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DOES EULERIAN PERCOLATION ON \mathbb{Z}^2 PERCOLATE ?

OLIVIER GARET, RÉGINE MARCHAND, AND IRÈNE MARCOVICI

ABSTRACT. Eulerian percolation on \mathbb{Z}^2 with parameter p is the classical Bernoulli bond percolation with parameter p conditioned on the fact that every site has an even degree. We first explain why Eulerian percolation with parameter p coincides with the contours of the Ising model for a well-chosen parameter $\beta(p)$. Then we study the percolation properties of Eulerian percolation.

1. INTRODUCTION

In this paper, we aim to study the percolation properties of the Eulerian (or even) percolation on the edges of \mathbb{Z}^2 . Eulerian percolation with parameter p on the edges of \mathbb{Z}^2 is heuristically the classical independent Bernoulli percolation with parameter p on the edges of \mathbb{Z}^2 , but conditioned to be even, *i.e.* conditioned to the fact that each vertex of \mathbb{Z}^2 has an even number of open edges touching it. This paper has two parts.

1. The event by which we want to condition has probability 0. The first step is thus to define properly the Eulerian percolation measures, by the mean of specifications in finite boxes and of Gibbs measures. Doing so, the Eulerian percolation measure with parameter p will be given by the contours of the Ising model on the sites of the dual $\mathbb{Z}_*^2 \sim \mathbb{Z}^2$ for a well-chosen parameter $\beta = \beta(p)$:

Theorem 1.1. *For every $p \in [0, 1]$, there exists a unique even percolation measure μ_p with opening parameter p . It is the image by the contour application of any Gibbs measure for the Ising model on the dual graph \mathbb{Z}_*^2 of \mathbb{Z}^2 , with parameter*

$$\beta = \frac{1}{2} \log \frac{1-p}{p} \quad \Leftrightarrow \quad p = \frac{1}{1 + \exp(2\beta)}.$$

Also, μ_p is invariant and ergodic under the natural action of \mathbb{Z}^d .

In the case of the \mathbb{Z}^2 lattice, this generalizes Theorem 5.2 of Grimmett and Janson [9], that studies random even subgraphs on finite planar graphs. In the same paper, they mention the existence of a thermodynamic limit, but the question of uniqueness is not asked.

2. We are interested in the probability, under the even percolation measure μ_p , of the percolation event

$$\mathcal{C} = \text{“there exists an infinite open cluster”}.$$

Our first result consists in proving the almost-sure uniqueness of the infinite cluster when it exists:

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Theorem 1.2. *For every $p \in [0, 1]$, we denote by μ_p the even percolation measure with opening parameter p . There exists μ_p -almost surely exactly one infinite cluster or μ_p -almost surely no infinite cluster.*

Note that the “even degree” condition induces correlations between states of edges, that break the classical finite energy property. However, we can adapt the classical proof by using the interpretation in terms of contours of the Ising model. To study the percolation itself, we have at our disposal the results proved for the Ising model on \mathbb{Z}^2 , especially in the ferromagnetic range. Remember that $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$ is the critical value of the Ising model in \mathbb{Z}^2 ; we introduce the corresponding percolation parameter

$$p_{c,\text{even}} = \frac{1}{1 + \exp(2\beta_c)} = 1 - \frac{1}{\sqrt{2}} < \frac{1}{2}.$$

We prove the following:

Theorem 1.3. *In terms of even percolation with parameter $p \in [0, 1]$, there exists p_1 and p_2 with $\frac{1}{2} < p_1 < p_2 \leq \frac{3}{4}$ such that*

- for $p < p_{c,\text{even}}$, $\mu_p(\mathcal{C}) = 0$,
- for $p_{c,\text{even}} < p \leq p_1$, $\mu_p(\mathcal{C}) = 1$,
- for $p > p_2$, $\mu_p(\mathcal{C}) = 1$.

In terms of the Ising model with parameter β , these results correspond to:

- for $\beta > \beta_c$, for every Gibbs measure with parameter β , contours a.s. do not percolate,
- for $0 \leq \beta < \beta_c$, for every Gibbs measure with parameter β , contours a.s. percolate,
- for $\beta < -(\log 3)/2$, for every Gibbs measure with parameter β , contours a.s. percolate.

These results are summarized in the following table:

p	0	$p_{c,\text{even}}$	$\frac{1}{2}$	p_1	p_2	1
$\beta(p)$	$+\infty$	β_c	0			$-\infty$
μ_p	no perco.	perco.	perco.	?	perco.	

We naturally conjecture that $p_{c,\text{even}}$ is indeed the unique percolation threshold for the Eulerian percolation on \mathbb{Z}^2 :

Conjecture 1.4.

- *In terms of even percolation: for every $p \leq p_{c,\text{even}}$, $\mu_p(\mathcal{C}) = 0$, while for every $p > p_{c,\text{even}}$, $\mu_p(\mathcal{C}) = 1$.*
- *In terms of the Ising model: for every $\beta \geq \beta_c$, for every Gibbs measure with parameter β , contours a.s. do not percolate, while for every $\beta \in (-\infty, \beta_c)$, for every Gibbs measure with parameter β , contours a.s. percolate.*

But despite our efforts, we did not manage to fill the gap $p \in [p_1, p_2]$. In independent Bernoulli bond percolation, $p \mapsto \mathbb{P}_p(\mathcal{C})$ is non-decreasing, and this follows from a natural coupling of percolation for all parameters $p \in [0, 1]$. The same monotonicity occurs for FK percolation with parameter $q \geq 1$. This is strongly related to the fact that FK percolation satisfies the FKG inequality. Here, conditioning by the Eulerian condition breaks the association, even if the underlying graph is Eulerian. See the appendix for an example of the strange things that may happen.

The percolation results for $p \leq 1/2$ essentially follow from the results about percolation of colors in the Ising model in the ferromagnetic case $\beta > 0$. The Ising model in the antiferromagnetic case has been much less studied, so other kinds of arguments are needed for $p > 1/2$. The case for large p follows from the link between the Ising model and FK percolation. A stochastic comparison between even percolation with a large parameter and independent percolation with a large parameter gives the result for $p \geq 3/4$. This bound may be improved by some argument derived from Beffara and Duminil-Copin [2], but it is not sufficient to fill the gap.

2. EULERIAN PERCOLATION PROBABILITY MEASURES

On \mathbb{Z}^2 , we consider the set of edges \mathbb{E}_2 between vertices at distance 1 for $\|\cdot\|_1$. An edge configuration is an element $\omega \in \{0, 1\}^{\mathbb{E}_2}$: if $\omega(e) = 1$, the edge e is present (or open) in the configuration ω , and if $\omega(e) = 0$, the edge is absent (or closed). For $x \in \mathbb{Z}^2$, we define the degree $d_\omega(x)$ of x in the configuration ω by setting

$$d_x(\omega) = \sum_{e \ni x} \omega(e).$$

An Eulerian edge configuration is then an element of

$$\Omega_{\text{EP}} = \{\omega \in \{0, 1\}^{\mathbb{E}_2} : \forall x \in \mathbb{Z}^d, d_x(\omega) = 0 \text{ [2]}\}.$$

If $\omega, \eta \in \Omega_{\text{EP}}$ and $\Lambda \subset \mathbb{E}_2$, we denote by $\eta_\Lambda \omega_{\Lambda^c}$ the concatenation of the configuration η restricted to Λ and of the configuration ω restricted to Λ^c .

