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# Global strong solution for the compressible Navier Stokes system with large rotationnally invariant initial data in dimension

$$N \geq 2$$

Boris Haspot \*

## Abstract

We show the existence of global strong solution for compressible Navier-Stokes equations with shallow water viscosity coefficients when the initial data are large and rotationally invariant. In dimension  $N = 2$  we consider general initial data of the form  $(\rho_0(|x|), u_0(x))$  with  $u_0(x) = \nabla\theta(|x|) + \nabla^\perp\psi(|x|)$  with  $\theta$  and  $\psi$  sufficiently regular. In dimension  $N = 3$  we deal with radially symmetric initial data  $(\rho_0(|x|), \nabla\theta(|x|))$ .

## 1 Introduction

The conservation of mass and of momentum write:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) D u) - \nabla(\lambda(\rho) \operatorname{div} u) + \nabla P(\rho) = 0, \\ (\rho(0, \cdot), u(0, \cdot)) = (\rho_0, u_0). \end{cases} \quad (1.1)$$

Here  $u = u(t, x) \in \mathbb{R}^N$  stands for the velocity field,  $\rho = \rho(t, x) \in \mathbb{R}^+$  is the density and  $D(u) = \frac{1}{2}(\nabla u + {}^t \nabla u)$  is the strain tensor. We denote by  $\lambda$  and  $\mu$  the two viscosity coefficients of the fluid, which are assumed to satisfy  $\mu > 0$  and  $\lambda + 2\mu > 0$ . Such a condition ensures ellipticity for the momentum equation and is satisfied in the physical cases where  $\lambda + \frac{2\mu}{N} > 0$ . We supplement the problem with initial condition  $(\rho_0, u_0)$ . Throughout the paper, we assume that the space variable  $x \in \mathbb{R}^N$ . We restrict ourselves to the case  $N \geq 2$ .

In the sequel we shall only consider the viscous shallow-water system which corresponds to:

$$\mu(\rho) = \mu\rho \quad \text{with } \mu > 0 \quad \text{and} \quad \lambda(\rho) = 0. \quad (1.2)$$

It is motivated by the physical consideration that in the derivation of the NavierStokes equations from the Boltzmann equation through the ChapmanEnskog expansion to the second order (see [20]), the viscosity coefficient is a function of the temperature. If we consider the case of isentropic fluids, this dependence is reduced to the dependence on the density function.

Following the new formulation proposed in [30] (see also [7, 26, 31]), we can rewrite the previous system as follows setting  $v = u + 2\mu\nabla \ln \rho$ :

$$\begin{cases} \partial_t \rho - 2\mu\Delta\rho + \operatorname{div}(\rho v) = 0, \\ \rho\partial_t v + \rho u \cdot \nabla v - \mu\operatorname{div}(\rho \operatorname{curl} v) + \nabla P(\rho) = 0. \end{cases} \quad (1.3)$$

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\*Université Paris Dauphine, PSL Research University, Ceremade, Umr Cnrs 7534, Place du Maréchal De Lattre De Tassigny 75775 Paris cedex 16 (France), haspot@ceremade.dauphine.fr

Let us express now the energy of the system (when  $P(\rho) = a\rho^\gamma$  with  $a > 0$  and  $\gamma > 1$ ) and when the density is close from a constant state  $\bar{\rho} > 0$ , multiplying the momentum equation by  $u$  we have:

$$\begin{aligned} \mathcal{E}(\rho, u)(t) &= \int_{\mathbb{R}^N} \left( \frac{1}{2} \rho(t, x) |u(t, x)|^2 + (\Pi(\rho)(t, x) - \Pi(\bar{\rho})) \right) dx \\ &+ \int_0^t \int_{\mathbb{R}^N} 2\mu \rho(t, x) |Du|^2(t, x) dt dx \leq \int_{\mathbb{R}^N} (\rho_0(x) |u_0(x)|^2 + \Pi(\rho_0)(x)) dx. \end{aligned} \quad (1.4)$$

with  $\Pi(\rho)$  defined as follows:

$$\Pi(s) = s \left( \int_{\bar{\rho}}^s \frac{P(z)}{z^2} dz - \frac{P(\bar{\rho})}{\bar{\rho}} \right),$$

so that  $P(s) = s\Pi'(s) - \Pi(s)$ ,  $\Pi'(\bar{\rho}) = 0$  and:

$$\partial_t \Pi(\rho) + \operatorname{div}(u\Pi(\rho)) + P(\rho) \operatorname{div} u = 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^N).$$

Notice that  $\Pi$  is convex in the case of the  $\gamma$  law. Multiplying now momentum equation of (1.3) by  $v$  when  $P(\rho) = a\rho^\gamma$  we obtain the so called BD entropy:

$$\begin{aligned} \mathcal{E}_1(\rho, v)(t) &= \int_{\mathbb{R}^N} \left( \frac{1}{2} \rho(t, x) |v(t, x)|^2 + (\Pi(\rho)(t, x) - \Pi(\bar{\rho})) \right) dx + \int_0^t \int_{\mathbb{R}^N} \mu \rho(t, x) |\operatorname{curl} v|^2(t, x) dt dx \\ &+ \frac{8\mu}{\gamma} \int_0^t \int_{\mathbb{R}^N} |\nabla \rho^{\frac{\gamma}{2}}|^2 dx dt \leq \int_{\mathbb{R}^N} (\rho_0(x) |v_0(x)|^2 + (\Pi(\rho_0)(x) - \Pi(\bar{\rho}))) dx. \end{aligned} \quad (1.5)$$

In the sequel we will assume that  $\bar{\rho} = 1$  in order to simplify the notations. Combining (1.5) and (1.4) allows to prove at least heuristically (if in particular the solution of (1.1) is sufficiently regular) that for any  $T > 0$  we have (see [44] for the definition of the Orlicz space):

- $\rho - 1 \in L^\infty((0, T), L_2^\gamma(\mathbb{R}^N))$ ,  $\nabla \sqrt{\rho} \in L^\infty((0, T), L^2(\mathbb{R}^N))$ ,  $\sqrt{\rho} - 1 \in L^\infty((0, T), H^1(\mathbb{R}^N))$ .
- $\nabla \rho^{\frac{\gamma}{2}} \in L^2((0, T), L^2(\mathbb{R}^N))$ .
- $\sqrt{\rho} u \in L^\infty((0, T), L^2(\mathbb{R}^N))$ ,  $\sqrt{\rho} \nabla u \in L^2((0, T), L^2(\mathbb{R}^N))$ .

In a first time we will consider the case of axisymmetric solution in dimension  $N = 2$ . In particular we assume that we can write  $(\rho, v)$  as follows:

$$v(t, x) = v_1(t, |x|) \frac{x}{|x|} + v_2(t, |x|) \frac{x^\perp}{|x|} \quad \text{and} \quad \rho(t, x) = \rho(t, |x|). \quad (1.6)$$

It implies that  $(\rho, v^1, u^2)$  verify the following system (see the appendix for the details of the calculus):

$$\begin{cases} \frac{\partial}{\partial t} \rho - 2\mu + \operatorname{div}(\rho v^1) = 0, \\ \rho \partial_t v^1 + \rho u^1 \cdot \nabla v + \nabla P(\rho) = \rho \frac{u_2^2}{|x|} \frac{x}{|x|}, \\ \rho \partial_t u^2 + \rho u^1 \cdot \nabla u^2 + \rho u^2 \cdot \nabla u^1 - 2\mu \operatorname{div}(\rho Du^2) = 0. \end{cases} \quad (1.7)$$

We will also consider the case of radially symmetric initial data when  $N \geq 3$ , it means:

$$u(t, x) = \frac{x}{|x|} u_1(t, |x|) \quad \text{and} \quad \rho(t, x) = \rho_1(t, |x|), \quad (1.8)$$

The system (1.3) can be written as follows since  $v$  is irrotational in this case:

$$\begin{cases} \partial_t \rho - 2\mu \Delta \rho + \operatorname{div}(\rho v) = 0, \\ \rho \partial_t v + \rho u \cdot \nabla v + \nabla P(\rho) = 0. \end{cases} \quad (1.9)$$

The main goals of the present paper are to prove global existence and uniqueness of axisymmetric solutions to the systems (1.7) and (1.9) with non-negative initial densities and large initial data. There is huge literature on the studies about global existence and behaviors of solutions to (1.1) in the case that the viscosity coefficients  $\mu$  and  $\lambda$  are both constants. The global existence of weak solution with large initial data of the system (1.1) has been proved for the first time by Lions in [44] using the so called notion of renormalized solution combined with the introduction of the effective pressure which allows to test the compactness of the solutions. Feireisl et al in [19] have been generalized the previous results in the case of general pressure of the form  $P(\rho) = \rho^\gamma$  with  $\gamma > \frac{N}{2}$  (see also Plotnikov and Weigant in [55] for the case  $N = 2$ ,  $\gamma = 1$ ). This last result have been generalized by Bresch and Jabin for anisotropic viscosity coefficient (see [6]). We refer also to [13, 15, 51, 10, 8, 23, 9, 50, 16] for the existence of global strong solution with small initial data in critical spaces for the scaling of the equations.

Let us focus now on the case of the system (1.1) with degenerate viscosity coefficients, we can first mention that the study is completely different from the constant viscosity case. Indeed the dependence of viscosity coefficients on the flow density makes difficult the analysis, in particular it is not clear how to obtain uniformly a-priori estimates for the velocity and trace the motion of particle paths near vacuum regions, which is the essential difference of (1.1) from compressible Navier-Stokes equations with constant viscosity coefficients where the particle path can be defined a-priori even near vacuum. Recently different authors have obtained significant progress on the existence of global weak solutions with degenerate viscosity coefficients. Bresch and Desjardins in [3, 4, 5] discovered a new entropy structure which ensures estimate on the gradient of the density. More precisely if  $\mu$  and  $\lambda$  verifies the following algebraic relation:

$$\lambda(\rho) = 2\mu'(\rho) - 2\mu(\rho), \quad (1.10)$$

then at least heuristically Bresch and Desjardins have observed that the quantity  $\frac{\mu'(\rho)}{\sqrt{\rho}} \nabla \rho$  is conserved in  $L_T^\infty(L^2(\mathbb{R}^N))$  norm for any  $T > 0$ . It enables us (see [3, 4, 5] and also [7, 52, 64]) to prove the existence of global weak solution with either a drag friction or a cold pressure term (a pressure that is singular at the vacuum). The fact to work with a friction term is crucial in the sense that it provides simply a gain of integrability on the velocity, combined with the BD entropy it allows to obtain the stability of the global weak solution and to construct global approximated solution which verify uniformly the energy and the BD entropy. The stability of the global weak solutions for a general  $\gamma$  law ( $P(\rho) = a\rho^\gamma$  with  $1\gamma < +\infty$  when  $N = 2$  and  $1 < \gamma < 3$  when  $N = 3$ ) was proved by Mellet and Vasseur in [48] using new energy estimate improving the integrability of the velocity. A particular choice of viscosity coefficients  $\mu(\rho) = \mu\rho$ ,  $\lambda(\rho) = 0$  satisfying (1.10) leads to the so-called *shallow water system*, for which the proof of existence of global weak solutions has been recently delivered independently by Vasseur and Yu [59, 60] and Li and Xin in [43].

Concerning the existence of global strong solution with large initial data, there is a huge literature for the one dimension case. We refer to [39, 32] for the case of constant viscosity coefficients. . In the case of degenerate viscosity coefficient of the form  $\mu(\rho) = \mu\rho^\alpha$  with  $\alpha > 0$ , when  $0 < \alpha < \frac{1}{2}$  the BD entropy allows to bound the density from below. Indeed, in [49] Mellet and Vasseur proved the existence of global strong solution to (1.1) for initial density

far away from the vacuum. The main ingredient of the proof was to use entropy from [3] to estimate  $\partial_x(\rho^{\alpha-\frac{1}{2}})$  in  $L^\infty(0, T; L^2(\mathbb{R}))$  for all  $T > 0$ . This implies boundedness of  $\rho^{-1}$  in  $L^\infty(0, T; L^\infty(\mathbb{R}))$  which allows to show sufficient regularity of solution in order to prove the uniqueness. We have recently extended this result in [29] to the general case. The main idea was to prove boundedness of  $v$  appearing in (1.3) by using the structure of the transport equation for  $v$ . It allows to bound  $\rho^{-1}$  by application of the maximum principle to the continuity equation.

Concerning the existence of global strong solution in the multi-d case, there is very few works and the problem remains essentially open. However Kazhikhov and Vaigant proved the existence of global strong solution with large initial data in two dimension for the torus with particular viscosity coefficient of the form:

$$\mu(\rho) = \mu \text{ and } \lambda(\rho) = \lambda\rho^\alpha, \quad (1.11)$$

with  $\lambda, \mu > 0$  and  $\alpha > 3$ . This result is remarkable and use strongly the structure of effective velocity (see [44]) which consists to estimate  $(\mu + \lambda(\rho))\operatorname{div}u - P(\rho) + P(\bar{\rho})$  with  $\bar{\rho}$ . Indeed roughly speaking it is possible for such viscosity coefficients by applying the operator  $(\Delta)^{-1}\operatorname{div}$  to the momentum equation. An additional key point is the form of the viscosity coefficient  $\lambda(\rho)$  which enables for  $\alpha > 3$  to show a gain of integrability on  $\rho$  and more precisely  $\rho$  is bounded in  $L_T^\infty(L^p(\mathbb{T}))$  for any  $T > 0$  and  $p \geq 1$ . This result has slightly been improved by weakening the condition on  $\alpha$  (see [25, 38, 35, 36]). Let us also mention [27] where we prove the existence of global strong solution for irrotational data with a smallness condition which is supercritical. It allows in particular to exhibit initial data which are large for the energy estimate in dimension two such that they furnish global strong solution.

