \dots, d$ , evolve according to the following stochastic differential equations

$$X_t^i = X_0^i + \int_0^t X_{s-}^i \left( dM_s^i + \sum_{j=1}^d \alpha_s^j d\langle M^i, M^j \rangle_s \right), \quad t \geq 0,$$

with  $X_0^i$  a positive constant,  $M^i$  an  $\mathbb{F}$ -locally square integrable martingale and  $\alpha$  a  $\mathcal{P}(\mathbb{F})$ -measurable process valued in  $\mathbb{R}^d$  such that

$$\mathbb{E} \left[ \int_0^T \alpha_s^\top d\langle M \rangle_s \alpha_s \right] < +\infty. \quad (3.1)$$

The ordinary agent has access to the information flow given by the filtration  $\mathbb{F}$ , while the information flow of the insider is represented by the filtration  $\mathbb{G}$ . The optimization for the ordinary agent is standard. For the insider, we follow [2, 3] to solve the problem. We first describe the insider's portfolio in the enlarged filtration  $\mathbb{G}$ . Recall that under Assumption 2.2, the process  $M$  is a  $\mathbb{G}$ -semimartingale with canonical decomposition given by Theorem 2.10:

$$M_t = \tilde{M}_t + \mathbb{1}_{\{\tau \leq t\}} \int_0^t \frac{d\langle M - M^\tau, p(x) \rangle_s}{p_{s-}(x)} \Big|_{x=G}, \quad t \geq 0, \quad (3.2)$$

where  $\tilde{M}$  is a  $\mathbb{G}$ -local martingale and  $M^\tau$  is the stopped process  $(M_{t \wedge \tau})_{t \geq 0}$ .

Applying Theorem 2.5 of [17] to the  $\mathbb{F}$ -locally square integrable martingale  $M - M^\tau$ , we have the following result.

**Lemma 3.1** *For  $i = 1, \dots, d$ , there exists a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ -measurable function  $m^i$  such that*

$$\langle p(x), M^i - (M^i)^\tau \rangle_t = \int_0^t m_s^i(x) p_{s-}(x) d\langle M^i - (M^i)^\tau \rangle_s$$

for all  $x \in E$  and all  $t \geq 0$ .

We now rewrite the integral of  $m$  w.r.t.  $\langle M - M^\tau \rangle$ .

**Lemma 3.2** *Under Assumption 2.1, there exists a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ -measurable process  $\mu$  valued in  $\mathbb{R}^d$  such that*

$$\int_0^t d\langle M - M^\tau \rangle_s \mu_s(x) = \begin{pmatrix} \int_0^t m_s^1(x) d\langle M^1 - (M^1)^\tau \rangle_s \\ \vdots \\ \int_0^t m_s^d(x) d\langle M^d - (M^d)^\tau \rangle_s \end{pmatrix}$$

for all  $t \geq 0$ .

**Proof.** The proof is the same as that of Lemma 2.8 in [3]. We therefore omit it.  $\square$

We can then rewrite the process  $M$  in (3.2) in the following way

$$M_t = \widetilde{M}_t + \mathbb{1}_{\{\tau \leq t\}} \int_0^t d\langle M - M^\tau \rangle_s \mu_s(G), \quad t \geq 0, \quad (3.3)$$

and the dynamics of the process  $X$  can be expressed with the  $\mathbb{G}$ -local martingale  $\widetilde{M}$  as follows

$$dX_t = \text{Diag}(X_{t-}) \left( d\widetilde{M}_t + d\langle M \rangle_t \alpha_t + d\langle M - M^\tau \rangle_t \mu_t(G) \right), \quad t \geq 0,$$

where  $\text{Diag}(X_{t-})$  stands for the  $d \times d$  diagonal matrix whose  $i$ -th diagonal term is  $X_{t-}^i$  for  $i = 1, \dots, d$ . We then introduce the following integrability assumption.