Gibbs measures for Eulerian percolation. For each finite subset Λ of \mathbb{E}_2 and each function f on Ω_{EP} , we can define

$$\begin{aligned} \forall \omega \in \Omega_{\text{EP}} \quad (M_\Lambda^p f)(\omega) &= \sum_{\eta_\Lambda \in \{0, 1\}^\Lambda} \mathbf{1}_{\Omega_{\text{EP}}}(\eta_\Lambda \omega_{\Lambda^c}) f(\eta_\Lambda \omega_{\Lambda^c}) \left(\frac{p}{1-p}\right)^{\sum_{e \in \Lambda} \eta_e}, \\ (1) \quad (\mu_\Lambda^p f)(\omega) &= \frac{(M_\Lambda^p f)(\omega)}{(M_\Lambda^p 1)(\omega)}. \end{aligned}$$

Note that μ_Λ^p is Feller, in the following sense: $\mu_\Lambda^p f$ is continuous (for the product topology) as soon as f is continuous. A standard calculation gives

$$(2) \quad \mu_\Delta^p \circ \mu_\Lambda^p = \mu_\Delta^p \text{ for } \Lambda \subset \Delta.$$

We denote by $\mu_{\Lambda, \omega}^p$ the probability measure on $\{0, 1\}^{\mathbb{E}_2}$ that is such that, for each bounded measurable function f ,

$$\int_{\Omega_{\text{EP}}} f d\mu_{\Lambda, \omega}^p = (\mu_\Lambda^p f)(\omega) = \frac{\sum_{\eta_\Lambda \in \{0, 1\}^\Lambda} \mathbf{1}_{\Omega_{\text{EP}}}(\eta_\Lambda \omega_{\Lambda^c}) f(\eta_\Lambda \omega_{\Lambda^c}) \left(\frac{p}{1-p}\right)^{\sum_{e \in \Lambda} \eta_e}}{\sum_{\eta_\Lambda \in \{0, 1\}^\Lambda} \mathbf{1}_{\Omega_{\text{EP}}}(\eta_\Lambda \omega_{\Lambda^c}) \left(\frac{p}{1-p}\right)^{\sum_{e \in \Lambda} \eta_e}}.$$

A measure probability μ on $(\{0, 1\}^{\mathbb{E}_2}, \mathcal{B}(\{0, 1\}^{\mathbb{E}_2}))$ is said to be a Gibbs measure for Eulerian percolation (or a Eulerian percolation probability measure) if one has

- $\mu(\Omega_{\text{EP}}) = 1$

- For each continuous function on $\{0, 1\}^{\mathbb{E}^2}$, for each finite subset Λ of \mathbb{E}_2 ,

$$(3) \quad \int_{\Omega_{\text{EP}}} f \, d\mu = \int_{\Omega_{\text{EP}}} (\mu_{\Lambda}^p f) d\mu.$$

We denote by $\mathcal{G}_{\text{EP}}(p)$ the set of Gibbs measures for Eulerian percolation with opening parameter p .

Colorings with two colors and Eulerian percolation. A natural way to obtain an Eulerian configuration of the edges of a planar graph is to take the contours of a coloring in two colours of the sites of its dual, and this is what we describe now in the \mathbb{Z}^2 case.

Let $\mathbb{Z}_*^2 = (1/2, 1/2) + \mathbb{Z}^2$ be the dual graph of \mathbb{Z}^2 . The set \mathbb{E}_*^2 of edges of \mathbb{Z}_*^2 is the image of \mathbb{E}^2 by the translation with respect to the vector $(1/2, 1/2)$. If $e \in \mathbb{E}^2$, we denote by e^\perp its dual edge, *i.e.* the only edge in \mathbb{E}_*^2 that intersects e . We can map any coloring of the sites of \mathbb{Z}_*^2 with the two colors -1 and 1 to its contour in the following way:

$$\Gamma : \begin{array}{ll} \{-1, 1\}^{\mathbb{Z}_*^2} & \longrightarrow \Omega_{\text{EP}} \\ \sigma = (\sigma_{i_*})_{i_* \in \mathbb{Z}_*^2} & \longmapsto (\eta_e)_{e \in \mathbb{E}^2}, \text{ with } \eta_e = \mathbf{1}_{\{\sigma_{i_*} \neq \sigma_{j_*}\}} \text{ if } e^\perp = \{i_*, j_*\} \end{array}.$$

Let us see that $\Gamma(\sigma) \in \Omega_{\text{EP}}$. Indeed, set $\eta = \Gamma(\sigma)$, and fix $x \in \mathbb{Z}^2$. Let a_*, b_*, c_*, d_* be the four corners of the square with length side 1 in \mathbb{Z}_*^2 whose center is x : then the four edges issued from x are the dual edges of $\{a_*, b_*\}$, $\{b_*, c_*\}$, $\{c_*, d_*\}$ and $\{d_*, a_*\}$. Thus

$$(-1)^{d_x(\eta)} = (-\sigma_{a_*} \sigma_{b_*})(-\sigma_{b_*} \sigma_{c_*})(-\sigma_{c_*} \sigma_{d_*})(-\sigma_{d_*} \sigma_{a_*}) = 1.$$

So $\Gamma(\sigma) \in \Omega_{\text{EP}}$.

Reciprocally, the dual of a planar Eulerian graph is bipartite (see for instance Wilson and Van Lint [18], th 34.4 p 481), and there are exactly two ways of coloring the sites of a connected bipartite graph with two colors in such a way that the extremities of every edge are in different colors. In our \mathbb{Z}^2 case, fix an Eulerian edge configuration η . By setting $c_\eta(0_*) = +1$, and for any $x_* \in \mathbb{Z}_*^2$, $c_\eta(x_*)$ equals $(-1)^{\text{number of edges in } \eta \text{ crossed by any path (in the dual) between } 0_* \text{ and } x_*}$, we properly define a coloring c_η of \mathbb{Z}_*^2 , and $\Gamma^{-1}(\eta) = \{c_\eta, -c_\eta\}$. Finally, the contour application Γ is surjective and two to one.

As we will see now, the Gibbs measures for Eulerian percolation can be obtained as the images by the contour application Γ of the Gibbs measures for the Ising model in \mathbb{Z}_*^2 .

Gibbs measures for the Ising model on \mathbb{Z}_*^2 . It is of course the same model as the Ising model on \mathbb{Z}^2 , but to avoid confusion between the initial graph \mathbb{Z}^2 and its dual \mathbb{Z}_*^2 in the sequel, we present it directly in the dual \mathbb{Z}_*^2 . Fix a parameter $\beta \in \mathbb{R}$. For a finite subset Λ of \mathbb{Z}_*^2 , the Hamiltonian on Λ is defined by

$$\forall \omega \in \{-1, +1\}^{\mathbb{Z}_*^2} \quad H_\Lambda(\omega) = - \sum_{\substack{e = \{x, y\} \in \mathbb{E}_*^2 \\ e \cap \Lambda \neq \emptyset}} \omega_x \omega_y.$$

Then, we can define, for each bounded measurable function f ,

$$\begin{aligned} \forall \omega \in \{-1, +1\}^{\mathbb{Z}_*^2} \quad \mathcal{Z}_\Lambda^\beta(\omega) &= \sum_{\eta \in \{-1, +1\}^\Lambda} \exp(-\beta H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})), \\ \Pi_\Lambda^\beta f(\omega) &= \frac{1}{\mathcal{Z}_\Lambda^\beta(\omega)} \sum_{\eta \in \{-1, +1\}^\Lambda} \exp(-\beta H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})) f(\eta_\Lambda \omega_{\Lambda^c}). \end{aligned}$$

For each ω , we denote by $\Pi_{\Lambda, \omega}^\beta$ the probability measure on $\{-1, +1\}^{\mathbb{Z}_*^2}$ which is associated to the map $f \mapsto \Pi_\Lambda^\beta f(\omega)$. When $\beta = 0$, colors of sites inside Λ are i.i.d. and follow the uniform law in $\{-1, +1\}$. When $\beta > 0$, neighbour sites prefer to be in the same color (ferromagnetic case), while when $\beta < 0$, neighbour sites prefer to be in different colors (anti-ferromagnetic case).