In all its generality, the problem of the existence of global strong solution with large initial data for the shallow. It is then a natural and interesting problem to investigate the existence of global strong solution by adding symmetry condition on the initial data. It is well known that in dimension  $N = 3$  (see [40, 57]) there exists global strong solution for the incompressible Navier-Stokes equation and the incompressible Euler equation for axisymmetric initial data without swirl. There exists a large literature on the existence of global weak or strong solution with smallness assumption involving symmetry condition on the initial data, let us point out that systematically the authors consider initial data which are radially symmetric in dimension  $N \geq 2$ . It yields that they study initial data such that  $u^2 = 0$  in (1.7) in dimension  $N = 2$ . The first existence result was proved by Hoff [34]; he proved the global existence of radially symmetric weak solutions to problem (1.1) with strictly positive initial densities in annular domains and with constant viscosity coefficients. Then it was extended by Jiang and Zhang in [37] to the Cauchy problem with general non-negative initial densities. Roughly speaking, they proved the global existence of radially symmetric weak solutions for general pressure  $P(\rho) = \rho^\gamma$  with  $\gamma > 1$  generalizing the results of Feireisl et al (see [19]). Choe and Kim in [12] proved the existence of global strong solution with large initial data for radially symmetric initial data when  $N \geq 2$  in an annular domain. Let us mention that it is important here to work in an annular domain since it avoids the difficulties related to the control of the vacuum around 0 which could blow-up.

Indeed as emphasized in many related papers, the possible appearance of vacuum is one of the main difficulties, which indeed leads to the singular behaviors of solutions in the presence of vacuum, such as the failure of continuous dependence of weak solutions on initial data [33] and the finite time blow-up of smooth solutions [62]. That is why, it is a natural and interesting problem to investigate the influence of the vacuum state on the existence and dynamics of

global solutions to (1.1). One of the prototype problems which has been largely studied is the time-evolution of the compressible viscous flow of finite mass expanding into infinite vacuum. This corresponds to free boundary value problems for the compressible Navier-Stokes equations (1.1) for general initial data and variant boundary conditions imposed on the free surface. The study is a fundamental issue of fluid mechanics and has attracted lots of research interests, we refer to [11, 17, 18, 47, 53, 54, 56, 61, 63, 65, 42] and the reference therein.

In this paper we are interested in proving the existence of global strong solution for the system (1.1) with large axisymmetric initial data verifying in dimension  $N = 2$  the condition (1.6) and (??) in dimension  $N = 3$ . The purpose is triple, indeed in comparison with the previous works cited below we would like to avoid any condition of smallness on the initial data, to work in the complete euclidean space  $\mathbb{R}^N$  (and not only in an annular which allows to avoid all the difficulties related to some phenomena of blow-up of the density around the origin) and to take into account the rotational component ( $u^2$  when  $N = 2$ , we refer to (1.6), ). Let us explain briefly now the strategy of the proof in the case  $N = 2$  (the case  $N = 3$  follows the same ideas up to small modifications). As we explained previously, one of the main difficulty in order to prove the existence of global strong solution consist in controlling the vacuum or in other word to estimate the  $L^\infty$  norm of  $\frac{1}{\rho}$ . If we look at the mass equation in (1.7), it seems natural to apply a maximum principle theorem since the density  $\rho$  verifies a parabolic equation as observed in [30]. We have then to estimate the effective velocity  $v^1$  in  $L^p((0, T), L^q(\mathbb{R}^N))$  space with  $(p, q)$  suitably chosen such that (see [41]):

$$\frac{2}{p} + \frac{N}{q} < 1. \quad (1.12)$$

In order to obtain this type of information, we are going to estimate the  $L^\infty((0, T), L^p(\mathbb{R}^N))$  norm of  $\rho^{\frac{1}{p}} v^1$  for any  $p \geq 2$  by multiplying the second equation of (1.7) by  $v^1 |v^1|^{p-2}$ . The main difficulty consists then in evaluating the remainder term in the second equation of (1.7)  $\frac{u_2^2}{|x|} \frac{x}{|x|}$ . Indeed this terms tends to blow-up in a neighborhood of the origin, in order to overcome this difficulty we are going to establish new estimates on  $u^2$  by involving new weight estimate combining with some maximum principle arguments on  $u^1$  in order to show that  $u^1$  is essentially the sum of a positive function and of an  $L_{t,x}^\infty$  function in space and time. It will enables us to show that  $\frac{u_2^2}{|x|} \frac{x}{|x|}$  is bounded in  $L^\infty((0, T), L^p(\mathbb{R}^N))$  for any  $p \geq 2$ . The last difficulty is to deal with the pressure term which can be rewritten as the sum of  $v^1$  and  $-u^1$ . It implies that the pressure produces a damping effect on the effective velocity  $v^1$  provided that we estimate  $u^1$  in  $L^\infty((0, T), L^p(\mathbb{R}^N))$  for any  $p \geq 2$  (it will be done by using a gain of integrability as in [48]). Below we are now giving our main results on the global existence of strong solution with large initial data for axisymmetric initial data. In a first time let us recall a general <sup>1</sup> theorem of strong solution in finite time in critical Besov space for the scaling of the equations (see [15, 22, 24, 9]). We set in the sequel  $\rho = q + 1$ .

**Theorem 1.1.** *Let  $N \geq 2$ , the viscosity coefficients verify (1.2). Assume now that  $(q_0, u_0) \in B_{p,1}^{\frac{N}{p}} \times B_{p,1}^{\frac{N}{p}-1}$  with  $1 \leq p < 2N$  and that there exists  $c$  such that  $0 < c \leq \rho_0$ . Then there exists a time  $T$  such that system (1.1) has a unique solution on  $[0, T]$  with:*

$$q \in \tilde{C}_T(B_{p,1}^{\frac{N}{p}}), \quad \frac{1}{\rho}, \rho \in L_T^\infty(L^\infty(\mathbb{R}^N)) \quad \text{and} \quad u \in \tilde{C}_T(B_{p,1}^{\frac{N}{p}-1}) \cap L_T^1(B_{p,1}^{\frac{N}{p}+1}). \quad (1.13)$$

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<sup>1</sup>It means that there is no geometric assumption on the initial data.

If in addition  $(q_0, u_0)$  belongs to  $B_{p,1}^s \times B_{p,1}^{s-1}$  for any  $s > \frac{N}{p}$  then we have:

$$q \in \tilde{C}_T(B_{p,1}^s) \text{ and } u \in \tilde{C}_T(B_{p,1}^{s-1}) \cap L_T^1(B_{p,1}^{s+1}). \quad (1.14)$$

Assume now that (1.1) has a solution  $(q, u) \in C([0, T^*), B_{p,1}^{\frac{N}{p}} \times B_{p,1}^{\frac{N}{p}-1})$  on the time interval  $[0, T^*)$  which satisfies the following three conditions:

1. the function  $q$  belongs to  $L^\infty([0, T^*], B_{p,1}^{\frac{N}{p}})$ ,
2.  $\frac{1}{\rho} \in L^\infty([0, T^*], L^\infty(\mathbb{R}^N))$ ,
3. we have  $\int_0^{T^*} \|\nabla u(s)\|_{L^\infty} ds < +\infty$ ,

Then  $(q, u)$  may be continued beyond  $T^*$ .

**Remark 1.** This theorem is a direct consequence of [27, 24, 13, 15]. In [27] the existence is proved for  $(q_0, u_0) \in B_{p,1}^{\frac{N}{p}} \times B_{p,1}^{\frac{N}{p}-1}$  for  $1 \leq p < 2N$  with uniqueness for  $1 \leq p \leq N$ . The uniqueness is extended in [15] to the case  $N \leq p < 2N$ .

The blow up criteria is given in [27] following an argument developed in [13].

Let us define now the maximal time  $T^*$  of existence:

$$T^* = \sup\{T \in \mathbb{R}; \text{there exists a strong solution } (q, u) \text{ of the system (1.1) on } [0, T] \text{ verifying (1.13)}\}$$

**Theorem 1.2.** Let  $N = 2$ ,  $P(\rho) = a\rho^\gamma$  with  $\gamma > 1$ , the viscosity coefficients verify (1.2) and assume that the initial data can be written under the form (1.6):

$$\rho_0(x) = (\rho_1)_0(|x|), \quad u_0 = u_0^1 + u_0^2 = \frac{x}{|x|}(u_1)_0(|x|) + \frac{x^\perp}{|x|}(u_2)_0(|x|).$$

In addition we assume that  $(q_0, v_0) \in (B_{p,1}^{\frac{N}{p}} \cap B_{p,1}^{\frac{N}{p}+\epsilon}) \times (B_{p,1}^{\frac{N}{p}-1} \cap B_{p,1}^{\frac{N}{p}-1+\epsilon})$  with  $\epsilon > 0$  such that  $\frac{N}{1-\epsilon} < p < 2N$  and  $0 < c \leq \rho_0$ . Furthermore we suppose that  $v_0^1 \in L^\infty$ ,  $u_0^1, u_0^2 \in L^\infty$ ,  $\frac{u_0^2}{|x|} \in L^\infty$  and:

$$\mathcal{E}(\rho_0, u_0) < +\infty, \quad \mathcal{E}_1(\rho_0, v_0) < +\infty,$$

then we have  $T^* = +\infty$ . Furthermore the global solution so defined  $(\rho, u)$  is unique and verifies locally in time (1.13).

**Remark 2.** As we explained previously there is essentially one result of global strong solution in dimension  $N = 2$  for the compressible Navier-Stokes system which is due to Vaigant and Kazhikhov [58] when the viscosity coefficients verify  $\mu(\rho) = \mu$  and  $\lambda(\rho) = \rho^\alpha$  with  $\alpha > 3$ . The proof is based on the structure of effective velocity which allows such a choice of coefficients of viscosity (see [25, 38, 35, 36] for some refinements). However the problem remains actually completely open for the shallow water system (it means with the viscosity coefficients verifying (1.2) or more generally for a system verifying the BD condition (1.10).

To the best of our knowledge, it is the first generic result of global strong solution in dimension  $N = 2$  for the shallow water system with large axisymmetric initial data in  $\mathbb{R}^2$ . Indeed the most of the results which deal with geometric initial data are restrict to the case  $u_0^2 = 0$  (the rotational part) which corresponds to radially symmetric initial data and are focused on annular domain in order to avoid all the possible blow-up around the origin.

We can obviously mention that the existence of global strong solution with general initial data (without any geometric assumption) remains open for the shallow water system.

To finish we conclude with our theorem on the system (1.9).

**Theorem 1.3.** *Let  $N = 3$ ,  $P(\rho) = a\rho^\gamma$  with  $1 \leq \gamma < \frac{7}{6}$  and assume that the initial data can be written under the form (1.8):*

$$u_0(x) = \frac{x}{|x|} u_0^1(|x|) \quad \text{and} \quad \rho_0(x) = (\rho_1)_0(r, z).$$

*In addition we assume that  $(q_0, v_0) \in (B_{p,1}^{\frac{N}{p}} \cap B_{p,1}^{\frac{N}{p}+\epsilon}) \times (B_{p,1}^{\frac{N}{p}-1} \cap B_{p,1}^{\frac{N}{p}-1+\epsilon})$  with  $\epsilon > 0$  such that  $\frac{N}{1-\epsilon} < p < 2N$  and  $0 < c \leq \rho_0$ . Furthermore we suppose that  $v_0 \in L^\infty$ ,  $u_0 \in L^\infty$  and:*

$$\mathcal{E}(\rho_0, u_0) < +\infty, \quad \mathcal{E}_1(\rho_0, v_0) < +\infty,$$

*then we have  $T^* = +\infty$ . Furthermore the global solution so defined  $(\rho, u)$  is unique and verifies locally in time (1.13).*

**Remark 3.** Let us mention that it would be easy to extend this result to the case of general dimension  $N \geq 4$ , the only modification would concern the exponent  $\gamma$  which should be small enough. The case  $\gamma = 1$  works (see [28]).

The paper unfolds as follows. In the next section, we introduce the homogeneous Besov spaces and the tools (the Littlewood-Paley decomposition and paradifferential calculus). In the section 3, we use the theorem 1.1 that we describe briefly to obtain the existence of strong solution in finite time. Furthermore we show that the solution remains axisymmetric. Section 4 is devoted to the proof of a gain of integrability on  $v^1$  which allows us to apply a maximum principle in order to bound in  $L^\infty$  norm  $\rho$  and  $\frac{1}{\rho}$ . The proof of the theorem 1.2 is carried over to the section 5. The proof of the theorem 1.3 is given in section 6. We postpone in appendix some tedious computations on the writing of the system (1.7) and (??).

