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CONVERGENCE OF THE MAC SCHEME FOR THE COMPRESSIBLE STATIONARY NAVIER-STOKES EQUATIONS

T. GALLOUËT, R. HERBIN, J.-C. LATCHÉ, AND D. MALTESE

ABSTRACT. We prove in this paper the convergence of the Marker and Cell (MAC) scheme for the discretization of the steady state compressible and isentropic Navier-Stokes equations on two or three-dimensional Cartesian grids. Existence of a solution to the scheme is proven, followed by estimates on approximate solutions, which yield the convergence of the approximate solutions, up to a subsequence, and in an appropriate sense. We then prove that the limit of the approximate solutions satisfies the mass and momentum balance equations, as well as the equation of state, which is the main difficulty of this study.

1. INTRODUCTION

The aim of this paper is to prove the convergence of the marker-and-cell (MAC) scheme for the discretization of the stationary and isentropic compressible Navier-Stokes system. These equations are posed on a bounded domain Ω of \mathbb{R}^d adapted to the MAC scheme (see section 3), $d = 2, 3$, and read:

$$\begin{aligned} (1.1a) \quad & \operatorname{div}(\varrho \mathbf{u}) = 0 \text{ in } \Omega, \\ (1.1b) \quad & \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega, \\ (1.1c) \quad & p = \varrho^\gamma \text{ in } \Omega, \quad \varrho \geq 0 \text{ in } \Omega, \quad \int_{\Omega} \varrho \, d\mathbf{x} = M, \end{aligned}$$

supplemented by the boundary condition

$$(1.2) \quad \mathbf{u}|_{\partial\Omega} = 0.$$

In the above equations, the unknown functions are the scalar density and pressure fields, denoted by $\varrho(\mathbf{x}) \geq 0$ and $p(\mathbf{x})$ respectively, and the vector velocity field $\mathbf{u} = (u_1, \dots, u_d)(\mathbf{x})$, where $\mathbf{x} \in \Omega$ denotes the space variable. The viscosity coefficients μ and λ are such that (see [10])

$$(1.3) \quad \mu > 0, \quad \lambda + \frac{2}{d}\mu \geq 0.$$

The function $\mathbf{f} \in L^2(\Omega)^d$ represents the resultant of the exterior forces acting on the fluid while the constant $M > 0$ stands for the total mass of the fluid. In the compressible barotropic Navier-Stokes equations, the pressure is a given function of the density. Here we assume that the fluid is a perfect gas obeying the Boyle's law:

$$(1.4) \quad p = a\varrho^\gamma \text{ in } \Omega,$$

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where $a > 0$ and where $\gamma > 1$ is termed the *adiabatic constant*. Typical values of γ range from a maximum $5/3$ for *monoatomic gases*, through $7/5$ for *diatomic gases* including air, to lower values close to 1 for *polyatomic gases* at high temperature. For the sake of simplicity, the constant a will be taken equal to 1. Unfortunately, for purely technical reasons, we will be forced to require that $\gamma > 3$ if $d = 3$ to prove the convergence of the MAC scheme. There is no restriction if $d = 2$ in the sense that we can choose $\gamma > 1$.

Remark 1 (Forcing term involving the density). *Instead of taking a given function \mathbf{f} in (1.1b), it is possible, in order to take the gravity effects into account, to take $\mathbf{f} = \rho\mathbf{g}$ with $\mathbf{g} \in L^\infty(\Omega)^d$.*

The mathematical analysis of numerical schemes for the discretization of the steady and/or time-dependent compressible Navier-Stokes and/or compressible Stokes equations has been the object of some recent works. The convergence of the discrete solutions to the weak solutions of the compressible stationary Stokes problem was shown for a finite volume–non conforming P1 finite element [9, 12, 14] and for the wellknown MAC scheme (see [8]) which was introduced in [20] and is widely used in computational fluid dynamics. The unsteady Stokes problem was also discretized using a FV-FE scheme (Finite Volumes and Finite Elements) on a reformulation of the problem, which were proven to be convergent [26]. The unsteady barotropic Navier-Stokes equations was also recently tackled in [27], with a FV-FE scheme, albeit only in the case $\gamma > 3$ (there is a real difficulty in the realistic case $\gamma \leq 3$ arising from the treatment of the non linear convection term). Some error estimates have been derived for this FV-FE scheme in [17].

Since the very beginning of the introduction of the Marker-and-Cell (MAC) scheme [20], it is claimed that this discretization is suitable for both incompressible and compressible flow problems (see [18, 19] for the seminal papers, [1–3, 23–25, 32–36] for subsequent developments and [37] for a review). The use of the MAC scheme in the incompressible case is now standard, and the convergence in this case has been recently tackled in [16].

The paper is organized as follows. After recalling the fundamental setting of the problem in the continuous case in Section 2, we present a simple way (which adapts to the discrete setting) to prove a known preliminary result, namely the convergence (up to a subsequence) of the weak solution of Problem (1.1)–(1.4) with \mathbf{f}_n and M_n (instead of \mathbf{f} and M) towards a weak solution of Problem (1.1)–(1.4) (with $M_n \rightarrow M$ and $\mathbf{f}_n \rightarrow \mathbf{f}$ weakly in $L^2(\Omega)^d$ as $n \rightarrow +\infty$). Then we proceed in Section 3 to the discretization: we introduce the discrete functional spaces and the definition of the numerical scheme, and state an existence result for this numerical scheme, the proof of which is given in Appendix A. The main result of this paper, that is the convergence theorem, is stated in Theorem 4. The remaining sections are devoted to the proof of Theorem 4. In Section 6, we derive estimates satisfied by the solutions of the scheme. In Section 7, we prove the convergence of the numerical scheme in the sense of Theorem 4 toward a weak solution of Problem (1.1)–(1.4).

2. THE CONTINUOUS PROBLEM

2.1. Definition of weak solution. In the sequel we explain what we mean by weak solution of Problem (1.1)–(1.4). Briefly, if $d = 2$ and $\gamma > 1$, it is possible to obtain a weak solution (\mathbf{u}, p, ϱ) of (1.1)–(1.4) in the space $(H_0^1(\Omega))^2 \times L^2(\Omega) \times L^{2\gamma}(\Omega)$ and to prove the convergence of a sequence of approximate solutions (up to a subsequence) towards a weak solution in the sense of Definition 1. If $d = 3$, the problem is much more difficult. For any $\gamma > 3/2$, a weak solution (\mathbf{u}, p, ϱ) may be defined (with the extra hypothesis that \mathbf{f} satisfies $\text{curl}\mathbf{f} = \mathbf{0}$ in the case $\gamma \in (\frac{3}{2}, \frac{5}{3}]$). However, this weak solution belongs to a functional space which depends on γ . Indeed, the function \mathbf{u} always

belongs to $H_0^1(\Omega)^3$, but the function p belongs to $L^2(\Omega)$ only if $\gamma \geq 3$ (and the function ϱ belongs to $L^2(\Omega)$ only if $\gamma \geq 5/3$). More precisely, for $d = 3$ and $\gamma < 3$, we only get an estimate on p in $L^\delta(\Omega)$, and an estimate on ϱ in $L^{\gamma\delta}(\Omega)$, with $\delta = \frac{3(\gamma-1)}{\gamma}$. Note that for $\gamma = \frac{3}{2}$, one has $\delta = \frac{3(\gamma-1)}{\gamma} = 1$, and $\gamma\delta = 3(\gamma-1) = \frac{3}{2}$, so that the natural spaces are $p \in L^1(\Omega)$ and $\varrho \in L^{\frac{3}{2}}(\Omega)$. Note that in the case of the compressible Stokes equations, an L^2 estimate on the pressure and an $L^{2\gamma}$ estimate on the density are obtained for $d = 2$ or 3 and there is no restriction on γ in the sense that we can take $\gamma > 1$ (see for instance [9] and [8]).

To be in accordance with the main theorem of this article (see Theorem 4), we then define the notion of weak solution only for the case $\gamma > 3$ if $d = 3$ and $\gamma > 1$ if $d = 2$. We refer the reader to [29] and [30] for further informations about the notion of weak solutions and their existence. We recall that a domain of \mathbb{R}^d is a connected open subset of \mathbb{R}^d . A Lipschitz domain is a domain with a Lipschitz boundary.

Definition 1. *Let $d = 2$ or 3 , Ω be a bounded Lipschitz domain of \mathbb{R}^d and let $\mathbf{f} \in L^2(\Omega)^d$, $M > 0$. Let $\gamma > 3$ if $d = 3$ or $\gamma > 1$ if $d = 2$. A weak solution of Problem (1.1)–(1.4) is a function $(\mathbf{u}, p, \varrho) \in (H_0^1(\Omega))^d \times L^2(\Omega) \times L^{2\gamma}(\Omega)$ satisfying:*

(1) *The continuity equation is satisfied in the following weak sense,*

$$(2.1) \quad \int_{\Omega} \varrho \mathbf{u} \cdot \nabla \varphi \, d\mathbf{x} = 0, \quad \forall \varphi \in W^{1,\infty}(\Omega).$$

(2) *The momentum equation (1.1b) is satisfied in the weak sense,*

$$(2.2) \quad - \int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x} + \mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x} + (\mu + \lambda) \int_{\Omega} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, d\mathbf{x} \\ - \int_{\Omega} p \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, \quad \forall \mathbf{v} \in C_c^\infty(\Omega)^d.$$

(3) *$\varrho \geq 0$ a.e in Ω , $\int_{\Omega} \varrho \, d\mathbf{x} = M$ and $p = \varrho^\gamma$ a.e in Ω .*

Remark 2. (1) *Any weak solution (\mathbf{u}, p, ϱ) in the sense of Definition 1 satisfies the following inequality*

$$(2.3) \quad \int_{\Omega} \left(\mu |\nabla \mathbf{u}|^2 + (\mu + \lambda) |\operatorname{div} \mathbf{u}|^2 \right) \, d\mathbf{x} \leq \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, d\mathbf{x}.$$

(2) *By a density argument, using $\gamma \geq 3$, one can take $\mathbf{v} \in H_0^1(\Omega)^d$ in (2.2).*

2.2. Passage to the limit with approximate data. In order to understand our strategy in the discrete case, we first prove here the following result (which states the continuity, up to a subsequence, of the weak solution of (1.1)–(1.4) with respect to the data). In the following, we set

$$q(d) = \begin{cases} +\infty & \text{if } d = 2, \\ 6 & \text{if } d = 3. \end{cases}$$

Theorem 1. *Let Ω be a bounded Lipschitz domain of \mathbb{R}^d , $d = 2$ or 3 . Let $\gamma > 1$ if $d = 2$ and $\gamma > 3$ if $d = 3$. Let $\mathbf{f} \in L^2(\Omega)^d$, $M > 0$ and $(\mathbf{f}_n)_{n \in \mathbb{N}} \subset L^2(\Omega)^d$, $(M_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+^*$ be some sequences satisfying $\mathbf{f}_n \rightharpoonup \mathbf{f}$ weakly in $(L^2(\Omega))^d$ and $M_n \rightarrow M$. For $n \in \mathbb{N}$, let $(\mathbf{u}_n, p_n, \varrho_n)$ be a weak solution*

of (1.1)-(1.4), in the sense of Definition 1, with \mathbf{f}_n and M_n (instead of \mathbf{f} and M), i.e. a solution to:

$$(2.4a) \quad \int_{\Omega} \varrho_n \mathbf{u}_n \cdot \nabla \varphi \, d\mathbf{x} = 0 \text{ for all } \varphi \in W^{1,\infty}(\Omega),$$

$$(2.4b) \quad - \int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \mu \nabla \mathbf{u}_n : \nabla \mathbf{v} \, d\mathbf{x} + (\mu + \lambda) \operatorname{div} \mathbf{u}_n \operatorname{div} \mathbf{v} \, d\mathbf{x} \\ - \int_{\Omega} p_n \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f}_n \cdot \mathbf{v} \, d\mathbf{x}, \quad \forall \mathbf{v} \in C_c^\infty(\Omega)^d.$$

$$(2.4c) \quad \varrho_n \geq 0 \text{ a.e. in } \Omega, \quad \int_{\Omega} \varrho_n \, d\mathbf{x} = M_n, \quad p_n = \varrho_n^\gamma \text{ a.e. in } \Omega.$$

Then, there exists $(\mathbf{u}, p, \varrho) \in (H_0^1(\Omega))^d \times L^2(\Omega) \times L^{2\gamma}(\Omega)$ such that, up to a subsequence, as $n \rightarrow +\infty$,

- $\mathbf{u}_n \rightarrow \mathbf{u}$ in $(L^q(\Omega))^d$ for $1 \leq q < q(d)$ and weakly in $H_0^1(\Omega)^d$,
- $p_n \rightarrow p$ in $L^q(\Omega)$ for $1 \leq q < 2$ and weakly in $L^2(\Omega)$,
- $\varrho_n \rightarrow \varrho$ in $L^q(\Omega)$ for $1 \leq q < 2\gamma$ and weakly in $L^{2\gamma}(\Omega)$,

and (\mathbf{u}, p, ϱ) is a weak solution of (1.1)-(1.4).