A Gibbs measure for the Ising model on \mathbb{Z}_*^2 with parameter β is any probability measure γ on $\{-1, +1\}^{\mathbb{Z}_*^2}$ such that for each continuous function, for each finite subset Λ of \mathbb{Z}_*^2 ,

$$\int_{\{-1, +1\}^{\mathbb{Z}_*^2}} f d\gamma = \int_{\{-1, +1\}^{\mathbb{Z}_*^2}} (\Pi_\Lambda^\beta f) d\gamma.$$

We denote by $\mathcal{G}(\beta)$ the set of Gibbs measures for the Ising model with parameter β . The Ising model presents a phase transition: set $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$ (see Onsager [14]), then

- if $0 \leq \beta \leq \beta_c$, then there is a unique Gibbs measure;
- if $\beta > \beta_c$ then there are infinitely many Gibbs measures. The set $\mathcal{G}(\beta)$ is the convex hull of the two extremal measures γ_β^+ and γ_β^- , that can be deduced one from the other by exchanging the two colors. This result has been obtained independently by Aizenmann [1] and Higuchi [10]. See also Georgii–Higuchi [7].

For $\beta < 0$, the Gibbs measures are obtained from $\mathcal{G}(-\beta)$ by changing the colors on the subset of even sites. In other words, if

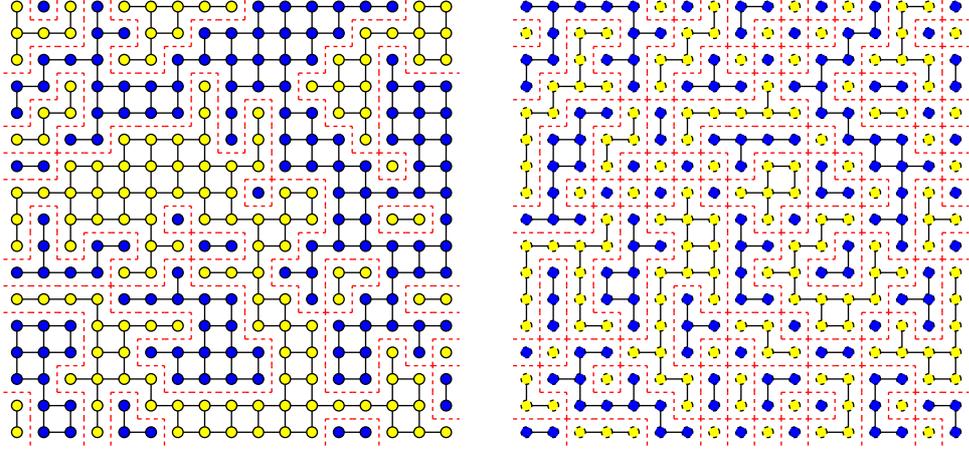
$$S((\omega)_{\omega \in \mathbb{Z}_*^2}) = ((-1)^{i+j} \omega_{(i,j)})_{(i,j) \in \mathbb{Z}_*^2},$$

then $\mu_S = (A \mapsto \mu(S^{-1}(A)))$ belongs to $\mathcal{G}(-\beta)$ if and only if $\mu \in \mathcal{G}(\beta)$. For the details, see chapter 6 in Georgii [6].

Proof of Theorem 1.1. We first prove the existence of Gibbs measure for Eulerian percolation. Let us define $\Lambda_n = (1/2, 1/2) + \{-n, \dots, n\}^2 \subset \mathbb{Z}_*^2$ and denote by $E(\Lambda_n)$ the set of edges e such that e^\perp has at least one end in Λ_n . Since Ω_{EP} is a closed subset of the compact set $\{0, 1\}^{\mathbb{E}^2}$, the sequence $(\mu_{E(\Lambda_n), 0}^p)_n$ has a limit point μ with $\mu(\Omega_{\text{EP}}) = 1$. Using Equation (1) and the fact that μ_Λ^p is Feller, it is easy to see that $\mu \in \mathcal{G}_{\text{EP}}(p)$, which is therefore not empty.

This proof is not surprising for people who are familiar to the general theory of Gibbs measures, as described in Georgii [6]. Nevertheless, it must be noticed that $\mu_\Lambda^p f$ is not defined on the whole set $\{0, 1\}^{\mathbb{E}^2}$ (it is not a specification in the realm of Georgii [6]), which leads us to mimic a standard proof.

Now, let's prove the uniqueness of the Eulerian percolation probability measure and characterize it. Fix $p \in (0, 1)$ and set $\beta = \beta(p) = \frac{1}{2} \log \frac{1-p}{p}$. We first need to prove the following lemma:

FIGURE 1. The mapping $\beta \longleftrightarrow -\beta$

Lemma 2.1. *Let $c \in \{-1, 1\}^{\mathbb{Z}_*^2}$ and $\eta \in \Omega_{EP}$ with $\eta = \Gamma(c)$. Suppose that Λ_* is a simply connected subset of \mathbb{Z}_*^2 , and denote by $E(\Lambda_*)$ the set of edges e such that e^\perp has at least one end in Λ_* .*

Then, the probability $\mu_{E(\Lambda_)}^p$ is the image of $\Pi_{\Lambda_*, c}^{\beta(p)}$ under the contour application $\omega \mapsto \Gamma(\omega)$.*

Proof. By construction, the image of $\Pi_{\Lambda_*, c}^\beta$ under the map $\omega \mapsto \Gamma(\omega)$ is concentrated on configurations that coincide with η outside $E(\Lambda_*)$. Obviously it is the same for $\mu_{E(\Lambda_*)}^p$, so we must focus on the behaviour of the edges in $E(\Lambda_*)$.

Let $\eta' \in \Omega_{EP}$ be such that η and η' coincide outside $E(\Lambda_*)$. There are exactly two colorings $c', -c'$ such that $\Gamma(c') = \Gamma(-c') = \eta'$. If x and y are two neighbours in $(\Lambda^*)^c$, then

$$c_x c_y = 1 - 2\eta_{(x,y)^\perp} = 1 - 2\eta'_{(x,y)^\perp} = c'_x c'_y,$$

so $c_x c'_x = c_y c'_y$. Since Λ_*^c is connected, it follows that one of two colorings, say c' , coincides with c on $(\Lambda_*)^c$ (and $-c'$ with $-c$). Thus $\Pi_{\Lambda_*, c}^\beta(-c') = 0$ and $\Pi_{\Lambda_*, c}^\beta(c') > 0$, and:

$$\begin{aligned} \Pi_{\Lambda_*, c}^\beta(\Gamma(\cdot) = \eta') &= \Pi_{\Lambda_*, c}^\beta(c') \\ &= \frac{1}{\mathcal{Z}_{\Lambda_*}^\beta(c)} \exp \left(\beta \sum_{\substack{e=\{x,y\} \in \mathbb{E}_*^2 \\ e \cap \Lambda_* \neq \emptyset}} c'_x c'_y \right) = \frac{1}{\mathcal{Z}_{\Lambda_*}^\beta(c)} \exp \left(\beta \sum_{\substack{e=\{x,y\} \in \mathbb{E}_*^2 \\ e \cap \Lambda_* \neq \emptyset}} (1 - 2\eta'_{(x,y)^\perp}) \right) \\ &= \frac{1}{\mathcal{Z}_{\Lambda_*}^\beta(c)} \exp \left(\beta \sum_{e \in E(\Lambda_*)} (1 - 2\eta'_e) \right) = \frac{\exp(\beta |E(\Lambda_*)|)}{\mathcal{Z}_{\Lambda_*}^\beta(c)} \left(\frac{p}{1-p} \right)^{\sum_{e \in E(\Lambda_*)} \eta'_e} \\ &= \alpha_{\Lambda_*, \eta} \mu_{E(\Lambda_*)}^p(\eta'). \end{aligned}$$