## 2 Littlewood-Paley theory and Besov spaces

Throughout the paper,  $C$  stands for a constant whose exact meaning depends on the context. The notation  $A \lesssim B$  means that  $A \leq CB$ . This section is intended to briefly recall some classical results on homogeneous Besov spaces which are defined via the so-called Littlewood-Paley decomposition (for a recent account of the theory we refer to the excellent book [1]). To build the Littlewood-Paley decomposition, one may fix a smooth radial function  $\chi$  supported in (say) the ball  $B(0, \frac{4}{3})$ , equal to 1 in a neighborhood of  $B(0, \frac{3}{4})$  and such that  $r \mapsto \chi(r.e_r)$  is nondecreasing over  $\mathbb{R}_+$ . So that if we define  $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$ , then  $\varphi$  is compactly supported in the annulus  $\{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  and we have

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1. \quad (2.15)$$

Then we can define the *dyadic blocks*  $(\Delta_l)_{l \in \mathbb{Z}}$  by  $\Delta_l := \varphi(2^{-l}D)$  (that is  $\widehat{\Delta_l u} = \varphi(2^{-l}\xi)\widehat{u}(\xi)$ ) so that, formally, we have

$$u = \sum_l \Delta_l u. \quad (2.16)$$

As (2.15) is satisfied for  $\xi \neq 0$ , the previous formal equality holds true for tempered distributions *modulo polynomials*. A way to avoid working modulo polynomials is to consider the set  $\mathcal{S}'_h$



of tempered distributions  $u$  such that  $\lim_{l \rightarrow -\infty} \|S_l u\|_{L^\infty} = 0$ , where  $S_l$  stands for the low frequency cut-off defined by  $S_l := \chi(2^{-l}D)$ . If  $u \in \mathcal{S}'_h$ , (2.16) becomes true and one may write  $S_l u = \sum_{q \leq l-1} \Delta_q u$ . One can now define the homogeneous Besov spaces used in this article:

**Definition 2.1.** For  $s \in \mathbb{R}$  and  $1 \leq p, r \leq \infty$ , we set

$$\|u\|_{B_{p,r}^s} := \left( \sum_l 2^{rls} \|\Delta_l u\|_{L^p}^r \right)^{\frac{1}{r}} \quad \text{if } r < \infty \quad \text{and} \quad \|u\|_{B_{p,\infty}^s} := \sup_l 2^{ls} \|\Delta_l u\|_{L^p}.$$

We then define the space  $B_{p,r}^s$  as the subset of distributions  $u \in \mathcal{S}'_h$  such that  $\|u\|_{B_{p,r}^s}$  is finite.

Bernstein lemma (describing how derivatives act on spectrally localized functions, see [1]) implies the following embedding result:

**Proposition 2.1.** For all  $s \in \mathbb{R}$ ,  $1 \leq p_1 \leq p_2 \leq \infty$  and  $1 \leq r_1 \leq r_2 \leq \infty$ , the space  $B_{p_1,r_1}^s$  is continuously embedded in the space  $B_{p_2,r_2}^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}$ . In addition we have:

$$B_{p,1}^{\frac{N}{p}} \hookrightarrow B_{\infty,1}^0 \hookrightarrow L^\infty \quad \text{and} \quad B_{p,1}^0 \hookrightarrow L^p \hookrightarrow B_{p,\infty}^0.$$

In this paper, we shall mainly work with functions or distributions depending on both the time variable  $t$  and the space variable  $x$ . We shall denote by  $\mathcal{C}(I; X)$  the set of continuous functions on  $I$  with values in  $X$ . For  $p \in [1, \infty]$ , the notation  $L^p(I; X)$  stands for the set of measurable functions on  $I$  with values in  $X$  such that  $t \mapsto \|f(t)\|_X$  belongs to  $L^p(I)$ . In the case where  $I = [0, T]$ , the space  $L^p([0, T]; X)$  (resp.  $\mathcal{C}([0, T]; X)$ ) will also be denoted by  $L_T^p X$  (resp.  $\mathcal{C}_T X$ ). Finally, if  $I = \mathbb{R}^+$  we shall alternately use the notation  $L^p X$ . The Littlewood-Paley decomposition enables us to work with spectrally localized (hence smooth) functions rather than with rough objects. We naturally obtain bounds for each dyadic block in spaces of type  $L_T^\rho L^p$ . Going from those type of bounds to estimates in  $L_T^\rho B_{p,r}^s$  requires to perform a summation in  $\ell^r(\mathbb{Z})$ . When doing so however, we *do not* bound the  $L_T^\rho B_{p,r}^s$  norm for the time integration has been performed *before* the  $\ell^r$  summation. This leads to the following notation:

**Definition 2.2.** For  $T > 0$ ,  $s \in \mathbb{R}$  and  $1 \leq r, \sigma \leq \infty$ , we set  $\|u\|_{\tilde{L}_T^\sigma(B_{p,r}^s)} := \|2^{js} \|\Delta_l u\|_{L_T^\sigma(L^p)}\|_{\ell^r(\mathbb{Z})}$ .

One can then define the space  $\tilde{L}_T^\sigma(B_{p,r}^s)$  as the set of tempered distributions  $u$  over  $(0, T) \times \mathbb{R}^d$  such that  $\lim_{l \rightarrow -\infty} S_l u = 0$  in  $L^\sigma([0, T]; L^\infty(\mathbb{R}^d))$  and  $\|u\|_{\tilde{L}_T^\sigma(B_{p,r}^s)} < \infty$ <sup>2</sup>. The spaces  $\tilde{L}_T^\sigma(B_{p,r}^s)$  may be compared with the spaces  $L_T^\sigma(B_{p,r}^s)$  using the Minkowski inequality:

$$\|u\|_{\tilde{L}_T^\sigma(B_{p,r}^s)} \leq \|u\|_{L_T^\sigma(B_{p,r}^s)} \quad \text{if } r \geq \sigma \quad \text{and} \quad \|u\|_{\tilde{L}_T^\sigma(B_{p,r}^s)} \geq \|u\|_{L_T^\sigma(B_{p,r}^s)} \quad \text{if } r \leq \sigma.$$

All the properties of continuity for the product and composition which are true in Besov spaces remain true in the above spaces. The time exponent just behaves according to Hölder's inequality.

Formally, any product of two distributions  $u$  and  $v$  may be decomposed into

$$uv = T_u v + T_v u + R(u, v), \tag{2.17}$$

<sup>2</sup> The letter  $T$  is omitted for functions defined over  $\mathbb{R}^+$ .

$$T_u v := \sum_l S_{l-1} u \Delta_l v, \quad T_v u := \sum_l S_{l-1} v \Delta_l u \quad \text{and} \quad R(u, v) := \sum_l \sum_{|l'-l| \leq 1} \Delta_l u \Delta_{l'} v.$$

The above operator  $T$  is called “paraproduct” whereas  $R$  is called “remainder”. The decomposition (2.17) has been introduced by Bony in [2] (see also [1]). In this article we will deal with the following classical paraproduct estimates (we refer to [1]).

**Proposition 2.2.** *Under the same assumptions there exists a constant  $C > 0$  such that if  $1/p_1 + 1/p_2 = 1/p$ , and  $1/r_1 + 1/r_2 = 1/r$ :*

$$\begin{aligned} \|T_u v\|_{B_{p,1}^s} &\leq C \|u\|_{L^\infty} \|v\|_{B_{p,1}^s}, \\ \|T_u v\|_{B_{p,r}^{s+t}} &\leq C \|u\|_{B_{p_1,r_1}^t} \|v\|_{\dot{B}_{p_2,r_2}^s} \quad (t < 0), \\ \|R(u, v)\|_{B_{p,r}^{s_1+s_2-\frac{d}{2}}} &\leq C \|u\|_{B_{p_1,r_1}^{s_1}} \|v\|_{B_{p_2,r_2}^{s_2}} \quad (s_1 + s_2 > 0). \end{aligned} \quad (2.18)$$

Let us now turn to the composition estimates (see [1]).

**Proposition 2.3.** *Let  $s > 0$ ,  $u \in B_{p,1}^s \cap L^\infty$  and  $F \in W_{loc}^{[s]+2,\infty}(\mathbb{R}^d)$  such that  $F(0) = 0$ . Then  $F(u) \in B_{p,1}^s$  and there exists a function of one variable  $C_0$  only depending on  $s, p, d$  and  $F$  such that*

$$\|F(u)\|_{B_{p,1}^s} \leq C_0 (\|u\|_{L^\infty}) \|u\|_{B_{p,1}^s}.$$

### A maximal regularity estimate for the heat equation

According to [1], the following result holds:

**Proposition 2.4.** *Let  $s \in \mathbb{R}$ ,  $(p, r) \in [1, +\infty]^2$  and  $1 \leq \rho_2 \leq \rho_1 \leq +\infty$ . Assume that  $u_0 \in B_{p,r}^s$  and  $f \in \tilde{L}_T^{\rho_2}(B_{p,r}^{s-2+2/\rho_2})$ . Let  $u$  be a solution of:*

$$\begin{cases} \partial_t u - \mu \Delta u = f, \\ u|_{t=0} = u_0, \end{cases}$$

where  $\mu > 0$ . Then there exists  $C > 0$  depending only on  $N, \mu, \rho_1$  and  $\rho_2$  such that:

$$\|u\|_{\tilde{L}_T^{\rho_1}(B_{p,r}^{s+2/\rho_1})} \leq C (\|u_0\|_{B_{p,r}^s} + \|f\|_{\tilde{L}_T^{\rho_2}(B_{p,r}^{s-2+2/\rho_2})}).$$

If in addition  $r$  is finite then  $u$  belongs to  $C([0, T], B_{p,r}^s)$ .

### 3 Strong solution in finite time

In this section we are interested in proving the existence of strong solution on a maximal time interval  $(0, T^*)$  for initial data verifying the assumptions of the theorem 1.2. Furthermore we will verify that the solution remains axisymmetric on  $(0, T^*)$ .

### 3.1 Existence of strong solution in finite time on a maximal time interval $(0, T^*)$

The theorem 1.1 is a direct consequence of [27, 24, 13, 15]. In [27] the existence is proved for  $(q_0, u_0) \in B_{p,1}^{\frac{N}{p}} \times B_{p,1}^{\frac{N}{p}-1}$  for  $1 \leq p < 2N$  with uniqueness for  $1 \leq p \leq N$ . The uniqueness is extended in [15] to the case  $N \leq p < 2N$ . The blow up criteria is given in [27] following an argument developed in [13]. We can simply recall that the condition  $p < 2N$  is a restriction due to the paraproduct law when we want to deal with the term of the form  $\nabla \ln \rho \cdot Du$ .

### 3.2 Existence of strong solution in finite time which remains axisymmetric

We have seen that there exists a strong solution  $(\rho, u)$  on a maximal time interval  $(0, T^*)$  of the system (1.1) for the initial data  $(\rho_0, u_0)$  verifying the assumptions of the theorem 1.2. We are now interested in proving that this solutions remains radially symmetric in the case  $N = 3$  or rotationally invariant in the case  $N = 2$  on the time interval  $(0, T^*)$  provided that the initial data are radially symmetric or rotationally invariant. Let us start with the first case. In other way we wish to verify that  $(\rho, u)$  can be written under the following form:

$$\rho(t, x) = \rho_1(t, |x|) \quad \text{and} \quad v(t, x) = \frac{x}{|x|} v_1(t, |x|). \quad (3.19)$$

To see this, it suffices to check that any isometry  $A \in \mathcal{M}_N(\mathbb{R}^N)$  (whith  $A^t A = A A^t = Id_N$ ) we have for all  $t \in (0, T^*)$ :

$$\rho(t, x) = \rho(t, Ax) \quad \text{and} \quad v(t, x) = Av(t, A^t x). \quad (3.20)$$

Indeed (3.20) implies that for every  $t \in (0, T^*)$ :

$$v(t, x) = x f(t, x), \quad (3.21)$$

with  $f$  a scalar function. By contradiction assume that it is not true, then there exists  $t \in (0, T^*)$ ,  $y_1$  and  $y_0$  which are not collinear such that:

$$v(t, y_0) = \lambda y_1,$$

with  $\lambda \in \mathbb{R}^*$ . (3.20) is true for any  $A$  an isometry, let us now choose  $A_0$  a rotation such that:

$$A_0 y_0 = \frac{y_1}{|y_1|} |y_0|.$$

Now we have for all the rotations  $A$  (we recall that the rotations are commutative), we have:

$$v(t, A y_0) = \lambda A y_1 = \lambda \frac{|y_1|}{|y_0|} A A_0 y_0 = \lambda \frac{|y_1|}{|y_0|} A_0 A y_0.$$

We deduce then that:

$$v(t, x) = \lambda \frac{|y_1|}{|y_0|} A_0 x.$$

We take now any isometry  $B$  and we deduce that:

$$B v(t, x) = \lambda \frac{|y_1|}{|y_0|} B A_0 x = \lambda \frac{|y_1|}{|y_0|} A_0 B x = v(t, B x).$$

It implies that for any isometry  $B$  we have  $BA_0 = A_0B$  which is false. Then by contradiction (3.21) is true. We now prove that  $f$  in (3.21) is necessary a radial function. From (3.20) we observe that for any isometry  $A$ :

$$H(t, x) = (v(t, x), x) = (Av(t, {}^t Ax), x) = (v(t, {}^t Ax), {}^t Ax) = H(t, {}^t Ax).$$

It implies that  $H$  is radial furthermore we have:

$$H(t, x) = |x|^2 f(x).$$

We deduce that  $f$  is radial and it concludes the fact that  $v$  remains radially symmetric.

