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ON LOGARITHMIC SOBOLEV INEQUALITIES FOR THE HEAT KERNEL ON THE HEISENBERG GROUP

MICHEL BONNEFONT, DJALIL CHAFAÏ, AND RONAN HERRY

ABSTRACT. In this note, we derive a new logarithmic Sobolev inequality for the heat kernel on the Heisenberg group. The proof is inspired from the historical method of Leonard Gross with the Central Limit Theorem for a random walk. Here the non commutativity of the increments produces a new gradient which naturally involves a Brownian bridge on the Heisenberg group. This new inequality contains the optimal logarithmic Sobolev inequality for the Gaussian distribution in two dimensions. We show that this new inequality is close to the symmetrized version of the sub-elliptic logarithmic Sobolev inequality of Hong-Quan Li on the Heisenberg group, seen as a weighted inequality. We show furthermore that a semigroup approach can produce such weighted inequalities.

1. INTRODUCTION

The Heisenberg group \mathbb{H} is a remarkable simple mathematical object, with rich algebraic, geometric, probabilistic, and analytic aspects. It is available in two versions, a discrete one and a continuous one. In this text, we focus on the continuous Heisenberg group \mathbb{H} , formed by the set of 3×3 matrices

$$M(x, y, z) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{R}.$$

Notations. The Heisenberg group \mathbb{H} is a non commutative sub-group of the general linear group, $M(a, b, c)M(a', b', c') = M(a + a', b + b', c + c' + ab')$ and $M(a, b, c)^{-1} = (-a, -b, -c + ab)$, and the discrete Heisenberg group is the discrete sub-group of \mathbb{H} formed by the elements of \mathbb{H} with integer coordinates. The Heisenberg group \mathbb{H} is a Lie group (manifold compatible with group structure). Its Lie algebra \mathfrak{L} (tangent space at the origin) is the sub-algebra of $\mathcal{M}_3(\mathbb{R})$ given by the 3×3 matrices of the form

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \quad x, y, z \in \mathbb{R}.$$

The matrix exponential map $\exp : A \in \mathfrak{L} \mapsto \exp(A) \in \mathbb{H}$ is a diffeomorphism. This allows to identify the group \mathbb{H} with the algebra \mathfrak{L} . Let us define

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad Z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have then

$$[X, Y] := XY - YX = Z \quad \text{and} \quad [X, Z] = [Y, Z] = 0.$$

The Lie algebra \mathfrak{L} is nilpotent of order 2:

$$\mathfrak{L} = \text{span}(X, Y) \oplus \text{span}(Z).$$

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This makes the Baker-Campbell-Hausdorff formula on \mathfrak{L} particularly simple:

$$\exp(A)\exp(B) = \exp\left(A + B + \frac{1}{2}[A, B]\right), \quad A, B \in \mathfrak{L}.$$

The names *Heisenberg group* and *Heisenberg algebra* come from the fact that in quantum physics, following Werner Heisenberg and Hermann Weyl, the algebra generated by the position operator and the momentum operator is \mathfrak{L} . The identification of \mathbb{H} with \mathfrak{L}

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \equiv \exp \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \exp(xX + yY + zZ)$$

allows to identify \mathbb{H} with \mathbb{R}^3 equipped with the group structure

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx'))$$

and $(x, y, z)^{-1} = (-x, -y, -z)$. The neutral element is the ‘‘origin’’ $e := (0, 0, 0)$. From now on, we use these ‘‘exponential coordinates’’. By translation with respect to the group operation, the vector fields X, Y, Z at point (x, y, z) are given by the formulas

$$X := \partial_x - \frac{y}{2}\partial_z, \quad Y := \partial_y + \frac{x}{2}\partial_z, \quad Z := \partial_z. \quad (1.1)$$

Geometrically, the quantity $\frac{1}{2}(xy' - yx')$ is the algebraic area in \mathbb{R}^2 between the piecewise linear path $[(0, 0), (x, y)] \cup [(x, y), (x + x', y + y')]$ and its chord $[(0, 0), (x + x', y + y')]$. This area is zero if (x, y) and (x', y') are collinear. This shows that the group product $(x, y, 0)(x', y', 0) = (x + x', y + y', \frac{1}{2}(xy' - yx'))$ in \mathbb{H} encodes the sum of increments in \mathbb{R}^2 and computes automatically the generated area.

The Heisenberg group \mathbb{H} is also a metric space for the Carnot-Carathéodory distance $d: \mathbb{H} \times \mathbb{H} \rightarrow [0, \infty)$, defined for any $h, g \in \mathbb{H}$ by

$$d(h, g) := \sup_f (f(h) - f(g))$$

where the supremum runs over all $f \in \mathcal{C}^\infty(\mathbb{H}, \mathbb{R})$ such that $\Gamma(f) := (Xf)^2 + (Yf)^2 \leq 1$. We have the following estimates (see [BGG, L1]): for any $g = (x, y, z) \in \mathbb{H}$,

$$r^2 := x^2 + y^2 \leq d(e, g)^2 \leq r^2 + 2\pi|z|. \quad (1.2)$$

There exists a constant $c \in (0, 2\pi)$ such that for any $g = (x, y, z) \in \mathbb{H}$ with $|z| \geq r^2$,

$$r^2 + c|z| \leq d(e, g)^2 \leq r^2 + 2\pi|z|. \quad (1.3)$$

The Heisenberg group is a Carnot group. Its Hausdorff dimension for the Carnot-Carathéodory metric is 4. Its dimension as a topological manifold which is 3.