**Assumption 3.3** *The process  $\mu(G)$  is square integrable w.r.t.  $d\langle M - M^\tau \rangle$ :*

$$\mathbb{E} \left[ \int_0^T \mu_t(G)^\top d\langle M - M^\tau \rangle_t \mu_t(G) \right] < \infty.$$

Denote by  $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$  the underlying filtration. We define an  $\mathbb{H}$ -portfolio as a couple  $(x, \pi)$  where  $x$  is a constant representing the initial wealth and  $\pi$  is an  $\mathbb{R}^d$ -valued  $\mathcal{P}(\mathbb{H})$ -measurable process  $\pi$  such that

$$\int_0^T \pi_t^\top d\langle M \rangle_t \pi_t < \infty, \quad \mathbb{P}\text{-a.s.}$$

and

$$\sum_{i=1}^d \pi_t^i \frac{\Delta X_t^i}{X_{t-}^i} > -1, \quad t \in [0, T]. \quad (3.4)$$

Here  $\pi_t^i$  represents the proportion of discounted wealth invested at time  $t$  in the asset  $X^i$ . For such an  $\mathbb{H}$ -portfolio, we define the associated discounted wealth process  $V(x, \pi)$  by

$$V_t(x, \pi) = x + \sum_{i=1}^d \int_0^t \pi_s^i V_{s-}(x, \pi) \frac{dX_s^i}{X_{s-}^i}, \quad t \geq 0.$$

By the condition (3.4), the wealth process is positive. We suppose that the agents preferences are described by the logarithmic utility function. For a given initial capital  $x$ , we define the set of admissible  $\mathbb{H}$ -portfolio processes by

$$\mathcal{A}_{\mathbb{H}}^{\log}(x) = \left\{ \pi : (x, \pi) \text{ is an } \mathbb{H}\text{-portfolio satisfying } \mathbb{E}[\log^- V_T(x, \pi)] < \infty \right\}$$

For an initial capital  $x$  we then consider the two optimization problems:

- the ordinary agent's problem consists in computing

$$V_{\mathbb{F}}^{\log} = \sup_{\pi \in \mathcal{A}_{\mathbb{F}}^{\log}(x)} \mathbb{E}[\log V_T(x, \pi)],$$

- the insider's problem consists in computing

$$V_{\mathbb{G}}^{\log} = \sup_{\pi \in \mathcal{A}_{\mathbb{G}}^{\log}(x)} \mathbb{E}[\log V_T(x, \pi)].$$

To solve these problems, we introduce the minimal martingale density processes  $\hat{Z}^{\mathbb{F}}$  and  $\hat{Z}^{\mathbb{G}}$  defined by

$$\hat{Z}_t^{\mathbb{F}} = \mathcal{E} \left( - \int_0^t \alpha_s^{\top} dM_s \right)_t$$

and

$$\hat{Z}_t^{\mathbb{G}} = \mathcal{E} \left( - \int_0^t (\alpha_s + \mathbb{1}_{\tau \leq s} \mu_s(G))_s^{\top} d\widetilde{M}_s \right)_t$$

for  $t \in [0, T]$ , where  $\mathcal{E}(\cdot)$  denotes the Doléans-Dade exponential. We first have the following result.

**Proposition 3.4** (i) *The processes  $\hat{Z}^{\mathbb{F}} X$  and  $\hat{Z}^{\mathbb{F}} V(x, \pi)$  are  $\mathbb{F}$ -local martingales for any portfolio  $(x, \pi)$  such that  $\pi \in \mathcal{A}_{\mathbb{F}}(x)$ .*

(ii) *The processes  $\hat{Z}^{\mathbb{G}} X$  and  $\hat{Z}^{\mathbb{G}} V(x, \pi)$  are  $\mathbb{G}$ -local martingales for any portfolio  $(x, \pi)$  such that  $\pi \in \mathcal{A}_{\mathbb{G}}(x)$ .*



**Proof.** We only prove assertion (ii). The same arguments can be applied to prove (i) by taking  $\mu(G) \equiv 0$ . From Ito's formula we have

$$d(\hat{Z}^G X) = X_- d\hat{Z}^G + \hat{Z}^G_- dX + d\langle Z^G, X \rangle + d([Z^G, X] - \langle Z^G, X \rangle).$$