Since we compare probability measures with the same support, $\alpha_{\Lambda^*, \eta} = 1$. \square

Let us now see that all Gibbs measures for the Ising model with parameter β have the same image by the application Γ . Let $\gamma \in \mathcal{G}(\beta)$: there exists $\alpha \in [0, 1]$ such that $\gamma = \alpha\gamma_\beta^+ + (1 - \alpha)\gamma_\beta^-$. Remember that γ_β^- is the image of γ_β^+ by the exchange of colors, that leaves the contours unchanged. So, if $A \in \mathcal{B}(\{0, 1\}^{\mathbb{E}^2})$,

$$\begin{aligned} \gamma(\Gamma \in A) &= \alpha\gamma_\beta^+(\Gamma \in A) + (1 - \alpha)\gamma_\beta^-(\Gamma \in A) \\ &= \alpha\gamma_\beta^+(\Gamma \in A) + (1 - \alpha)\gamma_\beta^+(\Gamma \in A) = \gamma_\beta^+(\Gamma \in A) \end{aligned}$$

Let $\mu \in \mathcal{G}_{EP}(p)$ and set as before $\Lambda_n = (1/2, 1/2) + \{-n, \dots, n\}^2$. Let f be a bounded continuous function on $\{-1, 1\}^{\mathbb{E}^2}$, and let us prove that for each $\eta \in \Omega_{EP}$,

$$(\mu_{E(\Lambda_n)}^p f)(\eta) \rightarrow \int f \circ \Gamma d\gamma_\beta^+.$$

With Equation (3), it will imply by dominated convergence that

$$\int_{\Omega_{EP}} f d\mu = \int f \circ \Gamma d\gamma_\beta^+,$$

and thus that μ is the image by the application Γ of γ_β^+ , or of any Gibbs measures for the Ising model with parameter β .

Let $\eta \in \Omega_{EP}$ be an Eulerian edge configuration, and let $c \in \{-1, +1\}^{\mathbb{Z}_*^2}$ be such that $\Gamma(c) = \eta$. Let x be a limiting value of $((\mu_{E(\Lambda_n)}^p f)(\eta))_{n \geq 1}$. By extracting a subsequence if necessary, we can assume that $(\Pi_{\Lambda_n, c})_{n \geq 1}$ converges to γ – which is then in $\mathcal{G}(\beta)$ – and that $x = \lim_{n \rightarrow +\infty} (\mu_{E(\Lambda_n)}^p f)(\eta)$. By Lemma 2.1,

$$\begin{aligned} (\mu_{E(\Lambda_n)}^p f)(\eta) &= \Pi_{\Lambda_n, c_n}^\beta(f \circ \Gamma), \\ \text{so } x &= \int_{\{-1, 1\}^{\mathbb{Z}_*^2}} (f \circ \Gamma) d\gamma = \int_{\{-1, 1\}^{\mathbb{Z}_*^2}} (f \circ \Gamma) d\gamma_\beta^+. \end{aligned}$$

To conclude, note that γ_β^+ is stationary and ergodic, and so does μ_p .

3. UNICITY OF THE INFINITE CLUSTER IN EULERIAN PERCOLATION

Proof of Theorem 1.2. Since μ_p is ergodic and \mathcal{C} is a translation-invariant event, it is obvious that $\mu_p(\mathcal{C}) \in \{0, 1\}$. To prove the unicity of the infinite cluster, we now follow the famous proof by Burton and Keane [3]. The main point is that the Eulerian percolation measure does not satisfy the finite energy property: once a configuration is fixed outside a box, the even degree condition forbids some configurations inside the box. But the Ising model has the finite energy property, and we will thus use the representation of even percolation in terms of contours of the Ising model.

The number N of infinite clusters is translation-invariant, so the ergodicity of μ_p implies that it is μ_p almost surely constant: there exists $k \in \mathbb{N} \cup \{\infty\}$ such that $\mu_p(N = k) = 1$. The first step consists in proving that $k \in \{0, 1, \infty\}$. So assume for contradiction that k is an integer larger than 2. Consider a finite box Λ , large enough to ensure that with positive probability (under μ_p), the box Λ intersects at least two infinite clusters. Using Theorem 1.1, this implies that with positive probability (under γ_β^+ for the parameter β corresponding to p), the contours of the Ising model present two infinite connected components that intersect Λ . But the Ising model has the finite energy property: by forcing the colors inside Λ to be a chessboard, we keep an event with positive probability, and we decrease the

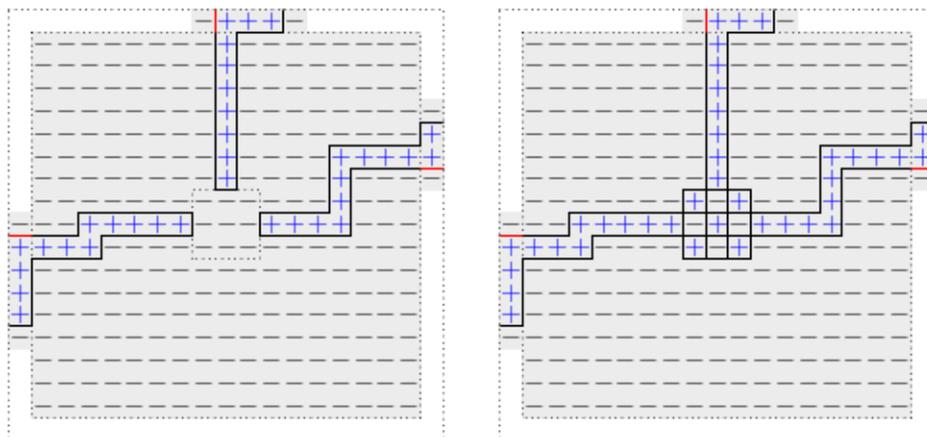


FIGURE 2. Construction of a trifurcation in B_{L_1} . Dotted squares are, from inside to outside, B_1 , B_{L-1} and B_L . Red edges are, on the left, in three distinct infinite clusters of open edges.

number of infinite clusters in the contours by at least one. Coming back to Eulerian percolation, this gives $\mu_p(N \leq k-1) > 0$, which is a contradiction. See [13, 12] for the first version of such an argument.

In the final step, we prove that $k = \infty$ is impossible. Assume by contradiction that $\mu_p(N = +\infty) = 1$. We work now with the colorings of the sites of \mathbb{Z}_*^2 , under $\gamma_{\beta(p)}$.

By taking $L \in \mathbb{N}$ large enough, we can assume that the event E_L “the box $B_L = [-L, L]^2$ intersects at least 30 infinite clusters” has positive probability. Let $\partial\eta_0$ be a coloring of the sites in $\partial_{int}B_L = B_L \setminus B_{L-1}$ such that

$$\gamma_{\beta(p)}(\eta \in E_L, \eta|_{\partial_{int}B_L} = \partial\eta_0) > 0.$$

Take ω in this event. Each infinite cluster intersecting B_L crosses $\partial_{int}B_L$ via an open edge, and this edge sits between a +1 site and a -1 site.