The Heisenberg group \mathbb{H} is topologically homeomorphic to \mathbb{R}^3 and the Lebesgue measure on \mathbb{R}^3 is a Haar measure of \mathbb{H} (translation invariant). However, as a manifold, the geometry of \mathbb{H} is *sub-Riemannian*: the tangent space (at the origin and thus everywhere) is of dimension 2 instead of 3, putting a constraint on the geodesics (due to the lack of vertical speed vector, some of them are helices instead of straight lines).

Let $\beta \geq 0$ be a real parameter. Let $(x_n, y_n, z_n)_{n \geq 0}$ be independent and identically distributed random variables on \mathbb{R}^3 with zero mean and covariance matrix $\text{diag}(1, 1, \beta^2)$. Now set $S_0 := 0$ and for any $n \geq 1$,

$$S_n := (X_n, Y_n, Z_n) := \left(\frac{x_1}{\sqrt{n}}, \frac{y_1}{\sqrt{n}}, \frac{z_1}{\sqrt{n}}\right) \cdots \left(\frac{x_n}{\sqrt{n}}, \frac{y_n}{\sqrt{n}}, \frac{z_n}{\sqrt{n}}\right). \quad (1.4)$$

The sequence $(S_n)_{n \geq 0}$ is a random walk on \mathbb{H} started from the origin and with i.i.d. ‘‘non commutative multiplicative increments’’ given by a triangular array. We have

$$X_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i, \quad Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n y_i, \quad Z_n = A_n + \frac{\beta}{\sqrt{n}} \sum_{i=1}^n z_i$$

where

$$A_n := \frac{1}{2n} \sum_{i=1}^n \sum_{j=1}^n x_i \epsilon_{ij} y_j \quad \text{and} \quad \epsilon_{i,j} := \mathbf{1}_{j>i} - \mathbf{1}_{j<i}.$$

The random variable A_n is the algebraic area between the path $(X_k, Y_k)_{0 \leq k \leq n}$ of a random walk in \mathbb{R}^2 and its chord $[(0, 0), (X_n, Y_n)]$. If we introduce the dilation operator $\text{dil}_t(x, y, z) = (tx, ty, t^2z)$ on \mathbb{H} then

$$(X_n, Y_n, A_n) = \text{dil}_{\frac{1}{\sqrt{n}}}((x_1, y_1, 0) \cdots (x_n, y_n, 0)).$$

According to a Functional Central Limit Theorem (or Invariance Principle) on Lie groups due to Daniel Stroock and Srinivasa Varadhan [SV] (see also Donald Wehn [W]),

$$(S_{\lfloor nt \rfloor})_{t \geq 0} \xrightarrow[n \rightarrow \infty]{\text{law}} (\mathbf{X}_t, \mathbf{Y}_t, \mathbf{Z}_t)_{t \geq 0} = (\mathbf{X}_t, \mathbf{Y}_t, \mathbf{A}_t + \beta \mathbf{W}_t)_{t \geq 0} \quad (1.5)$$

where $(\mathbf{X}_t, \mathbf{Y}_t)_{t \geq 0}$ is a standard Brownian motion on \mathbb{R}^2 started from the origin, where $(\mathbf{W}_t)_{t \geq 0}$ is a standard Brownian motion on \mathbb{R} started from the origin and independent of $(\mathbf{X}_t, \mathbf{Y}_t)_{t \geq 0}$, and where $(\mathbf{A}_t)_{t \geq 0}$ is the Lévy area of $(\mathbf{X}_t, \mathbf{Y}_t)_{t \geq 0}$, in other words the algebraic area between the Brownian path and its chord, seen as a stochastic integral:

$$\mathbf{A}_t := \frac{1}{2} \left(\int_0^t \mathbf{X}_s d\mathbf{Y}_s - \int_0^t \mathbf{Y}_s d\mathbf{X}_s \right).$$

The stochastic process $(\mathbf{H}_t)_{t \geq 0} = (h \cdot (\mathbf{X}_t, \mathbf{Y}_t, \mathbf{Z}_t))_{t \geq 0}$ started from $\mathbf{H}_0 = h$ is a Markov diffusion process on \mathbb{R}^3 admitting the Lebesgue measure as an invariant and reversible measure. The Markov semigroup $(P_t)_{t \geq 0}$ of this process is defined for any $t \geq 0$, $h \in \mathbb{H}$, and bounded measurable $f : \mathbb{H} \rightarrow \mathbb{R}$, by

$$P_t(f)(h) := \mathbf{E}(f(\mathbf{H}_t) \mid \mathbf{H}_0 = h).$$

For any $t > 0$ and $h \in \mathbb{H}$, the law of \mathbf{H}_t conditionally on $\mathbf{H}_0 = h$ admits a density and

$$P_t(f)(h) = \int_{\mathbb{H}} f(g) p_t(h, g) dg.$$

Estimates on the heat kernel p_t are available, see [BGG, L3]. For instance when $\beta = 0$, there exist constants $C_2 > C_1 > 0$ such that for any $g = (x, y, z) \in \mathbb{H}$ and $t > 0$,

$$\frac{C_1}{\sqrt{t^3 + t^4 r d(e, g)}} \exp\left(-\frac{d^2(e, g)}{4t}\right) \leq p_t(e, g) \leq \frac{C_2}{\sqrt{t^3 + t^4 r d(e, g)}} \exp\left(-\frac{d^2(e, g)}{4t}\right) \quad (1.6)$$

where d is the Carnot-Carathéodory distance and where $r^2 := x^2 + y^2$.