From the dynamics of  $\hat{Z}^G$  and  $X$  we have

$$\begin{aligned} d\langle \hat{Z}^G_-, X \rangle &= -\hat{Z}^G_- \text{Diag}(X_-) d\left\langle \int_0^\cdot (\alpha_s + \mathbb{1}_{\tau \leq s} \mu_s(G))_s^\top d\widetilde{M}_s, M \right\rangle \\ &= -\hat{Z}^G_- \text{Diag}(X_-) d\langle M \rangle (\alpha + \mathbb{1}_{[\tau, +\infty[} \mu(G)) \\ &= -\hat{Z}^G_- \text{Diag}(X_-) (d\langle M \rangle \alpha + d\langle M - M^\tau \rangle \mu(G)). \end{aligned}$$

Therefore we get

$$d(\hat{Z}^G X) = X_- d\hat{Z}^G + \hat{Z}^G_- \text{Diag}(X_-) d\widetilde{M} + d([\hat{Z}^G, X] - \langle \hat{Z}^G, X \rangle)$$

which shows that  $\hat{Z}^G X$  is a  $\mathbb{G}$ -local martingale.  $\square$

We are now able to compute  $V_{\mathbb{F}}$  and  $V_{\mathbb{G}}$  and provide optimal strategies.

**Theorem 3.5** (i) *An optimal strategy for the ordinary agent is given by*

$$\pi_t^{ord} = \alpha_t, \quad t \in [0, T],$$

and the maximal expected logarithmic utility is given by

$$V_{\mathbb{F}}^{\log} = \mathbb{E} \left[ \log V_T(x, \pi^{ord}) \right] = \log x + \frac{1}{2} \mathbb{E} \left[ \int_0^T \alpha_t^\top d\langle M \rangle_t \alpha_t \right].$$

(ii) *An optimal strategy for the insider is given by*

$$\pi_t^{ins} = \alpha_t + \mathbb{1}_{\tau \leq t} \mu_t(G), \quad t \in [0, T],$$

and the maximal expected logarithmic utility is given by

$$\begin{aligned} V_{\mathbb{G}}^{\log} &= \mathbb{E} \left[ \log V_T(x, \pi^{ins}) \right] \\ &= \log x + \frac{1}{2} \mathbb{E} \left[ \int_0^T \alpha_t^\top d\langle M \rangle_t \alpha_t \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T \mu_t(G)^\top d\langle M - M^\tau \rangle_t \mu_t(G) \right]. \end{aligned}$$

(iii) *The insider's additional expected utility is given by*

$$V_{\mathbb{G}}^{\log} - V_{\mathbb{F}}^{\log} = \frac{1}{2} \mathbb{E} \left[ \int_0^T \mu_t(G)^\top d\langle M - M^\tau \rangle_t \mu_t(G) \right].$$

**Proof.** We do not prove (i) since it relies on the same arguments as for (ii) with  $\mu(G) \equiv 0$  and  $\hat{Z}^{\mathbb{F}}$  in place of  $\hat{Z}^{\mathbb{G}}$ .  $\square$

(ii) We recall that for a  $C^1$  concave function  $u$  such that its derivative  $u'$  admits an inverse function  $I$  we have

$$u(a) \leq u(I(b)) - b(I(b) - a)$$

for all  $a, b \in \mathbb{R}$ . Applying this inequality with  $u = \log$ ,  $a = V_T(x, \pi)$  for  $\pi \in \mathcal{A}_{\mathbb{G}}(x)$  and  $b = y\hat{Z}_T^{\mathbb{G}}$  for some constant  $y > 0$  we get

$$\begin{aligned} \log V_T(x, \pi) &\leq \log \frac{1}{y\hat{Z}_T^{\mathbb{G}}} - y\hat{Z}_T^{\mathbb{G}} \left( \frac{1}{y\hat{Z}_T^{\mathbb{G}}} - V_T(x, \pi) \right) \\ &\leq -\log y - \log \hat{Z}_T^{\mathbb{G}} - 1 + y\hat{Z}_T^{\mathbb{G}} V_T(x, \pi) \end{aligned}$$