Proof. For the sake of simplicity, we will perform the proof for $\gamma > 3$ and $d = 3$. The case $d = 2$ and $\gamma > 1$ is simpler, and the modifications to be done to adapt the proof to the two-dimensional case are mostly due to the fact that Sobolev embeddings differ.

The proof consists in four steps. In Step 1, we obtain some estimates on $(\mathbf{u}_n, p_n, \varrho_n)$. These estimates imply the convergence, in an appropriate sense, of $(\mathbf{u}_n, p_n, \varrho_n)$ to some (\mathbf{u}, p, ϱ) , up to a subsequence. Then, it is quite easy to prove that (\mathbf{u}, p, ϱ) satisfies the weak form of (1.1)-(1.2) (this is Step 2) but it is not easy to prove that $p = \varrho^\gamma$ since, using the estimates of Step 1, the convergence of p_n and ϱ_n is only weak (and $\gamma \neq 1$). In Step 3, we prove the convergence of the integral of $p_n \varrho_n$ to the integral of $p \varrho$. This allows in Step 4 to obtain the ‘‘strong’’ convergence of ϱ_n (or p_n) and to conclude the proof.

We recall Lemma 2.1 of [9], which is crucial for Steps 1 and 3 of the proof. This lemma states that if $\varrho \in L^{2\gamma}(\Omega)$, $\gamma > 1$, $\varrho \geq 0$ a.e. in Ω , $\mathbf{u} \in (H_0^1(\Omega))^3$ and (ϱ, \mathbf{u}) satisfies (2.1), then we have:

$$(2.5) \quad \int_{\Omega} \varrho \operatorname{div} \mathbf{u} \, d\mathbf{x} = 0$$

and

$$(2.6) \quad \int_{\Omega} \varrho^\gamma \operatorname{div} \mathbf{u} \, d\mathbf{x} = 0.$$

Indeed, this result is also true for $\gamma = 1$ (see Lemma B1 of [12]). In Step 1 below, we use (2.6) (in fact, we only need $\int_{\Omega} \varrho^\gamma \operatorname{div} \mathbf{u} \, d\mathbf{x} \leq 0$ and it is this weaker result which will be adapted and used for the approximate solution obtained by a numerical scheme). In Step 3, we use (2.5).

Step 1. Estimates.

1.a *Estimate on the velocity.* Let $(\mathbf{u}_n, p_n, \varrho_n)$ be a solution to (2.4). Taking \mathbf{u}_n as a test function in the weak formulation of (2.4b), we get:

$$\begin{aligned} \mu \int_{\Omega} \nabla \mathbf{u}_n : \nabla \mathbf{u}_n \, d\mathbf{x} + (\mu + \lambda) \int_{\Omega} (\operatorname{div} \mathbf{u}_n)^2 \, d\mathbf{x} - \int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{u}_n \, d\mathbf{x} \\ - \int_{\Omega} p_n \operatorname{div} \mathbf{u}_n \, d\mathbf{x} = \int_{\Omega} \mathbf{f}_n \cdot \mathbf{u}_n \, d\mathbf{x}. \end{aligned}$$

Note that, since $\gamma > 3$, we have $\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n \in L^2(\Omega)^{3 \times 3}$, and, by density of $C_c^\infty(\Omega)^d$ in $L^2(\Omega)^d$, \mathbf{u}_n is indeed an admissible test function. But $p_n = \varrho_n^\gamma$ a.e. in Ω and $\operatorname{div}(\varrho_n \mathbf{u}_n) = 0$ (in the sense of (2.4a)), then using (2.6) (with ϱ_n and \mathbf{u}_n)

$$\int_{\Omega} p_n \operatorname{div} \mathbf{u}_n \, d\mathbf{x} = 0.$$

Thanks to the mass equation and $\varrho_n \in L^{2\gamma}(\Omega) \subset L^6(\Omega)$ a straightforward computation gives

$$\int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{u}_n \, d\mathbf{x} = 0.$$

Hence, there exists C_1 , only depending on the L^2 -bound of $(\mathbf{f}_n)_{n \in \mathbb{N}}$, on Ω and on μ , such that:

$$(2.7) \quad \|\mathbf{u}_n\|_{(H_0^1(\Omega))^3} \leq C_1.$$

1.b *Estimate on the pressure.* In order to obtain an estimate on p_n in $L^2(\Omega)$, we now use the two following lemmas. The first one is due to Bogovski, see e.g. [30, Section 3.3] or [11, Theorem 10.1] for a proof.

Lemma 1. *Let Ω be a bounded Lipschitz domain of \mathbb{R}^d ($d \geq 1$). Let $r \in (1, +\infty)$. Let $q \in L^r(\Omega)$ such that $\int_{\Omega} q \, d\mathbf{x} = 0$. Then, there exists $\mathbf{v} \in (W_0^{1,r}(\Omega))^d$ such that $\operatorname{div}(\mathbf{v}) = q$ a.e. in Ω and $\|\mathbf{v}\|_{(W_0^{1,r}(\Omega))^d} \leq C_2 \|q\|_{L^r(\Omega)}$ with C_2 only depending on Ω .*

The following lemma is a straightforward consequence of [13, Lemma 5.4].

Lemma 2. *Let Ω be a bounded Lipschitz domain of \mathbb{R}^d ($d \geq 1$) and $p \in L^2(\Omega)$ such that $p \geq 0$ a.e. in Ω . We assume that there exist $a > 0$, $b \in \mathbb{R}$ and $r \in (0, 1)$ such that*

$$\begin{cases} \|p - m(p)\|_{L^2(\Omega)} \leq a \|p\|_{L^2(\Omega)}^r + b, \\ \int_{\Omega} p^r \, d\mathbf{x} \leq c, \end{cases}$$

where $m(p) = \frac{1}{|\Omega|} \int_{\Omega} p \, d\mathbf{x}$ is the mean value of p . Then, there exists C only depending on Ω, a, b, c and r such that

$$\|p\|_{L^2(\Omega)} \leq C.$$

Let $m_n = \frac{1}{|\Omega|} \int_{\Omega} p_n \, d\mathbf{x}$; thanks to Lemma 1 with $r = 2$, there exists $\mathbf{v}_n \in H_0^1(\Omega)^3$ such that $\operatorname{div}(\mathbf{v}_n) = p_n - m_n$ and

$$(2.8) \quad \|\mathbf{v}_n\|_{(H_0^1(\Omega))^3} \leq C_2 \|p_n - m_n\|_{L^2(\Omega)}.$$

Taking \mathbf{v}_n as a test function in (2.4b) yields:

$$(2.9) \quad \mu \int_{\Omega} \nabla \mathbf{u}_n : \nabla \mathbf{v}_n \, d\mathbf{x} + (\mu + \lambda) \int_{\Omega} \operatorname{div} \mathbf{u}_n \operatorname{div} \mathbf{v}_n \, d\mathbf{x} - \int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{v}_n \, d\mathbf{x} \\ - \int_{\Omega} p_n \operatorname{div} \mathbf{v}_n \, d\mathbf{x} = \int_{\Omega} \mathbf{f}_n \cdot \mathbf{v}_n \, d\mathbf{x}.$$

Since $\int_{\Omega} \operatorname{div} \mathbf{u}_n \, d\mathbf{x} = \int_{\Omega} \operatorname{div} \mathbf{v}_n \, d\mathbf{x} = 0$, we get:

$$\int_{\Omega} (p_n - m_n)^2 \, d\mathbf{x} = \int_{\Omega} (-\mathbf{f}_n \cdot \mathbf{v}_n + \mu \nabla \mathbf{u}_n : \nabla \mathbf{v}_n + (\mu + \lambda) p_n \operatorname{div} \mathbf{u}_n - \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{v}_n) \, d\mathbf{x}.$$

Since $\|\mathbf{u}_n\|_{(H_0^1(\Omega))^3} \leq C_1$ and $H_0^1(\Omega)$ is continuously embedded in $L^6(\Omega)$, we get that:

$$(2.10) \quad \int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{v}_n \, d\mathbf{x} \leq \|\varrho_n\|_{L^6(\Omega)} \|\mathbf{u}_n\|_{L^6(\Omega)}^2 \|\mathbf{v}_n\|_{(H^1(\Omega))^3}.$$

From (2.9), (2.10) and (2.8), since $2\gamma \geq 6$ and $p_n = \varrho_n^\gamma$, we get:

$$\|p_n - m_n\|_{L^2(\Omega)} \leq C_3 (1 + \|\varrho_n\|_{L^6(\Omega)}) \leq C_4 (1 + \|\varrho_n\|_{L^{2\gamma}(\Omega)}) \leq C_4 (1 + \|p_n\|_{L^2(\Omega)}^{1/\gamma}).$$

Since $\int_{\Omega} p_n^{1/\gamma} \, d\mathbf{x} = \int_{\Omega} \varrho_n \, d\mathbf{x} \leq \sup\{M_k, k \in \mathbb{N}\}$, we get from Lemma 2 that

$$\|p_n\|_{L^2(\Omega)} \leq C_5,$$

where C_4 depends only on the L^2 -bound of $(\mathbf{f}_n)_{n \in \mathbb{N}}$, the bound of $(M_n)_{n \in \mathbb{N}}$, γ , μ , λ and Ω .

Thanks to the equation of state, we have $p_n = \varrho_n^\gamma$ a.e. in Ω , and therefore:

$$\|\varrho_n\|_{L^{2\gamma}(\Omega)} \leq C_6 = C_5^{1/\gamma}.$$

Step 2. Passing to the limit on the equations (1.1a) and (1.1b). The estimates obtained in Step 1 yield that, up to a subsequence, as $n \rightarrow +\infty$:

$$\begin{aligned} \mathbf{u}_n &\rightarrow u \text{ in } L^q(\Omega)^3 \text{ for any } 1 \leq q < 6 \text{ and weakly in } H_0^1(\Omega)^3, \\ p_n &\rightarrow p \text{ weakly in } L^2(\Omega), \\ \varrho_n &\rightarrow \varrho \text{ weakly in } L^{2\gamma}(\Omega). \end{aligned}$$

The weak momentum equation reads:

$$\begin{aligned} \mu \int_{\Omega} \nabla \mathbf{u}_n : \nabla \mathbf{v} \, d\mathbf{x} + (\mu + \lambda) \int_{\Omega} \operatorname{div} \mathbf{u}_n \operatorname{div} \mathbf{v} \, d\mathbf{x} - \int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{v} \, d\mathbf{x} \\ - \int_{\Omega} p_n \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f}_n \cdot \mathbf{v} \, d\mathbf{x}, \forall \mathbf{v} \in C_c^\infty(\Omega)^3. \end{aligned}$$

Since $\varrho_n \rightarrow \varrho$ weakly in $L^{2\gamma}(\Omega)$, with $2\gamma > \frac{3}{2}$, and $\mathbf{u}_n \rightarrow \mathbf{u}$ in $L^q(\Omega)$ for all $q < 6$ (and $\frac{2}{3} + \frac{1}{6} + \frac{1}{6} = 1$), we have that $\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n \rightarrow \varrho \mathbf{u} \otimes \mathbf{u}$ weakly in $L^1(\Omega)$. Moreover, $\nabla \mathbf{u}_n \rightarrow \nabla \mathbf{u}$ weakly in $L^2(\Omega)^3$,

$p_n \rightarrow p$ weakly in $L^2(\Omega)$ and $\mathbf{f}_n \rightarrow \mathbf{f}$ weakly in $L^2(\Omega)^3$. Therefore, passing to the limit in the preceding equation, we get:

$$\begin{aligned} \mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x} + (\mu + \lambda) \int_{\Omega} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, d\mathbf{x} - \int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x} - \int_{\Omega} p \operatorname{div} \mathbf{v} \, d\mathbf{x} \\ = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, \forall \mathbf{v} \in C_c^\infty(\Omega)^3. \end{aligned}$$

which shows that (\mathbf{u}, p, ϱ) satisfies (2.2).