Let us define the probability measure (which depends on the β parameter)

$$\gamma := \text{Law}(\mathbf{H}_1 \mid \mathbf{H}_0 = 0) = P_1(\cdot)(0).$$

The infinitesimal generator is the linear second order operator

$$L = \frac{1}{2}(X^2 + Y^2 + \beta Z^2)$$

where X, Y, Z are as in (1.1). The Schwartz space $\text{Schwartz}(\mathbb{H}, \mathbb{R})$ of rapidly decaying C^∞ functions from \mathbb{H} to \mathbb{R} is contained in the domain of L and is stable by L and by P_t for any $t \geq 0$. We have $[X, Y] = Z = \partial_z$ and $[X, Z] = [Y, Z] = 0$, the operator L is hypoelliptic, and by the Hörmander theorem P_t admits a C^∞ kernel. The operator L is elliptic if $\beta > 0$ and not elliptic if $\beta = 0$ (singular diffusion matrix).

The operator L acts like the two dimensional Laplacian on functions depending only on x, y and not on z . The one parameter family of operators obtain from L when β runs over the interval $[0, 1]$ interpolates between the sub-elliptic or sub-Riemannian Laplacian $\frac{1}{2}(X^2 + Y^2)$ (for $\beta = 0$) and the elliptic or Riemannian Laplacian $\frac{1}{2}(X^2 + Y^2 + Z^2)$ (for $\beta = 1$). The sub-Riemannian Brownian motion $(\mathbf{H}_t)_{t \geq 0}$ has independent and stationary (non commutative) increments and is a Lévy processes associated to a (non commutative)

convolution semigroups $(P_t)_{t \geq 0}$ on \mathbb{H} . When $\beta = 0$ then the probability measure γ_0 is infinitely divisible and can be seen as a Gaussian measure on \mathbb{H} and a formula (oscillatory integral) for the kernel of P_t was computed by Paul Lévy using Fourier analysis. We refer to the books [M, B1, N] and references therein for more details.

The entropy of $f : \mathbb{H} \rightarrow [0, \infty)$ with respect to a probability measure μ is defined by

$$\mathbf{Ent}_\mu(f) := \mathbf{E}_\mu(\Phi(f)) - \Phi(\mathbf{E}_\mu(f)) \quad \text{with} \quad \mathbf{E}_\mu(f) := \int f \, d\mu$$

where $\Phi(u) = u \log(u)$.

Main results. Our main results are the following.

Theorem 1.1 (Logarithmic Sobolev inequality). *For any $\beta \geq 0$ and $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$,*

$$\mathbf{Ent}_\gamma(f^2) \leq 2 \int_0^1 \mathbf{E}(g(\mathbf{H}_1, \mathbf{H}_t)) \, dt \quad (1.7)$$

where

$$\begin{aligned} g(h, h') &:= \left(\partial_x f(h) - \frac{x - 2x'}{2} \partial_z f(h) \right)^2 + \left(\partial_y f(h) + \frac{y - 2y'}{2} \partial_z f(h) \right)^2 + \beta^2 (\partial_z f(h))^2 \\ &= ((X + x'Z)f(h))^2 + ((Y - y'Z)f(h))^2 + \beta^2 (Zf(h))^2. \end{aligned}$$

We refer for instance to the book [BGL] for a general introduction to Sobolev type functional inequalities for diffusion processes.

The shape of the right hand side of (1.7) comes from the fact that the increments are not commutative: the sum in S_n produces along (1.5) the integral from 0 to 1.

The following corollary is obtained via Brownian Bridge and heat kernel estimates.

Corollary 1.2 (Weighted logarithmic Sobolev inequality). *Take $\beta = 0$. There exist constants $C > 0$ and $C' > 0$ such that for any $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$,*

$$\mathbf{Ent}_\gamma(f) \leq 2 \mathbf{E}_\gamma \left((\partial_x f)^2 + (\partial_y f)^2 + C(1 + x^2 + y^2 + |z|)(\partial_z f)^2 \right). \quad (1.8)$$

Leonard Gross used in [G1] and [G2] the Central Limit Theorem to proof logarithmic Sobolev inequality on *elliptic* Lie groups. In a surprising way, the very same method works on the sub-Riemannian setting of the Heisenberg group, and gives a non degenerate result. Indeed it was a common thought to believe that in the non-elliptic case, the Central Limit Theorem will produce asymptotically the collapse or the explosion of the right hand side.

Theorem 1.1 is proved in Section 2, by following the method of Gross. Corollary 1.2 is proved in Section 3, by using an expansion of (1.7), and probabilistic (Bayes formula), analytic (bounds for the heat kernel on \mathbb{H}), and geometric (bounds for the Carnot-Carathéodory distance) arguments for the control of the density of the Brownian bridge.

Baudoin and Garofalo developed in [BG] a generalization of the Bakry-Émery semigroup approach to the curvature adapted to the sub-Riemannian setting. This framework is well-suited for studying weighted functional inequalities such as (1.8) and (1.13) and allow us to derive the following result. Note that a similar result is already stated in the PhD thesis [Bo, Prop. 5.3.7 p. 129], see also [BB], and [B2, Prop. 4.13] more generally.

Theorem 1.3 (Yet another logarithmic Sobolev inequality on Heisenberg group). *Take $\beta = 0$. For any real number $\nu > 0$ and any function $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$,*

$$\mathbf{Ent}_\gamma(f^2) \leq 2\nu(e^{\frac{1}{\nu}} - 1) \mathbf{E}_\gamma \left((Xf)^2 + (Yf)^2 + \nu(Zf)^2 \right). \quad (1.9)$$

The symmetrized version of (1.9) is the following new weighted logarithmic Sobolev inequality on the Heisenberg group, lying between (1.8) and (1.13).

