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Eric Amar, Bernard Chevreau, Isabelle Chalendar. Subspaces of  $H^p$  linearly homeomorphic to  $l^p$ . *Studia Mathematica*, 2019, 248 (3), pp. 233-253. <hal-01343701>

**HAL Id: hal-01343701**

**<https://hal.science/hal-01343701v1>**

Submitted on 10 Jul 2016

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# SUBSPACES OF $H^p$ LINEARLY HOMEOMORPHIC TO $\ell^p$ .

AMAR E., CHALENDAR I., CHEVREAU B.

ABSTRACT. We present two fast constructions of weak\*-copies of  $\ell^\infty$  in  $H^\infty$  and show that such copies are necessarily weak\*-complemented. Moreover, via a Paley-Wiener type of stability theorem for bases, a connection can be made in some cases between the two types of construction, via interpolating sequences (in fact these are at the basis of the second construction). Our approach has natural generalizations where  $H^\infty$  is replaced by an arbitrary dual space and  $\ell^\infty$  by  $\ell^p$  ( $1 \leq p \leq \infty$ ) relying on the notions of generalized interpolating sequence and bounded linear extension. An old (very simple but unpublished so far) construction of bases which are Besselian but not Hilbertian finds a natural place in this development.

## 1. INTRODUCTION.

This note is inspired by the construction in [Ha1] of special weak\*-closed subspaces in the algebra  $H^\infty$  (of bounded analytic functions in the open unit disc  $\mathbb{D}$ ). The last two named authors realized that it could be used to obtain weak\* homeomorphic copies of  $\ell^\infty$  in  $H^\infty$  in a very elementary fashion (at the expense of "forgetting" several additional technicalities needed by the author for later purposes). Then, the first author observed that another fast construction could be achieved via interpolating sequences using a deep result of P. Beurling. Thus below (Section 3) we present both types of construction. In fact a (relatively easy) characterization of weak\*-continuous linear mappings from  $\ell^\infty$  to  $H^\infty$  (Prop. 2.3) enables us to give a convenient description of "weak\*-copies" of  $\ell^\infty$  in  $H^\infty$  and, in particular, showing further on (Theorem 3.13) that any such copy is weak\*-complemented in  $H^\infty$ .

The starting point of our first construction is a countable separated set (i.e. consisting of isolated points)  $S := \{a_n; n \in \mathbb{N}\}$  in the unit circle  $\mathbb{T}$ . When the closure of this set has arc-length measure 0 the Rudin-Carleson interpolating theorem (see Remark 2 following Theorem 3.2) enables us to prove the existence of a weak\*-closed subspace  $\mathcal{E}$  of  $H^\infty$  whose predual has an  $\ell^1$ -basis (see below Section 2 for the terminology) consisting of evaluations at the points  $a_n$ . Then, a straightforward application of a Paley-Wiener type of stability theorem for bases (in the predual of  $\mathcal{E}$ ) leads to the fact that any sequence  $(b_n)_n$  in  $\mathbb{D}$  with  $b_n$  "close enough" to  $a_n$ , is an interpolating sequence (i.e. the map  $h \rightarrow (h(b_n))_n$  from  $H^\infty$  is onto  $\ell^\infty$ ) (see Theorem 3.8 and Theorem 3.9). In fact, this profusion of interpolating sequences in the "vicinity" of  $(a_n)_n$  can be obtained directly from Carleson characterization of such sequences under the mere separation of the sequence  $(a_n)_n$ ; moreover an idea from [Am3] leads to an explicit construction of a weak\*-copy of  $\ell^\infty$  for any of these interpolating sequences (cf. Theorem 3.11).

The above theme leads quite naturally to the question (already considered in the literature) of how to get copies of  $\ell^p$  in the Hardy space  $H^p$  for  $1 \leq p < \infty$ ; this question is related to the notion of Besselian and Hilbertian sequences and we briefly discuss it in Section 4. Along the way we give an elementary construction (coming from the first named author's thesis but never published

elsewhere) of a Hilbertian basis which is not Besselian (the first such known example being due to Babenko [Bab]).

During the elaboration of this paper we discovered that certain of our results were already known, e.g., the construction of copies of  $\ell^\infty$  via interpolating sequences is in Chap. VII of [Gar] or particular cases of more general ones (e.g., Theorem 3.13 which is a corollary of a deep result of [BesPelc]). However, in line with the above mentioned elementary character of the first construction (and starting point of this work), we have chosen to make this note as self-contained as possible, thus illustrating in (we hope) an easy fashion the interplay between some pieces of Banach space geometry and function theory.

Some terminology and preliminary material are developed in Section 2.

## 2. NOTATIONS AND PRELIMINARIES.

As usual,  $A(\mathbb{D})$  ( $\mathbb{D}$  open unit disk) denotes the disk algebra consisting of those functions continuous on the closed unit disk and analytic in  $\mathbb{D}$ . For  $1 \leq p \leq \infty$  the Hardy space  $H^p$  can be defined as the norm-closure (weak\*-closure if  $p = \infty$ ) of  $A(\mathbb{D})$  in  $L^p(= L^p(m))$  (where  $m$  denotes the normalized Lebesgue measure on the unit circle  $\mathbb{T}$ ). Via Fatou's lemma the space  $H^p$  can (and will) be identified as the space of analytic functions  $f$  in  $\mathbb{D}$  satisfying

$$(\|f\|_p :=) \sup_{0 \leq r < 1} \|f_r\|_p < \infty$$

where  $f_r$  is defined on  $\mathbb{T}$  by  $f_r(w) = f(rw)$  and the above weak\*-closure refers to the duality  $L^\infty(= L^\infty(m)) = (L^1)^*$ .

In particular,  $H^\infty$  can be seen as the Banach algebra of bounded analytic functions in  $\mathbb{D}$  (equipped with the "sup" norm:  $\|h\|_\infty := \sup_{z \in \mathbb{D}} |h(z)|$ ) and, since  $H_0^1$  is the preannihilator of  $H^\infty$  in the above duality,  $H^\infty$  is also the dual of the quotient space  $L^1/H_0^1$ . If  $[f]$  denotes the equivalence class of  $f \in L^1$  and  $h \in H^\infty$  we have thus

$$\langle [f], h \rangle = \langle f, h \rangle = \int_{\mathbb{T}} fh \quad (:= \int_{\mathbb{T}} f h dm = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) h(e^{it}) dt).$$

Among the elements in the predual  $L^1/H_0^1$  of  $H^\infty$  we have the evaluations  $E_\lambda$  at points  $\lambda \in \mathbb{D}$  ( $\langle E_\lambda, h \rangle = h(\lambda)$ ); in fact  $E_\lambda = [P_\lambda]$  where  $P_\lambda$  is the Poisson kernel at  $\lambda$  ( $P_\lambda(z) = \frac{1-|\lambda|^2}{|1-\bar{\lambda}z|^2}$ ). Of course these evaluations define (by restriction) weak\*-continuous linear functionals on any weak\*-closed subspace  $\mathcal{E}$  of  $H^\infty$  and will thus be identified (keeping the same notation  $E_\lambda$ ) with elements in the predual  $\mathcal{Q}_\mathcal{E}$  of  $\mathcal{E}$  (this predual  $\mathcal{Q}_\mathcal{E}$  can be seen either as a quotient of  $L^1$  or of  $L^1/H_0^1$ ). In some instances (see again Remark 2 after Theorem 3.2) evaluations at points of  $\mathbb{T}$  make sense and are weak\*-continuous.

We recall that a sequence  $(h_n)$  in  $H^\infty$  converges weak\* to 0 if and only if it is bounded and converges pointwise to 0 on  $\mathbb{D}$ .

We will also need some general duality facts. Given two Banach spaces  $X, Y$ ,  $\mathcal{L}(X, Y)$  will denote the Banach space of bounded linear operators from  $X$  to  $Y$ . We recall the following fundamental and classical result (cf. [Ru], Theorem 4.14.).

**Theorem 2.1.** *For  $S \in \mathcal{L}(X, Y)$  the following three assertions are equivalent :*

- a)  $S$  has closed range;
- b)  $S^*$  has (norm-) closed range;
- c)  $S^*$  has weak\*-closed range.

As an almost immediate corollary we have the following useful proposition.

**Proposition 2.2.** *Let  $X, Y$  be Banach spaces and  $\mathcal{M}$  a (linear) subspace of  $X^*$ .*

- a) Suppose  $\mathcal{M}$  is weak\*-homeomorphic to  $Y^*$  via a (linear) map  $T$  then  $\mathcal{M}$  is weak\*-closed (and  $T$  is norm-invertible).
- b) Conversely if a linear map  $T : Y^* \rightarrow X^*$  is weak\*-continuous and bounded below then it implements a weak\*-homeomorphism between  $Y^*$  and the (weak\* as well as norm-closed) subspace  $\mathcal{M} = T(Y^*)$ .

Proof

Everything is an immediate consequence of the classical duality result recalled above except the first part of a) for which (by virtue of this same result) we just need to show that  $T$  is bounded below. Suppose not: then there exists an unbounded sequence  $(y_n)_n$  in  $Y^*$  such that  $\|Ty_n\| \rightarrow 0$ ; consequently the sequence  $Ty_n$  converges weak\* to 0 and so does the sequence  $(y_n)_n$  in contradiction with its unboundedness.  $\blacksquare$

Spaces of complex-valued sequences will intervene throughout this paper. We denote by  $\ell^0$  (resp.  $c_{00}$ ) the linear space of all complex sequences (resp. finitely supported sequences) and by  $(\epsilon_n)_{n \in \mathbb{N}}$ ,  $(\epsilon_n = (\delta_{n,j})_{j \in \mathbb{N}})$  the canonical (algebraic) basis of  $c_{00}$ . The Banach spaces  $c_0$ ,  $\ell^p$ ,  $1 \leq p \leq \infty$  have their usual meaning.