The approximate mass balance equation reads:

$$\int_{\Omega} \varrho_n \mathbf{u}_n \cdot \nabla \varphi \, d\mathbf{x} = 0, \forall \varphi \in W^{1,\infty}(\Omega).$$

Since $\varrho_n \rightarrow \varrho$ weakly in $L^{2\gamma}(\Omega)$, with $2\gamma > \frac{6}{5}$ and $\mathbf{u}_n \rightarrow \mathbf{u}$ in $L^q(\Omega)$ for all $q < 6$, we get that $\varrho_n \mathbf{u}_n \rightarrow \varrho \mathbf{u}$ weakly in $L^1(\Omega)$. This gives

$$\int_{\Omega} \varrho \mathbf{u} \cdot \nabla \varphi = 0, \forall \varphi \in W^{1,\infty}(\Omega).$$

The weak convergence of ϱ_n to ϱ and the fact that $\varrho_n \geq 0$ a.e. in Ω gives that $\varrho \geq 0$ a.e. in Ω (indeed, taking $\psi = 1_{\varrho < 0}$ as test function gives $\int_{\Omega} \varrho \psi \, d\mathbf{x} = \lim_{n \rightarrow +\infty} \int_{\Omega} \varrho_n \psi \, d\mathbf{x} \geq 0$, which proves that $\varrho \psi = 0$ a.e.).

The weak convergence of ϱ_n to ϱ gives also (taking $\psi = 1$ as test function) that $\int_{\Omega} \varrho \, d\mathbf{x} = M$.

Therefore, (\mathbf{u}, p, ϱ) is solution of the momentum equation and of the mass balance equation (we have also $\varrho \geq 0$ a.e in Ω and $\int_{\Omega} \varrho \, d\mathbf{x} = M$). Hence Theorem 1 is proved except for the fact that $p = \varrho^\gamma$ a.e. in Ω . This is the objective of the last two steps, where we also prove a ‘‘strong’’ convergence of ϱ_n and p_n . We need to prove that $p = \varrho^\gamma$ in Ω , even though we only have a weak convergence of p_n and ϱ_n , and $\gamma > 1$. The idea (for $d = 2$ or $d = 3$, $\gamma > 3$) is to prove $\int_{\Omega} p_n \varrho_n \rightarrow \int_{\Omega} p \varrho$ and deduce the a.e. convergence (of p_n and ϱ_n) and $p = \varrho^\gamma$.

Step 3. Proving $\int_{\Omega} \varrho_n p_n \, d\mathbf{x} \rightarrow \int_{\Omega} \varrho p \, d\mathbf{x}$.

Since the sequence $(\varrho_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\Omega)$, The result of [9, Lemma B.8] gives the existence of a bounded sequence $(\mathbf{v}_n)_{n \in \mathbb{N}}$ in $H^1(\Omega)^3$ such that $\operatorname{div}(\mathbf{v}_n) = \varrho_n$ and $\operatorname{curl}(\mathbf{v}_n) = 0$. It is possible to assume (up to a subsequence) that $\mathbf{v}_n \rightarrow \mathbf{v}$ in $L^2(\Omega)^3$ and weakly in $H^1(\Omega)^3$. Passing to the limit in the preceding equations gives $\operatorname{div}(\mathbf{v}) = \varrho$ and $\operatorname{curl}(\mathbf{v}) = 0$.

Let $\varphi \in C_c^\infty(\Omega)$ (so that $\varphi \mathbf{v}_n \in H_0^1(\Omega)^3$). Taking $\mathbf{v} = \varphi \mathbf{v}_n$ in (2.4b) leads to:

$$\begin{aligned} \mu \int_{\Omega} \nabla \mathbf{u}_n : \nabla(\varphi \mathbf{v}_n) \, d\mathbf{x} + (\mu + \lambda) \int_{\Omega} \operatorname{div} \mathbf{u}_n \operatorname{div}(\varphi \mathbf{v}_n) \, d\mathbf{x} - \int_{\Omega} p_n \operatorname{div}(\varphi \mathbf{v}_n) \, d\mathbf{x} \\ = \int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla(\varphi \mathbf{v}_n) \, d\mathbf{x} + \int_{\Omega} \mathbf{f}_n \cdot (\varphi \mathbf{v}_n) \, d\mathbf{x}. \end{aligned}$$

The choice of \mathbf{v}_n gives $\operatorname{div}(\varphi \mathbf{v}_n) = \varphi \varrho_n + \mathbf{v}_n \cdot \nabla \varphi$ and $\operatorname{curl}(\varphi \mathbf{v}_n) = L(\varphi) \mathbf{v}_n$, where $L(\varphi)$ is a matrix with entries involving the first order derivatives of φ . Then, the preceding equality leads to:

$$\begin{aligned} & \int_{\Omega} \left((2\mu + \lambda) \operatorname{div} \mathbf{u}_n - p_n \right) \varrho_n \varphi \, d\mathbf{x} + \int_{\Omega} \left((2\mu + \lambda) \operatorname{div} \mathbf{u}_n - p_n \right) \mathbf{v}_n \cdot \nabla \varphi \, d\mathbf{x} \\ & \quad + \mu \int_{\Omega} \operatorname{curl}(\mathbf{u}_n) \cdot L(\varphi) \mathbf{v}_n \, d\mathbf{x} = \int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla(\varphi \mathbf{v}_n) \, d\mathbf{x} + \int_{\Omega} \mathbf{f}_n \cdot (\varphi \mathbf{v}_n) \, d\mathbf{x}. \end{aligned}$$

where we have used the following formula, $\forall (\mathbf{u}, \mathbf{v}) \in H_0^1(\Omega)^3$:

$$(2.11) \quad \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} \, d\mathbf{x}.$$

Thanks to the weak convergence of \mathbf{u}_n in $H_0^1(\Omega)^d$ to \mathbf{u} , the weak convergence of p_n in $L^2(\Omega)$ to p , the weak convergence of \mathbf{f}_n in $L^2(\Omega)$ to \mathbf{f} and the convergence of \mathbf{v}_n in $L^2(\Omega)^d$ to \mathbf{v} , we obtain:

$$(2.12) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} \left(\left((2\mu + \lambda) \operatorname{div}(\mathbf{u}_n) - p_n \right) \varrho_n \varphi - \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla(\varphi \mathbf{v}_n) \right) \, d\mathbf{x} = \\ \int_{\Omega} \mathbf{f} \cdot (\varphi \mathbf{v}) \, d\mathbf{x} + \int_{\Omega} \left(p - (2\mu + \lambda) \operatorname{div} \mathbf{u} \right) \mathbf{v} \cdot \nabla \varphi \, d\mathbf{x} - \mu \int_{\Omega} \operatorname{curl}(\mathbf{u}) \cdot L(\varphi) \mathbf{v} \, d\mathbf{x}.$$

But, thanks to (2.4b), we have

$$\begin{aligned} \mu \int_{\Omega} \nabla \mathbf{u} : \nabla(\varphi \mathbf{v}) \, d\mathbf{x} + (\mu + \lambda) \int_{\Omega} \operatorname{div} \mathbf{u} \operatorname{div}(\varphi \mathbf{v}) \, d\mathbf{x} - \int_{\Omega} p \operatorname{div}(\varphi \mathbf{v}) \, d\mathbf{x} \\ = \int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla(\varphi \mathbf{v}) \, d\mathbf{x} + \int_{\Omega} \mathbf{f} \cdot (\varphi \mathbf{v}) \, d\mathbf{x}, \end{aligned}$$

or equivalently:

$$\begin{aligned} \int_{\Omega} \left((2\mu + \lambda) \operatorname{div} \mathbf{u} - p \right) \operatorname{div}(\varphi \mathbf{v}) \, d\mathbf{x} + \mu \int_{\Omega} \operatorname{curl}(\mathbf{u}) \cdot \operatorname{curl}(\varphi \mathbf{v}) \, d\mathbf{x} \\ = \int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla(\varphi \mathbf{v}) \, d\mathbf{x} + \int_{\Omega} \mathbf{f} \cdot (\varphi \mathbf{v}) \, d\mathbf{x}. \end{aligned}$$

where we have used (2.11). Using $\operatorname{div}(\mathbf{v}) = \varrho$ and $\operatorname{curl}(\mathbf{v}) = 0$, we obtain:

$$\begin{aligned} \int_{\Omega} \left((2\mu + \lambda) \operatorname{div} \mathbf{u} - p \right) \varrho \varphi \, d\mathbf{x} - \int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla(\varphi \mathbf{v}) \, d\mathbf{x} = \\ \int_{\Omega} \mathbf{f} \cdot (\varphi \mathbf{v}) \, d\mathbf{x} + \int_{\Omega} \left(p - (2\mu + \lambda) \operatorname{div} \mathbf{u} \right) \mathbf{v} \cdot \nabla \varphi \, d\mathbf{x} - \mu \int_{\Omega} \operatorname{curl}(\mathbf{u}) \cdot L(\varphi) \mathbf{v} \, d\mathbf{x}. \end{aligned}$$

Let us assume momentarily that:

$$(2.13) \quad \int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla(\varphi \mathbf{v}_n) \, d\mathbf{x} \rightarrow \int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla(\varphi \mathbf{v}) \, d\mathbf{x} \text{ as } n \rightarrow +\infty.$$

We then obtain thanks to (2.12):

$$(2.14) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} \left(p_n - (2\mu + \lambda) \operatorname{div} \mathbf{u}_n \right) \varrho_n \varphi \, d\mathbf{x} = \int_{\Omega} \left(p - (2\mu + \lambda) \operatorname{div} \mathbf{u} \right) \varrho \varphi \, d\mathbf{x}.$$

In (2.14), the function φ is an arbitrary element of $C_c^\infty(\Omega)$. Then as in [9], we remark that it is possible to take $\varphi = 1$ in (2.14), thanks to the fact that $(p_n - (2\mu + \lambda) \operatorname{div} \mathbf{u}_n) \varrho_n \in L^r(\Omega)$ for some $r > 1$ (see [9, Lemma B.2]).

Using (2.5), which holds by [9, Lemma 2.1] thanks to the fact that $\operatorname{div}(\varrho_n \mathbf{u}_n) = \operatorname{div}(\varrho \mathbf{u}) = 0$ (in the sense of (2.4a) and (2.1)), we have:

$$\int_{\Omega} \varrho_n \operatorname{div}(\mathbf{u}_n) \, d\mathbf{x} = \int_{\Omega} \varrho \operatorname{div}(\mathbf{u}) \, d\mathbf{x} = 0.$$

Therefore, (2.14) yields:

$$(2.15) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} p_n \varrho_n \, d\mathbf{x} = \int_{\Omega} p \varrho \, d\mathbf{x}.$$

Remark 3. For Step 4, the equality in (2.15) is not necessary. It is sufficient to have:

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} p_n \varrho_n \, d\mathbf{x} \leq \int_{\Omega} p \varrho \, d\mathbf{x}.$$

Then, instead of $\int_{\Omega} \varrho_n \operatorname{div}(\mathbf{u}_n) \, d\mathbf{x} = 0$, it is sufficient to have

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \varrho_n \operatorname{div}(\mathbf{u}_n) \, d\mathbf{x} \leq 0.$$

This will be the case in the framework of an approximation by a numerical scheme.

In order to conclude Step 3, it remains to show (2.13).