Corollary 1.4 (Weighted logarithmic Sobolev inequality on Heisenberg group). *Take $\beta = 0$. For any real number $\nu > 0$ and any function $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$,*

$$\mathbf{Ent}_\gamma(f^2) \leq 2\nu(e^{\frac{1}{\nu}} - 1) \mathbf{E}_\gamma\left((\partial_x f)^2 + (\partial_y f)^2 + \left(\nu + \frac{x^2 + y^2}{4}\right)(\partial_z f)^2\right). \quad (1.10)$$

The proof of Theorem 1.3 and Corollary 1.4 is given in Section 4.

Discussion.

Novelty. Taking $\beta = 0$ in (1.7) provides a new sub-elliptic logarithmic Sobolev inequality for the sub-Riemannian Gaussian law γ , namely, for any $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$,

$$\mathbf{Ent}_\gamma(f^2) \leq 2 \int_0^1 \mathbf{E}(g(\mathbf{H}_1, \mathbf{H}_t)) dt \quad (1.11)$$

where

$$g(h, h') := \left(\partial_x f(h) - \frac{x - 2x'}{2} \partial_z f(h)\right)^2 + \left(\partial_y f(h) + \frac{y - 2y'}{2} \partial_z f(h)\right)^2.$$

Horizontal optimality. The logarithmic Sobolev inequality (1.11), implies the optimal logarithmic Sobolev inequality for the standard Gaussian distribution $\mathcal{N}(0, I_2)$ on \mathbb{R}^2 with the Euclidean gradient, namely, for any $f \in \text{Schwartz}(\mathbb{R}^2, \mathbb{R})$,

$$\mathbf{Ent}_{\mathcal{N}(0, I_2)}(f^2) \leq 2 \mathbf{E}_{\mathcal{N}(0, I_2)}((\partial_x f)^2 + (\partial_y f)^2).$$

To see it, it suffices to express (1.11) with a function f that does not depend on the third coordinate z . This shows in particular the optimality (minimality) of the constant 2 in front of the right hand side in the inequality of Theorem 1.1 and in (1.11).

Poincaré inequality. Recall that the variance of $f : \mathbb{H} \rightarrow \mathbb{R}$ with respect to μ is

$$\mathbf{Var}_\mu(f) := \int \Phi(f) d\mu - \Phi\left(\int f d\mu\right) \quad \text{where this time } \Phi(u) = u^2.$$

As usual, the logarithmic Sobolev inequality (1.11) gives a Poincaré inequality by linearization. More precisely, replacing f by $1 + \varepsilon f$ in (1.11) gives, as $\varepsilon \rightarrow 0$,

$$\mathbf{Var}_\gamma(f) \leq \int_0^1 \mathbf{E}(g(\mathbf{H}_1, \mathbf{H}_t)) dt.$$

Comparison with H.-Q. Li inequality. For $\beta = 0$, Hong-Quan Li has obtained in [L1] (see also [BBBC, DM] for a Poincaré inequality) the following logarithmic Sobolev inequality: there exists a constant $C > 0$ such that for any $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$,

$$\mathbf{Ent}_\gamma(f^2) \leq C \mathbf{E}_\gamma((Xf)^2 + (Yf)^2). \quad (1.12)$$

The right hand side in (1.12) involves the ‘‘carré du champ’’ of L , namely the functional quadratic form $L(f^2) - 2fLf = (Xf)^2 + (Yf)^2$. Following the by now standard Bakry-Émery approach, the expansion of the scaled version shows that necessarily $C \geq 2$ but the optimal (minimal) constant is unknown.

The random variables $-\mathbf{H}_t$ and \mathbf{H}_t have the same law conditionally to $\{\mathbf{H}_0 = 0\}$. This symmetry property permits us to cancel out, in average, the cross terms involving $x\partial_x f \partial_z f$ and $y\partial_y f \partial_z f$ when expanding the right hand side of the sum of (1.12) and its rotated version. The symmetrized version of (1.12) that we obtained in this way appears as a weighted logarithmic Sobolev inequality: for any $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$,

$$\mathbf{Ent}_\gamma(f^2) \leq C \mathbf{E}_\gamma\left((\partial_x f)^2 + (\partial_y f)^2 + \frac{x^2 + y^2}{4}(\partial_z f)^2\right). \quad (1.13)$$

The inequality (1.13) should be compared with our inequalities (1.8) and (1.10), which are of the same nature, yet sub-optimal.

Open questions and extensions. The Heisenberg group is the simplest non-trivial version of Carnot (stratified nilpotent Lie groups). Those group have a strong geometric meaning both in standard and stochastic analysis (see for instance [B1] for the latter point). A version of Theorem 1.1 should be available on those groups even though the function g might not be computable explicitly due to the complexity of the group operation. The bounds on the distance and the heat kernel used to derive the weighted inequality (1.8) are not available on general Carnot groups and it should require more work to obtain an equivalent of Corollary 1.2. As a comparison, note that a version of (1.12) exists on groups with a so called H-structure (see [L3]) but a general version on Carnot groups is unknown due to the lack of general estimates for the heat kernel.

In the spirit of the work of Leonard Gross [G2] in the elliptic case, an approach at the level of paths space should be available.

It is also natural to ask about a direct analytic proof or semigroup proof of the inequality of Theorem 1.1, without using the Central Limit Theorem.