We recall that a Schauder basis in a Banach space  $X$  (resp. a w\*-Schauder basis in a dual space  $X = (\mathcal{Q}_X)^*$ ) is a sequence  $(x_n)_n$  in  $X$  such that any  $x \in X$  has a unique norm expansion  $x = \sum_n \alpha_n x_n$  (resp. weak\*-convergent expansion  $x = \sum_n^{w*} \alpha_n x_n$ , that is, for any  $L \in \mathcal{Q}_X$  we have  $\langle L, x \rangle = \sum_n \alpha_n \langle L, x_n \rangle$ ) with  $(\alpha_n)_n \in \ell^0$ . Of course  $(\epsilon_n)_n$  is a Schauder basis for  $c_0$  and  $\ell^p$  (for  $p < \infty$ ) and a weak\*-Schauder basis for  $\ell^\infty$ . (The reader is referred to [Heil], [Si] for more details on bases.)

We recall also that two Schauder bases (resp. two w\*-Schauder bases)  $(x_n)_n$  in  $X$  and  $(y_n)_n$  in  $Y$  are said to be equivalent (resp. w\*-equivalent) if there exists  $A \in \mathcal{L}(X, Y)$  invertible (resp. w\*-homeomorphism) such that  $Ax_n = y_n$  for all  $n$ . We will say that a Schauder basis in  $X$  is a  $c_0$ -basis (resp.  $\ell^p$ -basis) if it is equivalent to the basis  $(\epsilon_n)_n$  (in the corresponding space and, of course, equivalence meaning w\*-equivalence in the case  $p = \infty$ ). Clearly  $X$  has an  $\ell^1$ -basis if and only if  $X^*$  has an  $\ell^\infty$ -basis. Thus our initial goal can be reformulated as finding weak\*-closed subspaces with an  $\ell^\infty$ -basis (or whose predual has an  $\ell^1$ -basis).

A first step in that direction is the following useful characterization of weak\*-continuous linear maps from  $\ell^\infty$  into  $H^\infty$ .

**Proposition 2.3.** *Let  $(g_n)_{n \in \mathbb{N}}$  be a sequence of functions in  $H^\infty$ . The following 3 conditions are equivalent:*

- (i) *There exists a weak\*-continuous linear map  $T : \ell^\infty \rightarrow H^\infty$  such that  $T\epsilon_j = g_j$ ,  $j \in \mathbb{N}$ .*
- (ii)  *$M := \sup_{z \in \mathbb{D}} \sum_{n \in \mathbb{N}} |g_n(z)| < \infty$ .*
- (iii) *There exists a Borel subset  $B$  of  $\mathbb{T}$  of full measure (i.e.  $m(B) = 1$ ) such that*

$$\tilde{M} := \sup_{z \in \mathbb{D}} \sum_{n \in \mathbb{N}} |g_n(z)| < \infty.$$

*When these conditions are fulfilled we have  $M = \tilde{M}$  and  $T = S^*$  with  $S$  (from  $L^1/H_0^1$  into  $\ell^1$ ) defined by  $[f] \rightarrow (\int_{\mathbb{T}} f g_n)_{n \in \mathbb{N}}$  and  $M = \tilde{M} = \|S\| (= \|T\|)$ .*

*Furthermore, in case the  $g_n$ 's all belong to the disk algebra, they are equivalent to*

- (ii)'  *$M_1 := \sup_{z \in \mathbb{D}} \sum_{n \in \mathbb{N}} |g_n(z)| < \infty$  (and  $M_1 = M$ ).*

Proof

The last assertion (equivalence of, say, (ii) and (ii)') under membership of the  $g_n$ 's in the disk algebra) is obvious and stated here just for "the record".

- (i)  $\Rightarrow$  (ii)

Let  $T$  be a weak\*-continuous linear map from  $\ell^\infty$  in  $H^\infty$  such that  $T\epsilon_j = g_j$ ,  $j \in \mathbb{N}$ . It is straightforward to check that (as indicated at the (iii) of the proposition 2.3)  $T$  is the adjoint of the (bounded linear) map  $S$  (from  $L^1/H_0^1$  into  $\ell^1$ ) defined by  $[f] \rightarrow (\int_{\mathbb{T}} f g_n)_{n \in \mathbb{N}}$ ; thus, for  $\lambda \in \mathbb{D}$ ,  $S(E_\lambda) = (\int P_\lambda g_n) = (g_n(\lambda))_n$  and we have  $\sum_{n \in \mathbb{N}} |g_n(\lambda)| (= \|S(E_\lambda)\|_1) \leq \|S\|$  (since the  $-$ characters-  $E_\lambda$  are of norm 1). Hence we have (ii) with  $M \leq \|S\|$ .

(ii)  $\Rightarrow$  (iii)

For each  $n$  there exists a subset  $\omega_n$  of  $\mathbb{T}$ , of measure 0, such that at any  $w \in \mathbb{T} \setminus \omega_n$ ,  $g_n(w)$  is the limit of  $g_n(z)$  as  $z$  approaches  $w$  nontangentially,  $g_n(w) = NT - \lim_{z \rightarrow w, z \in \mathbb{D}} g_n(z)$ . The set  $\omega := \bigcup_n \omega_n$  is also of measure 0 and for  $z \in B := \mathbb{T} \setminus \omega$ ,  $n \in \mathbb{N}$  we have

$$\sum_{k \leq n} |g_k(w)| = NT - \lim_{z \rightarrow w, z \in \mathbb{D}} \sum_{k \leq n} |g_k(z)| \leq M.$$

Therefore the sequence  $(g_n)_{n \in \mathbb{N}}$  satisfies (iii) with  $\tilde{M} \leq M$ .

(iii)  $\Rightarrow$  (i)

Let  $c = (c_n)_n \in \ell^\infty$ ; the series  $\sum_n c_n g_n$  is absolutely convergent on  $B$  (indeed, for  $z \in B$ ,  $\sum_n |c_n g_n(z)| \leq \tilde{M} \|c\|_\infty$ ). Hence, the sequence of partial sums  $(G_N = \sum_{n \leq N} c_n g_n)$  is uniformly bounded and, via an application of the Lebesgue dominated convergence theorem to the equalities  $\langle f, G_N \rangle = \int f G_N$ ,  $f \in L^1$ ,  $N \in \mathbb{N}$  we obtain its weak\*-convergence in  $L^\infty$ . Since  $H^\infty$  is weak\*-closed in  $L^\infty$  the limit  $Tc := \sum_n c_n g_n$  belongs to  $H^\infty$ . Moreover the (linear) map  $S_o : [f] \rightarrow (\langle f, g_n \rangle)_n = (\int f g_n)_n$  from  $L^1/H_0^1$  in (a priori)  $\ell^0$  is in fact  $\ell^1$  valued and bounded; indeed

$$\sum_n \left| \int f g_n \right| \leq \sum_n \int |f g_n| \leq \int |f| \sum_n |g_n| \leq \tilde{M} \|f\|_1.$$

The duality relation

$$\langle Sf, c \rangle = \langle f, Tc \rangle \quad (f \in L^1, c \in \ell^\infty)$$

yields the equality  $T = S^*$  and the weak\*-continuity of  $T$ . Note also that  $\|T\| = \|S\| \leq \tilde{M}$ ; combined with the above inequalities  $M \leq \|S\|$  and  $\tilde{M} \leq M$  this proves the equality  $M = \tilde{M}$  and concludes the proof.  $\blacksquare$

Thus, having a sequence  $(g_n)_n$  satisfying Assertion (ii) in the above proposition we need "something more" to ensure the boundedness below of  $T$ . The idea is to have  $|g_n|$  equal or close to 1 at some point  $a_n$  or on some subset (these points or subsets being sufficiently "separated" for different values of  $n$ ) and small elsewhere, 0 if possible at the  $a_k$ 's for  $k \neq n$  (note that this idea clearly points the way towards interpolating sequences).

### 3. CONSTRUCTIONS OF WEAK\*-COPIES OF $l^\infty$ IN $H^\infty$ .

#### 3.1. An elementary construction.

It is based on the following simple lemma. Here  $\bar{\mathbb{D}}$  is the closed unit disk and for a subarc  $\Gamma$  of  $\mathbb{T}$  with (distinct) endpoints  $w_1, w_2$  we denote by  $\Delta_\Gamma$  the union of  $\Gamma$  with the open domain (contained in  $\mathbb{D}$ ) whose boundary consists of  $\Gamma$  together with the closed segment of endpoints  $w_1, w_2$  (in case  $\Gamma$  is a closed arc we exclude  $w_1, w_2$  from  $\Delta_\Gamma$ )

**Lemma 3.1.** *Let  $\Gamma$  be a proper subarc of  $\mathbb{T}$  whose complement is not a singleton,  $a$  a point of  $\Gamma$  and  $\delta > 0$ , then there exists a function  $g \in A(\mathbb{D})$  such that*

- (i)  $g(a) = 1$  and  $|g(z)| < 1$  for  $z \in \bar{\mathbb{D}} \setminus \{a\}$ , and
- (ii)  $|g| < \delta$  on  $\bar{\mathbb{D}} \setminus \Delta_\Gamma$ .

Proof

Take  $g(z) = (\frac{\bar{a}z+1}{2})^q$  (or  $(\frac{1}{2-\bar{a}z})^q$ ) with  $q$  sufficiently large. ■

We now give ourselves a sequence of open subarcs of  $\mathbb{T}$ ,  $(\Gamma_n)_{n \in \mathbb{N}}$  whose closures (in  $\mathbb{T}$ ) are pairwise disjoint, a sequence  $(a_n)_n$  such that, for each  $n$ ,  $a_n \in \Gamma_n$  and strictly positive scalars  $\epsilon$ ,  $\delta_n$  such that  $\sum_{n \in \mathbb{N}} \delta_n < \epsilon < 1$ . (We note that if we wish to start this procedure with a given sequence  $(a_n)_n$  the only condition required on this sequence is that it be separated.)

We claim that the sequence given by application of the lemma (that is,  $(g_n)_n$  in the disk algebra  $A(\mathbb{D})$  such that, for each  $n$ ,  $g_n(a_n) = 1$ ,  $|g_n| < 1$  on  $\mathbb{T} \setminus \{a_n\}$ , and  $|g_n| < \delta_n$  on  $\mathbb{T} \setminus \Gamma_n$ ) satisfies Assertion (i) of the proposition 2.3 with  $M = 1 + \epsilon$ .