We remark that, since $\operatorname{div}(\varrho_n \mathbf{u}_n) = 0$ and $(\varrho_n, \mathbf{u}_n) \in L^6(\Omega) \times H_0^1(\Omega)^3$,

$$\int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla(\varphi \mathbf{v}_n) \, d\mathbf{x} = - \int_{\Omega} (\varrho_n \mathbf{u}_n \cdot \nabla) \mathbf{u}_n \cdot (\varphi \mathbf{v}_n) \, d\mathbf{x}.$$

The sequence $((\varrho_n \mathbf{u}_n \cdot \nabla) \mathbf{u}_n)_{n \in \mathbb{N}}$ is bounded in $L^r(\Omega)^3$, with $\frac{1}{r} = \frac{1}{2} + \frac{1}{6} + \frac{1}{2\gamma}$. Since $\gamma > 3$, we have $r > \frac{6}{5}$.

Then, up to a subsequence $(\varrho_n \mathbf{u}_n \cdot \nabla) \mathbf{u}_n \rightarrow G$ weakly in $L^r(\Omega)^3$. Since $\mathbf{v}_n \rightarrow \mathbf{v}$ in $L^s(\Omega)^3$ for all $s < 6$ and therefore for $s = \frac{r}{r-1}$, we deduce that:

$$\int_{\Omega} (\varrho_n \mathbf{u}_n \cdot \nabla) \mathbf{u}_n \cdot (\varphi \mathbf{v}_n) \, d\mathbf{x} \rightarrow \int_{\Omega} G \cdot (\varphi \mathbf{v}) \, d\mathbf{x}.$$

Moreover, for a fixed $\mathbf{w} \in H_0^1(\Omega)^3$,

$$\int_{\Omega} (\varrho_n \mathbf{u}_n \cdot \nabla) \mathbf{u}_n \cdot \mathbf{w} \, d\mathbf{x} = - \int_{\Omega} \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{w} \, d\mathbf{x} \rightarrow - \int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{w} \, d\mathbf{x}.$$

But, since $\operatorname{div}(\varrho \mathbf{u}) = 0$ and $(\varrho, \mathbf{u}) \in L^6(\Omega) \times H_0^1(\Omega)^3$, we have

$$- \int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{w} \, d\mathbf{x} = \int_{\Omega} (\varrho \mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{w} \, d\mathbf{x}.$$

We thus get that $G = (\varrho \mathbf{u} \cdot \nabla) \mathbf{u}$, which concludes the proof of (2.13).

Step 4. Passing to the limit on the EOS and “strong” convergence of ϱ_n and p_n .

The end of the proof is exactly the same as Step 4 of [9, Proof of Theorem 2.2]; it is reproduced here for the sake of completeness. For $n \in \mathbb{N}$, let $G_n = (\rho_n^\gamma - \rho^\gamma)(\rho_n - \rho)$. For all $n \in \mathbb{N}$, the function G_n belongs to $L^1(\Omega)$ and $G_n \geq 0$ a.e. in Ω . Furthermore $G_n = (p_n - \rho^\gamma)(\rho_n - \rho) = p_n \rho_n - p_n \rho - \rho^\gamma \rho_n + \rho^\gamma \rho$ and:

$$\int_{\Omega} G_n \, d\mathbf{x} = \int_{\Omega} p_n \rho_n \, d\mathbf{x} - \int_{\Omega} p_n \rho \, d\mathbf{x} - \int_{\Omega} \rho^\gamma \rho_n \, d\mathbf{x} + \int_{\Omega} \rho^\gamma \rho \, d\mathbf{x}.$$

Using the weak convergence in $L^2(\Omega)$ of p_n to p and of ρ_n to ρ , the fact that $\rho, \rho^\gamma \in L^2(\Omega)$ and (2.15) gives:

$$\lim_{n \rightarrow +\infty} \int_{\Omega} G_n \, d\mathbf{x} = 0,$$

that is $G_n \rightarrow 0$ in $L^1(\Omega)$. Then, up to a subsequence, we have $G_n \rightarrow 0$ a.e. in Ω . Since $y \mapsto y^\gamma$ is an increasing function on \mathbb{R}_+ , we deduce that $\rho_n \rightarrow \rho$ a.e., as $n \rightarrow +\infty$. Then, we also have $p_n = \rho_n^\gamma \rightarrow \rho^\gamma$ a.e.. Since $(\rho_n)_{n \in \mathbb{N}}$ is bounded in $L^{2\gamma}(\Omega)$ and $(p_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\Omega)$, we obtain, as $n \rightarrow +\infty$:

$$\begin{aligned} \rho_n &\rightarrow \rho \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 2\gamma, \\ p_n &\rightarrow \rho^\gamma \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 2. \end{aligned}$$

Since we already know that $p_n \rightarrow p$ weakly in $L^2(\Omega)$, we necessarily have (by uniqueness of the weak limit in $L^q(\Omega)$) that $p = \rho^\gamma$ a.e. in Ω . The proof of Theorem 1 is now complete. \square

3. THE NUMERICAL SCHEME

3.1. Mesh and discrete spaces. We will now assume that the bounded domain Ω is such that $\bar{\Omega}$ is a finite union of (closed) rectangles ($d = 2$) or (closed) orthogonal parallelepipeds ($d = 3$) and, without loss of generality, we assume that the edges (or faces) of these rectangles (or parallelepipeds) are orthogonal to the canonical basis vectors, denoted by $(\mathbf{e}_1, \dots, \mathbf{e}_d)$.

Definition 2 (MAC grid). *A discretization of Ω with a MAC grid, denoted by \mathcal{D} , is defined by $\mathcal{D} = (\mathcal{M}, \mathcal{E})$, where:*

- \mathcal{M} stands for the primal grid, and consists in a regular structured partition of Ω in possibly non uniform rectangles ($d = 2$) or rectangular parallelepipeds ($d = 3$). A generic cell of this grid is denoted by K , and its mass center by \mathbf{x}_K . The scalar variables, namely the density and the pressure, are associated to this mesh, and \mathcal{M} is also sometimes referred as "the pressure mesh".
- The set of all faces of the mesh is denoted by \mathcal{E} ; we have $\mathcal{E} = \mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$, where \mathcal{E}_{int} (resp. \mathcal{E}_{ext}) are the edges of \mathcal{E} that lie in the interior (resp. on the boundary) of the domain. The set of faces that are orthogonal to the i^{th} unit vector \mathbf{e}_i of the canonical basis of \mathbb{R}^d is denoted by $\mathcal{E}^{(i)}$, for $i = 1, \dots, d$. We then have $\mathcal{E}^{(i)} = \mathcal{E}_{\text{int}}^{(i)} \cup \mathcal{E}_{\text{ext}}^{(i)}$, where $\mathcal{E}_{\text{int}}^{(i)}$ (resp. $\mathcal{E}_{\text{ext}}^{(i)}$) are the edges of $\mathcal{E}^{(i)}$ that lie in the interior (resp. on the boundary) of the domain.

For each $\sigma \in \mathcal{E}$, we write that $\sigma = K|L$ if $\sigma = \partial K \cap \partial L$. A dual cell D_σ associated to a face $\sigma \in \mathcal{E}$ is defined as follows:

- * if $\sigma = K|L \in \mathcal{E}_{\text{int}}$ then $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$, where $D_{K,\sigma}$ (resp. $D_{L,\sigma}$) is the half-part of K (resp. L) adjacent to σ (see Fig. 1 for the two-dimensional case);
- * if $\sigma \in \mathcal{E}_{\text{ext}}$ is adjacent to the cell K , then $D_\sigma = D_{K,\sigma}$.

We obtain d partitions of the computational domain Ω as follows:

$$\Omega = \cup_{\sigma \in \mathcal{E}^{(i)}} D_\sigma, \quad 1 \leq i \leq d,$$

and the i^{th} of these partitions is called i^{th} dual mesh, and is associated to the i^{th} velocity component, in a sense which is precised below. The set of the faces of the i^{th} dual mesh is denoted by $\tilde{\mathcal{E}}^{(i)}$ and is decomposed into the internal and boundary edges: $\tilde{\mathcal{E}}^{(i)} = \tilde{\mathcal{E}}_{\text{int}}^{(i)} \cup \tilde{\mathcal{E}}_{\text{ext}}^{(i)}$. The dual face separating two dual cells D_σ and $D_{\sigma'}$ is denoted by $\epsilon = \sigma|\sigma'$.

To define the scheme, we need some additional notations. The set of faces of primal cell K and a dual cell D_σ are denoted by $\mathcal{E}(K)$ and $\tilde{\mathcal{E}}(D_\sigma)$ respectively. For $\sigma \in \mathcal{E}$, we denote by \mathbf{x}_σ the mass center of σ .

In some cases, we need to specify the orientation of a geometrical quantity with respect to the axis:

- a primal cell K will be denoted $K = [\overrightarrow{\sigma\sigma'}]$ if there exists $i \in [1, d]$ and $\sigma, \sigma' \in \mathcal{E}^{(i)} \cap \mathcal{E}(K)$ such that $(\mathbf{x}_{\sigma'} - \mathbf{x}_\sigma) \cdot \mathbf{e}_i > 0$;
- we write $\sigma = \overrightarrow{K|L}$ if $\sigma \in \mathcal{E}^{(i)}$ and $\overrightarrow{\mathbf{x}_K \mathbf{x}_L} \cdot \mathbf{e}_i > 0$ for some $i \in [1, d]$;
- the dual face ϵ separating D_σ and $D_{\sigma'}$ is written $\epsilon = \overrightarrow{\sigma|\sigma'}$ if $\overrightarrow{\mathbf{x}_\sigma \mathbf{x}_{\sigma'}} \cdot \mathbf{e}_i > 0$ for some $i \in [1, d]$.

For the definition of the discrete momentum diffusion operator, we associate to any dual face ϵ a distance d_ϵ as sketched on Figure 1. For a dual face $\epsilon \in \tilde{\mathcal{E}}(D_\sigma), \sigma \in \mathcal{E}^{(i)}, i \in [1, d]$, the distance d_ϵ is defined by:

$$(3.1) \quad d_\epsilon = \begin{cases} d(\mathbf{x}_\sigma, \mathbf{x}_{\sigma'}) & \text{if } \epsilon = \overrightarrow{\sigma|\sigma'} \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ d(\mathbf{x}_\sigma, \epsilon) & \text{if } \epsilon \in \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \cap \tilde{\mathcal{E}}(D_\sigma) \end{cases}$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance in \mathbb{R}^d .

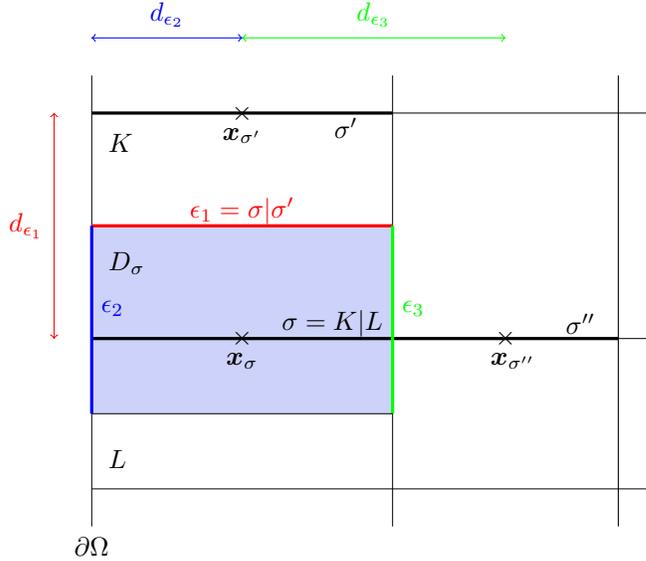


FIGURE 1. Notations for control volumes and dual cells (for the second component of the velocity).