Thus the 30 distinct infinite clusters intersecting B_L imply the existence of at least 15 +1-clusters in $\partial_{int}B_L$. To avoid geometric intricate details, we do not want to consider +1-clusters in $\partial_{int}B_L$ that are in the corners: we thus remove from our 15 clusters at most $12 = 3 \times 4$ clusters (the one containing the corner if it is a +1, and the nearest +1 cluster on each side). We are now left with at least 3 disjoint +1-clusters in $\partial_{int}B_L$, sitting near edges of distinct infinite clusters: they are far away enough so that we can draw, inside B_L , 3 paths of sites linking these three clusters to three of the four centers of the sides of $\partial_{int}B_L$, in such a way that two distinct paths are not $*$ -connected. See Figure 2.

Consider now the following coloring of B_{L-1} : all sites in the three paths are +1, all the other sites are -1. With this coloring, B_{L-1} intersects exactly three infinite clusters of open edges. If we change the coloring of B_1 in a chessboard, B_{L-1} intersects exactly one infinite cluster of open edges. In this case, we say that 0 is a trifurcation. As $\gamma_{\beta(p)}$ has finite energy, we see that the probability that 0 is a trifurcation has positive probability, and the end of the proof is as in Burton-Keane.

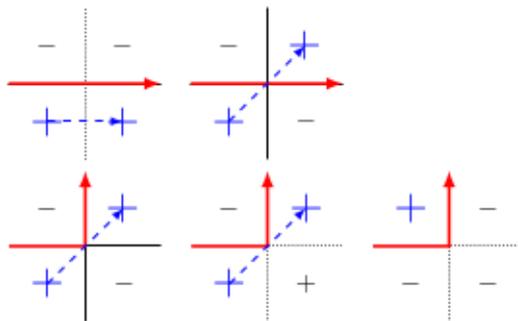


FIGURE 3. Construction of an $*$ -chain of spins $+1$ (dotted arrow) from a infinite path γ (full arrow)

4. PERCOLATION PROPERTIES OF EULERIAN PERCOLATION

The proof of Theorem 1.3 is split into three steps: Lemmas 4.1, 4.2 and 4.3.

4.1. The ferromagnetic zone of the Ising model: $p \leq 1/2$.

Lemma 4.1. For $p \in (0, p_{c,\text{even}})$, $\mu_p(\mathcal{C}) = 0$.

Proof. Let ω be a spin configuration of $\{+1, -1\}^{\mathbb{Z}_*^2}$, and let $\eta = \Gamma(\omega)$ be the even subgraph of \mathbb{Z}^2 made of the contours of ω .

We need here the notion of $*$ -neighbours: two sites $x_*, y_* \in \mathbb{Z}_*^2$ are $*$ -neighbours if and only if $\|x_* - y_*\|_\infty = 1$. A $*$ -chain is then a sequence of sites in \mathbb{Z}_*^2 such that two consecutive sites are $*$ -neighbours.

Let us assume that η contains an infinite path γ . For each edge along γ , there is a spin $+1$ in the configuration ω on one side of that edge, and a spin -1 on the other side. The set of spins $+1$ (resp. -1) in ω along γ constitutes an infinite $*$ -chain of spins $+1$ (resp. -1), as illustrated in Figure 3, which shows the evolution of the $*$ -chain of spins $+1$ for the different possible steps taken by γ . Set

$$\mathcal{C}_*^+ = \{\omega \in \{+1, -1\}^{\mathbb{Z}_*^2} : \text{there is an infinite } *\text{-chain of spins } +1 \text{ in } \omega\}.$$

It follows from Theorem 1.1 that for any $p \in (0, 1)$, $\mu_p(\mathcal{C}) \leq \gamma_\beta^-(\mathcal{C}_*^+)$, where p and β are related through the relation $\beta = \frac{1}{2} \log \frac{1-p}{p}$. By Proposition 1 in Russo [15], we know that if $\beta > \beta_c$, $\gamma_\beta^-(\mathcal{C}_*^+) = 0$. It follows that for $p < p_{c,\text{even}}$, $\mu_p(\mathcal{C}) = 0$. \square

Lemma 4.2. For $p \in (p_{c,\text{even}}, 1/2]$, $\mu_p(\mathcal{C}) = 1$.

Proof. Let us set

$$\mathcal{C}^+ = \{\omega \in \{+1, -1\}^{\mathbb{Z}_*^2} : \text{there is an infinite chain of spins } +1 \text{ in } \omega\}.$$

Let $\omega \in \mathcal{C}_*^+ \cap (\mathcal{C}^+)^c$, and let δ be an infinite $*$ -chain of spins $+1$ in ω . For each spin $+1$ along δ , let us consider the cluster of spins $+1$ to which it belongs. Since $\omega \notin \mathcal{C}^+$, these clusters are finite. The union of the contours of these clusters is an infinite connected subgraph of \mathbb{Z}^2 . Indeed, let $x_1, x_2 \in \mathbb{Z}_*^2$ be the coordinates of

two consecutive spins $+1$ of the $*$ -chain δ . If $\omega(x_1)$ and $\omega(x_2)$ are not in the same cluster of spins $+1$, it means that the step from x_1 to x_2 in δ is diagonal (with spins -1 in the opposite diagonal), and that the contours of the clusters of $\omega(x_1)$ and $\omega(x_2)$ meet at point $(x_1 + x_2)/2$. Thus, any two consecutive points of δ are such that the contours of their clusters are connected (or possibly the same). By induction, one can then prove that the union of the contours of all the clusters of spins $+1$ of δ is a connected subgraph of \mathbb{Z}^2 .

It follows from Theorem 1.1 that for any $p \in (0, 1)$, $\mu_p(\mathcal{C}) \geq \gamma_\beta^+(\mathcal{C}_*^+ \cap (\mathcal{C}^+)^c)$. For $\beta \in [0, \beta_c)$, we have $\gamma_\beta^+ = \gamma_\beta^- = \gamma_\beta$, and

- $\gamma_\beta^+(\mathcal{C}^+) = 0$, by Proposition 1 in [5],
- $\gamma_\beta^+(\mathcal{C}_*^+) = 1$, by Theorem 1 in [11].

Thus, $\gamma_\beta^+(\mathcal{C}_*^+ \cap (\mathcal{C}^+)^c) = 1$. It follows that for $p \in (p_c, 1/2]$, $\mu_p(\mathcal{C}) = 1$. \square

4.2. The antiferromagnetic zone of the Ising model: $p \geq 1/2$. This is the most complex case, because the geometry of the antiferromagnetic Ising model is not well known, so we only obtain partial results.

4.2.1. Percolation for $p > p_2$ for some $p_2 \leq 3/4$. In the following, we give a full proof of the fact that percolation occurs for $p \geq 3/4$, and give some hints about the way to prove that there is percolation for $p \geq 1 - \frac{1}{2+\sqrt{2}}$, which is a bit better, but also much too far to fill the gap.

The proof is based on a coupling between the Ising model and the random cluster (or FK-percolation) model. We just recall a few results on the random cluster model, and refer to Grimmett's book [8] for a complete survey on this model.

The random cluster measure with parameters p and q on a finite graph $G = (V, E)$ is the probability measure on $\{0, 1\}^E$ defined by:

$$\varphi_{p,q}^G(\eta) = \frac{1}{Z} p^{o(\eta)} (1-p)^{1-o(\eta)} q^{k(\eta)},$$

where $o(\eta)$ and $k(\eta)$ are respectively the number of open edges and the number of connected components in the subgraph of G given by η , and Z is a normalizing constant.