2. PROOF OF THEOREM 1.1

Fix a real $\beta \geq 0$. Consider $(x_n, y_n, z_n)_{n \geq 1}$ a sequence of independent and identically distributed random variables with standard Gaussian law $\mathcal{N}(0, I_3)$ and let S_n be as in (1.4). The Central Limit Theorem gives

$$S_n \xrightarrow[n \rightarrow \infty]{\text{law}} \gamma.$$

The law ν_n of S_n satisfies $\nu_n = (\mu_n)^{*n}$ where the convolution takes place in \mathbb{H} and where μ_n is the Gaussian law on \mathbb{R}^3 with covariance matrix $\text{diag}(1/n, 1/n, \beta^2/n)$.

For any $i = 1, \dots, n$, let us define

$$S_{n,i} := (X_n, Y_n, Z_n, X_{n,i}, Y_{n,i})$$

where

$$X_{n,i} := -\frac{1}{\sqrt{n}} \sum_{j=1}^n \epsilon_{ij} x_j \quad \text{and} \quad Y_{n,i} := -\frac{1}{\sqrt{n}} \sum_{j=1}^n \epsilon_{ij} y_j.$$

The optimal logarithmic Sobolev inequality for the standard Gaussian measure $\mathcal{N}(0, I_{3n})$ on \mathbb{R}^{3n} gives, for any $g \in \text{Schwartz}(\mathbb{R}^{3n}, \mathbb{R})$,

$$\mathbf{Ent}_{\mathcal{N}(0, I_{3n})}(g^2) \leq 2 \mathbf{E}_{\mathcal{N}(0, I_{3n})} \left(\sum_{i=1}^n (\partial_{x_i} g)^2 + (\partial_{y_i} g)^2 + (\partial_{z_i} g)^2 \right).$$

Let $s_n : \mathbb{R}^{3n} \rightarrow \mathbb{H}$ be the map such that $S_n = s_n((x_1, y_1, z_1), \dots, (x_n, y_n, z_n))$. For some $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$ the function $g = f(s_n)$ satisfies

$$\begin{aligned} \partial_{x_i} g(s_n) &= \frac{1}{\sqrt{n}} \left(\partial_x f - \frac{Y_{n,i}}{2} \partial_z f \right)(s_n), \\ \partial_{y_i} g(s_n) &= \frac{1}{\sqrt{n}} \left(\partial_y f + \frac{X_{n,i}}{2} \partial_z f \right)(s_n), \\ \partial_{z_i} g(s_n) &= \frac{\beta}{\sqrt{n}} (\partial_z f)(s_n). \end{aligned}$$

It follows that for any $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$, denoting $\nu_n := \text{Law}(S_n)$,

$$\mathbf{Ent}_{\nu_n}(f^2) \leq \frac{2}{n} \sum_{i=1}^n \mathbf{E}(h(S_n^i)) = 2 \mathbf{E} \left(\frac{1}{n} \sum_{i=1}^n h(S_n^i) \right) \quad (2.1)$$

where $h : \mathbb{R}^5 \rightarrow \mathbb{R}$ is defined from f by

$$h(x, y, z, x', y') := \left(\left(\partial_x - \frac{y'}{2} \partial_z \right) f(x, y, z) \right)^2 + \left(\left(\partial_y + \frac{x'}{2} \partial_z \right) f(x, y, z) \right)^2.$$

The right hand side of (2.1) as $n \rightarrow \infty$ is handled by explaining the law of S_n^i through a triangular array of increments of the process we anticipated in the limit. More precisely, let $((\mathbf{X}_t, \mathbf{Y}_t))_{t \geq 0}$ be a standard Brownian motion on \mathbb{R}^2 started from the origin, let $(\mathbf{A}_t)_{t \geq 0}$ be its Lévy area, and let $(\mathbf{W}_t)_{t \geq 0}$ be a Brownian motion on \mathbb{R} starting from the origin, independent of $(\mathbf{X}_t, \mathbf{Y}_t)_{t \geq 0}$. Let us define, for $n \geq 1$ and $1 \leq i \leq n$,

$$\xi_{n,i} := \sqrt{n} \left(\mathbf{X}_{\frac{i}{n}} - \mathbf{X}_{\frac{i-1}{n}} \right), \quad \eta_{n,i} := \sqrt{n} \left(\mathbf{Y}_{\frac{i}{n}} - \mathbf{Y}_{\frac{i-1}{n}} \right), \quad \zeta_{n,i} := \sqrt{n} \left(\mathbf{W}_{\frac{i}{n}} - \mathbf{W}_{\frac{i-1}{n}} \right).$$

For any fixed $n \geq 1$, the random variables $(\xi_{n,i})_{1 \leq i \leq n}$, $(\eta_{n,i})_{1 \leq i \leq n}$, and $(\zeta_{n,i})_{1 \leq i \leq n}$ are independent and identically distributed with Gaussian law $\mathcal{N}(0, 1)$. Let us define now

$$\begin{aligned} X_n &:= \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_{n,i}, & Y_n &:= \frac{1}{\sqrt{n}} \sum_{i=1}^n \eta_{n,i}, \\ A_n &:= \frac{1}{2n} \sum_{i=1}^n \sum_{j=1}^n \xi_{n,i} \epsilon_{i,j} \eta_{n,i}, & Z_n &:= \beta \frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_{n,i} + A_n, \\ X_{n,i} &:= -\frac{1}{\sqrt{n}} \sum_{j=1}^n \epsilon_{i,j} \eta_{n,j}, & Y_{n,i} &:= -\frac{1}{\sqrt{n}} \sum_{j=1}^n \epsilon_{i,j} \eta_{n,i}. \end{aligned}$$