We also define the size of the mesh by

$$h_{\mathcal{M}} = \max\{\text{diam}(K), K \in \mathcal{M}\}.$$

For the convergence we need to introduce the following quantity:

$$(3.2) \quad \eta_{\mathcal{M}} = \frac{\min_{K \in \mathcal{M}} \min_{1 \leq i \leq d} \{d(x_{\sigma}, x_{\sigma'}), \sigma, \sigma' \in \mathcal{E}^{(i)}(K)\}}{h_{\mathcal{M}}}.$$

In other words, $\eta_{\mathcal{M}}$ is such that

$$\eta_{\mathcal{M}} h_{\mathcal{M}} \leq d(x_{\sigma}, x_{\sigma'}) \leq h_{\mathcal{M}}, \quad \forall \sigma, \sigma' \in \mathcal{E}^{(i)}(K), \forall i = 1, \dots, d, \forall K \in \mathcal{M}.$$

The discrete velocity unknowns are associated to the velocity cells and are denoted by $(u_{\sigma})_{\sigma \in \mathcal{E}^{(i)}}$, $i = 1, \dots, d$, for each component u_i of the discrete velocity, while the discrete density and pressure unknowns are associated to the primal cells and are respectively denoted by $(\varrho_K)_{K \in \mathcal{M}}$ and $(p_K)_{K \in \mathcal{M}}$.

Definition 3 (Discrete spaces). *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid in the sense of Definition 2. The discrete density and pressure space $L_{\mathcal{M}}$ is defined as the set of piecewise constant functions over each of the grid cells K of \mathcal{M} , and the discrete i^{th} velocity space $H_{\mathcal{E}}^{(i)}$ as the set of piecewise constant functions over each of the grid cells D_{σ} , $\sigma \in \mathcal{E}^{(i)}$. The Dirichlet boundary conditions (1.2) are partly incorporated in the definition of the velocity spaces. To this purpose, we introduce $H_{\mathcal{E},0}^{(i)} \subset H_{\mathcal{E}}^{(i)}$, $i = 1, \dots, d$, defined as follows:*

$$H_{\mathcal{E},0}^{(i)} = \left\{ u \in H_{\mathcal{E}}^{(i)}, u(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in D_{\sigma}, \sigma \in \mathcal{E}_{\text{ext}}^{(i)} \right\}.$$

We then set $\mathbf{H}_{\mathcal{E},0} = \prod_{i=1}^d H_{\mathcal{E},0}^{(i)}$. Since we are dealing with piecewise constant functions, it is useful to introduce the characteristic functions χ_K , for $K \in \mathcal{M}$, and $\chi_{D_{\sigma}}$, for $\sigma \in \mathcal{E}$ of the density (or pressure) and velocity cells, defined by

$$\chi_K(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in K, \\ 0 & \text{if } \mathbf{x} \notin K, \end{cases} \quad \chi_{D_{\sigma}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in D_{\sigma}, \\ 0 & \text{if } \mathbf{x} \notin D_{\sigma}. \end{cases}$$

We can then write a function $\mathbf{u} \in \mathbf{H}_{\mathcal{E},0}$ and $p, \varrho \in L_{\mathcal{M}}$ as

$$\mathbf{u} = (u_1, \dots, u_d) \quad \text{with } u_i = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} u_{\sigma} \chi_{D_{\sigma}}, \quad \text{for } i \in [1, d],$$

$$p = \sum_{K \in \mathcal{M}} p_K \chi_K, \quad \varrho = \sum_{K \in \mathcal{M}} \varrho_K \chi_K.$$

3.2. The numerical scheme. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid of the computational domain $\Omega \subset \mathbb{R}^d$. Let $h_{\mathcal{M}}$ be the size of the mesh. Let $\alpha > 1$ and $C_s > 0$ be two given real numbers. Let $\mathbf{f} \in L^2(\Omega)^d$ and $M > 0$. Finally we denote $\varrho^* = M/|\Omega|$. We consider the following numerical scheme:

Find $(\mathbf{u}, p, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}} \times L_{\mathcal{M}}$ such that, a.e in Ω ,

$$(3.3a) \quad \text{div}_{\mathcal{M}}^{\text{up}}(\varrho \mathbf{u}) + C_s h_{\mathcal{M}}^{\alpha} (\varrho - \varrho^*) = 0,$$

$$(3.3b) \quad \text{div}_{\mathcal{E}}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathcal{E}} p - \mu \Delta_{\mathcal{E}} \mathbf{u} - (\mu + \lambda) \nabla_{\mathcal{E}} \text{div}_{\mathcal{M}} \mathbf{u} = \mathcal{P}_{\mathcal{E}} \mathbf{f},$$

$$(3.3c) \quad p = \varrho^{\gamma}, \quad \varrho \geq 0,$$

where the terms introduced for each discrete equation are defined hereafter and $\mathcal{P}_{\mathcal{E}}$ is the cell mean-value operator defined for $\mathbf{v} = (v_1, \dots, v_d) \in L^2(\Omega)^d$ by

$$\begin{aligned} \mathcal{P}_{\mathcal{E}}\mathbf{v} &= \left(\mathcal{P}_{\mathcal{E}}^{(1)}v_1, \dots, \mathcal{P}_{\mathcal{E}}^{(d)}v_d \right) \in H_{\mathcal{E},0}^{(1)} \times \dots \times H_{\mathcal{E},0}^{(d)}, \text{ where for } i = 1, \dots, d, \\ (3.4) \quad \mathcal{P}_{\mathcal{E}}^{(i)} : L^2(\Omega) &\longrightarrow H_{\mathcal{E},0}^{(i)} \\ v_i &\longmapsto \mathcal{P}_{\mathcal{E}}^{(i)}v_i = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \left(\frac{1}{|D_{\sigma}|} \int_{D_{\sigma}} v_i(\mathbf{x}) \, d\mathbf{x} \right) \chi_{D_{\sigma}}. \end{aligned}$$

3.2.1. *The mass balance equation.* Equation (3.3a) is a finite volume discretization of the mass balance (1.1a) over the primal mesh. The discrete function $\text{div}_{\mathcal{M}}^{\text{up}}(\varrho\mathbf{u}) \in L_{\mathcal{M}}$ is defined by

$$\text{div}_{\mathcal{M}}^{\text{up}}(\varrho\mathbf{u})(\mathbf{x}) = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}, \quad \forall \mathbf{x} \in K,$$

where $F_{K,\sigma} = F_{K,\sigma}(\varrho, \mathbf{u})$ stands for the mass flux across σ outward K , which reads:

$$(3.5) \quad \forall \sigma \in \mathcal{E}(K), \quad F_{K,\sigma} = |\sigma| \varrho_{\sigma}^{\text{up}} u_{K,\sigma}.$$

The quantity $u_{K,\sigma}$ is an approximation of the normal velocity to the face σ outward K , defined by:

$$(3.6) \quad u_{K,\sigma} = u_{\sigma} \mathbf{e}_i \cdot \mathbf{n}_{K,\sigma} \text{ for } \sigma \in \mathcal{E}^{(i)} \cap \mathcal{E}(K),$$

where $\mathbf{n}_{K,\sigma}$ denotes the unit normal vector to σ outward K . Thanks to the boundary conditions, $u_{K,\sigma}$ vanishes for any external face σ , and so does $F_{K,\sigma}$. The density at the internal face $\sigma = K|L$ is obtained by an upwind technique:

$$(3.7) \quad \varrho_{\sigma}^{\text{up}} = \begin{cases} \varrho_K & \text{if } u_{K,\sigma} \geq 0, \\ \varrho_L & \text{otherwise.} \end{cases}$$

Note that any solution $(\varrho, \mathbf{u}) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ to (3.3a) satisfies $\varrho_K > 0$, $\forall K \in \mathcal{M}$ and in particular (3.3c) makes sense. Moreover the positivity of the density ϱ in (3.3a) is not enforced in the scheme but results from the above upwind choice. Indeed, for any velocity field, the upwinding ensures that the discrete mass balance (3.3a) is a linear system for ϱ whose matrix is invertible and has a non negative inverse [12, Lemma C.3].

Note also that, with this definition, we have the usual finite volume property of local conservativity of the flux through a primal face $\sigma = K|L$ (i.e. $F_{K,\sigma} = -F_{L,\sigma}$). For $\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}$ we also define

$$(3.8) \quad [\varrho]_{\sigma} = \varrho_L - \varrho_K.$$

The artificial term $C_s h_{\mathcal{M}}^{\alpha} (\varrho - \varrho^*)$ guarantees that the integral of the density over the computational domain is always M . Indeed, summing (3.3a) over $K \in \mathcal{M}$, and using the conservativity of the flux through a primal face, immediately yields the total conservation of mass, which reads:

$$(3.9) \quad \int_{\Omega} \varrho \, d\mathbf{x} = M.$$

To conclude this section, the constant C_s is chosen in order to obtain uniform (with respect to the mesh) bounds on the solutions to (3.3); these bounds are stated in Proposition 1. The proof of this proposition shows that C_s has to be chosen sufficiently small with respect to the data (see (6.1)).

Remark 4. *It is possible to take $C_s = 1$. In this case, the uniform bounds stated in Proposition 1 will hold for $h_{\mathcal{M}}$ sufficiently small with respect to the data.*

3.2.2. *The momentum balance equation.* We now turn to the discrete momentum balances (3.3b), which are obtained by discretizing the momentum balance equation (1.1b) on the dual cells associated to the faces of the mesh.

The discrete convective operator - The discrete divergence of $\varrho \mathbf{u} \otimes \mathbf{u}$ is defined by

$$(3.10) \quad \operatorname{div}_{\mathcal{E}}(\varrho \mathbf{u} \otimes \mathbf{u}) = (\operatorname{div}_{\mathcal{E}}^{(1)}(\varrho \mathbf{u} u_1), \dots, \operatorname{div}_{\mathcal{E}}^{(d)}(\varrho \mathbf{u} u_d)) \in \mathbf{H}_{\mathcal{E},0},$$

where the i^{th} component of the above operator reads:

$$\operatorname{div}_{\mathcal{E}}^{(i)}(\varrho \mathbf{u} u_i)(\mathbf{x}) = \frac{1}{|D_{\sigma}|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_{\sigma})} F_{\sigma,\epsilon}(\varrho, \mathbf{u}) u_{\epsilon}, \quad \forall \mathbf{x} \in D_{\sigma}, \quad \sigma \in \mathcal{E}_{\text{int}}^{(i)}.$$

The expression $F_{\sigma,\epsilon} = F_{\sigma,\epsilon}(\varrho, \mathbf{u})$ stands for the mass flux through the dual face ϵ , and u_{ϵ} is an approximation of i^{th} component of the velocity over ϵ .

Let us consider the momentum balance equation for the i^{th} component of the velocity, and $\sigma \in \mathcal{E}_{\text{int}}^{(i)}$, $\sigma = K|L$. We have to distinguish two cases (see Figure 3.2.2):

- First case – The vector \mathbf{e}_i is normal to ϵ , in which case ϵ is included in a primal cell K ; we then denote by σ' the second face of K which is also normal to \mathbf{e}_i . We thus have $\epsilon = D_{\sigma}|D_{\sigma'}$. Then the mass flux through ϵ is given by:

$$(3.11) \quad F_{\sigma,\epsilon}(\varrho, \mathbf{u}) = \frac{1}{2} [F_{K,\sigma}(\varrho, \mathbf{u}) \mathbf{n}_{K,\sigma} + F_{K,\sigma'}(\varrho, \mathbf{u}) \mathbf{n}_{K,\sigma'}] \cdot \mathbf{n}_{D_{\sigma},\epsilon},$$

where $\mathbf{n}_{D_{\sigma},\epsilon}$ stands for the unit normal vector to ϵ outward D_{σ} .

- Second case – The vector \mathbf{e}_i is tangent to ϵ , and ϵ is the union of the halves of two primal faces τ and τ' such that $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. The mass flux through ϵ is then given by:

$$(3.12) \quad F_{\sigma,\epsilon}(\varrho, \mathbf{u}) = \frac{1}{2} [F_{K,\tau}(\varrho, \mathbf{u}) + F_{L,\tau'}(\varrho, \mathbf{u})].$$

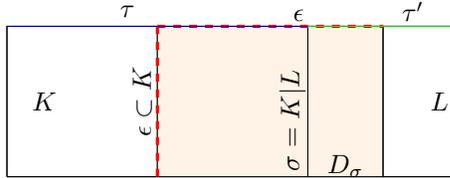


FIGURE 2. Notations for the dual fluxes of the first component of the velocity.