Let us prove now (3.20), since  $\rho$  and  $v$  are regular on  $(0, T^*)$  and solution of (1.3) we can easily verify that:

$$\begin{cases} \partial_t(\rho(t, A^t x)) - 2\mu\Delta(\rho(t, A^t x)) + \operatorname{div}(\rho(t, A^t x)Av(\rho(t, A^t x))) = 0 \\ \rho(t, A^t x)\partial_t(Av(t, A^t x)) + \rho(t, A^t x)Au(t, A^t x) \cdot \nabla(Av(t, A^t x)) \\ - \mu\operatorname{div}(\rho(t, A^t x)\operatorname{curl}[Av(t, A^t x)]) + \nabla[P(\rho)(t, A^t x)] = 0 \end{cases} \quad (3.22)$$

It implies that  $(\rho(t, A^t x), Av(t, A^t x))$  is a solution of (1.3) with initial data  $(\rho_0(x), v_0(x))$  since  $\rho_0$  is radial and  $v_0$  radially symmetric. By uniqueness of the strong solution we deduce the result (3.20).

We prove similar result for the case of rotationally invariant initial data in dimension  $N = 2$  (it means initial data verifying the condition 1.6). To do this it suffices to rewrite the system in function of  $\operatorname{div}u$  and  $\operatorname{curl}u$  and to observe that these functions are radial. More precisely for initial data verifying (1.6) we consider solution of the form:

$$v(t, x) = {}^t Av^1(t, Ax) + A^\perp v^2(t, Ax),$$

with:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad A^\perp = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

and  ${}^t AA = A^t A = Id_2$ .

## 4 Maximum principle on the density $\rho$ and control of the vacuum

In the sequel we are interested in dealing with radially symmetric solutions such that the initial data  $(\rho(0, \cdot), u(0, \cdot))$  verify:

$$\rho_0(x) = \rho_0^1(|x|) \quad \text{and} \quad u_0(x) = \frac{x}{|x|}u_0^1(|x|) + \frac{x^\perp}{|x|}u_0^2(|x|).$$

It is then natural to search solution to the system (1.3) under the form:

$$\rho(t, x) = \rho_1(t, |x|) \quad \text{et} \quad v(t, x) = \frac{x}{|x|}v_1(t, |x|) + \frac{x^\perp}{|x|}v_2(t, |x|).$$

It implies that  $(\rho, v^1, u^2)$  has to solve the following system:

$$\begin{cases} \partial_t \rho - 2\mu \Delta \rho + \operatorname{div}(\rho v^1) = 0 \\ \rho \partial_t v^1 + \rho u^1 \cdot \nabla v^1 + \nabla P(\rho) - \frac{u_2^2}{|x|} \frac{x}{|x|} = 0 \\ \rho \partial_t u^2 + \rho u^1 \cdot \nabla u^2 + \rho u^2 \cdot \nabla u^1 - 2\mu \operatorname{div}(\rho D u^2) = 0 \\ (\rho, v^1, u^2)_{t=0} = (\rho_0^1, v_0^1, u_0^2). \end{cases} \quad (4.23)$$

We have seen in the previous section that for initial data verifying the assumption of the theorem 1.2 there exists strong solution  $(\rho, v^1, u^2)$  on a finite time interval  $(0, T^*)$ . We are interested now in proving that  $T^* = +\infty$ . By absurd we assume now that  $T^* < +\infty$ . In order to prove a contradiction, we are going to show that the norm  $\|(q, u)(T, \cdot)_X$  does not blow up when  $T$  goes to  $T^*$ .

In a first time we have to show that  $\frac{1}{\rho}$  and  $\rho$  are bounded in  $L^\infty([0, T^*], L^\infty(\mathbb{R}^N))$ . In order to show this result, we are going to prove a gain of integrability on  $v^1$  which allows us to apply a maximum principle on the first equation of (4.23).

#### 4.0.1 Estimate on $u_1$

We recall that  $P(\rho) = a\rho^\gamma$  in addition since  $u^1$  is irrotational we have  $Du^1 = \nabla u^1$ , it implies that on  $(0, T^*)$  we have:

$$\rho \partial_t u^1 + \rho u^1 \cdot \nabla u^1 - 2\mu \operatorname{div}(\rho \nabla u^1) + \nabla P(\rho) = \frac{\rho u_2^2}{|x|} \frac{x}{|x|}. \quad (4.24)$$

It gives in particular since  $\frac{1}{\rho}$  is bounded in  $L^\infty((0, T^*) \times \mathbb{R}^N)$ :

$$\partial_t u^1 + (u^1 - 2\mu \nabla \ln \rho) \cdot \nabla u^1 - 2\mu \Delta u^1 + \nabla F(\rho) = \frac{u_2^2}{|x|} \frac{x}{|x|}. \quad (4.25)$$

We now define by  $S(t)$  the semi-group associated to parabolic the equation:

$$\partial_t u^1 + w \cdot \nabla u^1 - 2\mu \Delta u^1 = 0,$$

with  $w = u^1 - \mu \nabla \ln \rho$ . From Duhamel formula, it yields:

$$u^1(t, \cdot) = S(t)u_0^1 - \int_0^t S(t-s) \nabla F(\rho)(s, \cdot) ds + \int_0^t S(t-s) \left( \frac{u_2^2(s, \cdot)}{|\cdot|} \frac{\cdot}{|\cdot|} \right) ds.$$

Let us estimate now  $w^1$  the solution of the problem:

$$\begin{cases} \partial_t w^1 + w \cdot \nabla w^1 - 2\mu \Delta w^1 + \nabla F(\rho) = 0 \\ w^1(0, \cdot) = u_0^1, \end{cases} \quad (4.26)$$

then we show that for any  $p \geq 2$ ,  $\rho^{\frac{1}{p}} w^1$  is uniformly bounded in  $L_T^\infty(L^p(\mathbb{R}^N))$  which implies that  $w^1$  belongs to  $L_T^\infty(L^\infty(\mathbb{R}^N))$ . Indeed we observe that:

$$\begin{cases} \rho \partial_t w^1 + \rho u^1 \cdot \nabla w^1 - 2\mu \operatorname{div}(\rho \nabla w^1) + \nabla P(\rho) = 0 \\ w^1(0, \cdot) = u_0^1, \end{cases} \quad (4.27)$$

Multiplying the previous equation by  $w^1|w^1|^{p-2}$  with  $p \geq 2$  and integrating over  $(0, t) \times \mathbb{R}^N$  with  $0 < t < T^*$  we obtain:

$$\begin{aligned} & \frac{1}{p} \int_{\mathbb{R}^N} (\rho|w^1|^p)(t, x) dx + \int_0^t \int_{\mathbb{R}^N} \rho|w^1|^{p-2} |\nabla w^1|^2(s, x) ds dx \\ & + \frac{(p-2)}{4} \int_0^t \int_{\mathbb{R}^N} \rho |\nabla(|w^1|^2)|^2 |w^1|^{p-4}(s, x) ds dx \leq \frac{1}{p} \int_{\mathbb{R}^N} (\rho_0|u_0^1|^p)(x) dx \\ & + \left| \int_0^t \int_{\mathbb{R}^N} |w^1|^{p-2} w^1 \cdot \nabla a \rho^\gamma(s, x) ds dx \right|. \end{aligned} \quad (4.28)$$

Following the same procedure than in [28] we prove that exists a continuous function  $C$  such that for any  $0 < T < T^*$  and  $2 \leq p < +\infty$  we have:

$$\|\rho^{\frac{1}{p}} w^1\|_{L_T^\infty(L^p(\mathbb{R}^N))} \leq C(T). \quad (4.29)$$

In addition we can observe that  $w^1$  is bounded via the maximum principle in  $C([0, T], L^\infty(\mathbb{R}^N))$  when  $0 \leq T < T^*$  (with eventually  $\|w^1\|_{L^\infty([0, T^*], L^\infty(\mathbb{R}^N))} = +\infty$ ). We have now for any  $t \in [0, T]$  and  $\epsilon > 0$ :

$$\begin{aligned} & \left( \int_{\{x, |w(t, w)| \geq \|w(t, \cdot)\|_{L^\infty} - \epsilon\}} \rho(t, x) |w(t, x)|^p dx \right)^{\frac{1}{p}} \\ & \geq (\|w(t, \cdot)\|_{L^\infty} - \epsilon) \left( \min_{x \in \mathbb{R}^N} |\rho(t, x)| \lambda(\{x, |w(t, w)| \geq \|w(t, \cdot)\|_{L^\infty} - \epsilon\}) \right)^{\frac{1}{p}}, \end{aligned}$$

with  $\lambda$  the Lebesgue measure. We deduce that since  $\min_{x \in \mathbb{R}^N} |\rho(t, x)| \lambda(\{x, |w(t, w)| \geq \|w(t, \cdot)\|_{L^\infty} - \epsilon\}) \neq 0$ , we have then for any  $\epsilon > 0$ :

$$\liminf_{p \rightarrow +\infty} \|\rho^{\frac{1}{p}} w^1\|_{L_T^\infty(L^p(\mathbb{R}^N))} \geq (\|w(t, \cdot)\|_{L^\infty} - \epsilon).$$

Using (4.29), we deduce that for any  $T \in (0, T^*)$  we have:

$$\|w^1\|_{L^\infty([0, T], L^\infty(\mathbb{R}^N))} \leq C(T). \quad (4.30)$$

We are now interested in proving that:

$$\frac{x}{|x|} \cdot \int_0^t S(t-s) \left( \frac{u_2^2(s, \cdot)}{|\cdot|} \cdot \frac{\cdot}{|\cdot|} \right) ds \geq 0.$$

To do this it suffices to study the solution  $w^2(t, x) = \frac{x}{|x|} w_2(t, |x|)$  of the linear equation:

$$\begin{cases} \partial_t w^2 + w \cdot \nabla w^2 - 2\mu \Delta w^2 = \frac{u_2^2}{|x|} \frac{x}{|x|} \\ w^2(0, \cdot) = 0. \end{cases} \quad (4.31)$$

Simple calculus gives:

$$\begin{cases} (w \cdot \nabla w^2) \cdot \frac{x}{|x|} = w \cdot \nabla (w^2 \cdot \frac{x}{|x|}) \\ \Delta w^2 = \frac{x}{|x|} (3\partial_r(\frac{w_2}{r}) + |x| \partial_{rr}(\frac{w_2}{r})) \\ \Delta w^2 \cdot \frac{x}{|x|} = \partial_{rr} w_2 + \frac{\partial_r w_2}{r} - \frac{w_2}{r^2} \\ \Delta w_2 = \partial_{rr} w_2 + \frac{\partial_r w_2}{r}. \end{cases}$$

It means in particular that:

$$\begin{cases} \partial_t w_2 + w \cdot \nabla w_2 - 2\mu \Delta w_2 + 2\mu \frac{w_2}{|x|^2} = \frac{u_2^2}{|x|}. \\ w_2(0, \cdot) = 0. \end{cases} \quad (4.32)$$

Let us consider now the unknown  $w_2^k(t, x) = \max(0, \max_{x \in \mathbb{R}^N}(-w_2(t, x)))$ . Multiplying (4.32) by  $w_2^k$  and integrating over  $(0, t) \times \mathbb{R}^N$  for  $t \in (0, T^*)$  we obtain:

$$\frac{1}{2} \|w_2^k(t, \cdot)\|_{L^2}^2 + 2\mu \int_0^t \int_{\mathbb{R}^N} |\nabla w_2^k|(s, x) ds dx \leq - \int_0^t \int_{\mathbb{R}^N} w_i \partial_i w_2^k(s, x) ds dx. \quad (4.33)$$

The theorem 1.1 ensures that on  $[0, t]$  we have  $w \in L^{2r}([0, t], L^{2q}(\mathbb{R}^N))$  with  $\frac{1}{r} + \frac{N}{2q} = 1 - \kappa$  with  $0 < \kappa < 1$  and  $q \in [\frac{N}{2(1-\kappa)}, +\infty)$ ,  $r \in [\frac{1}{1-\kappa}, +\infty)$ . By Hölder inequality, Young inequality and Gagliardo Nirenberg inequalities, we show that:

$$\begin{aligned} | \int_0^t \int_{\mathbb{R}^N} w_i \partial_i w_2^k(s, x) ds dx | &\leq \int_0^t \int_{\mathbb{R}^N} \left( \frac{\epsilon}{2} |\nabla w_2^k|(s, x) + \frac{1}{2\epsilon} \sum_{i=1}^N w_i^2(s, x) (w_2^k)^2(s, x) \right) ds dx \\ &\leq \frac{\epsilon}{2} \int_0^t \int_{\mathbb{R}^N} |\nabla w_2^k|(s, x) ds dx + \frac{C}{2\epsilon} \left\| \sum_{i=1}^N w_i^2 \right\|_{L^r((0,t), L^q(\mathbb{R}^N))} (\|w_2^k(t, \cdot)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^N} |\nabla w_2^k|(s, x) ds dx). \end{aligned} \quad (4.34)$$

Plugging (4.34) in (4.33), we obtain for  $\epsilon$  small enough, there exists  $C > 0$  such that:

$$\|w_2^k(t, \cdot)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^N} |\nabla w_2^k|(s, x) ds dx \leq C \left\| \sum_{i=1}^N w_i^2 \right\|_{L^r((0,t), L^q(\mathbb{R}^N))} (\|w_2^k(t, \cdot)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^N} |\nabla w_2^k|(s, x) ds dx). \quad (4.35)$$

Taking  $t$  sufficiently small we obtain that:

$$\|w_2^k(t, \cdot)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^N} |\nabla w_2^k|(s, x) ds dx \leq 0. \quad (4.36)$$

It implies that  $w_2(s, x) \geq 0$  for  $(s, x) \in (0, t) \times \mathbb{R}^N$ . Repeating the procedure we show that:

$$w_2(t, x) \geq 0 \text{ on } (0, T^*) \times \mathbb{R}^N. \quad (4.37)$$

Combining (4.30) and (4.37), we deduce that for all  $t \in (0, T^*)$  we have:

$$u_1(t, \cdot) = f_1(t, \cdot) + f_2(t, \cdot), \quad (4.38)$$

with  $f_1 \in L^\infty([0, T^*], L^\infty(\mathbb{R}^N))$  and  $f_2 \geq 0$ .