Since  $V(x, \pi)$  is a non-negative process and  $\hat{Z}^{\mathbb{G}}V(x, \pi)$  is a  $\mathbb{G}$ -local martingale it is a  $\mathbb{G}$ -supermartingale. therefore, we get

$$\mathbb{E} \log V_T(x, \pi) \leq -1 - \log y - \mathbb{E} \log \hat{Z}_T^{\mathbb{G}} + xy .$$

Since this inequality holds for any  $\pi \in \mathcal{A}_{\mathbb{G}}(x)$ , we obtain by taking  $y = \frac{1}{x}$

$$V_{\mathbb{G}}^{\log} \leq \log x - \mathbb{E} \log \hat{Z}_T^{\mathbb{G}} .$$

Moreover, we have

$$\begin{aligned} \log V_T(x, \pi^{ins}) &= \log x + \int_0^T \pi_t^{ins \top} d\widetilde{M}_t + \int_0^T \pi_t^{ins \top} d\langle M \rangle_t \alpha_t + \int_0^T \pi_t^{ins \top} d\langle M - M^\tau \rangle_t \mu_t(G) \\ &= \log x + \int_0^T \pi_t^{ins \top} d\widetilde{M}_t + \int_0^T \pi_t^{ins \top} d\langle M \rangle_t \alpha_t + \int_0^T \pi_t^{ins \top} d\langle M \rangle_t \mu_t(G) \mathbb{1}_{\tau \leq t} \\ &= \log x + \int_0^T (\alpha_t + \mu_t(G))^\top d\widetilde{M}_t \\ &\quad + \int_0^T (\alpha_t + \mu_t(G) \mathbb{1}_{\tau \leq t})^\top d\langle M \rangle_t (\alpha_t + \mu_t(G) \mathbb{1}_{\tau \leq t}) \\ &= \log x + \log \hat{Z}_T^{\mathbb{G}} \end{aligned}$$

From (3.1) and Assumption 3.3, we get  $\pi^{ins} \in \mathcal{A}_{\mathbb{G}}^{\log}(x)$ . Therefore  $\pi^{ins}$  is an optimal strategy for the insider's problem.

Using (3.1), we get that  $\int_0^\cdot \alpha^\top dM$  and  $\int_0^\cdot \alpha^\top d\widetilde{M}$  are respectively  $\mathbb{F}$  and  $\mathbb{G}$  martingales. Therefore we have

$$0 = \mathbb{E} \left[ \int_0^T \alpha^\top d\widetilde{M} \right] - \mathbb{E} \left[ \int_0^T \alpha^\top dM \right] = \mathbb{E} \left[ \int_0^T \alpha^\top d\langle M \rangle \mu(G) \mathbb{1}_{[\tau, +\infty)} \right],$$

which gives

$$\mathbb{E} \left[ \log V_T(x, \pi^{ins}) \right] = \log x + \frac{1}{2} \mathbb{E} \left[ \int_0^T \alpha_t^\top d\langle M \rangle_t \alpha_t \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T \mu_t(G)^\top d\langle M - M^\tau \rangle_t \mu_t(G) \right].$$

(iii) The result is a consequence of (i) and (ii).  $\square$

## 4 Example of a hybrid model

In this section, we consider an explicit example where the random default time  $\tau$  is given by a hybrid model as in [8, 9] and the information flow  $G$  is supposed to depend on the asset values at a horizon time which is similar to [16].