Note that, with this definition, we have the usual finite volume property of local conservativity of the flux through an a dual face $D_{\sigma}|D_{\sigma'}$ (i.e. $F_{\sigma,\epsilon}(\varrho, \mathbf{u}) = -F_{\sigma',\epsilon}(\varrho, \mathbf{u})$), and that the flux through a dual face included in the boundary still vanishes.

The density on a dual cell is given by:

$$(3.13) \quad \begin{aligned} &\text{for } \sigma \in \mathcal{E}_{\text{int}}, \quad \sigma = K|L && |D_{\sigma}| \varrho_{D_{\sigma}} = |D_{K,\sigma}| \varrho_K + |D_{L,\sigma}| \varrho_L, \\ &\text{for } \sigma \in \mathcal{E}_{\text{ext}}, \quad \sigma \in \mathcal{E}(K), && \varrho_{D_{\sigma}} = \varrho_K. \end{aligned}$$

These definitions of the dual mass fluxes and the dual densities ensure that a finite volume discretization of the mass balance equation over the diamond cells holds:

$$(3.14) \quad \text{for } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \quad \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} + C_s h_{\mathcal{M}}^\alpha (\varrho_{D_\sigma} - \varrho^*) = 0.$$

This condition is essential to derive a discrete kinetic energy balance (see Proposition 1).

Since the flux across a dual face lying on the boundary is zero, the values u_ϵ are only needed at the internal dual faces, and we make the centered choice for their discretization, *i.e.*,

$$\text{for } \epsilon = D_\sigma | D_{\sigma'} \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \quad u_\epsilon = \frac{u_\sigma + u_{\sigma'}}{2}.$$

Discrete divergence and gradient - The discrete divergence operator $\text{div}_{\mathcal{M}}$ is defined by:

$$(3.15) \quad \text{div}_{\mathcal{M}} : \begin{cases} \mathbf{H}_{\mathcal{E}} \longrightarrow L_{\mathcal{M}} \\ \mathbf{u} \longmapsto \text{div}_{\mathcal{M}} \mathbf{u} = \sum_{K \in \mathcal{M}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_{K,\sigma} \chi_K, \end{cases}$$

where $u_{K,\sigma}$ is defined in (3.6). Once again, we have the usual finite volume property of local conservativity of the flux through an interface $\sigma = K|L$ between the cells $K, L \in \mathcal{M}$, *i.e.*

$$(3.16) \quad u_{K,\sigma} = -u_{L,\sigma}, \quad \forall \sigma = K|L \in \mathcal{E}_{\text{int}}.$$

The discrete divergence of $\mathbf{u} = (u_1, \dots, u_d) \in \mathbf{H}_{\mathcal{E},0}$ may also be written as

$$(3.17) \quad \text{div}_{\mathcal{M}}(\mathbf{u}) = \sum_{i=1}^d \sum_{K \in \mathcal{M}} (\tilde{\partial}_i u_i)_K \chi_K,$$

where the discrete derivative $(\tilde{\partial}_i u_i)_K$ of u_i on K is defined by

$$(3.18) \quad (\tilde{\partial}_i u_i)_K = \frac{|\sigma|}{|K|} (u_{\sigma'} - u_\sigma) \text{ with } K = \overrightarrow{[\sigma\sigma']}, \sigma, \sigma' \in \mathcal{E}^{(i)}.$$

The gradient in the discrete momentum balance is defined as follows:

$$(3.19) \quad \nabla_{\mathcal{E}} : \begin{cases} L_{\mathcal{M}} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ p \longmapsto \nabla_{\mathcal{E}} p = (\tilde{\partial}_1 p, \dots, \tilde{\partial}_d p)^t, \end{cases}$$

where $\tilde{\partial}_i p \in H_{\mathcal{E},0}^{(i)}$ is the discrete derivative of p in the i^{th} direction, defined by:

$$(3.20) \quad \tilde{\partial}_i p(\mathbf{x}) = \frac{|\sigma|}{|D_\sigma|} (p_L - p_K) \quad \forall \mathbf{x} \in D_\sigma, \text{ for } \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}^{(i)}, i = 1, \dots, d.$$

Note that, in fact, the discrete gradient of a function of $L_{\mathcal{M}}$ should only be defined on the internal faces, and does not need to be defined on the external faces; we set it here in $\mathbf{H}_{\mathcal{E},0}$ (that is zero on the external faces) in order to be coherent with (3.3b). This gradient is built as the dual operator of the discrete divergence, which means:

Lemma 3 (Discrete $\text{div} - \nabla$ duality). *Let Ω be a bounded domain of \mathbb{R}^d , $d = 2$ or $d = 3$, adapted to the MAC-scheme. Let $q \in L_{\mathcal{M}}$ and $\mathbf{v} \in \mathbf{H}_{\mathcal{E},0}$. Then we have:*

$$(3.21) \quad \int_{\Omega} q \text{div}_{\mathcal{M}} \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \nabla_{\mathcal{E}} q \cdot \mathbf{v} \, d\mathbf{x} = 0.$$

Discrete Laplace operator - For $i = 1 \dots, d$, we classically define the discrete Laplace operator on the i^{th} velocity grid by:

$$(3.22) \quad \begin{aligned} -\Delta_{\mathcal{E}}^{(i)} : & \quad \left\{ \begin{array}{l} H_{\mathcal{E},0}^{(i)} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ u_i \longmapsto -\Delta_{\mathcal{E}}^{(i)} u_i \end{array} \right. \\ -\Delta_{\mathcal{E}}^{(i)} u_i(\mathbf{x}) = & \quad \frac{1}{|D_{\sigma}|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_{\sigma})} \phi_{\sigma,\epsilon}(u_i), \quad \forall \mathbf{x} \in D_{\sigma}, \quad \text{for } \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \end{aligned}$$

where

$$(3.23) \quad \phi_{\sigma,\epsilon}(u_i) = \begin{cases} \frac{|\epsilon|}{d_{\epsilon}}(u_{\sigma} - u_{\sigma'}) & \text{if } \epsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ \frac{|\epsilon|}{d_{\epsilon}}u_{\sigma} & \text{if } \epsilon \in \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \cap \tilde{\mathcal{E}}(D_{\sigma}) \end{cases}$$

with d_{ϵ} given by (3.1). The fluxes $\phi_{\sigma,\epsilon}$ satisfy the local conservativity property:

$$(3.24) \quad \phi_{\sigma,\epsilon}(u_i) = -\phi_{\sigma',\epsilon}(u_i), \quad \forall \epsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}.$$

Then the discrete Laplace operator of the full velocity vector is defined by

$$(3.25) \quad \begin{aligned} -\Delta_{\mathcal{E}} : & \quad \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ \mathbf{u} \mapsto & \quad -\Delta_{\mathcal{E}} \mathbf{u} = (-\Delta_{\mathcal{E}}^{(1)} u_1, \dots, -\Delta_{\mathcal{E}}^{(d)} u_d)^t. \end{aligned}$$

Let us now recall the definition of the discrete H_0^1 inner product [6]; it is obtained by taking the inner product of the discrete Laplace operator and a test function $\mathbf{v} \in \mathbf{H}_{\mathcal{E},0}$ and integrating over the computational domain. A simple reordering of the sums (which may be seen as a discrete integration by parts) yields, thanks to the conservativity of the diffusion flux (3.24):

$$(3.26) \quad \begin{aligned} \forall (\mathbf{u}, \mathbf{v}) \in \mathbf{H}_{\mathcal{E},0}^2, \quad & \int_{\Omega} -\Delta_{\mathcal{E}} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} = [\mathbf{u}, \mathbf{v}]_{1,\mathcal{E},0} = \sum_{i=1}^d [u_i, v_i]_{1,\mathcal{E}^{(i)},0}, \\ \text{with } [u_i, v_i]_{1,\mathcal{E}^{(i)},0} = & \quad \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{int}}^{(i)} \\ \epsilon = \sigma|\sigma'}} \frac{|\epsilon|}{d_{\epsilon}} (u_{\sigma} - u_{\sigma'}) (v_{\sigma} - v_{\sigma'}) + \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \\ \epsilon \in \tilde{\mathcal{E}}(D_{\sigma})}} \frac{|\epsilon|}{d_{\epsilon}} u_{\sigma} v_{\sigma}. \end{aligned}$$

The bilinear forms $\left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \times H_{\mathcal{E},0}^{(i)} \rightarrow \mathbb{R} \\ (u, v) \mapsto [u_i, v_i]_{1,\mathcal{E}^{(i)},0} \end{array} \right.$ and $\left| \begin{array}{l} \mathbf{H}_{\mathcal{E},0} \times \mathbf{H}_{\mathcal{E},0} \rightarrow \mathbb{R} \\ (\mathbf{u}, \mathbf{v}) \mapsto [\mathbf{u}, \mathbf{v}]_{1,\mathcal{E},0} \end{array} \right.$ are inner products on $H_{\mathcal{E},0}^{(i)}$, for $i = 1, \dots, d$, and on $\mathbf{H}_{\mathcal{E},0}$ respectively, which induce the following discrete H_0^1 norms:

$$(3.27a) \quad \|u_i\|_{1,\mathcal{E}^{(i)},0}^2 = [u_i, u_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{int}}^{(i)} \\ \epsilon = \sigma|\sigma'}} \frac{|\epsilon|}{d_{\epsilon}} (u_{\sigma} - u_{\sigma'})^2 + \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \\ \epsilon \in \tilde{\mathcal{E}}(D_{\sigma})}} \frac{|\epsilon|}{d_{\epsilon}} u_{\sigma}^2$$

$$(3.27b) \quad \|\mathbf{u}\|_{1,\mathcal{E},0}^2 = [\mathbf{u}, \mathbf{u}]_{1,\mathcal{E},0} = \sum_{i=1}^d \|u_i\|_{1,\mathcal{E}^{(i)},0}^2.$$

Since we are working on Cartesian grids, this inner product may be formulated as the L^2 inner

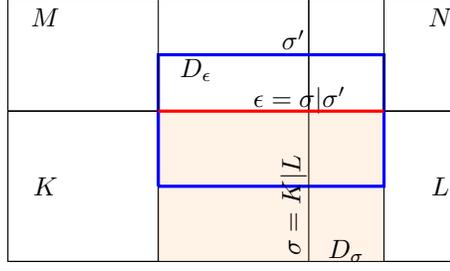


FIGURE 3. Full grid for definition of the derivative of the velocity.

product of discrete gradients. Indeed, consider the following discrete gradient of each velocity component u_i .

$$(3.28) \quad \nabla_{\mathcal{E}^{(i)}} u_i = (\tilde{\partial}_1 u_i, \dots, \tilde{\partial}_d u_i) \text{ with } \tilde{\partial}_j u_i = \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{int}}^{(i)} \\ \epsilon \perp e_j}} (\tilde{\partial}_j u_i)_{D_\epsilon} \chi_{D_\epsilon} + \sum_{\epsilon \in \tilde{\mathcal{E}}_{\text{ext}}^{(i)}} (\tilde{\partial}_j u_i)_{D_\epsilon} \chi_{D_\epsilon},$$

where $(\tilde{\partial}_j u_i)_{D_\epsilon} = \frac{u_{\sigma'} - u_\sigma}{d_\epsilon}$ with $\epsilon = \overrightarrow{\sigma|\sigma'}$, and $D_\epsilon = \epsilon \times \mathbf{x}_\sigma \mathbf{x}_{\sigma'}$ (see Figure 3, note also that $u_\sigma = 0$ if $\sigma \in \mathcal{E}_{\text{ext}}^{(i)}$). This definition is compatible with the definition of the discrete derivative $(\tilde{\partial}_i u_i)_K$ given by (3.18), since, if $\epsilon \subset K$, then $D_\epsilon = K$. If $\epsilon \in \tilde{\mathcal{E}}_{\text{ext}}^{(i)} \cap \tilde{\mathcal{E}}(D_\sigma)$, we set $(\tilde{\partial}_j u_i)_{D_\epsilon} = \frac{-u_\sigma}{d_\epsilon} \mathbf{n}_{D_\sigma, \epsilon} \cdot e_j$ with $D_\epsilon = \epsilon \times \mathbf{x}_\sigma \mathbf{x}_{\sigma, b}$, where $\mathbf{x}_{\sigma, b} = \sigma \cap \partial\Omega$.