#### 4.0.2 Estimate on $v_1$

We can rewrite the equation on  $v^1$  as follows:

$$\rho \partial_t v^1 + \rho u^1 \cdot \nabla v^1 + \frac{a}{2\mu} \rho^\gamma v^1 - \frac{\rho u_2^2}{|x|} \frac{x}{|x|} = \frac{a}{2\mu} \rho^\gamma u^1. \quad (4.39)$$

It gives that:

$$\partial_t v^1 + u^1 \cdot \nabla v^1 + \frac{a}{2\mu} \rho^{\gamma-1} v^1 = \left( \frac{a}{2\mu} \rho^{\gamma-1} u_1 + \frac{u_2^2}{|x|} \right) \frac{x}{|x|}. \quad (4.40)$$

Applying the characteristic method, we now observe that:

$$\begin{aligned} \partial_t[v^1(t, \chi(t, x))] + \frac{a}{2\mu}\rho^{\gamma-1}(t, \chi(t, x))v^1(t, \chi(t, x)) &= \left(\frac{a}{2\mu}\rho^{\gamma-1}(t, \chi(t, x))u_1(t, \chi(t, x))\right. \\ &\left. + \frac{\rho u_2^2(t, \chi(t, x))}{|(t, \chi(t, x))|}\right) \frac{\chi(t, x)}{|\chi(t, x)|}, \end{aligned} \quad (4.41)$$

with:

$$\partial_t \chi(t, x) = u^1(t, \chi(t, x)).$$

We have now:

$$\partial_t[v^1(t, \chi(t, x))] = \partial_t[v_1(t, \chi(t, x))] \frac{\chi(t, x)}{|\chi(t, x)|} + v_1(t, \chi(t, x)) \left[ \frac{\partial_t \chi(t, x)}{|\chi(t, x)|} + \chi(t, x) \partial_t \left( \frac{1}{|\chi(t, x)|} \right) \right]$$

Next we have:

$$\begin{aligned} \partial_t \left( \frac{1}{|\chi(t, x)|} \right) &= -\frac{1}{2} \frac{1}{|\chi(t, x)|^3} (2\chi_1(t, x) \partial_t \chi_1(t, x) + 2\chi_2(t, x) \partial_t \chi_2(t, x)) \\ &= -\frac{1}{2} \frac{1}{|\chi(t, x)|^3} 2u_1(t, \chi(t, x)) |\chi(t, x)| = -\frac{1}{|\chi(t, x)|^2} u_1(t, \chi(t, x)). \end{aligned}$$

It implies that:

$$\partial_t[v^1(t, \chi(t, x))] = \partial_t[v_1(t, \chi(t, x))] \frac{\chi(t, x)}{|\chi(t, x)|}$$

Finally we have obtained that:

$$\begin{aligned} \partial_t[v_1(t, \chi(t, x))] + \frac{a}{2\mu}\rho^{\gamma-1}(t, \chi(t, x))v_1(t, \chi(t, x)) &= \left(\frac{a}{2\mu}\rho^{\gamma}(t, \chi(t, x))u_1(t, \chi(t, x))\right. \\ &\left. + \frac{\rho u_2^2(t, \chi(t, x))}{|(t, \chi(t, x))|}\right). \end{aligned} \quad (4.42)$$

It implies that according to (4.38) and since  $(v_1)_0$  belongs to  $L^\infty(\mathbb{R}^N)$  that for any  $(t, x)$  we have:

$$v_1 = g_1 + g_2, \quad (4.43)$$

with  $g_1 \in L^\infty([0, T^*], L^\infty(\mathbb{R}^N))$  and  $g_2(t, x) \geq 0$  for any  $(t, x) \in (0, T^*) \times \mathbb{R}^N$ .

## 4.1 Energy estimate

We recall now that we have:

$$\begin{cases} \rho \partial_t v^1 + \rho u^1 \cdot \nabla v^1 + \nabla P(\rho) + \frac{\rho u_2^2}{|x|} \frac{x}{|x|} = 0 \\ \rho \partial_t u^2 + \rho u^1 \cdot \nabla u^2 - \mu \operatorname{div}(\rho \nabla u^2) + \frac{\rho u_1}{2|x|} u^2 + \frac{\rho v_1}{2|x|} u^2 = 0 \end{cases}$$

We multiply the first equation by  $v^1$ , and the second by  $2u^2$  and we obtain:

$$\frac{1}{2} \int_{\mathbb{R}^N} \partial_t (\rho |v^1|^2 + \rho |u^2|^2) dx + \int_{\mathbb{R}^N} \nabla P(\rho) \cdot v^1 dx + 2\mu \int_{\mathbb{R}^N} \rho |\nabla u^2|^2 dx + \int \rho \frac{u_1}{|x|} |u^2|^2 dx = 0.$$



We can check now that ( $w = \text{curl}u^2 = \partial_1 u_2^2 - \partial_2 u_1^2$ ):

$$|\nabla u^2|^2 = (\partial_r u_2)^2 + \frac{(u_2)^2}{|x|^2}, \quad w = \frac{u_2}{|x|} + \partial_r u_2 \quad \text{and} \quad \text{div}u = \frac{u_1}{|x|} + \partial_r u_1.$$

We have finally:

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N} \partial_t(\rho|v^1|^2 + \rho|u^2|^2 + j_\gamma(\rho))dx + 2\mu \int_{\mathbb{R}^N} \nabla P(\rho) \cdot \nabla \ln \rho dx + 2\mu \int_{\mathbb{R}^N} \rho(\partial_r u_2)^2 dx + 2\mu \int_{\mathbb{R}^N} \rho \frac{(u_2)^2}{|x|^2} dx \\ + \int_{\mathbb{R}^N} \rho \frac{u_1}{|x|} |u^2|^2 = 0. \end{aligned}$$

## 4.2 Weight energy estimate on $u^2$

We recall that:

$$\rho \partial_t u^2 + \rho u^1 \cdot \nabla u^2 - \mu \text{div}(\rho \nabla u^2) + \frac{\rho(u_1 + \mu \partial_r \ln \rho)}{|x|} u^2 = 0. \quad (4.44)$$

Let  $f$  a radial function that we will determine later, we verify that:

$$f(|x|) \text{div}(\rho \nabla u^2) = \text{div}(\rho \nabla(fu^2)) - \rho u^2(\Delta f + f' \partial_r \ln \rho) - 2\rho \nabla \ln f \cdot \nabla(fu^2) + 2\rho f |\nabla \ln f|^2 u^2.$$

And:

$$f \rho u^1 \cdot \nabla u^2 = \rho u^1 \cdot \nabla(fu^2) - \rho u^2 u^1 \cdot \nabla f.$$

We can then rewrite (4.44) as follows:

$$\begin{aligned} \rho \partial_t(fu^2) + \rho(u^1 + 2\mu \nabla \ln f) \cdot \nabla(fu^2) - \mu \text{div}(\rho \nabla(fu^2)) + \frac{\rho(fu_1 + \mu f \partial_r \ln \rho)}{|x|} u^2 \\ + \mu \rho u^2(\mu \Delta f + f' \partial_r \ln \rho - 2f |\nabla \ln f|^2) - \rho u^2 u^1 \cdot \nabla f = 0. \end{aligned}$$

It gives in particular:

$$\begin{aligned} \rho \partial_t(fu^2) + \rho(u^1 + 2\mu \nabla \ln f) \cdot \nabla(fu^2) - \mu \text{div}(\rho \nabla(fu^2)) \\ + \mu \rho u^2(\Delta f - 2f |\nabla \ln f|^2) + \rho u^2(u_1(\frac{f}{|x|} - f') + \mu \partial_r \ln \rho(\frac{f}{|x|} + f')) = 0 \end{aligned} \quad (4.45)$$

Multiplying the previous equation by  $fu^2$  we get:

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N} \partial_t(\rho|fu^2|^2)dx - \mu \int_{\mathbb{R}^N} \rho(\Delta \ln f + \partial_r \ln \rho \frac{f'}{f})|fu^2|^2 dx + \mu \int_{\mathbb{R}^N} \rho |\nabla(fu^2)|^2 dx \\ + \mu \int_{\mathbb{R}^N} \rho f |u^2|^2 (\Delta f - 2\frac{(f')^2}{f}) dx + \int_{\mathbb{R}^N} \rho |fu^2|^2 (u_1(\frac{1}{|x|} - \frac{f'}{f}) + \mu \partial_r \ln \rho(\frac{1}{|x|} + \frac{f'}{f})) dx = 0. \end{aligned}$$

It gives:

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N} \partial_t(\rho|fu^2|^2)dx + \mu \int_{\mathbb{R}^N} \rho |\nabla(fu^2)|^2 dx \\ + \mu \int_{\mathbb{R}^N} \rho f |u^2|^2 (\Delta f - 2\frac{(f')^2}{f} - f \Delta \ln f) dx + \int_{\mathbb{R}^N} \rho |fu^2|^2 (u_1(\frac{1}{|x|} - \frac{f'}{f}) + \mu \partial_r \ln \rho(\frac{1}{|x|})) dx = 0. \end{aligned}$$

We have in particular:

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} \partial_t(\rho|fu^2|^2)dx + \mu \int_{\mathbb{R}^N} \rho|\nabla(fu^2)|^2 dx \\ & + \mu \int_{\mathbb{R}^N} \rho f|u^2|^2(\Delta f - 2\frac{(f')^2}{f} - f\Delta \ln f)dx + \int_{\mathbb{R}^N} \rho|fu^2|^2(u_1(\frac{1}{2|x|} - \frac{f'}{f}) + \frac{1}{2|x|}v_1)dx = 0. \end{aligned}$$

Furthermore a basic calculus yields:

$$\Delta f - 2\frac{(f')^2}{f} - f\Delta \ln f = \Delta f - 2\frac{(f')^2}{f} - (\Delta f - \frac{|\nabla f|^2}{f}) = -\frac{(f')^2}{f}. \quad (4.46)$$

Using the fact that  $|\nabla u^2|^2 = (\partial_r u_2)^2 + \frac{(u_2)^2}{|x|^2}$ , we deduce from (4.46):

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} \partial_t(\rho|fu^2|^2)dx + \mu \int_{\mathbb{R}^N} \rho|\partial_r(fu_2)|^2 dx + \mu \int_{\mathbb{R}^N} \rho \frac{|fu^2|^2}{|x|^2} dx \\ & - \mu \int_{\mathbb{R}^N} \rho|fu^2|^2 \frac{(f')^2}{f^2} dx + \int_{\mathbb{R}^N} \rho|fu^2|^2(u_1(\frac{1}{2|x|} - \frac{f'}{f}) + \frac{1}{2|x|}v_1)dx = 0. \end{aligned} \quad (4.47)$$

Now taking  $f(|x|) = |x|^{\epsilon-1}$  with  $\epsilon > 0$  small enough we have:

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} \partial_t(\rho \frac{|u^2|^2}{|x|^{2-2\epsilon}})dx + \mu \int_{\mathbb{R}^N} \rho|\partial_r(fu_2)|^2 dx + \mu(2\epsilon - \epsilon^2) \int_{\mathbb{R}^N} \rho \frac{|u^2|^2}{|x|^{4-2\epsilon}} dx \\ & + \int_{\mathbb{R}^N} \rho \frac{|u^2|^2}{|x|^{2-2\epsilon}}(u_1(\frac{3}{2} - \epsilon) \frac{1}{|x|} + \frac{1}{2|x|}v_1)dx = 0. \end{aligned} \quad (4.48)$$