Let  $B = (B_t, t \geq 0)$  be a standard Brownian motion and  $N^P = (N_t^P, t \geq 0)$  be a Poisson process with intensity  $\lambda \in \mathbb{R}_+$ . We suppose that  $B$  and  $N^P$  are independent. Let  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  be the complete and right-continuous filtration generated by the processes  $B$  and  $N^P$  where  $\mathcal{F}_t = \cap_{s > t} \sigma\{B_u, N_u^P; u \leq s\}$ . We define the default time  $\tau$  by a hybrid model. More precisely, consider a first asset process  $S_t^1 = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$  where  $\sigma > 0$  and let  $\tau_1 = \inf\{t > 0, S_t^1 \leq l\}$  where  $l$  is a given constant threshold such that  $l < S_0^1$ . In a similar way, consider a second asset process  $S_t^2 = \exp(\lambda t - N_t^P)$  and define  $\tau_2 = \inf\{t > 0, N_t^P = 1\}$ . Let the default time be given by  $\tau = \tau_1 \wedge \tau_2$  which is an  $\mathbb{F}$ -stopping time with a predictable component  $\tau_1$  and a totally inaccessible component  $\tau_2$ . Let the information flow  $G$  be given by the vector  $G = (S_{T'}^1, S_{T'}^2)$  where  $T' > T$  is a horizon time.

We first give the density of  $G$  which is defined in (2.4). By direct computations,

$$p_t(x_1, x_2) = \sqrt{\frac{T'}{T' - t}} \frac{\phi\left(\frac{\ln(x_1) + \frac{1}{2}\sigma^2 T' - \sigma B_t}{\sigma\sqrt{T' - t}}\right)}{\phi\left(\frac{\ln(x_1) + \frac{1}{2}\sigma^2 T'}{\sigma\sqrt{T'}}\right)} \cdot e^{\lambda t} \frac{(\lambda(T' - t))^{\lambda T' - \ln(x_2) - N_t^P}}{(\lambda T')^{\lambda T' - \ln(x_2)}} \frac{(\lambda T' - \ln(x_2))!}{(\lambda T' - \ln(x_2) - N_t^P)!} \mathbb{1}_{\mathbb{N}}(\lambda T' - \ln(x_2) - N_t^P)$$

where  $\phi$  is the density function of the standard normal distribution  $N(0, 1)$ , i.e.  $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ .

Denote by  $\tilde{N}^P$  the compensated Poisson process defined by

$$\tilde{N}_t^P = N_t^P - \lambda t, \quad t \geq 0.$$

The dynamics of the assets processes are then given by

$$\begin{aligned} dS_t^1 &= S_t^1 \sigma dB_t, \\ dS_t^2 &= S_t^2 \left( (e^{-1} - 1) d\tilde{N}_t^P + e^{-1} \lambda dt \right). \end{aligned}$$

This leads to consider the driving martingale  $M$  defined by  $M = (\sigma B, (e^{-1} - 1)\tilde{N}^P)^\top$ . Its oblique bracket of  $M$  is then given by

$$\langle M \rangle_t = \begin{pmatrix} \sigma^2 t & 0 \\ 0 & (e^{-1} - 1)^2 \lambda t \end{pmatrix}, \quad t \geq 0.$$

Then, we can write the dynamics of the asset processes using the notations of the previous section:

$$dS_t^i = S_t^i \left( dM_t^i + \alpha_t^1 d\langle M^i, M^1 \rangle_t + \alpha_t^2 d\langle M^i, M^2 \rangle_t \right)$$

with

$$\begin{aligned}\alpha_t^1 &= 0 \\ \alpha_t^2 &= \frac{e^{-1}}{(e^{-1} - 1)^2}\end{aligned}$$

for all  $t \geq 0$ . We can then compute the terms  $m^1$  and  $m^2$  appearing in Lemma 3.1 and we get

$$\begin{aligned}m_t^1(x) &= -\frac{1}{\sigma\sqrt{T'-t}} \frac{\phi'}{\phi} \left( \frac{\ln(x_1) + \frac{1}{2}\sigma^2 T' - \sigma B_t}{\sigma\sqrt{T'-t}} \right) \\ &= \frac{\ln(x_1) + \frac{1}{2}\sigma^2 T' - \sigma B_t}{\sigma^2(T'-t)}\end{aligned}$$

and

$$m_t^2(x) = \frac{1}{(e^{-1} - 1)} \left( \frac{\lambda T' - \ln(x_2) - N_{t-}^P}{\lambda(T'-t)} - 1 \right)$$

for  $t \geq 0$ . Since the matrix  $\langle M \rangle$  is diagonal the process  $\mu$  given by Lemma 3.2 can be taken such that  $\mu = (m^1, m^2)^\top$ . We easily check that Assumption 3.3 is satisfied. We can then apply Theorem 3.5 to the optimization problem with maturity  $T$  and we get