With this definition, it is easily seen that

$$(3.29) \quad \int_{\Omega} \nabla_{\mathcal{E}^{(i)}} u \cdot \nabla_{\mathcal{E}^{(i)}} v \, d\mathbf{x} = [u, v]_{1, \mathcal{E}^{(i)}, 0}, \quad \forall u, v \in H_{\mathcal{E}, 0}^{(i)}, \text{ for } i = 1, \dots, d.$$

where $[u, v]_{1, \mathcal{E}^{(i)}, 0}$ is the discrete H_0^1 inner product defined by (3.26). We may then define

$$\nabla_{\mathcal{E}} \mathbf{u} = (\nabla_{\mathcal{E}^{(1)}} u_1, \dots, \nabla_{\mathcal{E}^{(d)}} u_d),$$

so that

$$\int_{\Omega} \nabla_{\mathcal{E}} \mathbf{u} : \nabla_{\mathcal{E}} \mathbf{v} \, d\mathbf{x} = [\mathbf{u}, \mathbf{v}]_{1, \mathcal{E}, 0}.$$

We are now in position to introduce an equivalent formulation of the discrete momentum balance (3.3b), which reads:

$$(3.30) \quad \int_{\Omega} \operatorname{div}_{\mathcal{E}} (\varrho \mathbf{u} \otimes \mathbf{u}) \cdot \mathbf{v} \, d\mathbf{x} + \mu \int_{\Omega} \nabla_{\mathcal{E}} \mathbf{u} : \nabla_{\mathcal{E}} \mathbf{v} \, d\mathbf{x} + (\mu + \lambda) \int_{\Omega} \operatorname{div}_{\mathcal{M}} \mathbf{u} \operatorname{div}_{\mathcal{M}} \mathbf{v} \, d\mathbf{x} \\ - \int_{\Omega} p \operatorname{div}_{\mathcal{M}} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathcal{P}_{\mathcal{E}} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{H}_{\mathcal{E}, 0}$$

and which is equivalent to (3.3b).

4. SOME ANALYSIS RESULTS FOR DISCRETE FUNCTIONS

In the theory developed in this paper, we will need discrete Sobolev inequalities for the discrete approximations. The following result is proved in [6, Lemma 9.5].

Theorem 2 (Discrete Sobolev inequalities). *Let Ω be a bounded domain of \mathbb{R}^d , $d = 2$ or $d = 3$, adapted to the MAC-scheme ($\bar{\Omega}$ is a finite union of rectangles in 2D or rectangular parallelepipeds in 3D), and let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid of Ω . Let $q < +\infty$ if $d = 2$ and $q = 6$ if $d = 3$. Then there exists $C = C(q, \Omega, \eta_{\mathcal{M}})$, nonincreasing with respect to $\eta_{\mathcal{M}}$, such that, for all $\mathbf{u} \in \mathbf{H}_{\mathcal{E},0}$,*

$$\|\mathbf{u}\|_{L^q(\Omega)} \leq C \|\mathbf{u}\|_{1,\mathcal{E},0}.$$

The following compactness theorem is a consequence of [6, Theorem 9.1 and Lemma 9.5] and [7, Lemma 5.7].

Theorem 3. *Let Ω be a bounded domain of \mathbb{R}^d , $d = 2$ or $d = 3$, adapted to the MAC-scheme. Consider a sequence of MAC grids $(\mathcal{M}_n, \mathcal{E}_n)_{n \in \mathbb{N}}$, with step size $h_{\mathcal{M}_n}$ tending to zero as $n \rightarrow +\infty$. Let $(\mathbf{u}_n)_{n \in \mathbb{N}}$ be a sequence of discrete functions such that each element of the sequence belongs to $\mathbf{H}_{\mathcal{E}_n,0}$ and such that the sequence $(\|\mathbf{u}_n\|_{1,\mathcal{E}_n,0})_{n \in \mathbb{N}}$ is bounded. Then, up to the extraction of a subsequence, the sequence $(\mathbf{u}_n)_{n \in \mathbb{N}}$ converges in $L^2(\Omega)^d$ to a limit \mathbf{u} and this limit satisfies $\mathbf{u} \in (\mathbf{H}_0^1(\Omega))^d$. Furthermore, one has $\nabla_{\mathcal{E}_n} \mathbf{u}_n \rightharpoonup \nabla \mathbf{u}$ weakly in $L^2(\Omega)^{d \times d}$ as $n \rightarrow +\infty$. If $\eta_{\mathcal{M}} \geq \eta > 0$, one has also $\mathbf{u}_n \rightarrow \mathbf{u}$ in $L^q(\Omega)$ for all $q < q(d)$.*

We now recall a discrete analogue of the identity (2.11) linking the gradient, divergence and curl operators, which is proved in [8]. First of all, we modify the definition of the discrete gradient ($\nabla_{\mathcal{E}}$) of an element of $L_{\mathcal{M}}$ in some dual cells near the boundary, in order to take into account a null boundary condition at the external faces. It reads:

$$(4.1) \quad \bar{\nabla}_{\mathcal{E}} : \left\{ \begin{array}{l} L_{\mathcal{M}} \longrightarrow \mathbf{H}_{\mathcal{E}} \\ w \longmapsto \bar{\nabla}_{\mathcal{E}} w = (\bar{\partial}_1 w, \dots, \bar{\partial}_d w)^t, \end{array} \right.$$

where $\bar{\partial}_i w \in H_{\mathcal{E}}^{(i)}$ is the discrete derivative of w in the i^{th} direction, defined, for $i = 1, \dots, d$, by:

$$(4.2) \quad \bar{\partial}_i w(\mathbf{x}) = \begin{cases} \bar{\partial}_i w(\mathbf{x}) = \frac{|\sigma|}{|D_{\sigma}|} (w_L - w_K), & \forall \mathbf{x} \in D_{\sigma}, \text{ for } \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}^{(i)}, \\ -\frac{|\sigma|}{|D_{\sigma}|} w_K \mathbf{n}_{\sigma,K} \cdot \mathbf{e}_i, & \forall \mathbf{x} \in D_{\sigma}, \text{ for } \sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\text{ext}}^{(i)}. \end{cases}$$

In order to define the discrete curl operator of a function $\mathbf{v} = (v_1, \dots, v_d) \in \mathbf{H}_{\mathcal{E}}$, we use the functions $(\bar{\partial}_j u_i)_{1 \leq i, j \leq d}$ defined in (3.28). This definition is the same for $\mathbf{v} \in \mathbf{H}_{\mathcal{E},0}$ and $\mathbf{v} \in \mathbf{H}_{\mathcal{E}}$, the only difference is that we may have $u_{\sigma} \neq 0$ if $\sigma \in \mathcal{E}_{\text{ext}}^{(i)}$ and $\mathbf{v} \in \mathbf{H}_{\mathcal{E}}$. Then, the discrete curl operator of a function $\mathbf{v} = (v_1, \dots, v_d) \in \mathbf{H}_{\mathcal{E}}$ is defined by

$$(4.3) \quad \text{curl}_{\mathcal{M}} \mathbf{v} = \begin{cases} \bar{\partial}_1 v_2 - \bar{\partial}_2 v_1 & \text{if } d = 2, \\ \left(\bar{\partial}_2 v_3 - \bar{\partial}_3 v_2, \bar{\partial}_3 v_1 - \bar{\partial}_1 v_3, \bar{\partial}_1 v_2 - \bar{\partial}_2 v_1 \right) & \text{if } d = 3, \end{cases}$$

The following algebraic identity is nothing than a discrete version of (2.11).

Lemma 4. *Let Ω be a bounded domain of \mathbb{R}^d , $d = 2$ or $d = 3$, adapted to the MAC-scheme. Let \mathcal{M} be a MAC grid, $(\mathbf{v}, \mathbf{w}) \in (\mathbf{H}_{\mathcal{E},0})^2$. Then the following discrete identity holds:*

$$(4.4) \quad \int_{\Omega} \nabla_{\mathcal{E}} \mathbf{v} : \nabla_{\mathcal{E}} \mathbf{w} \, d\mathbf{x} = \int_{\Omega} \text{div}_{\mathcal{M}} \mathbf{v} \, \text{div}_{\mathcal{M}} \mathbf{w} \, d\mathbf{x} + \int_{\Omega} \text{curl}_{\mathcal{M}} \mathbf{v} \cdot \text{curl}_{\mathcal{M}} \mathbf{w} \, d\mathbf{x}.$$

We finish this section by introducing a discrete construction of the test function used in Step 3 of the proof of theorem 1 to obtain the convergence of the so-called effective viscous flux. We recall that this test function is the product of a scalar regular function with a velocity field whose divergence is the density; we need here to show the existence, at the discrete level, of such a velocity field, and then some regularity estimates for the resulting test function. To this goal, we first introduce the discrete Laplace operator on the primal mesh. For $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$, let d_σ be defined as the distance between the mass center of K and L , *i.e.* $d_\sigma = d(\mathbf{x}_K, \mathbf{x}_L)$; for an external face $\sigma \in \mathcal{E}_{\text{ext}}$ adjacent to the primal cell K , let $d_\sigma = d(\mathbf{x}_K, \sigma)$. Then, with this notation, we obtain a discretization of the Laplace operator with homogeneous Dirichet boundary conditions on the primal mesh by:

$$(4.5) \quad \begin{aligned} -\Delta_{\mathcal{M}} : & \quad \left| \begin{array}{l} L_{\mathcal{M}} \longrightarrow L_{\mathcal{M}} \\ w \longmapsto -\Delta_{\mathcal{M}} w \end{array} \right. \\ -\Delta_{\mathcal{M}} w(\mathbf{x}) = & \quad \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} \phi_{K,\sigma}(w), \quad \forall \mathbf{x} \in K, \text{ for } K \in \mathcal{M}, \end{aligned}$$

where

$$(4.6) \quad \phi_{K,\sigma}(w) = \begin{cases} \frac{|\sigma|}{d_\sigma} (w_K - w_L) & \text{if } \sigma = K|L \in \mathcal{E}_{\text{int}}, \\ \frac{|\sigma|}{d_\sigma} w_K & \text{if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K). \end{cases}$$

The following lemma [8] clarifies the relations between this Laplace operator and the already defined gradient divergence and curl operators.

Lemma 5. *Let Ω be a bounded domain of \mathbb{R}^d , $d = 2$ or $d = 3$, adapted to the MAC-scheme. Let $w \in L_{\mathcal{M}}$. Let $\mathbf{v} = -\overline{\nabla}_{\mathcal{E}} w \in \mathbf{H}_{\mathcal{E}}$ be defined by (4.1). Then, with the discrete curl operator defined by (4.3), we have $\text{curl}_{\mathcal{M}} \mathbf{v} = 0$. Furthermore, for any $\varrho \in L_{\mathcal{M}}$, there exists one and only one w in $L_{\mathcal{M}}$ such that $-\Delta_{\mathcal{M}} w = \varrho$, and, in this case, $\text{div}_{\mathcal{M}} \mathbf{v} = \varrho$.*

Now, to any regular function $\varphi \in C_c^\infty(\Omega)$, we associate an interpolant $\varphi_{\mathcal{M}} \in L_{\mathcal{M}}$ defined by:

$$(4.7) \quad \varphi_{\mathcal{M}}(\mathbf{x}) = \varphi(\mathbf{x}_K) \text{ for all } \mathbf{x} \in K, \forall K \in \mathcal{M}.$$

We are now in position to state the following discrete regularity result (see [8] for a proof).