We have then the equality in distribution

$$(X_n, Y_n, Z_n - A_n, X_{n,i}, Y_{n,i}) \stackrel{d}{=} (\mathbf{X}_1, \mathbf{Y}_1, \beta \mathbf{W}_1, \mathbf{X}_1 - (\mathbf{X}_{\frac{i-1}{n}} + \mathbf{X}_{\frac{i}{n}}), \mathbf{Y}_1 - (\mathbf{Y}_{\frac{i-1}{n}} + \mathbf{Y}_{\frac{i}{n}})).$$

Moreover, as $i/n \rightarrow s \in [0, 1]$, we have the convergence in distribution

$$S_{n,i} := (X_n, Y_n, Z_n, X_{n,i}, Y_{n,i}) \xrightarrow[n \rightarrow \infty]{d} (\mathbf{X}_1, \mathbf{Y}_1, \mathbf{A}_1 + \beta \mathbf{W}_1, \mathbf{X}_1 - 2\mathbf{X}_s, \mathbf{Y}_1 - 2\mathbf{Y}_s).$$

It follows that for any continuous and bounded $h : \mathbb{R}^5 \rightarrow \mathbb{R}$,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \mathbf{E}(h(S_{n,i})) &= \int_0^1 \mathbf{E}(h(X_n, Y_n, Z_n, X_{n, \lfloor tn \rfloor / n}, Y_{n, \lfloor tn \rfloor / n})) dt \\ &\xrightarrow[n \rightarrow \infty]{} \int_0^1 \mathbf{E}(h(\mathbf{X}_1, \mathbf{Y}_1, \mathbf{A}_1 + \beta \mathbf{W}_1, \mathbf{X}_1 - 2\mathbf{X}_t, \mathbf{Y}_1 - 2\mathbf{Y}_t)) dt. \end{aligned}$$

3. PROOF OF COROLLARY 1.2

Let us consider (1.7) with $\beta = 0$. By expanding the right-hand side, and using the fact that the conditional law of \mathbf{H}_1 given $\{\mathbf{H}_1 = 0\}$ is invariant by central symmetry, we get a symmetrized weighted logarithmic Sobolev inequality: for any $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$,

$$\begin{aligned} \text{Ent}_\gamma(f^2) &\leq 2 \mathbf{E}((\partial_x f)^2(\mathbf{H}_1) + (\partial_y f)^2(\mathbf{H}_1)) \\ &\quad + \frac{1}{2} \int_0^1 \mathbf{E}(\mathbf{E}((\mathbf{X}_1 - 2\mathbf{X}_t)^2 + (\mathbf{Y}_1 - 2\mathbf{Y}_t)^2 \mid \mathbf{H}_1) (\partial_z f)^2(\mathbf{H}_1)) dt \\ &\leq 2 \mathbf{E}((\partial_x f)^2(\mathbf{H}_1) + (\partial_y f)^2(\mathbf{H}_1)) \\ &\quad + \mathbf{E}((\mathbf{X}_1^2 + \mathbf{Y}_1^2) (\partial_z f)^2(\mathbf{H}_1)) + 4 \mathbf{E}((\partial_z f)^2(\mathbf{H}_1) \int_0^1 \mathbf{E}(\mathbf{X}_t^2 + \mathbf{Y}_t^2 \mid \mathbf{H}_1) dt). \end{aligned}$$

The desired result is a direct consequence of the following lemma.

Lemma 3.1 (Bridge control). *There exists a constant $C > 0$ such that for any $0 \leq t \leq 1$ and $h = (x, y, z) \in \mathbb{H}$,*

$$\mathbf{E}(\mathbf{X}_t^2 + \mathbf{Y}_t^2 \mid \mathbf{H}_1 = h) \leq C(t^2 d^2(e, h) + t)$$

and

$$\int_0^1 \mathbf{E}(\mathbf{X}_t^2 + \mathbf{Y}_t^2 \mid \mathbf{H}_1 = h) dt \leq C(1 + x^2 + y^2 + |z|).$$

Note that for the classical euclidean Brownian motion

$$\mathbf{E}(\|\mathbf{B}_t\|^2 \mid \mathbf{B}_1) = t^2\|\mathbf{B}_1\|^2 + nt(1-t).$$

Proof of Lemma 3.1. For any random variables U, V , we denote by φ_U the density of U and by $\varphi_{U|V=v}$ the conditional density of U given $\{V=v\}$. For any $0 < t \leq 1$ and $k \in \mathbb{H}$,

$$\mathbf{E}(\mathbf{X}_t^2 + \mathbf{Y}_t^2 \mid \mathbf{H}_1 = k) = \int_{\mathbb{H}} r_g^2 \varphi_{\mathbf{H}_t | \mathbf{H}_1 = k}(g) dg$$

where $r_g^2 = x_g^2 + y_g^2$ and $g = (x_g, y_g, z_g)$. Recall that $p_t(h, g) := \varphi_{\mathbf{H}_t = g | \mathbf{H}_0 = h}$ and that $e := (0, 0, 0)$ is the origin in \mathbb{H} . Thanks to the Bayes formula, for any $g, k \in \mathbb{H}$,

$$\varphi_{\mathbf{H}_t | \mathbf{H}_1 = k}(g) = \frac{\varphi_{(\mathbf{H}_t, \mathbf{H}_1)}(g, k)}{\varphi_{\mathbf{H}_1}(k)} = \frac{\varphi_{\mathbf{H}_t}(g) \varphi_{\mathbf{H}_1 | \mathbf{H}_t = g}(k)}{\varphi_{\mathbf{H}_1}(k)} = \frac{p_t(e, g) p_{1-t}(g, k)}{p_1(e, k)}$$