- an optimal strategy for the ordinary agent given by

$$\pi_t^{ord} = \alpha_t, \quad t \in [0, T],$$

and the maximal expected utility

$$V_{\mathbb{F}}^{\log} = \log x + \frac{1}{2} \mathbb{E} \left[ \int_0^T \alpha_t^\top d\langle M \rangle_t \alpha_t \right] = \log x + \frac{e^{-1}}{(e^{-1} - 1)^2} \frac{\lambda T}{4},$$

- an optimal strategy for the insider given by

$$\pi_t^{ins} = \alpha_t + \mathbb{1}_{\tau \leq t} \mu_t(G), \quad t \in [0, T],$$

and the maximal expected logarithmic utility

$$\begin{aligned}V_{\mathbb{G}}^{\log} &= \log x + \frac{1}{2} \mathbb{E} \left[ \int_0^T \alpha_t^\top d\langle M \rangle_t \alpha_t \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T \mu_t(G)^\top d\langle M - M^\tau \rangle_t \mu_t(G) \right] \\ &= \log x + \frac{e^{-1}}{(e^{-1} - 1)^2} \frac{\lambda T}{4} + \frac{1}{2} \mathbb{E} \left[ \int_0^T \mathbb{1}_{t \geq \tau} \frac{(\ln(S_{T'}^1) + \frac{1}{2}\sigma^2 T' - \sigma B_t)^2}{\sigma^3(T'-t)^2} dt \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[ \int_0^T \mathbb{1}_{t \geq \tau} \left( \frac{\lambda T' - \ln(S_{T'}^2) - N_t^P}{\lambda(T'-t)} - 1 \right)^2 \lambda dt \right],\end{aligned}$$

- the insider's additional expected utility

$$\begin{aligned}V_{\mathbb{G}}^{\log} - V_{\mathbb{F}}^{\log} &= \frac{1}{2} \mathbb{E} \left[ \int_0^T \mathbb{1}_{t \geq \tau} \frac{(\ln(S_{T'}^1) + \frac{1}{2}\sigma^2 T' - \sigma B_t)^2}{\sigma^3(T'-t)^2} dt \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[ \int_0^T \mathbb{1}_{t \geq \tau} \left( \frac{\lambda T' - \ln(S_{T'}^2) - N_t^P}{\lambda(T'-t)} - 1 \right)^2 \lambda dt \right]\end{aligned}$$

where

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T \mathbb{1}_{t \geq \tau} \frac{(\ln(S_{T'}^1) + \frac{1}{2}\sigma^2 T' - \sigma B_t)^2}{\sigma^3 (T' - t)^2} dt \right] = \mathbb{E} \left[ \int_0^T \mathbb{1}_{t \geq \tau} \frac{(B_{T'} - B_t)^2}{\sigma (T' - t)^2} dt \right] \\ & = \mathbb{E} \left[ \int_{\tau \wedge T}^T \frac{1}{\sigma (T' - t)} dt \right] = \sigma^{-1} \mathbb{E} \left[ \ln \left( \frac{T' - \tau \wedge T}{T' - T} \right) \right] \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T \mathbb{1}_{t \geq \tau} \left( \frac{\lambda T' - \ln(S_{T'}^2) - N_t^P}{\lambda (T' - t)} - 1 \right)^2 \lambda dt \right] = \mathbb{E} \left[ \int_0^T \mathbb{1}_{t \geq \tau} \left( \frac{N_{T'}^P - N_t^P}{\lambda (T' - t)} - 1 \right)^2 \lambda dt \right] \\ & = \mathbb{E} \left[ \int_{\tau \wedge T}^T \frac{dt}{T' - t} \right] = \mathbb{E} \left[ \ln \left( \frac{T' - \tau \wedge T}{T' - T} \right) \right]. \end{aligned}$$