Indeed for  $z \in \Gamma_n$  we have  $|g_n(z)| \leq 1$  and for  $k \neq n$   $|g_k(z)| \leq \delta_k$ ; hence

$$\sum_{k \in \mathbb{N}} |g_k(z)| \leq (1 + \sum_{k \neq n} \delta_k) \leq (1 + \epsilon)$$

and this inequality is also satisfied for  $z \in \mathbb{T} \setminus \bigcup_{n \in \mathbb{N}} \Gamma_n$  (since there we have  $\sum_{k \in \mathbb{N}} |g_k(z)| \leq \epsilon$ ).

We now show that  $T$  is bounded below:

Let  $c = (c_n)_n \in \ell^\infty$  and  $h := Tc = \sum_n^{w^*} c_n g_n$ ; for a given  $n$  we have

a)  $|c_n| = \max_{z \in \Delta_n} |c_n g_n(z)|$

and, for any  $z \in \mathbb{D}$ , we have

b)  $|c_n g_n(z)| \leq |h(z)| + \sum_{j \neq n} |c_j g_j(z)|$ .

Specializing to  $z \in \Delta_{\Gamma_n}$  (in which case  $|g_j(z)| < \delta_j$  for  $j \neq n$ ) we obtain

$$|c_n g_n(z)| \leq \|h\| + \sum_{j \neq n} \delta_j |c_j| \leq \|h\| + \epsilon \|c\|$$

and, via a),  $|c_n| \leq \|h\| + \epsilon \|c\|$  for all  $n$ , leading easily to the desired result:

$$(1 - \epsilon) \|c\| \leq \|T(c)\|.$$

Let us summarize what we have shown.

**Theorem 3.2.** *The linear mapping  $T : c = (c_n)_{n \geq N} \rightarrow T(c) = \sum_{n \geq N} c_n g_n$  implements a weak\* and norm-homeomorphism between  $\ell^\infty$  and  $\mathcal{E} = T(\ell^\infty)$  such that  $\|T\| \leq 1 + \epsilon$ ,  $\|T^{-1}\| \leq (1 - \epsilon)^{-1}$ .*

### Remarks

1. Observe that it is possible to choose  $\epsilon$  so that the Banach-Mazur distance between  $\ell^\infty$  and  $\mathcal{E} = \mathcal{E}_\epsilon$ , i.e.,  $d_{BM}(\ell^\infty, \mathcal{E}_\epsilon) := \inf \{ \|U\| \|U^{-1}\| ; U \text{ isomorphism} \}$ , be arbitrarily close to 1.

2. Note also that we can replace the functions  $g_n$  by  $h_n = u_n g_n$  where the  $u_n$  satisfy  $\|u_n\|_\infty = 1 = |u_n(a_n)|$  without affecting the result (that is the map  $\tilde{T} : c = (c_n) \rightarrow \sum_n^{w^*} c_n h_n$  still implements a weak\*-homeomorphism between  $\ell^\infty$  and  $\tilde{T}(\ell^\infty)$ ). Let us illustrate this in the case where the closure  $F$  of the set  $\{a_n; n \in \mathbb{N}\}$  has measure 0 (this will happen for instance when the sequence  $(a_n)_n$  is convergent -which is in fact the situation in [Ha1]). This closed set  $F$  is thus a peak set for  $A(\mathbb{D})$ ; the  $(u_n)_n$  can then be chosen in  $A(\mathbb{D})$  by virtue of the Rudin-Carleson interpolation theorem (cf. [Gam] Chap. 2, Theorem 6.12) so as to satisfy  $u_n(a_k) = \delta_{n,k}$ . This will lead to  $h_n(a_k) = \delta_{n,k}$  and  $\tilde{T}$  bounded below by one.

We observe that in this case the corresponding  $\ell^1$ -basis in  $\mathcal{Q}_{\mathcal{E}}$  consists of evaluations  $E_{a_n}$  at the points  $a_n$  (i.e.,  $\langle E_{a_n}, h \rangle = h(a_n)$ ,  $h \in H^\infty$ ) which, though not defined on all of  $H^\infty$ , are well-defined and weak\*-continuous on  $\tilde{T}(\ell^\infty)$ .

3. The construction in [Ha1] (in fact already in [Ha2]) was inspired by [Wo] which develops a general technique of producing subspaces of  $H^p$  ( $1 \leq p \leq \infty$ ) which are (isomorphic to) a direct

sum of infinitely many copies of  $H^p$ . This technique relies on constructing subspaces  $\mathcal{M}$  which are  $\epsilon$ - supported on a subarc  $J$  of  $\mathbb{T}$ , that is,

$$\sup_{z \in \mathbb{T} \setminus J} |f(z)| \leq \epsilon \|f\|_\infty \text{ for } f \in \mathcal{M};$$

it is clear that in our construction above, each (one-dimensional) subspace  $\mathbb{C}g_n$  is  $\delta_n$ - supported on the corresponding  $\Gamma_n$  and hence our space  $\mathcal{E}$  above can be seen to be norm-isomorphic to  $\ell^\infty$  (with a control on  $d_{BM}(\ell^\infty, \mathcal{E})$  similar to ours) as a consequence of [Wo], Lemma 1 (but weak\*-topologies are not discussed there).

4. Since for each  $z \in \bar{\mathbb{D}}$  the sequence  $(g_n(z))_n$  is in  $\ell^1$ , the expansion for  $h = T(c) = \sum_n c_n g_n(z)$  shows that  $E_z$ , the evaluation at  $z$ , is well-defined on  $\mathcal{E}$  and weak\*-continuous, in fact  $E_z = S^{-1}(g_n(z))_n$ . The following continuity properties of the map  $z \rightarrow E_z$ , besides their intrinsic interest, will be useful later on. Here we denote by  $\mathcal{D}$  the derived set of  $\{a_n; n \in \mathbb{N}\}$  (that is, the set of accumulation points).

**Proposition 3.3.** *The map  $z \rightarrow E_z$  is continuous on  $\bar{\mathbb{D}} \setminus \mathcal{D}$  and has nontangential limit at every point of  $\mathbb{T}$ ; of course these limit properties transfer to any  $h$  in  $\mathcal{E}$  individually.*

Proof.

The last statement results from the inequality

$$\|h(z) - h(w)\| \leq \|E_z - E_w\| \|h\| \quad (h \in \mathcal{E}, z, w \in \bar{\mathbb{D}}).$$

Since  $S$  is bicontinuous, proving the first statement amounts to prove the same continuity properties for the map  $\Phi : \bar{\mathbb{D}} \ni z \rightarrow S(E_z) = (g_n(z))_n \in \ell^1$ .

*Continuity of  $\Phi$  on  $\bar{\mathbb{D}} \setminus \mathcal{D}$ :*

Let  $a \in \bar{\mathbb{D}} \setminus \mathcal{D}$  and  $\tau > 0$ ; by the definition of  $\mathcal{D}$  and the fact that  $\{a_n; n \in \mathbb{N}\}$  consists of isolated points there exists  $r > 0$  such that the open disk  $D_{a,r}$  intersects at most one  $\Delta_{\Gamma_n}$ . Consequently we can choose an integer  $N$  such that  $\sum_{n \geq N} \delta_n < \tau/3$  and, for  $n \geq N$ ,  $D_{a,r} \cap \Delta_{\Gamma_n} = \emptyset$  (hence

$$|g_n(a) - g_n(z)| < 2\delta_n \text{ for } z \in D_{a,r}.$$

Thus starting with the inequality

$$\|\Phi(a) - \Phi(z)\| \leq \sum_{n < N} |g_n(a) - g_n(z)| + \sum_{n \geq N} |g_n(a) - g_n(z)|$$

we obtain, for  $z \in D_{a,r}$

$$\|\Phi(a) - \Phi(z)\| \leq \sum_{n < N} |g_n(a) - g_n(z)| + 2\tau/3.$$

Now the continuity (with respect to  $z$ ) of the first term on the right hand side of this inequality enables us to pick  $\eta$  ( $0 < \eta < r$ ) such that, for  $|z - a| < \eta$ ,  $\sum_{n < N} |g_n(a) - g_n(z)| < \tau/3$ , yielding for such  $z$ ,  $\|\Phi(a) - \Phi(z)\| < \tau$ .

*Nontangential continuity of  $\Phi$ :*

Let  $a \in \mathbb{T}$  and, for  $0 < r < 1$ , let  $St_r$  the Stoltz domain defined by  $a$  and  $r$  (that is  $St_r$  is the interior of the convex hull of  $r\mathbb{D} \cup \{a\}$ ); there is at most one  $n$  (say  $n_0$ ) such that  $a \in \Delta_{\Gamma_n}$ ; elementary geometric considerations show that, as soon as the length of  $\Gamma_n$  is smaller than  $2\sqrt{1-r^2}$  (and  $n \neq n_0$ ) we have  $St_r \cap \Delta_{\Gamma_n} = \emptyset$ . This done we proceed as before to conclude.  $\blacksquare$

5. We note also that this construction provides (by restriction of the map  $T$  to  $c_0$ ) a norm-isomorphism between  $c_0$  and a subspace of the disk algebra, in other words a subspace of  $A(\mathbb{D})$  with a  $c_0$ - basis.

6. Finally in the case when  $(a_n)_n$  is convergent the set  $\mathcal{D}$  is reduced to a singleton, say  $\{a\}$ , we observe that the subspace  $\mathcal{E}$  does not seem to be "missing by much" its inclusion in the disk algebra  $A(\mathbb{D})$  : indeed only continuity at the point  $a$  might be lacking. Nevertheless this is enough to substantially "enlarge" the space  $\mathcal{E}$  since (being isomorphic to  $\ell^\infty$ ) it is not even separable.

### 3.2. A construction with interpolating sequences.

As is well-known, a sequence  $(b_n)_n$  in  $\mathbb{D}$  is said to be an interpolating sequence if the mapping  $\mathcal{J} : h \in H^\infty \rightarrow (h(b_n))_n \in \ell^\infty$  is onto.

Here the construction goes even faster thanks to the following result of Pehr Beurling.