Back to our objective, we have

$$\begin{aligned} \mathbf{E}(\mathbf{X}_t^2 + \mathbf{Y}_t^2 \mid \mathbf{H}_1 = k) &= \frac{\int p_t(e, g) r_g^2 p_{1-t}(g, k) dg}{p_1(e, k)} \\ &\leq \frac{\int p_t(e, g) d_g^2 p_{1-t}(g, k) dg}{\int p_t(e, g) p_{1-t}(g, k) dg}. \end{aligned}$$

In what follows, the constant C may change from line to line. The idea is to kill the polynomial term d^2 in the numerator by using the exponential decay of the heat kernel, at the price of a slight time change. Namely, using (1.6), we get, for any $0 < \varepsilon < 1$,

$$\begin{aligned} \int p_t(e, g) r^2(e, g) p_{1-t}(g, k) dg &\leq \int \frac{C}{\sqrt{t^3 + t^4 r d(e, g)}} d^2(e, g) \exp\left(-\frac{d^2(e, g)}{4t}\right) p_{1-t}(g, k) dg \\ &\leq \int \frac{Ct}{\varepsilon \sqrt{t^3 + t^4 r d(e, g)}} \exp\left(-\frac{(1-\varepsilon)d^2(e, g)}{4t}\right) p_{1-t}(g, k) dg \\ &\leq \int \frac{Ct}{\varepsilon} \sqrt{\frac{\left(\frac{t}{1-\varepsilon}\right)^3 + \left(\frac{t}{1-\varepsilon}\right)^4 r d(e, g)}{t^3 + t^4 r d(e, g)}} p_{\frac{t}{1-\varepsilon}}(e, g) p_{1-t}(g, k) dg \\ &\leq \frac{Ct}{\varepsilon(1-\varepsilon)^2} \int p_{\frac{t}{1-\varepsilon}}(e, g) p_{1-t}(g, k) dg \\ &\leq \frac{Ct}{\varepsilon(1-\varepsilon)^2} p_{1+\frac{\varepsilon t}{1-\varepsilon}}(e, k). \end{aligned}$$

where we used $x \exp(-x) \leq \frac{1}{e\varepsilon} \exp(-(1-\varepsilon)x)$. Therefore, for all $0 < \varepsilon \leq 1/2$,

$$\begin{aligned} \mathbf{E}(\mathbf{X}_t^2 + \mathbf{Y}_t^2 \mid \mathbf{H}_1) &\leq \frac{Ct}{\varepsilon(1-\varepsilon)^2} \frac{p_{1+\frac{\varepsilon t}{1-\varepsilon}}(e, k)}{p_1(e, k)} \\ &\leq \frac{Ct}{\varepsilon(1-\varepsilon)^2} \exp\left(\frac{\varepsilon t}{1+\varepsilon t} \frac{d^2(e, k)}{4}\right) \\ &\leq \frac{Ct}{\varepsilon} \exp\left(\varepsilon t \frac{d^2(e, k)}{4}\right). \end{aligned}$$

Now if $td^2(e, k) \geq 1$, then we take $\varepsilon = 1/(2td^2(e, k))$ which gives

$$\mathbf{E}(\mathbf{X}_t^2 + \mathbf{Y}_t^2 \mid \mathbf{H}_1) \leq Ct^2 d^2(e, k),$$

while if $td^2(e, k) < 1$, then we take $\varepsilon = 1/2$ which gives

$$\mathbf{E}(\mathbf{X}_t^2 + \mathbf{Y}_t^2 \mid \mathbf{H}_1) \leq Ct.$$

This provides the first desired inequality. We get the second using (1.2). \square

4. PROOF OF THEOREM 1.3 AND COROLLARY 1.4

In this section, we provide the proofs of Theorem 1.3 and Corollary 1.4. Note that the method remains valid beyond the assumption $\beta = 0$.

Proof of Theorem 1.3. For simplicity, we change L by a factor 2 and set $L = X^2 + Y^2$ and $P_t = e^{tL}$, and in particular $\gamma = P_{1/2}(\cdot)(0)$. For any $f, g \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$, let us define

$$\begin{aligned}\Gamma^{\text{hori}}(f, g) &:= \frac{1}{2}(L(fg) - fLg - gLf) = X(f)X(g) + Y(f)Y(g), \\ \Gamma^{\text{vert}}(f, g) &:= Z(f)Z(g), \\ \Gamma^{\text{elli}}(f, g) &:= \Gamma^{\text{hori}}(f, g) + \nu\Gamma^{\text{vert}}(f, g).\end{aligned}$$

Let us also denote

$$\begin{aligned}\Gamma_2^{\text{hori}}(f, f) &:= \frac{1}{2}(L\Gamma^{\text{hori}}(f, f) - 2\Gamma^{\text{hori}}(f, Lf)), \\ \Gamma_2^{\text{vert}}(f, f) &:= \frac{1}{2}(L\Gamma^{\text{vert}}(f, f) - 2\Gamma^{\text{vert}}(f, Lf)), \\ \Gamma_2^{\text{mix}}(f, f) &:= \frac{1}{2}(L\Gamma^{\text{elli}}(f, f) - 2\Gamma^{\text{elli}}(f, Lf)).\end{aligned}$$

In the sequel, we also denote $\Gamma(f) = \Gamma(f, f)$ and $\Gamma_2(f) = \Gamma_2(f, f)$.