On \mathbb{Z}^2 , it is known that at least for $p \neq \frac{\sqrt{q}}{1+\sqrt{q}}$, there exists a unique infinite volume random cluster measure, that we denote by $\varphi_{p,q}$. It is a probability measure on $\{0, 1\}^{\mathbb{E}^2}$. We denote by $\varphi_{p,q}^*$ the random cluster measure, but considered on the dual $(\mathbb{Z}_*^2, \mathbb{E}_*^2)$ of $(\mathbb{Z}^2, \mathbb{E}^2)$.

The Ising model for $\beta > 0$ is closely related to the random cluster model. For $\beta > 0$, $\beta \neq \beta_c$, let us set $f(\beta) = 1 - \exp(-2\beta)$. From a spin configuration $\omega \in \{+1, -1\}^{\mathbb{Z}^2}$ whose distribution is any Gibbs measure γ_β with parameter β , one obtains a subgraph $\eta \in \{0, 1\}^{\mathbb{E}_*^2}$ with distribution $\varphi_{f(\beta), 2}$ by keeping independently each edge between identical spins with probability p , and erasing all the edges between different spins.

For a subgraph $\eta \in \{0, 1\}^{\mathbb{E}^2}$, we denote by $\eta^c \in \{0, 1\}^{\mathbb{E}^2}$ the complementary subgraph of \mathbb{Z}^2 , meaning that the open edges of η^c are exactly the closed edges of η . We denote by $\eta_* \in \{0, 1\}^{\mathbb{E}_*^2}$ the dual graph of η : in η_* , the edge e_* is open if and only if e is closed. Let us point out that $(\eta^c)_* = (\eta_*)^c$: we thus simply denote this graph by η_*^c .

The random cluster model has the following duality property: if η is distributed according to $\varphi_{p,2}$, then the distribution of η_* is $\varphi_{p^*,2}^*$, where:

$$\frac{p^*}{1-p^*} = 2 \frac{1-p}{p} \Leftrightarrow p^* = \frac{2-2p}{2-p}.$$

Let us define

$$\begin{aligned} \mathcal{C}_0 &= \{\eta \in \{0,1\}^{\mathbb{E}^2} : \text{the origin belongs to an infinite cluster in } \eta^c\} \\ &= \{\eta \in \{0,1\}^{\mathbb{E}^2} : \text{there is no cycle surrounding the origin in } \eta_*^c\}, \\ \text{and } \mathcal{C}_0^* &= \{\eta \in \{0,1\}^{\mathbb{E}^2} : \text{there is no cycle surrounding the origin in } \eta^c\} \\ &= \{\eta \in \{0,1\}^{\mathbb{E}^2} : \text{the origin belongs to an infinite cluster in } \eta_*^c\}. \end{aligned}$$

Note that the map $\eta \mapsto (\eta_*^c)$ is increasing. Since $p \mapsto \varphi_{p,2}$ is stochastically increasing, the map $p \mapsto \varphi_{p,2}^*(\mathcal{C}_0^*)$ is increasing and there exists a critical $p_c^* \in [0,1]$ such that $\varphi_{p,2}^*(\mathcal{C}_0^*) = 0$ for $p < p_c^*$ and $\varphi_{p,2}^*(\mathcal{C}_0^*) > 0$ for $p > p_c^*$.

Lemma 4.1. *For $p < 1/2$, $\varphi_{p,2}(\mathcal{C}_0) > 0$.*

Proof. It follows from the fact that $\varphi_{p,2}$ is dominated by a product of Bernoulli measures with parameter p , and that for $p < 1/2$, the event \mathcal{C}_0 has a positive probability under product of Bernoulli measures with parameter p . \square

Lemma 4.2. *For $p > 2/3$, $\varphi_{p,2}^*(\mathcal{C}_0^*) > 0$. In other words, $p_c^* \leq 2/3$.*

Proof. We have:

$$\begin{aligned} \varphi_{p,2}^*(\mathcal{C}_0^*) &= \varphi_{p,2}^*(\{\eta \in \{0,1\}^{\mathbb{E}^2} : \text{the origin belongs to an infinite cluster in } \eta_*^c\}) \\ &= \varphi_{p^*,2}(\{\eta_* \in \{0,1\}^{\mathbb{E}^2} : \text{the origin belongs to an infinite cluster in } \eta_*^c\}) \\ &= \varphi_{p^*,2}(\mathcal{C}_0). \end{aligned}$$

The result then follows from the previous lemma and the observation that for $p < 1/2 \Leftrightarrow p^* > 2/3$. \square

Lemma 4.3. *For $p \in (\frac{1}{2-p_c^*}, 1)$, $\mu_p(\mathcal{C}) = 1$.*

Proof. Let $p \in (\frac{1}{2-p_c^*}, 1)$, and let $\beta = g(p) = -\frac{1}{2} \log \frac{1-p}{p} > 0$, so that the measure μ_p corresponds to the contours of any antiferromagnetic Ising measure $\gamma_{-\beta}$. We have (comments on the equalities are given just after):

$$\begin{aligned} \mu_p(\mathcal{C}) &\geq \mu_p(\{\eta \in \{0,1\}^{\mathbb{E}^2} : \text{the origin } 0_{\mathbb{Z}^2} \text{ belongs to the infinite cluster in } \eta\}) \\ &= \mu_p(\{\eta \in \{0,1\}^{\mathbb{E}^2} : \text{there is no cycle surrounding } 0_{\mathbb{Z}^2} \text{ in } \eta_*\}) \\ (4) \quad &= \gamma_{-\beta}(\{\omega \in \{\pm 1\}^{\mathbb{Z}^2} : \text{no cycle of identical spins surrounding } 0_{\mathbb{Z}^2} \text{ in } \omega\}) \\ (5) \quad &= \gamma_{\beta}^+(\{\omega \in \{\pm 1\}^{\mathbb{Z}^2} : \text{no cycle of alternating spins surrounding } 0_{\mathbb{Z}^2} \text{ in } \omega\}) \\ (6) \quad &\geq \varphi_{f(\beta),2}^d(\mathcal{C}_0^*). \end{aligned}$$

Comments: for (4), remember that

$$e_* = \{x_*, y_*\} \text{ is open} \Leftrightarrow e \text{ is closed} \Leftrightarrow \omega_{x_*} \omega_{y_*} = 1.$$

For (5), we use the following facts about Gibbs measures for the Ising model:

- Any Gibbs measure γ_{β} for the parameter β is a convex combination of the two extremal Gibbs measures for the parameter β .

- The two extremal Gibbs measures for the parameter β are the images one of the other by the spin inversion, and this transformation leaves the contours unchanged.
- For $\beta > 0$, the Gibbs measures for parameter $-\beta$ are the images of Gibbs measures for parameter β by the inversion of colors on the subset of even sites.

For (6), we use the coupling between the Ising model and the random cluster model. The event “there is a cycle of alternating spins surrounding $0_{\mathbb{Z}^2}$ ” implies, in the random cluster model, the event “there is a cycle of closed edges surrounding $0_{\mathbb{Z}^2}$ ”; this is not an equivalence as there can exist closed edges between identical spins.

Now, by Lemma 4.2, if $f(\beta) > p_c^*$, then $\varphi_{f(\beta),2}^*(\mathcal{C}_0^*) > 0$ and the condition $f(\beta) > p_c^*$ is fulfilled as soon as

$$\beta > f^{-1}(p_c^*) \iff p = g^{-1}(\beta) > g^{-1}(f^{-1}(p_c^*)) = \frac{1}{2 - p_c^*}.$$

Thus, for $p > \frac{1}{2 - p_c^*}$, $\mu_p(\mathcal{C}) > 0$, and by the 0–1 law, $\mu_p(\mathcal{C}) = 1$. \square

The inequality $p_c^* \leq 2/3$ then imply that $\mu_p(\mathcal{C}) = 1$ for $p > \frac{1}{2 - 2/3} = \frac{3}{4}$. In fact, using the technique by Duminil-Copin and Beffara, it is possible to prove that $p_c^* = \frac{\sqrt{2}}{1 + \sqrt{2}}$, which gives $\mu_p(\mathcal{C}) = 1$ for $p > 1 - \frac{1}{2 + \sqrt{2}} \sim 0,7071$.