**Theorem 3.4.** [Be] *Given an interpolating sequence,  $(b_n)_n$  in  $\mathbb{D}$ , there exists a sequence of bounded analytic functions  $(\beta_n)_{n \in \mathbb{N}}$  having the following properties:*

- (i)  $\beta_k(b_j) = \delta_{k,j}$  for all  $j, k \in \mathbb{N}$  and
- (ii)  $\sup_{z \in \mathbb{D}} \sum_{n \in \mathbb{N}} |\beta_n(z)| < M$  (where  $M$  is the interpolating constant).

Indeed, since Hypothesis (ii) is exactly Assertion (ii) of Proposition 2.3 we have already a weak\*-continuous linear map  $T$  from  $\ell^\infty$  in  $H^\infty$  such that  $\forall n \in \mathbb{N}$ ,  $T\epsilon_n = \beta_n$ . Moreover, for any  $c \in \ell^\infty$  and any  $n \in \mathbb{N}$ , we have

$$|c_n| = |c_n \beta_n(a_n)| = \left| \sum_k c_k \beta_k(a_n) \right| \leq \|T(c)\|_\infty$$

which shows that the map  $T$  is bounded below by 1. Thus, by Prop. 2.2,  $\mathcal{E} := T(\ell^\infty)$  is weak\*-closed and  $T$ , seen as a map from  $\ell^\infty$  into  $\mathcal{E}$ , is a weak\*-homeomorphism whose inverse is obviously the restriction to the subspace  $\mathcal{E}$  of the map  $\mathcal{J}$  (defined at the beginning of this subsection). Moreover by (i) we can write for any  $a = (a_n)_n \in \ell^1$  and  $c = (c_n)_n \in \ell^\infty$

$$\langle a, c \rangle = \sum_n a_n c_n = \sum_{j,k} a_j c_k \delta_{j,k} = \left\langle \sum_j a_j E_{b_j}, T(c) \right\rangle.$$

Thus the "preadjoint"  $S (\in \mathcal{L}(\mathcal{Q}_\mathcal{E}, \ell^1))$  of  $T$  is defined by  $S(\sum_j a_j E_{b_j}) = a$  : in other words, the predual  $\mathcal{Q}_\mathcal{E}$  admits  $(E_{b_n})_n$  as an  $\ell^1$ -basis and  $\mathcal{E}$  admits  $(\beta_n)_n$  as an  $\ell^\infty$ -basis.

Summing up we have shown:

**Theorem 3.5.** *Let  $(b_n)_n$  be an interpolating sequence in  $\mathbb{D}$  and  $(\beta_n)_n$  a sequence of  $H^\infty$  functions satisfying (i) and (ii) of Theorem 3.4 ; then the linear mapping*

$$T : c = (c_n)_{n \in \mathbb{N}} \rightarrow T(c) = \sum_{n \in \mathbb{N}} c_n \beta_n$$

*implements a weak\* and norm-homeomorphism between  $\ell^\infty$  and  $\mathcal{E} = T(\ell^\infty)$  (with inverse  $\mathcal{J}|_\mathcal{E}$ ) such that  $\|T\| \leq M$ ,  $\|T^{-1}\| \leq 1$  ; the sequence  $(E_{b_n})_n$  is an  $\ell^1$ -basis for  $\mathcal{Q}_\mathcal{E}$  and  $(\beta_n)_n$  an  $\ell^\infty$ -basis for  $\mathcal{E}$ .*

It turns out that the reference [Be] is a preprint which is far from widely available. Thus we give below a proof of a weaker version (bound  $M^2$  instead of  $M$ ) based on a lemma of Drury [Dru] and a more classical formulation of results on interpolating sequences.

We start with a "finite" version of what is needed. This is in [Am1], Prop. 2.2, (with  $n = 1$ ) where Drury's lemma is used along the lines of [Ber]. We include the proof for the sake of completeness.

**Lemma 3.6.** *Suppose the sequence  $(b_j)_{1 \leq j \leq n}$  consisting of pairwise distinct elements of  $\mathbb{D}$  is interpolating with constant  $M$  ; then we can find bounded analytic functions  $(\beta_j)_{1 \leq j \leq n}$  such that*

- (i)  $\beta_k(b_j) = \delta_{k,j}$   $1 \leq k \neq j \leq n$  and
- (ii)  $\sup_{z \in \mathbb{D}} \sum_{1 \leq j \leq n} |\beta_j(z)| < M^2$ .

Proof.

Let  $\lambda$  be a primitive  $n$ -th roots of unity in  $\mathbb{C}$ ; the hypothesis  $((b_j)_{1 \leq j \leq n}$  interpolating with constant  $M$ ) ensures the existence of functions  $(v_j)_{1 \leq j \leq n}$  such that  $v_k(b_j) = \lambda^{kj}$   $1 \leq j \leq n$  and  $\max_{1 \leq j \leq n} \|v_j\|_\infty \leq M$ .

Define the functions  $\varphi_j$  on  $\mathbb{D}$  by

$$\varphi_j(z) = \frac{1}{n} \sum_{k=1}^n \lambda^{-kj} v_k(z), \quad 1 \leq j \leq n,$$

the "Fourier transform" of the function  $v(\cdot, z) := v_k(z)$ , where  $z \in \mathbb{D}$  is a parameter, on the group  $G$  of the  $n$ -th root of unity with the measure  $\frac{1}{n} \sum_{k=1}^n \delta_k$ .

Observe that

$$\varphi_j(b_l) = \frac{1}{n} \sum_{k=1}^n \lambda^{-kj} v_k(b_l) = \frac{1}{n} \sum_{k=1}^n \lambda^{(l-j)k} = \delta_{j,l}, \quad 1 \leq j \neq l \leq n.$$

Clearly, the  $\varphi_j$ 's belong to  $H^\infty$  (and  $\|\varphi_j\|_\infty \leq M$ ); a straightforward computation shows that, for any  $z$  in  $\mathbb{D}$ ,

$$|\varphi_j(z)|^2 = \frac{1}{n^2} \left( \sum_{k=1}^n |\varphi_k(z)|^2 + \sum_{1 \leq k \neq l \leq n} \lambda^{j(l-k)} A_{k,l} \right)$$

with  $A_{k,l} = v_k(z) \overline{v_l(z)}$ .

Since, for given  $l \neq k$  (as already observed before), it follows that

$$\sum_{k=1}^n |\varphi_k(z)|^2 = \frac{1}{n^2} \sum_{j=1}^n \sum_{k=1}^n |v_k(z)|^2 \leq M^2.$$

The above equality can also be easily seen as a consequence of Plancherel theorem between the group on  $n$ -th roots of unity in  $\mathbb{C}$  and its dual.

The proof of the lemma is thus completed by setting  $\beta_j = (\varphi_j)^2$ . ■

Thus, given an interpolating sequence  $(b_n)_{n \in \mathbb{N}}$  with constant  $M$ , for each  $n$  we have a finite sequence of analytic functions  $(\beta_{j,n})_{1 \leq j \leq n}$  satisfying (i) and (ii) of lemma 3.6. Now a standard procedure completes the proof of (the weak version of) P. Beurling's theorem.

Since this procedure requires successive extractions of subsequences it is convenient to introduce the following notation for (strictly increasing in our case) maps  $\varphi$  and  $\psi$  of  $\mathbb{N}$  in itself:

$\varphi \prec \psi$  if there exists another strictly increasing sequence  $j : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\psi = \varphi \circ j$  (thus for any sequence whatsoever  $(u_n)_n$  the sequence  $(u_{\psi(n)})_n$  is a subsequence of the sequence  $(u_{\varphi(n)})_n$  (itself subsequence of  $(u_n)_n$ )).

Note that all the  $\beta_{j,n}$  are of norm bounded by  $M^2$ . As a first step we extract from the bounded sequence  $(\beta_{1,n})_n$  a subsequence  $(\beta_{1,\varphi_1(n)})_n$  which is weak\*-convergent to say  $\beta_1$ . Without loss of generality we may and do assume that  $\varphi_1(1) \geq 2$  so that the subsequence  $(\beta_{2,\varphi_1(n)})_n$  of  $(\beta_{2,n})_n$  is well-defined. As a second step we extract from  $(\beta_{2,\varphi_1(n)})_n$  a subsequence  $(\beta_{2,\varphi_2(n)})_n$  (that is,  $\varphi_1 \prec \varphi_2$ , with again, as precaution for the next step,  $\varphi_2(1) \geq 3$ ) which is weak\*-convergent to say  $\beta_2$ . Continuing in this fashion we clearly obtain a sequence of strictly increasing maps  $\varphi_k$  from  $\mathbb{N}$  to  $\mathbb{N}$  such that  $\varphi_1 \prec \varphi_2 \prec \dots \prec \varphi_k \prec \dots$  with, for all  $k$ ,  $\varphi_k(1) \geq k+1$  and the sequence  $(\beta_{k,\varphi_k(n)})_n$  is weak\*-convergent to say  $\beta_k$ .

We now show that the sequence  $(\beta_j)_j$  satisfies the desired properties:

let  $j, l$  in  $\mathbb{N}$ ; by the initial definition of the (finite) sequence  $\{\beta_{j,\varphi_j(n)}\}_{1 \leq j \leq \varphi_j(n)}$ , we have  $\beta_{j,\varphi_j(n)}(b_l) = \delta_{j,l}$ , an equality which is of course preserved by taking the limit as  $n \rightarrow \infty$ , yielding thus  $\beta_j(b_l) = \delta_{j,l}$ .