Curvature inequality. The following inequality holds: for any $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$,

$$\Gamma_2^{\text{mix}}(f, f) \geq -\frac{1}{\nu}\Gamma^{\text{elli}}(f, f). \quad (4.1)$$

Indeed, an easy computation gives

$$\begin{aligned}\Gamma_2^{\text{hori}}(f, f) &= (X^2f)^2 + (Y^2f)^2 + (XYf)^2 + (YXf)^2 - 2(Xf)(YZf) + 2(Yf)(XZf) \\ \Gamma_2^{\text{vert}}(f, f) &= (XZf)^2 + (YZf)^2.\end{aligned}$$

Since $\Gamma_2^{\text{mix}} = \Gamma_2^{\text{hori}} + \nu\Gamma_2^{\text{vert}}$ and $Zf = XYf - YXf$, the Cauchy-Schwarz's inequality gives

$$\Gamma_2^{\text{mix}}(f, f) \geq \frac{1}{2}(Lf)^2 + \frac{1}{2}(XYf + YXf)^2 + \frac{1}{2}(Zf)^2 - \frac{1}{\nu}X(f)^2 - \frac{1}{\nu}Y(f)^2,$$

which implies the desired curvature inequality (4.1).

Semigroup inequality. Let $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$ with $f \geq 0$. For any $0 \leq s \leq t$, set

$$U(s) := (P_{t-s}f) \log(P_{t-s}f) \quad \text{and} \quad V(s) := (P_{t-s}f) \Gamma^{\text{elli}}(\log(P_{t-s}f)).$$

Then for $0 \leq s \leq t$,

$$LU + \partial_s U = (P_{t-s}f) \Gamma^{\text{hori}}(\log(P_{t-s}f)) \leq (P_{t-s}f) \Gamma^{\text{elli}}(\log(P_{t-s}f)) = V(s) \quad (4.2)$$

$$LV + \partial_s V = 2(P_{t-s}f) \Gamma_2^{\text{mix}}(\log(P_{t-s}f)) \geq -\frac{2}{\nu}V(s). \quad (4.3)$$

The first equality in (4.2) holds since Γ^{hori} is the ‘‘carré du champ’’ associated to L , the inequality holds because $\Gamma^{\text{vert}}(f, f) \geq 0$ and the second equality is the definition of V .

In (4.3) we have used that $\Gamma^{\text{elli}} = \Gamma^{\text{hori}} + \nu\Gamma^{\text{vert}}$, that the horizontal part of Γ^{elli} will produce Γ_2^{hori} (same kind of computation as in the first inequality of (4.2)) and that vertical part commute to the horizontal one $\Gamma^{\text{hori}}(f, \Gamma^{\text{elli}}(f, f)) = \Gamma^{\text{elli}}(f, \Gamma^{\text{vert}}(f, f))$, the inequality comes from (4.1).

Final step. Since by (4.3), $L(e^{\frac{2s}{\nu}} V(s)) + \partial_s(e^{\frac{2s}{\nu}} V(s)) \geq 0$, a parabolic comparison such as [BG, Prop. 4.5] or a simple semigroup interpolation implies that for $t \geq 0$,

$$e^{\frac{2t}{\nu}} P_t(f \Gamma^{\text{elli}}(\log f)) = e^{\frac{2t}{\nu}} V(t) \geq V(0) = (P_t f) \Gamma^{\text{elli}}(\log P_t f).$$

In particular,

$$V(s) \leq e^{\frac{2(t-s)}{\nu}} P_{t-s}(f\Gamma^{\text{elli}}(\log f)). \quad (4.4)$$

Now from (4.2) another application of the parabolic comparison theorem and the last estimate (4.4) give

$$P_t(U(t)) \leq U(0) + \int_0^t P_s(V(s)) ds \leq U(0) + \int_0^t e^{\frac{2(t-s)}{\nu}} ds P_t(f\Gamma^{\text{elli}}(\log f));$$

that is:

$$P_t(f \log f)(x) - P_t(f)(x) \log P_t(f)(x) \leq \frac{\nu}{2} \left(e^{\frac{2t}{\nu}} - 1 \right) P_t \left(\frac{\Gamma^{\text{elli}}(f, f)}{f} \right) (x).$$

The conclusion follows by taking $t = 1/2$ and $x = 0$ since $\gamma = P_{1/2}(\cdot)(0)$. \square

Proof of Corollary 1.4. Let us consider the right (instead of left) invariant vector fields

$$\hat{X} := \partial_x + \frac{y}{2} \partial_z \quad \text{and} \quad \hat{Y} := \partial_y - \frac{x}{2} \partial_z$$

and $\hat{L} = \hat{X} + \hat{Y}$ and $\hat{P}_t = e^{t\hat{L}}$ the corresponding generator and semi-group. The semi-group is bi-invariant in the sense that $P_t f(0) = \hat{P}_t f(0)$, see for instance [BBBC]. Recall that $\gamma = P_{1/2}(\cdot)(0) = \hat{P}_{1/2}(\cdot)(0)$. The method of proof of Theorem 1.9 remains valid if one replaces X, Y, L, P_t by their right invariant counter parts and yields that for any $f \in \text{Schwartz}(\mathbb{H}, \mathbb{R})$,

$$\mathbf{Ent}_\gamma(f^2) \leq 2\nu(e^{\frac{1}{\nu}} - 1) \mathbf{E}_\gamma \left((\hat{X}f)^2 + (\hat{Y}f)^2 + \nu(-Zf)^2 \right). \quad (4.5)$$

The conclusion follows by the summation of the inequalities (1.9) and (4.5). \square

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