Applying Young inequality we have now for  $\alpha_1 > 0$  small enough and  $C > 0$ :

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} \partial_t(\rho \frac{|u^2|^2}{|x|^{2-2\epsilon}})dx + \mu \int_{\mathbb{R}^N} \rho|\partial_r(fu_2)|^2 dx + \mu(2\epsilon - \epsilon^2) \int_{\mathbb{R}^N} \rho \frac{|u^2|^2}{|x|^{4-2\epsilon}} dx \\ & + \int_{\mathbb{R}^N} \rho \frac{|u^2|^2}{|x|^{2-2\epsilon}}(f_2(\frac{3}{2} - \epsilon) \frac{1}{|x|} + \frac{1}{2|x|}g_2)dx \leq \frac{\alpha_1}{2} \int_{\mathbb{R}^N} \rho \frac{|u^2|^2}{|x|^{4-2\epsilon}} dx \\ & + \frac{1}{2\alpha_1} \int_{\mathbb{R}^N} \rho \frac{|u^2|^2}{|x|^{2-2\epsilon}} C(|f_1|^2 + |g_2|^2)dx. \end{aligned} \quad (4.49)$$

Since  $f_2$  and  $g_2$  are bounded in  $L^\infty([0, T^*], L^\infty(\mathbb{R}^N))$  we deduce by Gronwall lemma that:

$$\begin{aligned} & \sqrt{\rho} \frac{u^2}{|x|^{1-\epsilon}} \in L^\infty([0, T^*], L^2(\mathbb{R}^N)). \\ & \sqrt{\rho} \frac{u^2}{|x|^{2-\epsilon}} \in L^2([0, T^*], L^2(\mathbb{R}^N)). \end{aligned} \quad (4.50)$$

### 4.3 $L^p$ weight estimate on $u^2$

Multiplying the previous equation (4.45) by  $fu^2|fu^2|^p$ , we have:

$$\begin{aligned} & \frac{1}{p+2} \int_{\mathbb{R}^N} \partial_t(\rho|fu^2|^{p+2})dx + \mu \int_{\mathbb{R}^N} \rho|fu^2|^p|\nabla(fu^2)|^2 + \frac{\mu p}{4} \int_{\mathbb{R}^N} \rho|\nabla|fu^2|^2|^2|fu^2|^{p-4}dx \\ & + \mu \int_{\mathbb{R}^N} \rho|fu^2|^{p+2}(\frac{\Delta f}{f} - 2|\nabla \ln f|^2) + \rho|fu^2|^{p+2}(u_1(\frac{1}{2|x|} - \frac{3f'}{2f}) + \frac{v_1}{2}(\frac{1}{|x|} + \frac{f'}{f}))dx \\ & + 2\mu \int_{\mathbb{R}^N} \rho \nabla \ln f \cdot \nabla(fu^2)(fu^2)|fu^2|^p dx = 0. \end{aligned}$$

Next we have:

$$\int_{\mathbb{R}^N} \rho \nabla \ln f \cdot \nabla (fu^2) (fu^2) |fu^2|^p dx = -\frac{1}{p+2} \int_{\mathbb{R}^N} \rho (\partial_r \ln \rho \frac{f'}{f} + \Delta \ln f) |fu^2|^{p+2} dx.$$

We deduce that:

$$\begin{aligned} & \frac{1}{p+2} \int_{\mathbb{R}^N} \partial_t (\rho |fu^2|^{p+2}) dx + \mu \int_{\mathbb{R}^N} \rho |fu^2|^p |\nabla (fu^2)|^2 + \frac{\mu p}{4} \int_{\mathbb{R}^N} \rho |\nabla |fu^2|^2|^2 |fu^2|^{p-4} dx \\ & + \mu \int_{\mathbb{R}^N} \rho |fu^2|^{p+2} \left( \frac{\Delta f}{f} - 2|\nabla \ln f|^2 - \frac{2}{p+2} \Delta \ln f \right) + \rho |fu^2|^{p+2} \left( u_1 \left( \frac{1}{|x|} - \frac{f'}{f} \right) \right. \\ & \left. + \mu \partial_r \ln \rho \left( \frac{1}{|x|} + \frac{f'}{f} \left( 1 - \frac{2}{p+2} \right) \right) \right) dx = 0. \end{aligned}$$

We have in particular:

$$\begin{aligned} & \frac{1}{p+2} \int_{\mathbb{R}^N} \partial_t (\rho |fu^2|^{p+2}) dx + \mu \int_{\mathbb{R}^N} \rho |fu^2|^p |\partial_r (fu_2)|^2 dx + \mu \int_{\mathbb{R}^N} \rho \frac{|fu^2|^{p+2}}{|x|^2} dx \\ & + \frac{\mu p}{4} \int_{\mathbb{R}^N} \rho |\nabla |fu^2|^2|^2 |fu^2|^{p-4} dx + \mu \int_{\mathbb{R}^N} \rho |fu^2|^{p+2} \left( \frac{\Delta f}{f} \left( 1 - \frac{2}{p+2} \right) - 2 \left( 1 - \frac{1}{p+2} \right) |\nabla \ln f|^2 \right) \\ & + \rho |fu^2|^{p+2} \left( u_1 \left( \frac{1}{2|x|} - \frac{f'}{f} \left( \frac{3}{2} - \frac{1}{p+2} \right) \right) + \frac{v_1}{2} \left( \frac{1}{|x|} + \frac{f'}{f} \left( 1 - \frac{2}{p+2} \right) \right) \right) dx = 0. \end{aligned} \tag{4.51}$$

Taking  $f(|x|) = |x|^\alpha$  we have:

$$\begin{aligned} \Delta f(x) &= \alpha^2 |x|^{\alpha-2}. \\ \frac{\Delta f}{f} \left( 1 - \frac{2}{p+2} \right) - 2 \left( 1 - \frac{1}{p+2} \right) |\nabla \ln f|^2 &= -\frac{\alpha^2}{|x|^2}. \\ \left( \frac{1}{2|x|} - \frac{f'}{f} \left( \frac{3}{2} - \frac{1}{p+2} \right) \right) &= \frac{1}{|x|} \left( \frac{1}{2} - \frac{3\alpha}{2} + \frac{\alpha}{p+2} \right) \\ \frac{1}{|x|} + \frac{f'}{f} \left( 1 - \frac{2}{p+2} \right) &= \frac{1}{|x|} \left( 1 + \alpha - \frac{2\alpha}{p+2} \right). \end{aligned}$$

Let us deal now with the case  $\alpha = -1 + \epsilon$  with  $\epsilon > 0$  small enough, it implies that:

$$\begin{aligned} & \frac{1}{p+2} \int_{\mathbb{R}^N} \partial_t \left( \rho \frac{|u^2|^{p+2}}{|x|^{(p+2)(1-\epsilon)}} \right) dx + \mu \int_{\mathbb{R}^N} \rho |fu^2|^p |\partial_r (fu_2)|^2 + \mu (2\epsilon - \epsilon^2) \int_{\mathbb{R}^N} \rho \frac{|u^2|^{p+2}}{|x|^{2+(p+2)(1-\epsilon)}} dx \\ & + \frac{\mu p}{4} \int_{\mathbb{R}^N} \rho |\nabla |fu^2|^2|^2 |fu^2|^{p-4} dx + \mu \int_{\mathbb{R}^N} \rho \frac{|u^2|^{p+2}}{|x|^{(p+2)(1-\epsilon)}} \left( \frac{u_1}{|x|} \left( 2 - \frac{3\epsilon}{2} - \frac{1-\epsilon}{p+2} \right) \right. \\ & \left. + \frac{v_1}{2|x|} \left( \epsilon + \frac{2-\epsilon}{p+2} \right) \right) dx = 0. \end{aligned} \tag{4.52}$$

Using (4.38), (4.43) and Young inequality there exists  $C > 0$  and  $\alpha_1 > 0$  small enough such that:

$$\begin{aligned} & \frac{1}{p+2} \int_{\mathbb{R}^N} \partial_t \left( \rho \frac{|u^2|^{p+2}}{|x|^{(p+2)(1-\epsilon)}} \right) dx + \mu \int_{\mathbb{R}^N} \rho |fu^2|^p |\partial_r (fu_2)|^2 + \mu (2\epsilon - \epsilon^2) \int_{\mathbb{R}^N} \rho \frac{|u^2|^{p+2}}{|x|^{2+(p+2)(1-\epsilon)}} dx \\ & + \frac{\mu p}{4} \int_{\mathbb{R}^N} \rho |\nabla |fu^2|^2|^2 |fu^2|^{p-4} dx + \mu \int_{\mathbb{R}^N} \rho \frac{|u^2|^{p+2}}{|x|^{(p+2)(1-\epsilon)}} \left( \frac{f_2}{|x|} \left( 2 - \frac{3\epsilon}{2} - \frac{1-\epsilon}{p+2} \right) \right. \\ & \left. + \frac{g_2}{2|x|} \left( \epsilon + \frac{2-\epsilon}{p+2} \right) \right) dx \leq \frac{\alpha_1}{2} \int_{\mathbb{R}^N} \rho \frac{|u^2|^{p+2}}{|x|^{2+(p+2)(1-\epsilon)}} dx + \frac{C}{2\alpha_1} \int_{\mathbb{R}^N} \rho \frac{|u^2|^{p+2}}{|x|^{(p+2)(1-\epsilon)}} (|f_1|^2 + |g_1|^2) dx \end{aligned} \tag{4.53}$$

By Gronwall lemma we deduce that for any  $\epsilon > 0$  small enough and any  $p \geq 2$ , there exists a continuous function  $C$  independent on  $p$  such that for any  $T \in (0, T^*)$ :

$$\|\rho^{\frac{1}{p}} \frac{u^2}{|x|^{1-\epsilon}}\|_{L^\infty(0,T),L^p(\mathbb{R}^N)} \leq C(T). \quad (4.54)$$

#### 4.4 Gain of integrability on $u^2$

We recall now that  $u^2$  verifies the following equation:

$$\rho \partial_t u^2 + \rho u^1 \cdot \nabla u^2 - \mu \operatorname{div}(\rho \nabla u^2) + \frac{\rho u_1}{2|x|} u^2 + \frac{\rho v_1}{2|x|} u^2 = 0$$

We multiply this equation by  $u^2 |u^2|^p$  and we integrate over  $\mathbb{R}^N$ :

$$\begin{aligned} & \frac{1}{p+2} \int_{\mathbb{R}^N} \partial_t (\rho |u^2|^{p+2}) dx + \mu \int_{\mathbb{R}^N} \rho |u^2|^p |\partial_r u_2|^2 dx + \mu \int_{\mathbb{R}^N} \rho |u^2|^p \frac{|u^2|^2}{|x|^2} dx \\ & + \frac{\mu p}{4} \int_{\mathbb{R}^N} \rho |\nabla |u^2|^2|^2 |u^2|^{p-4} dx + \int_{\mathbb{R}^N} \frac{\rho u_1}{2|x|} |u^2|^{p+2} dx + \int_{\mathbb{R}^N} \frac{\rho v_1}{2|x|} |u^2|^{p+2} dx = 0. \end{aligned}$$

Using (4.38), (4.43) and Young inequality there exists  $C > 0$  and  $\alpha_1 > 0$  small enough such that:

$$\begin{aligned} & \frac{1}{p+2} \int_{\mathbb{R}^N} \partial_t (\rho |u^2|^{p+2}) dx + \mu \int_{\mathbb{R}^N} \rho |u^2|^p |\partial_r u_2|^2 dx + \mu \int_{\mathbb{R}^N} \rho |u^2|^p \frac{|u^2|^2}{|x|^2} dx \\ & + \frac{\mu p}{4} \int_{\mathbb{R}^N} \rho |\nabla |u^2|^2|^2 |f u^2|^{p-4} dx + \int_{\mathbb{R}^N} \frac{\rho f_2}{2|x|} |u^2|^{p+2} dx + \int_{\mathbb{R}^N} \frac{\rho g_2}{2|x|} |u^2|^{p+2} dx \leq \frac{\alpha_1}{2} \int_{\mathbb{R}^N} \rho |u^2|^p |\partial_r u_2|^2 dx \\ & + \frac{C}{2\alpha_1} \int_{\mathbb{R}^N} \rho |u^2|^{p+2} (|f_1|^2 + |g_1|^2) dx. \end{aligned}$$

By Gronwall lemma we deduce that for any  $\epsilon > 0$  small enough and any  $p \geq 2$ , there exists a continuous function  $C$  independent on  $p$  such that for any  $T \in (0, T^*)$ :

$$\|\rho^{\frac{1}{p}} u^2\|_{L^\infty((0,T),L^p(\mathbb{R}^N))} \leq C(T). \quad (4.55)$$