Similarly we have for any given  $z \in \mathbb{D}$ ,  $\sum_{1 \leq j \leq \varphi_j(n)} |\beta_{j, \varphi_j(n)}| < M^2$  ; in particular for any integer  $N$  we will have,

$$\sum_{1 \leq j \leq N} |\beta_{j, \varphi_j(n)}| < M^2 \text{ as soon as } \varphi_j(n) \geq N,$$

and taking the limit as  $n \rightarrow \infty$  we get  $\sum_{1 \leq j \leq N} |\beta_j| < M^2$  from which the proof is easily concluded. ■

**Remark 3.7.** *Of course, the above construction is more "existential" than "explicit". However, as indicated in the introduction, in some situations it is possible to obtain a "concrete" construction (see Theorem 3.11).*

### 3.3. Additional remarks.

When the elementary construction (developed in section 3.1) can be modified via the Rudin-Carleson interpolation theorem we obtain a weak\*-copy  $\mathcal{E}$  of  $\ell^\infty$  whose predual  $\mathcal{Q}_\mathcal{E}$  admits an  $\ell^1$ -basis consisting of point evaluations at the points  $a_n$  (cf. Remark 2 following Theorem 3.2 above). It is interesting to observe that in such a case the predual  $\mathcal{Q}_\mathcal{E}$  admits also  $\ell^1$ -bases of the form  $(E_{b_n})_n$  with the  $b_n$ 's in  $\mathbb{D}$ . Thus  $\mathcal{E}$  admits the system of functions biorthogonal to  $(E_{b_n})_n$ , say  $(\beta_n)_n$ , as an  $\ell^\infty$ -basis and it follows easily that  $(b_n)_n$  is an interpolating sequence. Before stating a precise result we recall a Paley-Wiener type stability theorem that will be the key to its proof.

**Theorem 3.8.** ([Si] Theorem 9.1, p. 84) *Let  $(x_n)_n$  be a Schauder basis in a Banach space  $X$ .*

a) *A sequence  $(y_n)_n$  in  $X$  for which there exists a constant  $\lambda \in ]0, 1[$  such that*

$$\forall \alpha = (\alpha_j)_j \in c_{00}, \left\| \sum_j \alpha_j (x_j - y_j) \right\| \leq \lambda \left\| \sum_j \alpha_j x_j \right\|$$

*is a basis (equivalent to the basis  $(x_n)_n$ ).*

b) *If in addition  $(x_n)_n$  is an  $\ell^1$ -basis then there exists  $\eta > 0$  such that  $(y_n)_n$  is an  $\ell^1$ -basis whenever  $\|x_n - y_n\| < \eta$  for all  $n$ .*

Proof.

a) The hypothesis implies that the linear map  $\sum_j \alpha_j x_j \rightarrow \sum_j \alpha_j (x_j - y_j)$  defined on the linear span of the  $x_n$ 's extends into a bounded linear operator  $U \in \mathcal{L}(X)$  satisfying  $\|U\| \leq \lambda < 1$ . Consequently the operator  $A := I_X - U$  is invertible and since  $Ax_n = y_n$  for all  $n$  we have the desired conclusion.

b) Since  $(x_n)_n$  is an  $\ell^1$ -basis there exists a constant  $\nu$  such that for all  $\alpha = (\alpha_j)_j \in c_{00}$  we have  $\|\alpha\|_1 \leq \nu \left\| \sum_j \alpha_j x_j \right\|$ ; for  $\alpha \in c_{00}$  and  $(y_n)_n$  such that  $\|x_j - y_j\| \leq \eta$  for all  $j$ , the inequalities

$$\left\| \sum_j \alpha_j (x_j - y_j) \right\| \leq \eta \|\alpha\|_1 \leq \eta \nu \left\| \sum_j \alpha_j x_j \right\|$$

enable us to apply a) whenever  $\eta \nu < 1$ . ■

**Theorem 3.9.** *Let  $\mathcal{E}$  be a weak\*-closed subspace of  $H^\infty$  consisting of functions continuously extendable to a subset  $\Omega$  of  $\mathbb{D}$  ( $\Omega \supset \mathbb{D}$ ) such that the map  $\Omega \ni z \rightarrow E_z$  is valued in  $\mathcal{Q}_\mathcal{E}$  and continuous. Suppose furthermore that  $\mathcal{Q}_\mathcal{E}$  admits an  $\ell^1$ -basis of the form  $(E_{a_n})_n$  with the  $a_n$ 's in  $\mathbb{T} \cap \Omega$ . Then there exists (strictly) positive numbers  $\tau_n$  such that*

(i) *the closed disks  $\mathbb{D}_n := \mathbb{D}_{a_n, \tau_n}$  are pairwise disjoint, and*

(ii) for any sequence  $(b_n)_n$  chosen so that (for all  $n$ )  $b_n \in \mathbb{D} \cap \mathbb{D}_n$  the sequence  $(E_{b_n})_n$  is an  $\ell^1$ -basis for  $\mathcal{Q}_{\mathcal{E}}$  (and consequently  $(b_n)_n$  is an interpolating sequence).

(Of course in the context of this theorem the system  $(g_n)_n$  in  $\mathcal{E}$ , biorthogonal to  $(E_{a_n})_n$  is an  $\ell^\infty$ -basis for  $\mathcal{E}$  and the map implementing the weak\*-homeomorphism between  $\ell^\infty$  and  $\mathcal{E}$  is given by  $T(\gamma) = \sum_j \gamma_j g_j \cdot$ )

Proof.

(i) Since  $(E_{a_n})_n$  is an  $\ell^1$ -basis in  $\mathcal{Q}_{\mathcal{E}}$ , we have  $\inf_{n \neq m} \|E_{a_n} - E_{a_m}\| > 0$ ; this inequality combined with the continuity of the map  $z \rightarrow E_z$  at the points  $a_k$  ensures that for all  $n$ ,  $d(a_n, \{a_m ; m \neq n\}) > 0$ .

(ii) : Immediate consequence of the continuity of the map  $z \rightarrow E_z$  combined with the above Theorem 3.8 (with  $X = \mathcal{Q}_{\mathcal{E}}$ ,  $(x_n)_n = (E_{a_n})_n$ ,  $(y_n)_n = (E_{b_n})_n$ ).  $\blacksquare$

**Remark 3.10.** *This last argument is in fact a particular case of the Krein-Milman-Rutman stability theorem (cf. [Si] Theorem 10.3 p. 98) simplified here because we deal with the  $\ell^1$ -norm.*

As announced in the introduction this rich supply of interpolating sequences in the vicinity of  $(a_n)_n$  can be obtained under the mere hypothesis of separation of the  $a_n$ 's on the unit circle. This is probably known to specialists in this area but for the sake of completeness we give a precise statement and prove it using the Carleson's characterization of interpolating sequences, namely a sequence  $(b_n)_n$  in  $\mathbb{D}$  is interpolating if there exists  $m > 0$  such that

$$\inf_{k \in \mathbb{N}} \prod_{j \neq k} \left| \frac{b_k - b_j}{1 - \bar{b}_k b_j} \right| > m.$$

We will even exhibit a concrete linear extension operator associated to any such sequence  $(b_n)_n$  close to  $(a_n)_n$  (but of course the weak\*-closed subspace of  $H^\infty$ -image of this operator- depends on the sequence  $(b_n)_n$  while in Theorem 3.9 we obtain the same subspace). We will use the standard terminology  $B$ ,  $B_k$ ,  $k = 1, 2, \dots$  denote the usual Blaschke products,

$$B(z) := \prod_n \frac{b_n - z}{1 - \bar{b}_n z}, \quad B_k(z) := \prod_{n \neq k} \frac{b_n - z}{1 - \bar{b}_n z}, \quad k \in \mathbb{N}$$

and, following [Am3] formula 7.6, p.48, we introduce the functions  $g_k$  :

$$g_k(z) = \frac{1 - |b_k|^2}{1 - \bar{b}_k z} \frac{B_k(z)}{B_k(b_k)}, \quad k \in \mathbb{N}.$$

**Theorem 3.11.** *Let  $(a_n)_n$  be a separated sequence in  $\mathbb{T}$  and let  $m \in ]0, 1[$ ; then, there exists a sequence  $(\eta_n)_n$  of strictly positive numbers such that any sequence  $(b_n)_n$  in  $\mathbb{D}$  satisfying  $|b_n - a_n| < \eta_n$  for all  $n$  is an interpolating sequence with constant no smaller than  $m$ . Moreover the functions  $(g_n)_n$  associated to the sequence  $(b_n)_n$  by the above formula generate a weak\*-copy of  $\ell^\infty$  (with  $(g_n)_n$  as an  $\ell^\infty$ -basis).*

Proof.

We choose first positive numbers  $d_n$  such that the closed disks  $\bar{\mathbb{D}}_{a_n, d_n}$  are pairwise disjoint.

*Notational simplification for this proof:*  $\bar{\mathbb{D}}_{a,d}$  means  $\mathbb{D} \cap \bar{\mathbb{D}}_{a,d}$ .

We get by induction on  $N$  a sequence of positive numbers  $\tau_n$  and a sequence of compact sets  $F_n$  in  $\bar{\mathbb{D}}^n$  in the following fashion.

Set  $\tau_1 = d_1$  and  $F_1 = \bar{\mathbb{D}}_{a_1, \tau_1}$ ; the map

$$\Theta : \bar{\mathbb{D}}_{a_2, d_2} \times F_1 \ni (t, s) \rightarrow \left| \frac{t-s}{1-\bar{t}s} \right|$$

is continuous and satisfies  $\Theta(a_2, s) (= 1) > m$  for all  $s$ . Thus the open set  $\Theta^{-1}(]m, \infty[)$  contains the (compact) set  $\{a_2\} \times F_1$ ; therefore, by a standard (and elementary) compactness argument, it contains a (compact) set  $F_2$  of the form  $F_2 = \bar{\mathbb{D}}_{a_2, \tau_2}$  with  $0 < \tau_2 (\leq d_2)$ .

Suppose that  $\tau_1, \tau_2, \dots, \tau_N$  have been found such that for any  $s = (s_1, \dots, s_N) \in F_N := \prod_{1 \leq n \leq N} \bar{\mathbb{D}}_{a_n, d_n}$ , we have

$$\Theta_N(s) := \prod_{j \neq k, j, k \leq N} \left| \frac{s_j - s_k}{1 - \bar{s}_j s_k} \right| > m.$$

The function

$$\Theta_{N+1} : \bar{\mathbb{D}}_{a_{N+1}, d_{N+1}} \times F_N \ni (t, s) \rightarrow \left( \prod_{1 \leq j \leq N} \left| \frac{t - s_j}{1 - \bar{t}s_j} \right| \right) \times \Theta_N(s),$$

is continuous and satisfies  $\Theta_{N+1}(a_{N+1}, s) = \Theta_N(s) > m$  for all  $s$  in  $F_N$ . The same continuity-compactness argument as above yields the existence of a (compact) set  $F_{N+1} = \bar{\mathbb{D}}_{a_{N+1}, \tau_{N+1}} \times F_N$  such that, for any  $(t, s) \in F_{N+1}$ ,  $\Theta_{N+1}(t, s) (= 1) > m$ . Thus the first statement of our theorem holds for any choice of the  $\eta_n$ 's satisfying  $\eta_n \leq \tau_n$ .