Hence we get

$$V_{\mathbb{G}}^{\log} - V_{\mathbb{F}}^{\log} = (\sigma^{-1} + 1) \mathbb{E} \left[ \ln \left( \frac{T' - \tau \wedge T}{T' - T} \right) \right].$$

We note that the gain of the insider is strictly positive. In the limit case where  $T' = T$ , the insider may achieve a terminal wealth that is not bounded due to possible arbitrage strategies.

## 5 Conclusion

We study in this paper an optimal investment problem under default risk where related information is considered as an exogenous risk added at the default time. The framework we present can also be easily adapted to information risk modelling for other sources of risks. The main contributions are twofold. First, the information flow is added at a random stopping time rather than at the initial time. Second, we consider in the optimization problem a random time which does not necessarily satisfy the standard intensity nor density hypothesis in the credit risk. From the theoretical point of view, we study the associated enlargement of filtrations and prove that Jacod's (H')-hypothesis holds in this setting. From the financial point of view, we obtain explicit logarithmic utility maximization results and compute the gain of the insider due to additional information.

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## Appendix

### Proof of Proposition 2.1

**Proof.** We begin with the proof of the “if” part. Assume that  $Z$  can be written in the form (2.3) such that  $Y$  is  $\mathbb{F}$ -predictable and  $Y(\cdot)$  is  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ -measurable. Since  $\tau$  is an  $\mathbb{F}$ -stopping time, the stochastic interval  $\llbracket 0, \tau \rrbracket$  is a  $\mathcal{P}(\mathbb{F})$ -measurable set. Hence the process  $\mathbb{1}_{\llbracket 0, \tau \rrbracket} Y$  is  $\mathbb{F}$ -predictable and hence is  $\mathbb{G}$ -predictable. It remains to prove that the process  $\mathbb{1}_{\llbracket \tau, +\infty \rrbracket} Y(G)$  is  $\mathbb{G}$ -predictable. By a monotone class argument (see e.g. Dellacherie and Meyer [10] Chapter I.19-24), we may assume that  $Y(G)$  is of the form  $Xf(G)$ , where  $X$  is a left-continuous  $\mathbb{F}$ -adapted process, and  $f$  is a Borel function on  $E$ . Thus  $\mathbb{1}_{\llbracket \tau, +\infty \rrbracket} Xf(G)$  is a left-continuous  $\mathbb{G}$ -adapted process, hence is  $\mathbb{G}$ -predictable. Therefore, we obtain that the process  $Z$  is  $\mathbb{G}$ -predictable.

In the following, we proceed with the proof of the “only if” part. Let  $Z$  be a  $\mathbb{G}$ -predictable process. We first show that the process  $Z\mathbb{1}_{\llbracket 0, \tau \rrbracket}$  is an  $\mathbb{F}$ -predictable process. Again by a monotone class argument, we may assume that  $Z$  is left continuous. In this case the process  $Z\mathbb{1}_{\llbracket 0, \tau \rrbracket}$  is also left continuous. Moreover, by the left continuity of  $Z$  one has

$$Z_t\mathbb{1}_{\{\tau \geq t\}} = \lim_{\varepsilon \rightarrow 0^+} Z_{t-\varepsilon}\mathbb{1}_{\{\tau > t-\varepsilon\}}, \quad t > 0.$$

Since each random variable  $Z_{t-\varepsilon}\mathbb{1}_{\{\tau > t-\varepsilon\}}$  is  $\mathcal{F}_t$ -measurable, we obtain that  $Z_t\mathbb{1}_{\{\tau \geq t > 0\}}$  is also  $\mathcal{F}_t$ -measurable, so that the process  $Y = Z\mathbb{1}_{\llbracket 0, \tau \rrbracket}$  is  $\mathbb{F}$ -adapted and hence  $\mathbb{F}$ -predictable (since it is left continuous). Moreover, by definition one has  $Z_t\mathbb{1}_{\{\tau \geq t > 0\}} = Y_t\mathbb{1}_{\{\tau \geq t > 0\}}$ .