#### 4.5 Gain of integrability on $u^1$

Combining (4.54) and (4.55), we deduce that for any  $T \in (0, T^*)$  and any  $p \geq 2$ :

$$\|\rho^{\frac{2}{p}} \frac{u_2^2}{|x|} \frac{x}{|x|}(T, \cdot)\|_{L^{\frac{p}{2}}} \leq \|\rho^{\frac{1}{p}} \frac{u_2}{|x|^{1-\epsilon}} \frac{x}{|x|}(T, \cdot)\|_{L^p}^p + \|\rho^{\frac{1}{p}} u^2(T, \cdot)\|_{L^p}^p.$$

It implies that there exists a continuous function  $C$  such that for any  $p \geq 2$  we have:

$$\|\rho^{\frac{2}{p}} \frac{u_2^2}{|x|} \frac{x}{|x|}(T, \cdot)\|_{L^{\frac{p}{2}}} \leq C(T). \quad (4.56)$$

We recall now that  $u^1$  verifies the following equation:

$$\rho \partial_t u^1 + \rho u^1 \cdot \nabla u^1 - 2\mu \operatorname{div}(\rho \nabla u^1) + \nabla P(\rho) = \frac{\rho u_2^2}{|x|} \frac{x}{|x|}. \quad (4.57)$$

Multiplying the previous equation by  $w^1|w^1|^{p-2}$  with  $p \geq 2$  and integrating over  $(0, t) \times \mathbb{R}^N$  with  $0 < t < T^*$  we obtain:

$$\begin{aligned} & \frac{1}{p} \int_{\mathbb{R}^N} (\rho|u^1|^p)(t, x) dx + \int_0^t \int_{\mathbb{R}^N} \rho|u^1|^{p-2} |\nabla u^1|^2(s, x) ds dx \\ & + \frac{(p-2)}{4} \int_0^t \int_{\mathbb{R}^N} \rho |\nabla(|u^1|^2)|^2 |u^1|^{p-4}(s, x) ds dx \leq \frac{1}{p} \int_{\mathbb{R}^N} (\rho_0|u_0^1|^p)(x) dx \\ & + \left| \int_0^t \int_{\mathbb{R}^N} |u^1|^{p-2} w^1 \cdot \nabla a \rho^\gamma(s, x) ds dx \right| + \left| \int_0^t \int_{\mathbb{R}^N} \rho \frac{|u^2|^2}{|x|} u^1 |u^1|^{p-2} dx ds \right|. \end{aligned} \quad (4.58)$$

The pressure term can be treated as in [28], let us focus on the second term in the right hand side of (4.62). By Hölder inequality and Young inequality, we have when  $q = 2p$  for  $\epsilon > 0$  small enough and  $C > 0$ :

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^N} \rho \frac{|u^2|^2}{|x|} u^1 |u^1|^{p-2} dx ds \right| & \leq \int_0^t \left\| \rho^{\frac{p-1}{p}} u^1(s) \right\|_{L^{\frac{p}{p-1}}} \left\| \rho^{\frac{1}{p}}(s, \cdot) \frac{|u^2|^2(s, \cdot)}{|\cdot|} \right\|_{L^p} ds \\ & \leq \int_0^t \left\| \rho^{\frac{p-1}{p}} u^1(s) \right\|_{L^{\frac{p}{p-1}}} \left\| \rho^{\frac{2}{q}}(s, \cdot) \frac{|u^2|^2(s, \cdot)}{|\cdot|} \right\|_{L^{\frac{q}{2}}} ds \\ & \leq \int_0^t \left\| \rho^{\frac{1}{p}} u^1(s) \right\|_{L^p}^{\frac{p}{p-1}} \left\| \rho^{\frac{2}{q}}(s, \cdot) \frac{|u^2|^2(s, \cdot)}{|\cdot|} \right\|_{L^{\frac{q}{2}}} ds \\ & \leq \epsilon \int_0^t \left\| \rho^{\frac{1}{p}} u^1(s) \right\|_{L^p}^p ds + C \int_0^t \left\| \rho^{\frac{2}{q}}(s, \cdot) \frac{|u^2|^2(s, \cdot)}{|\cdot|} \right\|_{L^{\frac{q}{2}}}^{\frac{p}{p-1}} ds \end{aligned} \quad (4.59)$$

From (4.58) and (4.63) we deduce that there exists a continuous function  $C$  such that for any  $p \geq 2$  and  $T \in (0, T^*)$  we have:

$$\left\| \rho^{\frac{1}{p}} u^1 \right\|_{L^\infty((0, T), L^p(\mathbb{R}^N))} \leq C(T). \quad (4.60)$$

## 4.6 Gain of integrability on $v^1$

Let us recall now that  $v^1$  verifies the following equation:

$$\rho \partial_t v^1 + \rho u^1 \cdot \nabla v + \frac{a}{2\mu} \rho^\gamma v^1 = \frac{a}{2\mu} \rho^\gamma u^1 + \rho \frac{u_2^2}{|x|} \frac{x}{|x|}. \quad (4.61)$$

Multiplying the previous equation (4.61) by  $v|v|^{p-2}$  for  $p \geq 2$ , we have:

$$\begin{aligned} \frac{1}{p} \int_{\mathbb{R}^N} \partial_t (\rho|v^1|^p) dx + \frac{a}{2\mu} \int_{\mathbb{R}^N} \rho^\gamma |v^1|^p dx & = \frac{a}{2\mu} \int_{\mathbb{R}^N} \rho^{\frac{p-1}{p}} |v^1|^{p-2} v^1 \cdot \rho^{\frac{\gamma}{p}} u^1 dx \\ & + \int_{\mathbb{R}^N} \rho \frac{u_2^2}{|x|} v^1 |v^1|^{p-2} dx. \end{aligned} \quad (4.62)$$

We have now by Hölder inequalities and Young inequality:

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^N} \rho \frac{|u^2|^2}{|x|} v^1 |v^1|^{p-2} dx ds \right| & \leq \int_0^t \left\| \rho^{\frac{p-1}{p}} v^1(s) \right\|_{L^{\frac{p}{p-1}}} \left\| \rho^{\frac{1}{p}}(s, \cdot) \frac{|u^2|^2(s, \cdot)}{|\cdot|} \right\|_{L^p} ds \\ & \leq \epsilon \int_0^t \left\| \rho^{\frac{1}{p}} v^1(s) \right\|_{L^p}^p ds + C \int_0^t \left\| \rho^{\frac{2}{q}}(s, \cdot) \frac{|u^2|^2(s, \cdot)}{|\cdot|} \right\|_{L^{\frac{q}{2}}}^{\frac{p}{p-1}} ds \end{aligned} \quad (4.63)$$

Next by Höder, Young inequality and energy inequality we have for  $\epsilon > 0$  small enough:

$$\begin{aligned}
& \left| \int_{\mathbb{R}^N} \rho^{\gamma \frac{p-1}{p}} |v^1|^{p-2} v^1 \cdot \rho^{\frac{\gamma}{p}} u^1 dx \right| \leq \epsilon \frac{p-1}{p} \int_{\mathbb{R}^N} \rho^\gamma |v^1|^p dx + \frac{1}{\epsilon p} \int_{\mathbb{R}^N} \rho^\gamma |u^1|^p dx \\
& \leq \epsilon \frac{p-1}{p} \int_{\mathbb{R}^N} \rho^\gamma |v^1|^p dx + \frac{1}{\epsilon p} [\|\rho^{\frac{1}{p}} u^1\|_{L^p(\mathbb{R}^N)}^p + \|\sqrt{\rho}|u^1|^p\|_{L^2(\mathbb{R}^N)} \|\rho^{\gamma-\frac{1}{2}} - \sqrt{\rho}\|_{L^2(\mathbb{R}^N)}] \quad (4.64) \\
& \leq \epsilon \frac{p-1}{p} \int_{\mathbb{R}^N} \rho^\gamma |v^1|^p dx + \frac{1}{\epsilon p} [\|\rho^{\frac{1}{p}} u^1\|_{L^p(\mathbb{R}^N)}^p + C \|\rho^{\frac{1}{2p}} u^1\|_{L^{2p}(\mathbb{R}^N)}^2].
\end{aligned}$$

Combining (4.63), (4.63) and (4.64) It yields that for any  $p \geq 2$  and  $T \in (0, T^*)$  there exists a continuous increasing function  $C$  such that we have:

$$\|\rho^{\frac{1}{p}} v\|_{L_t^\infty(L^p)} \lesssim C(T). \quad (4.65)$$

## 4.7 Maximum principle

Let us recall the different estimates that we have obtained, from (4.65), (4.60), (4.55) there exists a continuous increasing function  $C$  such that for any  $p \geq 2$  and  $T \in (0, T^*)$  we have:

$$\begin{aligned}
& \|\rho^{\frac{1}{p}} u^1\|_{L^\infty((0,T),L^p(\mathbb{R}^N))} \leq C(T), \\
& \|\rho^{\frac{1}{p}} u^2\|_{L^\infty((0,T),L^p(\mathbb{R}^N))} \leq C(T), \quad (4.66) \\
& \|\rho^{\frac{1}{p}} v^1\|_{L^\infty((0,T),L^p(\mathbb{R}^N))} \leq C(T).
\end{aligned}$$

Our goal is now to show that we can extend the solution  $(\rho, u)$  beyond  $T^*$  using the estimates (4.66). From the previous estimates (4.66) we deduce using the maximum principle the following proposition (see [28] for a detailed proof).

**Proposition 4.5.** *Under the assumption of theorem 1.2, the density  $\rho$  verifies for any  $T \in (0, T^*)$ :*

$$\left\| \frac{1}{\rho} \right\|_{L_T^\infty(L^\infty(\mathbb{R}^N))} \leq C(T), \quad (4.67)$$

with  $C$  an increasing continuous function in  $T$ .

## 5 Proof of the theorem 1.2

We are going to show that we can extend the solution  $(\rho, u)$  beyond  $T^*$  using the estimates (4.66).

### 5.1 Estimate on $\|u(T, \cdot)\|_{B_{p,1}^{\frac{N}{p}-1+\epsilon'}}$ for any $T < T^*$ and $\epsilon' > 0$

Let us recall that we have proved according to (4.60), (4.55) and proposition 4.5 that for any  $p \geq 2$  there exists increasing continuous functions  $C$  and  $C_1$  depending only on the initial data  $(\rho_0, u_0)$  such that:

$$\|\rho^{\frac{1}{p}} u\|_{L_T^\infty(L^p)} \leq C(T) \quad \text{and} \quad \left\| \frac{1}{\rho} \right\|_{L_T^\infty(L^\infty)} \leq C_1^{\frac{1}{p}}(T). \quad (5.68)$$

By Besov embedding we observe that for  $p \geq 2$  there exists  $C > 0$  such that:

$$\|u\|_{L_T^\infty(B_{p,\infty}^0)} \leq CC(T)C_1^{\frac{1}{p}}(T) \quad \text{and} \quad \|u\|_{L_T^\infty(B_{p,\infty}^{-N(\frac{1}{2}-\frac{1}{p})})} \leq CC(T)C_1^{\frac{1}{2}}(T). \quad (5.69)$$

For  $p > \frac{N}{1-\epsilon'}$  we have  $-\frac{N}{2} + \frac{N}{p} < \frac{N}{p} - 1 + \epsilon' < 0$ ; we deduce then by interpolation in the Besov space that there exists  $M$  an increasing continuous function depending only on the initial data  $(\rho_0, u_0)$  verifying for any  $0 < T < T^*$  <sup>3</sup>:

$$\|u(T, \cdot)\|_{B_{p,1}^{\frac{N}{p}-1+\epsilon'}} \leq M(T). \quad (5.70)$$

## 5.2 Estimate on $\|q_1(T, \cdot)\|_{B_{p,1}^{\frac{N}{p}+\epsilon'}}$ for any $T < T^*$ and $\epsilon' > 0$

Using the first equation in (1.3) and the proposition 2.4, there exists  $C > 0$  such that for any  $T \in (0, T^*)$  we have (where we set  $q_1 = \rho - \bar{\rho}$ ):

$$\|q_1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}+\epsilon'})} \leq C(\|q_0^1\|_{B_{p,1}^{\frac{N}{p}+\epsilon'}} + \|\rho v^1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1+\epsilon'})}). \quad (5.71)$$

In the same way as in the previous section we have for any  $p \geq 2$ :

$$\begin{aligned} \|\rho v^1\|_{\tilde{L}_T^\infty(B_{p,\infty}^0)} &= \|\rho v^1\|_{L_T^\infty(B_{p,\infty}^0)} \leq \|\rho\|_{L_T^\infty(L^\infty)}^{1-\frac{1}{p}} \|\rho^{\frac{1}{p}} v^1\|_{L_T^\infty(L^p)}, \\ \|\rho v^1\|_{\tilde{L}_T^\infty(B_{p,\infty}^{-N(\frac{1}{2}-\frac{1}{p})})} &= \|\rho v^1\|_{L_T^\infty(B_{p,\infty}^{-N(\frac{1}{2}-\frac{1}{p})})} \leq \|\rho\|_{L_T^\infty(L^\infty)}^{\frac{1}{2}} \|\rho^{\frac{1}{2}} v^1\|_{L_T^\infty(L^2)}. \end{aligned} \quad (5.72)$$