To prove the last assertion we just have to show that the functions  $(g_k)_k$  satisfy Condition (ii) of Theorem 3.4 (indeed, as well as the functions  $(B_k(z_k(b_k)))_k$ , they satisfy Condition (i) which in turn ensures that the associated map  $(c_n)_n \rightarrow \sum_n c_n g_n$  is bounded below). By the Carleson condition we have  $|B_k(b_k)| \geq m$  (we assume of course that all sequences  $(b_n)_n$  considered here satisfy  $|b_n - a_n| < \tau_n$ ) hence  $|B_k(z_k(b_k))| \leq 1/m$  for all  $k$  in  $\mathbb{N}$  and  $z$  in  $\mathbb{D}$ .

Therefore, we just need to perform some work on the additional factor  $\frac{1-|b_n|^2}{|1-\bar{b}_n z|}$  to ensure (ii).

Firstly we note that for any  $z, t$  in  $\mathbb{D}$  we have by geometrical observation the inequality  $|1 - \bar{b}_n z| \geq 1 - |b_n|$  which leads to  $\frac{1-|b_n|^2}{|1-\bar{b}_n z|} \leq 2$ . Next we let  $(\epsilon_n)_n$  another sequence of positive numbers such that  $\sum_j \epsilon_j = \epsilon < m$ . We apply again the standard continuity-compactness argument to the function  $\varphi : K_n \times \bar{\mathbb{D}}_{a_n, \tau_n} \ni (z, t) \rightarrow \frac{1-|t|^2}{|1-\bar{t}z|}$  (obviously continuous and with value 0 on  $K_n \times \{a_n\}$ ) to obtain  $\eta_n > 0$  (and of course  $\eta_n < \tau_n$ ) such that  $\varphi$  is smaller than  $\epsilon_n$  on  $K_n \times \bar{\mathbb{D}}_{a_n, \eta_n}$ .

Now let  $z \in \mathbb{D}$  :

a) if  $z$  does not belong to the union of the disks  $\bar{\mathbb{D}}_{a_n, \tau_n}$  then, for each  $n$ ,  $|g_n(z)| \leq \epsilon_n/m$  (recall that  $|B_n(b_n)| > m$ ) and hence  $\sum_j |g_j(z)| < \epsilon$ ;

b) otherwise  $z$  belongs to exactly one of these disks say  $\bar{\mathbb{D}}_{a_k, \tau_k}$ ; in this case we get

$$\sum_j |g_j(z)| = \sum_{j \neq k} |g_j(z)| + |g_k(z)| \leq \sum_{j \neq k} \epsilon_j + 2/m.$$

Thus Condition (ii) is clearly satisfied; this concludes the proof. ■

In the context of Theorem 3.9 one might think that the closure of the set  $\{a_n; n \in \mathbb{N}\}$  is necessarily of measure 0. In fact it is easy to sketch examples showing that this idea is wrong.

**Example 3.12.** We start with a closed interval  $J = [\alpha, \beta]$  of the real line and build in this interval a Cantor type set  $K = \bigcap_{k \geq 1} J_k$  of positive measure in the standard fashion, that is,  $J_k = J_{k-1} \setminus \bigcup_{n=2^{k-1}}^{2^k-1} I_n$  (with  $J_0 = J$ ) where, for  $2^{k-1} \leq n < 2^k$ ,  $I_n = ]b_n, c_n[$  is an open interval of length  $\rho_k$  (with  $\sum_{k > 0} \rho_k < \beta - \alpha$  to ensure that  $K$  has positive measure) removed in the usual "centered" way from each of the successive  $2^{k-1}$  intervals whose union is  $J_{k-1}$ . It follows easily

from this classical construction that for any sequence  $(x_n)_n$  satisfying  $b_n \leq x_n \leq c_n$  ( $n \geq 1$ ) the endpoints  $b_n, c_n$  are in the closure of the set  $\{x_j; j \geq 1\}$ ; thus this closure contains the set  $K$  (in fact it is equal to  $K \cup \{x_j; j \geq 1\}$ ). Now choosing  $x_n$  in the open interval  $]b_n, c_n[$  we have  $d(x_n, \{x_j; j \neq n\}) (= \min\{x_n - b_n, c_n - x_n\}) > 0$ . Taking  $\beta - \alpha < 2\pi$  the sequence  $(a_n)_n := (e^{ix_n})_n$  will be one of the desired examples.

### 3.4. Weak\*-copies of $\ell^\infty$ are w\*-complemented.

As shown in the theorem 3.5, in the second type of construction the map  $T^{-1}(\mathcal{E} \rightarrow \ell^\infty)$  is exactly the restriction to  $\mathcal{E}$  of the interpolating map  $\mathcal{J} : H^\infty \rightarrow \ell^\infty$   $\mathcal{J}(h) := (h(b_n))_n$ . Observe also that the map  $P : H^\infty \rightarrow H^\infty$  defined by  $P(h) := T(\mathcal{J}(h))$  is a weak\*-continuous projection (norm-bounded by  $M^2$ ) whose range is the space  $\mathcal{E}$  and, consequently,  $\mathcal{E}$  is weak\*-complemented. This fact is no "accident" and is valid in a more general setting which we now describe. Let  $X$  denote an arbitrary Banach space  $X$  and let  $Z = X^*$ . Note first that the map  $\mathcal{J}$  above (where  $X = L^1/H_0^1$ ) can be defined by  $\mathcal{J}(h) = (\langle E_{b_j}, h \rangle)_j$  (where  $E_{b_j}$  is the element in  $L^1/H_0^1$  evaluation at  $b_j$ )

We generalize this map by associating to any (bounded) sequence  $K = (k_j)_j$  of elements in  $X$  an "interpolation map"  $R (= R_K)$  from  $Z$  in  $\ell^\infty$  defined by  $R(h) = (\langle k_j, h \rangle)_j$ ,  $h \in Z$ .

We also associate to the sequence  $K$  the (bounded linear) map  $V (= V_K)$  from  $\ell^1$  into  $X$  defined by  $V(\gamma) = \sum_n \gamma_n k_n$ ,  $\gamma \in \ell^1$  and note that  $V^* = R$ .

We will say that  $K$  is an  $\ell^\infty$ -interpolating sequence ( $\ell^\infty$ -IS for short) if the map  $R_K$  is onto (or equivalently if the map  $V_K$  is bounded below). We will denote by  $X_K$  the closure of the range of  $V_K$ .

In this setting we have the following general result which is, in fact, with a slightly different formulation, the equivalence of Assertions 2 and 3 of Theorem 10 in [Di], page 48, (see below, Remark 3).

**Theorem 3.13.** *a) Let  $\mathcal{E}$  be a (linear) subspace of  $Z$  weak\*-homeomorphic to  $\ell^\infty$  (via  $T$ ). Then*

*(i) there exists a (bounded)  $\ell^\infty$ -interpolating sequence  $K = (k_j)_j$  such that  $TR_K = I_Z$  (and  $V_K$  is bounded below);*

*(ii) the space  $\mathcal{E}$  is weak\*-complemented in  $Z$  and  $X_K$  is (norm-) complemented in  $X$ .*

*b) Conversely if  $K = (k_j)_j$  is an  $\ell^\infty$ -interpolating sequence such that  $X_K$  is complemented in  $X$ , say  $X = X_K \oplus Y$  then  $Y^\perp$  is weak\*-homeomorphic to  $\ell^\infty$ .*

Of course the case  $X = L^1/H_0^1$  gives the result announced by the title of this subsection.

Proof.

a) By Proposition 2.3 we can view  $T$  as a weak\*-continuous, bounded below map from  $\ell^\infty$  into  $Z$  with range  $\mathcal{E}$ . Thus  $T = S^*$  and  $S (\in \mathcal{L}(X, \ell^1))$  is onto (by virtue of the classical duality result recalled in the introduction). Hence there exists a sequence  $K = (k_n)_n$  in  $X$  such that  $S(k_n) = \epsilon_n$ ,  $n \in \mathbb{N}$ . Note that such a sequence is necessarily biorthogonal to the sequence  $(g_n)_n = T(\epsilon_n)_n$  in  $Z$ ; indeed

$$\delta_{j,n} = \langle \epsilon_j, \epsilon_n \rangle = \langle S(k_j), \epsilon_n \rangle = \langle k_j, T\epsilon_n \rangle = \langle k_j, g_n \rangle.$$

Moreover, by the open mapping theorem, we may (and do) choose the sequence  $K = (k_n)_n$  to be bounded. Then  $k_n = V_K(\epsilon_n)$  for all  $n$ , and hence  $SV_K = I_{\ell^1}$ ; consequently,  $V_K S (= V_K S V_K S)$  is a bounded projection in  $X$  whose range  $X_K$  is thus norm-complemented in  $X$ . Similarly,  $R_K T (= V_K^* S^* = (S V_K)^* = I_{\ell^\infty})$  and  $TV_K$  is a weak\*-continuous projection with range  $\mathcal{E}$  which is therefore weak\*-complemented in  $Z$ .

b) Here, since  $K$  is an  $\ell^\infty$ -IS,  $V_K$  implements an isomorphism between  $\ell^1$  and  $X_K$ ; also we know, from the topological direct sum  $X = X_K \oplus Y$ , that  $X_K$  is isomorphic to the quotient of  $X/Y$ . Therefore, by standard duality facts, the dual of  $X_K$  is isomorphic to  $Y^\perp$ .