4.2.2. *Percolation for $p \in [1/2, p_1[$ for some $p_1 > 1/2$.*

Lemma 4.4. *Let us denote by p_c^{site} the critical parameter for Bernoulli site percolation on \mathbb{Z}_*^2 (or \mathbb{Z}^2) and define*

$$p_1 = \frac{1}{1 + ((p_c^{\text{site}})^{-1} - 1)^{1/4}}.$$

Then, $\mu_p(\mathcal{C}) = 1$ for $p \in [1/2, p_1[$.

Proof. Fix $\beta \leq 0$ and consider an Ising model measure γ_β at inverse temperature β and fix $i \in \mathbb{Z}_*^2$. It is easy to see that

$$\begin{aligned} \gamma_\beta(\omega_i = 1 | \sigma(\omega_j, j \neq i)) &= \frac{\exp(-\beta S)}{\exp(-\beta S) + \exp(\beta S)} = \frac{1}{1 + \exp(2\beta S)} \\ &= \frac{1}{1 + \left(\frac{1-p}{p}\right)^S}, \end{aligned}$$

with $S = \sum_{j: \|j-i\|=1} \omega_j$. Since $-4 \leq S \leq 4$ and $p \geq 1/2$, we have

$$\frac{(1-p)^4}{p^4 + (1-p)^4} = \frac{1}{1 + \left(\frac{1-p}{p}\right)^{-4}} \leq \gamma_\beta(\omega_i = 1 | \sigma(\omega_j, j \neq i)) \leq \frac{1}{1 + \left(\frac{1-p}{p}\right)^4} = \frac{p^4}{p^4 + (1-p)^4}.$$

Denote $e(p) = \frac{p^4 - (1-p)^4}{p^4 + (1-p)^4}$, so that $\frac{(1-p)^4}{p^4 + (1-p)^4} = \frac{1 - e(p)}{2}$ and $\frac{p^4}{p^4 + (1-p)^4} = \frac{1 + e(p)}{2}$. Then, the Russo lemma (see [16]) gives the stochastic ordering

$$\left(\frac{1 + e(p)}{2} \delta_{-1} + \frac{1 - e(p)}{2} \delta_1\right)^{\otimes \mathbb{Z}_*^2} \preceq \gamma_\beta \preceq \left(\frac{1 - e(p)}{2} \delta_{-1} + \frac{1 + e(p)}{2} \delta_1\right)^{\otimes \mathbb{Z}_*^2}.$$

Suppose now that $\frac{1 + e(p)}{2} < p_c(\text{site})$.

Then, the +1 and the -1 are both in the subcritical phase for site percolation, so the probability for having a cluster of +1 (resp. -1) with size n decreases

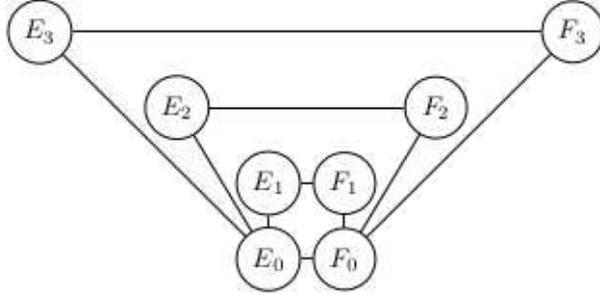


FIGURE 4. A simple finite Eulerian graph

exponentially fast with n . Now, with a Peierls argument, the probability that the box $\{-n, n\}^2$ is surrounded by some monochrome cluster exponentially decreases with n . Particularly, it is strictly less than one if n is large enough, which proves that there is a positive probability to have an infinite path in the model of even percolation. \square

Of course, this bound is useful because $p_c^{\text{site}} > 1/2$. Using the estimate by van den Berg and Ermakov [17]: $p_c^{\text{site}} \geq 0,556$, we can take $p_1 = 0,514$.

5. ASSOCIATION AND MONOTONICITY VERSUS THE EULERIAN CONDITION

The study of Bernoulli bond percolation on a graph $G = (V, E)$ intensively uses the following properties of the product measure $\text{Ber}(p)^{\otimes E}$:

- monotonicity: for every increasing event A , the map $p \mapsto \text{Ber}(p)^{\otimes E}(A)$ is non-decreasing.
- association: for every pair of increasing event A, B ,

$$\text{Ber}(p)^{\otimes E}(A \cap B) \geq \text{Ber}(p)^{\otimes E}(A) \text{Ber}(p)^{\otimes E}(B),$$

or, equivalently, for every pair of non-decreasing bounded functions F, G , we have $\text{Cov}_{\text{Ber}(p)^{\otimes E}}(F, G) \geq 0$.

It is natural to ask if these properties could be preserved for the measure

$$\mu_p(G) = \text{Ber}(p)^{\otimes E}(\cdot | \text{the subgraph of open vertices is Eulerian}).$$

In the following, we investigate the case of a particular finite Eulerian graph G . In this case, we show that the monotonicity property is preserved whereas the association property is lost.

Consider the undirected graph given by Figure 4. Note that every edge has even degree. We simply denote by μ_p the random even subgraph measure with parameter p for the graph given by Figure 4.

For each $i \in \{0, \dots, 3\}$, note $X_i = \mathbb{1}_{\{(E_i, F_i) \text{ is open}\}}$. For each $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{0, 1\}^3$, we have

$$(7) \quad \mu_p(X_1 = \varepsilon_1, X_2 = \varepsilon_2, X_3 = \varepsilon_3) = Z_p^{-1} \left(\frac{p}{1-p} \right)^{N(\varepsilon_1, \varepsilon_2, \varepsilon_3)},$$

with $N(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 3(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + \mathbb{1}_{\{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \text{ odd}\}}$. If we note $q = p/(1-p)$, we have

$$Z_p = 1 + 3q^4 + 3q^6 + q^{10}.$$

For $i \in \{1, 2, 3\}$, note $C_i = \{X_0 = X_i = 1\}$. We have

$$\mu_p(C_i) = \mu_p(C_1) = \mu_p((\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 0, 0)) + \mu_p((\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, 1)),$$

so $\mu_p(C_i) = \frac{q^4 + q^{10}}{Z_p}$. The events $\{X_0 = X_1 = X_2 = X_3 = 1\}$, $\{X_0 = X_1 = X_2 = 1\}$, $\{X_0 = X_2 = X_3 = 1\}$, $\{X_0 = X_1 = X_3 = 1\}$ coincide μ_p -almost-surely, so $C_1 \cap C_2$ and C_3 are positively correlated. But

$$\begin{aligned} \mu_p(C_1 \cap C_2) - \mu_p(C_1)\mu_p(C_2) &= \frac{q^{10}}{Z_p} - \left(\frac{q^4 + q^{10}}{Z_p}\right)^2 \\ &= \frac{q^{10}Z_p - (q^4 + q^{10})^2}{Z_p^2} = \frac{-q^8 + q^{10} + q^{14} + 3q^{16}}{Z_p^2} < 0 \end{aligned}$$

for $q < 0,74$, so C_1 and C_2 are negatively correlated for each $p < 0,42$.

However, the sequence $(\mu_p)_{p \in [0,1]}$ is non-decreasing for the stochastic order.