Since the inequality  $-\frac{N}{2} + \frac{N}{p} < \frac{N}{p} - 1 + \epsilon' < 0$  holds, we get by interpolation with  $\frac{N}{p} - 1 + \epsilon' = \theta(-\frac{N}{2} + \frac{N}{p})$ :

$$\begin{aligned} \|\rho v^1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1+\epsilon'})} &\leq \|\rho\|_{L_T^\infty(L^\infty)}^{\frac{\theta}{2}} \|\rho^{\frac{1}{2}} v^1\|_{L_T^\infty(L^2)}^\theta \|\rho\|_{L_T^\infty(L^\infty)}^{(1-\theta)(1-\frac{1}{p})} \|\rho^{\frac{1}{p}} v^1\|_{L_T^\infty(L^p)}^{1-\theta} \\ &\leq \|\rho\|_{L_T^\infty(L^\infty)}^{1-\frac{1-\epsilon'}{N}} \|\rho^{\frac{1}{2}} v^1\|_{L_T^\infty(L^2)}^\theta \|\rho^{\frac{1}{p}} v^1\|_{L_T^\infty(L^p)}^{1-\theta}. \end{aligned} \quad (5.73)$$

According to (4.65) and the energy inequality, there exists a continuous increasing function  $M_1$  such that:

$$\|\rho v\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1+\epsilon'})} \leq \|\rho\|_{L_T^\infty(L^\infty)}^{1-\frac{1-\epsilon'}{N}} M_1(T). \quad (5.74)$$

Plugging the previous estimate in (5.72) it gives:

$$\|q_1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}+\epsilon'})} \leq C(\|q_0^1\|_{B_{p,1}^{\frac{N}{p}+\epsilon'}} + (\|q_1\|_{L_T^\infty(L^\infty)} + \bar{\rho})^{1-\frac{1-\epsilon'}{N}} M_1(T)). \quad (5.75)$$

From Besov embedding and interpolation, we know that there exists  $C, C' > 0$  such that:

$$\|q_1\|_{L_T^\infty(L^\infty)} \leq C\|q_1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq C'\|q_1\|_{\tilde{L}_T^\infty(B_{p,\infty}^{-N(\frac{1}{2}-\frac{1}{p})})}^\theta \|q_1\|_{\tilde{L}_T^\infty(B_{p,\infty}^{\frac{N}{p}+\epsilon'})}^{1-\theta}, \quad (5.76)$$

---

<sup>3</sup> $\|u(T, \cdot)\|_{B_{p,1}^{\frac{N}{p}-1+\epsilon'}}$  is well defined since  $u$  belongs to  $\tilde{C}_{T^*}^{\frac{N}{p}-1+\epsilon'}$ .

with  $\frac{N}{p} = -\theta N(\frac{1}{2} - \frac{1}{p}) + (1 - \theta)(\frac{N}{p} + \epsilon')$ . Since  $q_1 = \rho - \bar{\rho}$  is bounded in  $L^\infty((0, T), L^2(\mathbb{R}^N))$  for any  $T \in (0, T^*)$  from the energy estimate and by Sobolev embedding, we can now show that by Young inequality and (5.76) there exists  $M'$  an increasing function such that:

$$\|q_1\|_{L_T^\infty(L^\infty)} \leq M'(T) + \|q_1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p} + \epsilon'})}. \quad (5.77)$$

From (5.75), (5.77) and by Young inequality we have for  $C > 0$  and any  $\epsilon > 0$ :

$$\begin{aligned} \|q_1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p} + \epsilon'})} &\leq C(\|q_0^1\|_{B_{p,1}^{\frac{N}{p} + \epsilon'}} + (\|q_1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p} + \epsilon'})} + M'(T) + \bar{\rho})^{1 - \frac{1 - \epsilon'}{N}} M_1(T)) \\ &\leq C(\|q_0^1\|_{B_{p,1}^{\frac{N}{p} + \epsilon'}} + \frac{\epsilon(N - 1 + \epsilon')}{N} (\|q_1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p} + \epsilon'})} + M'(T) + \bar{\rho}) + \frac{1 - \epsilon'}{\epsilon N} M_1(T)^{\frac{N}{1 - \epsilon'}}). \end{aligned} \quad (5.78)$$

Choosing  $\epsilon$  sufficiently small we deduce that there exists  $C > 0$  such that:

$$\|q_1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p} + \epsilon'})} \leq C(1 + \|q_0^1\|_{B_{p,1}^{\frac{N}{p} + \epsilon'}} + M'(T) + M_1(T)^{\frac{N}{1 - \epsilon'}}). \quad (5.79)$$

From (5.77) it implies in particular that the  $L_T^\infty(L^\infty)$  norm of  $\rho$  depends only on  $T$  and on the initial data.

### 5.3 Lipschitz control on the velocity $u$

We recall that we have on  $(0, T^*)$ :

$$\partial_t u - \mu \Delta u - \mu \nabla \operatorname{div} u = -u \cdot \nabla u - 2\mu \nabla \ln \rho \cdot Du - \frac{\nabla P(\rho)}{\rho}.$$

Applying the proposition 2.4 we have for any  $T \in (0, T^*)$ :

$$\|u\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p} - 1})} + \|u\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p} + 1})} \leq C(\|u_0\|_{B_{p,1}^{\frac{N}{2} - 1}} + \|F_1\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p} - 1})}), \quad (5.80)$$

with:

$$F_1 = -u \cdot \nabla u - 2\mu \nabla \ln \rho \cdot Du - \nabla F(\rho).$$



with  $F'(\rho) = \frac{P'(\rho)}{\rho}$ . It remains now to estimate  $\|F\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}-1})}$ , by proposition 2.2,2.3, interpolation and Young inequality we have for  $C$  a continuous function and with  $\theta \in (0, 1)$ :

$$\begin{aligned}
& \left\| \frac{\nabla P(\rho)}{\rho} \right\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}-1})} \leq T \|F(\rho) - 1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq T (\|q\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}+\epsilon'})} + \|q\|_{L_T^\infty(L^2)}) \\
& \|u \cdot \nabla u\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}-1})} \leq \int_0^T \|u(s)\|_{B_{p,1}^{\frac{N}{p}-1+\epsilon'}} \|\nabla u(s)\|_{B_{p,1}^{\frac{N}{p}-\epsilon'}} ds \\
& \leq \int_0^T \|u(s)\|_{B_{p,1}^{\frac{N}{p}-1+\epsilon'}} \|u(s)\|_{B_{p,1}^{\frac{N}{p}+1}}^\theta \|u(s)\|_{B_{p,1}^{\frac{N}{p}+1}}^{1-\theta} ds \\
& \leq C_\epsilon \int_0^T \|u(s)\|_{B_{p,1}^{\frac{N}{p}-1+\epsilon'}}^{\frac{1}{1-\theta}} \|u(s)\|_{B_{p,1}^{\frac{N}{p}+1}}^{1-\theta} ds + \epsilon \int_0^T \|u(s)\|_{B_{p,1}^{\frac{N}{p}+1}} ds \\
& \|\nabla \ln \rho \cdot Du\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}-1})} \leq C_\epsilon \int_0^T \|\nabla \ln \rho(s)\|_{B_{p,1}^{\frac{N}{p}-1+\epsilon'}}^{\frac{1}{1-\theta}} \|u(s)\|_{B_{p,1}^{\frac{N}{p}+1}}^{1-\theta} ds + \epsilon \int_0^T \|u(s)\|_{B_{p,1}^{\frac{N}{p}+1}} ds \\
& \leq C_\epsilon \int_0^T C \left( \frac{1}{\rho} \|L_T^\infty(L^\infty), \|\rho\|_{L_T^\infty(L^\infty)} \right) \|q(s)\|_{B_{p,1}^{\frac{N}{p}+\epsilon'}}^{\frac{1}{1-\theta}} \|u(s)\|_{B_{p,1}^{\frac{N}{p}+1}}^{1-\theta} ds + \epsilon \int_0^T \|u(s)\|_{B_{p,1}^{\frac{N}{p}+1}} ds
\end{aligned} \tag{5.81}$$

From proposition 4.5 and according to (5.70) and (5.79), plugging the previous estimate (5.81) and using Gronwall lemma we prove that there exists a function  $C$  continuous and increasing such that:

$$\|u\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1})} + \|u\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}+1})} \leq C(T). \tag{5.82}$$

We have proved in particular that for any  $T \in (0, T^*)$  we have:

$$\|\nabla u\|_{L^1((0,T), L^\infty(\mathbb{R}^N))} \leq C(T). \tag{5.83}$$

In addition by classical estimate on transport equation we prove also that there exists a function  $C$  continuous and increasing such that for any  $T \in (0, T^*)$ :

$$\|q\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq C(T). \tag{5.84}$$

Proposition (4.5), (5.83) and (5.84) allows to apply the blow-up criterion of the theorem 1.1. In particular it contradicts the assumption  $T^* < +\infty$  and we have then seen that necessary  $T^* = +\infty$ . It implies that the solution  $(q, u)$  is unique and verify locally in time (1.13). In particular since the proof of the uniqueness in [15] requires only local estimates in time, we have proved that  $(q, u)$  is a global strong solution of the system (1.1).

## 6 Proof of the theorem 1.3

The proof follows exactly the same lines than the previous section and is even simpler. We are going to explain how to adopt the previous proof. First since radially the solution is radially symmetric we observe that the strong solution  $(\rho, v)$  of the system (1.3) verifies on the maximal time interval  $(0, T^*)$ :

$$\rho \partial_t v + \rho u \cdot \nabla v + \nabla P(\rho) = 0. \tag{6.85}$$

Using an argument developed in [29], we can rewrite the previous equation as follows:

$$\rho \partial_t v + \rho u \cdot \nabla v + \frac{a\gamma}{2\mu} v = \frac{a\gamma}{2\mu} u. \quad (6.86)$$

The rest of the proof consists in getting  $L_T^\infty(L^p(\mathbb{R}^N))$  estimates on  $\rho^{\frac{1}{p}} v$  for any  $p \geq 2$  and  $T \in (0, T^*)$ . It is then sufficient as in the previous section and [28] to apply a maximum principle and to prove that  $\rho$  and  $\frac{1}{\rho}$  is bounded in  $L^\infty([0, T^*], L^\infty(\mathbb{R}^N))$ . To do this, it suffices to multiply the equation (6.86) by  $v|v|^p$  and to integrate over  $(0, T) \times \mathbb{R}^N$ . It gives:

$$\frac{1}{p+2} \int_{\mathbb{R}^N} \partial_t(\rho|v|^{p+2}(t, x)) dx + \frac{a\gamma}{2\mu} \int_{\mathbb{R}^N} |v|^{p+2}(t, x) dx = \frac{a\gamma}{2\mu} \int_{\mathbb{R}^N} u \cdot v|v|^p(t, x) dx. \quad (6.87)$$

In order to conclude, it remains to bound  $\rho^{\frac{1}{q}} u$  in any  $L^\infty([0, T^*], L^q(\mathbb{R}^N))$  with  $q \geq 2$ . We recall now that since  $u$  is radially symmetric, we have  $Du = \nabla u$  and we can rewrite the momentum equation on  $u$  as follows:

$$\rho \partial_t u + \rho u \cdot \nabla u - 2\mu \operatorname{div}(\rho \nabla u) + a \nabla \rho^\gamma = 0. \quad (6.88)$$

Multiplying the previous equation by  $u|u|^p$  and integrating over  $(0, T) \times \mathbb{R}^N$  we get:

$$\begin{aligned} \frac{1}{p+2} \int_{\mathbb{R}^N} \partial_t(\rho|u|^{p+2}(t, x)) dx + 2\mu \int_{\mathbb{R}^N} |\nabla u|^2 |u|^p(t, x) dx + \frac{\mu p}{2} \int_{\mathbb{R}^N} \rho |\nabla(|u|^2)|^2 |v|^{p-2}(t, x) dx \\ \leq \left| \int_{\mathbb{R}^N} a \nabla \rho^\gamma \cdot u |u|^p(t, x) dx \right|. \end{aligned} \quad (6.89)$$

Using same arguments than in [28], we show that we obtain  $L^\infty([0, T^*], L^q(\mathbb{R}^N))$  on  $\rho^{\frac{1}{q}} u$  for any  $q \geq 2$  when  $1 \leq \gamma < \frac{7}{6}$ . The rest of the proof follows the same arguments than the previous section.

## 7 Appendix

Assume that  $v$  has the following form:

$$v = \frac{x^\perp}{|x|} v_2(t, |x|) + \frac{x}{|x|} v_1(t, |x|) = v^2 + v^1.$$

We verify then that:

$$\begin{aligned} u^1 \cdot \nabla v^1 &= \frac{x^\perp}{|x|} (u_1 \partial_r v_1), \\ u^2 \cdot \nabla v^2 &= -\frac{x^\perp}{|x|} \left( \frac{u_2^2}{r} \right), \\ u^1 \cdot \nabla v^2 &= \frac{x^\perp}{|x|} (u_1 \partial_r v_2), \\ u^2 \cdot \nabla v^1 &= \frac{x^\perp}{|x|} \left( \frac{v_1 u_2}{r} \right). \end{aligned}$$

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