Putting all this together :  $\ell^\infty = (\ell^1)^* \cong X_K^* \simeq Y^\perp$  we obtain the desired result. ■

### Remarks

1. We come back briefly to the context of theorem 3.9 to point out that in this case the subspace  $\mathcal{E}$  is also norm-complemented in the norm-closed subalgebra  $\mathcal{A}_\Omega$  consisting of all the functions in  $H^\infty$  continuously extendable to  $\mathbb{D} \cup \Omega$ . Indeed the map  $\mathcal{J}$  (associated to  $(a_n)_n$ ) makes sense on  $\mathcal{A}_\Omega$  and so does the (norm-bounded) projection  $P$ .

2. The map  $T$  (from  $\ell^\infty$  into  $H^\infty$ ) implementing the weak\*-homeomorphism is usually called the (bounded) linear extension map (cf. [Gar] , Chap. VII, Section 2).

Thus in the more general setting above we will say that the sequence  $K = (k_j)_j$  in  $X$  is a boundedly linearly extendable sequence (notation: *BLES*) in  $X$  for  $Z$  if  $R_K$  admits a right inverse (i.e. there exists  $T \in \mathcal{L}(\ell^\infty, Z)$  such that  $RT = I_{\ell^\infty}$ ) (or equivalently if  $V_K$  admits a left inverse).

It follows immediately from these definitions and characterizations that an  $\ell^\infty$ -IS  $K$  such that  $X_K = X$  is a *BLES*. Thus (via standard facts about duality of subspaces and quotient spaces) a sequence  $K$  is an  $\ell^\infty$ -IS in  $X$  for  $Z = X^*$  if and only if it is a *BLES* in  $X_K$  for  $\tilde{Z} = (X_K)^* = Z/({}^\perp X_K)$ .

One more important fact coming out right away from the above is that there exists an  $\ell^\infty$ -IS in  $X$  for  $Z = X^*$  if and only if  $X$  contains (isomorphically) a copy of  $\ell^1$ , that is, in turn by virtue of the celebrated Rosenthal's  $\ell^1$ - theorem (cf. [Ros] or [Chap.1] in [NiPo]), if and only if there exists in  $X$  a bounded sequence  $(x_n)_n$  which admits no weakly Cauchy subsequence. Recall that a sequence  $(y_n)_n$  is weakly Cauchy if for any  $h$  in  $X^*$  the numerical sequence  $(\langle y_n, h \rangle)_n$  converges. Rosenthal's  $\ell^1$ - theorem says more precisely that a bounded sequence without weakly Cauchy subsequence admits a subsequence equivalent to the canonical basis of  $\ell^1$ . In particular we deduce from the above that there are such sequences in  $L^1/H_0^1$ .

3. Theorem 10 in Chapter V of [Di] (which, as mentioned there, is essentially Theorem 4 of [BesPelc] though the latter does not consider weak\*-topology) states also (again, up to minor variations in the formulation) that Assertions a) and b) in Theorem 3.13 above are equivalent to the existence a copy of  $c_0$  in  $X^*$ . This is done via the (nice and rather deep) Bessaga-Pelczynski Selection principle and we refer to [Di], Chap. V, for the details. We limit ourselves to pointing out that any bounded linear map  $T : c_0 \rightarrow X^*$  admits a bounded linear extension  $\tilde{T} : \ell^\infty \rightarrow X^*$  (just take  $\tilde{T} = S^*$  with  $S = (T^*)|_{c_0}$ ). (Of course, there is no reason why  $T$  bounded below would imply  $\tilde{T}$  bounded below.)

## 4. ABOUT $\ell^p$ - INTERPOLATING SEQUENCES ( $1 \leq p < \infty$ ).

### 4.1. The case $p = 1$ ; Sidon sets.

From a theoretical point of view (and in terms of norm-isomorphism) the question of the existence of copies of  $\ell^1$  in a given Banach space  $Y$  is settled by the celebrated Rosenthal's  $\ell^1$ - Theorem recalled above.

Concretely, we need a bounded sequence  $G = (g_j)_j$  in  $Y$  such that the associated map  $V_G : \lambda = (\lambda_n)_n \in \ell^1 \rightarrow \sum_j \lambda_j g_j$  is bounded below.

The Sidon sets provide a whole bunch of them "practically for free" in the disk algebra.

Indeed, *one definition of a Sidon set of integers* is the following:

An infinite set  $\Gamma \subset \mathbb{Z}$  is a Sidon set (here, for  $n \in \mathbb{Z}$ ,  $\epsilon_n$  is the function  $t \rightarrow e^{int}$  also identified as a function on the unit circle  $\mathbb{T}$ ) if there exists a constant  $M$  such that, for any  $\alpha \in c_{00}(\Gamma)$ ,  $\sum_j |\alpha_j| \leq M \left\| \sum_j \alpha_j \epsilon_j \right\|_\infty$  (where  $c_{00}(\Gamma)$  denotes the set of sequences with finite support in  $\Gamma$ ).

We have the following result.

**Theorem 4.1.** *Let  $\Gamma \subset \mathbb{N}$  a Sidon set ; then the (closed linear) subspace  $\mathcal{E}_\Gamma$  generated by the  $\epsilon_n$ ,  $n \in \Gamma$  in the disc algebra  $A(\mathbb{D})$  is isomorphic to  $\ell^1$  ; moreover this isomorphism is a weak\*-homeomorphism (considering  $\mathcal{E}_\Gamma$  as a subspace of  $H^\infty$ ).*

Proof.

The fact that  $\Gamma$  is a Sidon set means (with the above definition) that the (clearly valued in  $A(\mathbb{D})$  since here  $\Gamma \subset \mathbb{N}$ ) linear map  $V_\Gamma : \alpha \rightarrow \sum_{j \in \Gamma} \alpha_j \epsilon_j$  is well-defined, bounded and bounded below. This proves the first assertion. To conclude one checks that  $V_\Gamma$ , seen as a map from  $\ell^1$  in  $H^\infty$ , is the adjoint of the map  $S$  defined, for  $[f] \in L^1/H_0^1$ , by  $S([f]) = (c_{-n}(f))_{n \in \Gamma}$ . Indeed  $S$  is well-defined since if  $[f] = [g]$  then  $(f - g) \in H_0^1$  and consequently  $c_{-n}(f - g) (= \int (f - g) \epsilon_n) = 0$  for all  $n \in \mathbb{N}$  and, in particular, for all  $n \in \Gamma$  ; in addition  $S$  is clearly linear, bounded and valued in  $c_0(\Gamma)$ . Thus  $V_\Gamma$  is weak\*-continuous and implements a weak\*-homeomorphism from  $\ell^1(\Gamma)$  onto  $\mathcal{E}_\Gamma = V_\Gamma(\ell^1(\Gamma))$ .

**Example 4.2.** *Among the Sidon sets contained in  $\mathbb{N}$  we have the (set of values) of  $q$ -lacunary sequences, i.e., sets of the form  $\{\lambda_k; k \in \mathbb{N}\}$  where the integers  $\lambda_k$  satisfy  $\frac{\lambda_{k+1}}{\lambda_k} \geq q$  for some (fixed)  $q > 1$ .*

**Remark 4.3.** *In view of the result in Subsection 3.4, it is natural to ask if one can obtain a weak\*-copy, say  $\mathcal{E}$ , of  $\ell^1 (= (c_0)^*)$  in  $H^\infty (= (L^1/H_0^1)^*)$  with, in addition,  $\mathcal{E}$  weak\*-complemented in  $H^\infty$ . Following the same general and standard duality argument as in the proof of Theorem 3.13, if this were the case, one would obtain easily that the subspace  ${}^\perp \mathcal{E}$  has a complement (in  $L^1/H_0^1$ ) which is isomorphic to  $c_0$  (everything so far is valid if we consider, instead of  $H^\infty (= (L^1/H_0^1)^*)$ , an arbitrary dual Banach space  $Z = X^*$ ) ; but the space  $L^1/H_0^1$ , being of cotype 2 (cf. [Bou]), cannot contain a copy of  $c_0$ . Thus we cannot have a weak\*, weak\*-complemented copy of  $\ell^1$  in  $L^1/H_0^1$ .*

#### 4.2. The case $1 < p < \infty$ .

In this section  $X$  is a separable reflexive Banach space,  $Z = X^*$  and  $Z^*$  is identified with  $X$ . We denote by  $p'$  the conjugate of  $p$  ( $\frac{1}{p} + \frac{1}{p'} = 1$ ).

The following definitions are natural adaptations of the  $p = \infty$  case.

**Definition 4.4.** *A sequence  $K = (k_j)_j$  in  $X$  is called:*

- a) *an  $\ell^p$ -interpolating sequence for  $Z$  ( $\ell^p$ -IS for short) if the range of the linear map  $R(= R_K) : g \in Z \rightarrow (\langle k_j, g \rangle)_j \in \ell^0$  contains  $\ell^p$  ;*
- b) *an  $\ell^p$ -boundedly linearly extendable sequence for  $Z$  ( $\ell^p$ -BLES for short) if there exists  $T \in \mathcal{L}(\ell^p, Z)$  such that  $\forall \lambda = (\lambda_j)_j \in \ell^p$ ,  $R \circ T(\lambda) = \lambda$ .*

We list a few elementary observations and some examples before going on.

1. An  $\ell^p$ -BLES is an  $\ell^p$ -IS.

2. If  $K = (k_j)_j$  is an  $\ell^p$ -BLES via  $T \in \mathcal{L}(\ell^p, Z)$  then the sequence  $(g_j)_j$  defined by  $g_j = T\epsilon_j$  for  $j \in \mathbb{N}$ , is biorthogonal to the sequence  $K$  and is a so-called  $p$ -Hilbertian sequence, that is, there exists a constant  $C$  such that

$$\left\| \sum_j \alpha_j g_j \right\| \leq C \|\alpha\|_p, \quad \alpha = (\alpha_j)_j \in c_{00} \quad \dagger_h.$$

Note also that, conversely, if  $(g_j)_j$  is a  $p$ -Hilbertian sequence in  $Z$  then there exists  $T \in \mathcal{L}(\ell^p, Z)$  such that  $T\epsilon_j = g_j$ .