Theorem 5.1. *Let $G = (V, E)$ be the graph illustrated by Figure 4. Let $A \in \mathcal{P}(\{0, 1\}^E)$, where A is an increasing event. Then, $p \mapsto \mu_p(A)$ is non-decreasing. Equivalently, if F is a monotonic boolean function on $\{0, 1\}^{|E|}$, $p \mapsto \int F d\mu_p$ is non-decreasing.*

Proof. As previously, we can reduce to the case when the measure is described by (X_0, X_1, X_2, X_3) : μ_p -almost surely $X_1 = e_{E_0, E_1} = e_{E_1, F_1} = e_{E_1, F_0}$ and $X_2 = e_{E_0, E_2} = e_{E_2, F_2} = e_{E_2, F_0}$ and $X_3 = e_{E_0, E_3} = e_{E_3, F_3} = e_{E_3, F_0}$. So, if F is a non-decreasing function on $\{0, 1\}^E$, we have μ_p a.s. :

$$\begin{aligned} &F(e_{E_0, F_0}, e_{E_0, E_1}, e_{E_1, F_1}, e_{E_1, F_0}, e_{E_0, E_2}, e_{E_2, F_2}, e_{E_2, F_0}, e_{E_0, E_3}, e_{E_3, F_3}, e_{E_3, F_0}) \\ &= F_1(X_0, X_1, X_2, X_3), \text{ with } F_1(x, y, z, t) = F(x, y, y, y, z, z, z, t, t, t). \end{aligned}$$

By construction, F_1 is a non-decreasing function, so it is sufficient to prove that for any non-decreasing function $F: \{0, 1\}^4 \rightarrow \{0, 1\}$, the map $p \mapsto \int F(X_0, X_1, X_2, X_3) d\mu_p$ is non-decreasing. The law of (X_0, X_1, X_2, X_3) under μ_p is easy to express: we have

$$\begin{aligned} &\mu_p(X_0 = \varepsilon_0, X_1 = \varepsilon_1, X_2 = \varepsilon_2, X_3 = \varepsilon_3) \\ &= \mathbb{1}_{\{\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \text{ even}\}} \mu_p(X_1 = \varepsilon_1, X_2 = \varepsilon_2, X_3 = \varepsilon_3). \end{aligned}$$

With (7), it is easy to see that $\int G(X_0, X_1, X_2, X_3) d\mu_p$ can be expressed as a rational function of $q = \frac{p}{1-p}$: $\int F(X_0, X_1, X_2, X_3) d\mu_p = \frac{P_F(p/(1-p))}{Z(p/(1-p))}$, so it is sufficient to check that the polynomial $R_F = P'_F Z - P_F Z'$ has no positive root, which can be easily performed with a modern computer. In fact, it happens that for each of the 168 monotonic boolean function F ,

$$R_F \in \left\{ \begin{array}{ll} 0, 10q^9 + 18q^5 + 12q^3, & 12q^{13} + 22q^9 + 18q^5 + 4q^3, \\ 12q^{15} + 12q^{13} + 4q^9 + 4q^3, & 12q^{15} + 18q^{13} + 10q^9, \\ 12q^{15} + 6q^{13} - 2q^9 + 8q^3, & 12q^{15} - 8q^9 + 12q^3, \\ 18q^{13} + 28q^9 + 18q^5, & 4q^{15} + 12q^{13} + 16q^9 + 12q^5 + 4q^3, \\ 4q^{15} + 18q^{13} + 22q^9 + 12q^5, & 4q^{15} + 4q^9 + 12q^5 + 12q^3, \\ 4q^{15} + 6q^{13} + 10q^9 + 12q^5 + 8q^3, & 6q^{13} + 16q^9 + 18q^5 + 8q^3, \\ 8q^{15} + 12q^{13} + 10q^9 + 6q^5 + 4q^3, & 8q^{15} + 18q^{13} + 16q^9 + 6q^5, \\ 8q^{15} - 2q^9 + 6q^5 + 12q^3, & 8q^{15} + 6q^{13} + 4q^9 + 6q^5 + 8q^3 \end{array} \right\}$$

In most cases, the coefficients of R_G are non-negative; in any case, it is easy to prove that R_G has no positive root.

We obtain the list of the 168 functions by a brute-force algorithm based on the following remark: if M_n denotes the set of monotonic boolean functions on $\{0, 1\}^n$, there is a natural one-to-one correspondance between M_{n+1} and $\{(f, g) \in M_n^2; f \leq g\}$: a function G of $n + 1$ variables (x_1, \dots, x_{n+1}) is associated to the pair of functions $((x_1, \dots, x_n) \mapsto F(x_1, \dots, x_n, 0), (x_1, \dots, x_n) \mapsto F(x_1, \dots, x_n, 1))$. The number $|M_n|$ of monotonic boolean functions is known as the Dedekind number. The sequence $(|M_n|)_{n \geq 1}$ increases very fast and is not easy to compute. In fact, the exact values are only known for $n \leq 8$ (see Wiedemann [19]). \square

We conjecture that this result should be more general:

Conjecture 5.2. *Let $G = (V, E)$ be an Eulerian graph. Then, the sequence of Eulerian percolation measures $(\mu_p)_{p \in [0,1]}$ on $\{0, 1\}^E$ is stochastically non-decreasing.*

Note that Cammarota and Russo [4] proved related results supporting this conjecture.

APPENDIX: CODE OF THE JULIA PROGRAM

```
using SymPy

function valeur(a,b,c,d)
    q=Sym("q")
    if (((a+b+c+d)%2)==0)
        n=3*(b+c+d)+a
        if (n>0)
            return poly(q^n)
        else return 1 end
    else return 0
end

end

function zp()
    q=Sym("q")
    z=poly(q)*0
    for a=0:1
        for b=0:1
            for c=0:1
                for d=0:1
                    z+=valeur(a,b,c,d)
                end
            end
        end
    end
    return z
end

function evalue(numero,t)
    taille=length(t)
    if (taille==0) return(Int32(numero))
```

```

else
    haute=Int32(numero) & (2^(2^(taille-1))-1)
    basse=div((Int32(numero)-haute),2^(2^(taille-1)))
    if (t[taille]==1)
        return evaluate(haute,t[(1:taille-1)'])
    else
        return evaluate(basse,t[(1:taille-1)'])
    end
end
end

function integrale_num(numero)
    p=Sym("q")
    z=poly(p)*0
    for a=0:1
        for b=0:1
            for c=0:1
                for d=0:1
                    z+=evaluate(numero,[a b c d])*valeur(a,b,c,d)
                end
            end
        end
    end
    return z
end

function variation(numero)
    Z=zp()
    dZ=diff(Z)
    N=integrale_num(numero)
    dN=diff(N)
    nder=dN*Z-dZ*N
    return nder
end

function compte(numero)
    d=variation(numero)
    p=Sym("q")
    return count_roots(d,0)-1*(subs(d,p,0)==0)
end

t=[Int64(0);Int64(1)]
nbit=Int32(1)
for i=1:4
    s=[]
    for a=t
        for b=t
            if ((a & b)==a)

```

```

    s=[s; a*(2^nbit)+b]
    end
end
end
nbit=nbit*2

print("Pour n=",i,", il y en a ",length(s),"Voilà la liste:")
println(s)
t=s
end

bad=0
for i=t
    j=compte(i)
    if (j>0)
        bad+=1
    end
    print(" ",integrale_num(i)," ",compte(i),"\\n")
end

end
print("Z(q)=",zp(),"\\n")
print("sur les ", length(t), " fonctions croissantes,",bad, " ne marchent pas.")

```

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