For the study of the process  $Z$  on  $\llbracket \tau, +\infty \rrbracket$ , we use the following characterization of the predictable  $\sigma$ -algebra  $\mathcal{P}(\mathbb{G})$ . The  $\sigma$ -algebra  $\mathcal{P}(\mathbb{G})$  is generated by sets of the form  $B \times [0, +\infty)$  with  $B \in \mathcal{G}_0$  and sets of the form  $B' \times [s, s')$  with  $0 < s < s' < +\infty$  and  $B' \in \mathcal{G}_{s-} := \bigcup_{0 \leq u < s} \mathcal{G}_u$ . It suffices to show that, if  $Z$  is the indicator function of such a set, then  $\mathbb{1}_{\llbracket \tau, +\infty \rrbracket} Z$  can be written as  $\mathbb{1}_{\llbracket \tau, +\infty \rrbracket} Y(G)$  with  $Y(\cdot)$  being a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ -measurable function.

By (2.1),  $\mathcal{G}_0$  is generated by  $\mathcal{F}_0$  and sets of the form  $A \cap \{\tau = 0\}$ , where  $A \in \sigma(G)$ . Clearly for any  $B \in \mathcal{F}_0$ , the function  $\mathbb{1}_{B \times [0, +\infty)}$  is already  $\mathbb{F}$ -predictable process. Let  $U$  be a Borel subset of  $E$  and  $B = G^{-1}(U) \cap \{\tau = 0\}$ . Let  $Y(\cdot)$  be the  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ -measurable function sending  $(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times E$  to  $\mathbb{1}_{\{\tau(\omega)=0\}}\mathbb{1}_U(x)$ . Then one has  $\mathbb{1}_{B \times [0, +\infty)} = Y(G)$ . By a monotone class argument, we obtain that, if  $Z$  is of the form  $\mathbb{1}_{B \times [0, +\infty)}$  with  $B \in \mathcal{G}_0$ , then there exists a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ -measurable function  $Y(\cdot)$  such that  $\mathbb{1}_{\llbracket \tau, +\infty \rrbracket} Z = \mathbb{1}_{\llbracket \tau, +\infty \rrbracket} Y(G)$ .

In a similar way, let  $s, s' \in (0, +\infty)$ ,  $s < s'$ . By (2.1),  $\mathcal{G}_{s-}$  is generated by  $\mathcal{F}_{s-}$  and sets of the form  $A \cap \{\tau \leq u\}$  with  $u < s$  and  $A \in \sigma(G)$ . If  $B' \in \mathcal{F}_{s-}$ , then the function  $\mathbb{1}_{B' \times [s, s')}$  is already an  $\mathbb{F}$ -predictable process. Let  $U$  be a Borel subset of  $E$  and  $B' = G^{-1}(U) \cap \{\tau \leq u\}$ . Let  $Y(\cdot)$  be the  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ -measurable function sending  $(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times E$  to  $\mathbb{1}_{\{\tau(\omega) \leq u\}}\mathbb{1}_{[s, s')}(t)\mathbb{1}_U(x)$ , then one has  $\mathbb{1}_{B' \times [s, s')} = Y(G)$ . Therefore, for any process  $Z$  of the form  $\mathbb{1}_{B \times [s, s')}$  with  $B \in \mathcal{F}_{s-}$ , there exists a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ -measurable function  $Y(\cdot)$  such that  $\mathbb{1}_{\llbracket \tau, +\infty \rrbracket} Z = \mathbb{1}_{\llbracket \tau, +\infty \rrbracket} Y(G)$ . The proposition is thus proved. □