3. In case  $X$  is an Hilbert space then of course  $X_K$  is (orthogonally) complemented,  $Z$  may be identified with  $X$  and  $\tilde{Z}$  with  $X_K$  itself, and (since any bounded linear map from a subspace can be boundedly linearly extended to the whole space) the notions of  $IS$  and  $BLES$  coincide.

Like in the previous case a fundamental class of examples is provided by standard interpolation associated to sequences in the open unit disk for the Hardy spaces  $H^p$ .

We briefly review this context:

The duality  $H^p = (H^{p'})^*$  can be expressed by the mutual action

$$\Phi_g(f) =: \langle f, g \rangle = \int_{\mathbb{T}} f \bar{g} \quad (:= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \bar{g}(e^{it}) dt), \quad f \in H^p, \quad g \in H^{p'}.$$

Note that in this approach  $g \rightarrow \Phi_g$  is a conjugate linear isomorphism (isometric if  $p = 2$ ) from  $H^{p'}$  onto  $(H^p)^*$ .

Starting with a sequence  $(z_n)_n$  in  $\mathbb{D}$ , we define the sequence  $K = (k_j)_j$  in  $H^{p'}$  by

$$k_j(z) = (1 - |z_j|^2)^{1/p} \frac{1}{1 - \bar{z}_j z}.$$

Thus our map  $R_K : H^p \rightarrow \ell^0$  is here given by  $R_K(g) = ((1 - |z_j|^2)^{1/p} g(z_j))$  that is exactly the map  $T_p$  introduced in the development of interpolation theory in [ShSh].

The following proposition is essentially a reformulation of observation 2 above.

**Proposition 4.5.** *A sequence  $K = (k_j)_j$  in  $X$  is an  $\ell^p$ -BLES for  $Z$  iff it admits a biorthogonal  $p$ -Hilbertian sequence.*

Once we have a bounded linear map from  $\ell^p$  into  $Z$  we just need this map to be bounded below to have an isomorphic copy of  $\ell^p$  in  $Z$  or in other words (we need) the existence of a constant  $c > 0$  such that

$$\left\| \sum_j \alpha_j g_j \right\| \geq c \|\alpha\|_p, \quad \alpha = (\alpha_j)_j \in c_{00} \quad \dagger_b$$

which can be also expressed by saying that the sequence  $(g_n)_n$  is a  $p$ -Besselian sequence.

(We advert the reader that there is no universal agreement on the terminology regarding the notions of Hilbertian and Besselian sequences.)

A sequence  $(g_n)_n$  which is both  $p$ -Besselian and  $p$ -Hilbertian is called a  $p$ -Riesz sequence. In other words, a basis (in any Banach space) is a  $p$ -Riesz basis iff it is equivalent to the canonical basis of  $\ell^p$ .

Thus the above translates immediately in the following characterization of copies of  $\ell^p$  in a Banach space.

**Theorem 4.6.** *A (closed) subspace  $\mathcal{M}$  of  $Z$  is isomorphic to  $\ell^p$  if and only if it admits a  $p$ -Riesz basis.*

The previous discussion might lead one to think that the existence of a  $p$ -Hilbertian sequence  $(g_n)_n$  in  $Z$  which in addition is basic (that is, is a basis for the closed subspace  $\mathcal{M}$  generated by this sequence) is enough to make  $\mathcal{M}$  isomorphic to  $\ell^p$ .

We now describe elementary examples showing that this is not the case.

**Proposition 4.7.**  $(1 < p < \infty)$  *In the space  $\ell^p$  there exists*

- (i) *a basis  $(u_n)_n$  which is  $p$ - Hilbertian but non  $p$ - Besselian (hence non  $p$ - Riesz) ;*
- (ii) *a basis  $(v_n)_n$  which is  $p$ - Besselian but non  $p$ - Hilbertian;*
- (iii) *a basis which is neither  $p$ - Hilbertian nor  $p$ - Besselian.*

The construction is based on the following finite dimensional computational lemma.

**Lemma 4.8.**  $(1 < p < \infty)$  *In the space  $(\mathbb{C}^{N+1}, \|\cdot\|_p)$  there exists a basis  $(e_0, \dots, e_N)$  such that, denoting  $(f_0, \dots, f_N)$  its dual basis, we have:*

- a) *it is, as well as its dual basis in  $(\mathbb{C}^{N+1}, \|\cdot\|_{p'})$ , norm bounded by 2 ;*
- b) *for  $\lambda \in \mathbb{C}^{N+1}$   $\left\| \sum_{j=0}^N \lambda_j e_j \right\|_p \leq 3 \|\lambda\|_p$  ;*
- c) *there exists  $\lambda \in \mathbb{C}^{N+1}$  such that  $\left\| \sum_{j=0}^N \lambda_j e_j \right\|_p < \rho_N \|\lambda\|_p$  with  $\lim_{N \rightarrow \infty} \rho_N = 0$ .*

Proof (of lemma).

Let  $(\epsilon_j)_{0 \leq j \leq N}$  denote the canonical basis in  $\mathbb{C}^{N+1}$ .

Setting

$$e_j = \epsilon_j + \frac{1}{N+1}v, \quad j = 1, \dots, N, \quad e_0 = \rho v, \quad f_0 = \rho'v, \quad f_j = \epsilon_0 + \epsilon_j$$

with  $\rho = (N+1)^{-1/p}$  and  $\rho' = (N+1)^{-1/p'}$  ensures that

$$1 = \langle e_0, f_0 \rangle = \|e_0\|_p = \|f_0\|_{p'} = 1$$

and that  $(e_j)_{0 \leq j \leq N}$ ,  $(f_j)_{0 \leq j \leq N}$  are dual bases in  $\mathbb{C}^{N+1}$  (of course here,  $\mathbb{C}^{N+1}$  is -algebraically-identified with its dual via the canonical bilinear form  $\langle a, b \rangle = \sum_{j=0}^N a_j b_j$ )

Now, the inequalities in a) are obvious for the  $f_j$ 's and  $e_0$  and follow from the following ones for the  $e_j$ 's ( $j = 1, \dots, N$ ) :

$$\|e_j\|_p \leq 1 + \left\| \frac{1}{N+1}v \right\|_p = 1 + (N+1)^{-1/p'} \leq 2.$$

For  $\lambda \in \mathbb{C}^{N+1}$  we have

$$\sum_{j=0}^N \lambda_j e_j = \sum_{j=1}^N \lambda_j \epsilon_j + \lambda_0 e_0 + \frac{1}{N+1} \left( \sum_{j=1}^N \lambda_j \right) v.$$

Since  $\left\| \frac{1}{N+1}v \right\|_p = (N+1)^{-1/p'}$  and (via Holder's inequality)  $\left| \sum_{j=1}^N \lambda_j \right| \leq N^{1/p'} \|\lambda\|_p$ , it follows easily (using the triangular inequality)

$$\left\| \sum_{j=0}^N \lambda_j e_j \right\|_p \leq 3 \|\lambda\|_p.$$

Finally with regard to c) , with  $\lambda = (0, 1, 1, \dots, 1)$  we get

$$\|\lambda\|_p = N^{1/p}, \quad \left\| \sum_{j=0}^N \lambda_j e_j \right\|_p = \frac{1}{N+1} \|(N, 1, \dots, 1)\|_p = \frac{1}{N+1} N^{1/p} (1 + N^{p-1})^{1/p},$$

and hence  $\rho_N := \frac{\left\| \sum_{j=0}^N \lambda_j e_j \right\|_p}{\|\lambda\|_p} = \frac{(1 + N^{p-1})^{1/p}}{1 + N} \rightarrow 0$  when  $N \rightarrow \infty$ , leading to the desired conclusion. ■

We now finish the proof of Proposition 4.7.

Proof (of proposition).

Consider (with as usual  $(\epsilon_j)_{j \geq 0}$  the canonical basis in  $\ell^p$ ) the standard decomposition of  $\ell^p$  into the  $(\ell^p)$  direct sum of its  $(N + 1)$  dimensional  $\ell_{N+1}^p$  subspaces:

$$\ell_2^p = \bigvee_{k=0}^1 \epsilon_k, \dots, \ell_{N+1}^p = \bigvee_{a_N \leq k < a_{N+1}} \epsilon_k,$$

where  $a_1 = 0$  and for  $N > 1$   $a_N (= a_{N-1} + N) = \frac{(N+2)(N-1)}{2}$ .

To each of these subspaces we apply the above lemma and denote  $(e_k)_{a_N \leq k < a_{N+1}}$  the basis thus obtained and  $(f_k)_{a_N \leq k < a_{N+1}}$  the dual one.

It is clear that  $(e_k)_{k \geq 0}$  is a basis in  $\ell^p$  which is  $p$ -Hilbertian (indeed for all  $\lambda \in c_{00}$ ,  $\left\| \sum_j \lambda_j e_j \right\|_p \leq 3 \|\lambda\|_p$ ) with a dual bounded basis (namely  $(f_k)_{k \geq 0}$ ) and this basis  $(e_k)_{k \geq 0}$  cannot be  $p$ -Besselian: indeed, from c) in the previous lemma; in fact rather from its proof) we get for each  $N$  an element  $\lambda^{(N)}$  (in  $\ell_{N+1}^p$  hence in  $c_{00}$ ) such that

$$\left\| \lambda^{(N)} \right\|_p = N^{1/p} \rightarrow \infty \text{ as } N \rightarrow \infty,$$

while  $\left\| \sum_j \lambda_j^{(N)} \right\|_p$  is bounded.

Thus  $(e_k)_{k \geq 0}$  is an example for (i) and, by duality,  $(f_k)_{k \geq 0}$  is an example for (ii). As to (iii), we can just take  $w_k = e_k$  for  $a_N \leq k < a_{N+1}$  when  $N$  is even and  $w_k = f_k$  for  $a_N \leq k < a_{N+1}$  when  $N$  is odd.

**Remark 4.9.** *The first example of a Hilbertian basis not Besselian was provided by Babenko [Bab] in a Hilbert space context; the above one (again in the Hilbert space setting) comes from the first named author's thesis [Am2] but was never published elsewhere.*

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