Lemma 6. *Let Ω be a bounded domain of \mathbb{R}^d , $d = 2$ or $d = 3$, adapted to the MAC-scheme. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid. Let $\varrho \in L_{\mathcal{M}}$ and $w \in L_{\mathcal{M}}$ be defined by*

$$(4.8) \quad -\Delta_{\mathcal{M}} w = \varrho.$$

Let $\varphi \in C_c^\infty(\Omega)$ and $\overline{\nabla}(w\varphi_{\mathcal{M}})$ be the gradient of the function $w\varphi_{\mathcal{M}}$ as defined in (4.1). Then there exists C_φ only depending on φ , Ω and on $\eta_{\mathcal{M}}$ in a nonincreasing way such that $\|\overline{\nabla}_{\mathcal{E}}(w\varphi_{\mathcal{M}})\|_{1,\mathcal{E},0} \leq C_\varphi \|\varrho\|_{L^2(\Omega)}$, where $\|\cdot\|_{1,\mathcal{E},0}$ is defined in (3.27b).

5. MAIN THEOREM

Now, we are ready to state the main result of this paper. We recall the notation:

$$q(d) = \begin{cases} +\infty & \text{if } d = 2, \\ 6 & \text{if } d = 3. \end{cases}$$

Theorem 4. *Let Ω be a bounded domain of \mathbb{R}^d , $d = 2$ or $d = 3$, adapted to the MAC-scheme ($\bar{\Omega}$ is any finite union of rectangles in 2D or rectangular parallelepipeds in 3D). Let $\mathbf{f} \in (L^2(\Omega))^d$, $M > 0$, and $\alpha > 1$. Let $\gamma > 3$ if $d = 3$ and $\gamma > 1$ if $d = 2$.*

Consider a sequence of MAC grids $(\mathcal{D}_n = (\mathcal{M}_n, \mathcal{E}_n))_{n \in \mathbb{N}}$, with step size $h_{\mathcal{M}_n}$ going to zero as $n \rightarrow +\infty$. Assume that there exists $\eta > 0$ such that $\eta \leq \eta_{\mathcal{M}_n}$ for all $n \in \mathbb{N}$, where $\eta_{\mathcal{M}_n}$ is defined by (3.2).

For a value of the constant C_s independent of $n \in \mathbb{N}$ and sufficiently small with respect to the data (see (6.1)), there exists a discrete solution to the scheme (3.3), with all the discretizations $(\mathcal{D}_n)_{n \in \mathbb{N}}$; in addition, the obtained density and pressure are positive a.e. in Ω . For $n \in \mathbb{N}$, Let $(\mathbf{u}_n, p_n, \varrho_n) \in \mathbf{H}_{\mathcal{E}_n, 0} \times L_{\mathcal{M}_n}(\Omega) \times L_{\mathcal{M}_n}(\Omega)$ be the solution obtained with \mathcal{D}_n . Then, up to the extraction of a subsequence:

- *the sequence $(\mathbf{u}_n)_{n \in \mathbb{N}}$ converges in $(L^q(\Omega))^d$ for any $q \in [1, q(d))$ to a function $\mathbf{u} \in H_0^1(\Omega)^d$, and $(\nabla_{\mathcal{E}_n} \mathbf{u}_n)_{n \in \mathbb{N}}$ converges weakly to $\nabla \mathbf{u}$ in $L^2(\Omega)^{d \times d}$,*
- *the sequence $(\varrho_n)_{n \in \mathbb{N}}$ converges in $L^p(\Omega)$ for any p such that $1 \leq p < 2\gamma$ and weakly in $L^{2\gamma}(\Omega)$ to a function ϱ of $L^{2\gamma}(\Omega)$,*
- *the sequence $(p_n)_{n \in \mathbb{N}}$ converges in $L^p(\Omega)$ for any p such that $1 \leq p < 2$ and weakly in $L^2(\Omega)$ to a function p of $L^2(\Omega)$,*
- *(\mathbf{u}, p, ϱ) is a weak solution of Problem (1.1)–(1.4) in the sense of Definition 1.*

Theorem 4 is also true with a fixed value of C_s (for instance, $C_s = 1$). The only difference is that the estimates on the approximated solutions are valid only for $h_{\mathcal{M}}$ small enough with respect to the data.

The following sections are devoted to the proof of Theorem 4. For the sake of clarity, we shall perform the proofs only in the most interesting case, *i.e.* the three-dimensional case (and then $\gamma > 3$). The modifications to be done for the two-dimensional case, which is in fact simpler, are mostly due to the different Sobolev embeddings and are left to the interested reader. Throughout the proof of this theorem, we adapt to the discrete case the strategy followed to prove Theorem 1.

6. MESH INDEPENDENT ESTIMATES

6.1. Notations. From now on, we assume that Ω is a bounded domain of \mathbb{R}^d , $d = 2$ or $d = 3$, adapted to the MAC-scheme and we assume that all the considered meshes satisfy $\eta \leq \eta_{\mathcal{M}}$, for a given $\eta > 0$ and with $\eta_{\mathcal{M}}$ defined by (3.2). The constant C_s in (3.3a) is chosen sufficiently small with respect to the data. In dimension three, a possible choice is:

$$(6.1) \quad 0 < C_s < \frac{\mu \eta^6}{M \text{diam}(\Omega)^{\alpha-1}}.$$

Finally, the letter C denotes positive real numbers that may tacitly depend on $|\Omega|$, $\text{diam}(\Omega)$, γ , λ , μ , M , \mathbf{f} , α , η and on other parameters; the dependency on these other parameters (if any) is always explicitly indicated in the arguments of these numbers. These numbers can take different values, even in the same formula. They are always independent of the size of the discretisation $h_{\mathcal{M}}$.

6.2. Existence. Let us now state that the discrete problem (3.3) admits at least one solution. This existence result follows from standard arguments of the topological degree theory (see [4] for the theory, [5] for the first application to a nonlinear numerical scheme). We refer to Appendix A for a proof.

Theorem 5. *There exists a solution $(\mathbf{u}, p, \varrho) \in \mathbf{H}_{\mathcal{E}, 0} \times L_{\mathcal{M}} \times L_{\mathcal{M}}$ of Problem (3.3). Moreover any solution is such that $\varrho > 0$ a.e in Ω (in the sense that $\varrho_K > 0$, $\forall K \in \mathcal{M}$).*

6.3. Energy Inequality. Let us now turn to stability issues: in order to prove the convergence of the scheme, we wish to obtain some uniform (with respect to the mesh) bounds on the solutions to (3.3); these bounds are stated in Proposition 1 below. We begin by a technical lemma (see [8, Lemma 5.4] for a proof). In fact, Lemma 7 is not useful only for stability issues. It is useful for three reasons. First, it allows an estimate on \mathbf{u} in a discrete H_0^1 norm (Proposition 1), as in [8, Proposition 5.5]. Second, it yields a so called weak BV estimate. These weak BV estimates depend on the mesh and do not give a direct compactness result on the sequence of approximate solutions; however they are useful in the passage to the limit in the mass equation, in the discrete convective term and in the equation of state. Third, Lemma 7 gives (with $\beta = 1$) a crucial inequality which is also used in order to pass to the limit in the equation of state.

Lemma 7. *Let $\varrho \in L_{\mathcal{M}}$ and $\mathbf{u} \in \mathbf{H}_{\mathcal{E},0}$ satisfy (3.3a). Then, for any $\beta \geq 1$:*

$$\int_{\Omega} \varrho^{\beta} \operatorname{div}_{\mathcal{M}} \mathbf{u} \, d\mathbf{x} + \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\text{int}}} \beta |\sigma| \varrho_{\sigma, \beta} |u_{\sigma}| [\varrho]_{\sigma}^2 \leq CC_s h_{\mathcal{M}}^{\alpha},$$

where C depends only on M , β , μ , α , Ω and η , and, for any $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$,

$$(6.2) \quad \varrho_{\sigma, \beta} = \min(\varrho_K^{\beta-2}, \varrho_L^{\beta-2}).$$

In order to obtain an estimate on the pressure, we need to introduce a so-called Fortin interpolation operator, *i.e.* an operator which maps velocity functions to discrete functions and preserves the divergence. The following lemma is given in [15, Theorem 1], and we repeat it here with our notations for the sake of clarity. We will use this Lemma later on with $p = 2$.

Lemma 8 (Fortin interpolation operator). *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid of Ω . Let $1 \leq p < \infty$. For $\mathbf{v} = (v_1, \dots, v_d) \in (W_0^{1,p}(\Omega))^d$ we define $\tilde{\mathcal{P}}_{\mathcal{E}} \mathbf{v}$ by*

$$\begin{aligned} \tilde{\mathcal{P}}_{\mathcal{E}} \mathbf{v} &= \left(\tilde{\mathcal{P}}_{\mathcal{E}}^{(1)} v_1, \dots, \tilde{\mathcal{P}}_{\mathcal{E}}^{(1)} v_d \right) \in \mathbf{H}_{\mathcal{E},0}, \text{ where for } i = 1, \dots, d, \\ \tilde{\mathcal{P}}_{\mathcal{E}}^{(i)} &: W_0^{1,p}(\Omega) \longrightarrow H_{\mathcal{E},0}^{(i)} \\ (6.3) \quad v_i &\longmapsto \tilde{\mathcal{P}}_{\mathcal{E}} v_i \text{ defined by} \\ &\tilde{\mathcal{P}}_{\mathcal{E}}^{(i)} v_i(\mathbf{x}) = (\tilde{\mathcal{P}}_{\mathcal{E}}^{(i)} v_i)_{\sigma} = \frac{1}{|\sigma|} \int_{\sigma} v_i(\mathbf{x}) \, d\gamma(\mathbf{x}), \quad \forall \mathbf{x} \in D_{\sigma}, \quad \sigma \in \mathcal{E}^{(i)}. \end{aligned}$$

Then $\tilde{\mathcal{P}}_{\mathcal{E}}$ satisfies:

$$(6.4) \quad \|\tilde{\mathcal{P}}_{\mathcal{E}} \varphi - \varphi\|_{L^{\infty}(\Omega)} \leq C_{\varphi} h_{\mathcal{M}}, \quad \forall \varphi \in C_c^{\infty}(\Omega)^d.$$

For $q \in L^1(\Omega)$, we define $\mathcal{P}_{\mathcal{M}} q \in L_{\mathcal{M}}$ by:

$$(6.5) \quad \mathcal{P}_{\mathcal{M}} q(\mathbf{x}) = \frac{1}{|K|} \int_K q(\mathbf{x}) \, d\mathbf{x}.$$

Let $\eta_{\mathcal{M}} > 0$ be defined by (3.2). Then, for $\varphi \in (W_0^{1,p}(\Omega))^d$,

$$(6.6a) \quad \operatorname{div}_{\mathcal{M}}(\tilde{\mathcal{P}}_{\mathcal{E}} \varphi) = \mathcal{P}_{\mathcal{M}}(\operatorname{div} \varphi),$$

$$(6.6b) \quad \|\nabla_{\mathcal{E}} \tilde{\mathcal{P}}_{\mathcal{E}} \varphi\|_{(L^p(\Omega))^{d \times d}} \leq C_{\eta_{\mathcal{M}}} \|\nabla \varphi\|_{(L^p(\Omega))^d},$$

where $C_{\eta_{\mathcal{M}}}$ depends only on Ω , p and on $\eta_{\mathcal{M}}$ in a decreasing way.

We are now in position to state and prove the estimates of a discrete solution that we are seeking. These estimates may be seen as an equivalent for the discrete case of Step 1 of the proof of Theorem 1.

Proposition 1. *Let $(\mathbf{u}, p, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}} \times L_{\mathcal{M}}$ be a solution to the scheme, i.e. system (3.3). Taking C_s small enough with respect to the data (namely $\mu, M, \Omega, \alpha, \eta$) there exists C_1 depending only on $\mathbf{f}, \mu, M, \Omega, \gamma, \alpha$ and on η such that:*

$$(6.7) \quad \|\mathbf{u}\|_{1,\mathcal{E},0} + \|p\|_{L^2(\Omega)} + \|\varrho\|_{L^{2\gamma}(\Omega)} \leq C_1.$$

Moreover, for any $\beta \in [1, \gamma]$, there exists C_2 depending only on $\mathbf{f}, M, \Omega, \gamma, \mu, \alpha, \beta$ and η such that

$$(6.8) \quad \sum_{\sigma \in \mathcal{E}_{\text{int}}} |\sigma| \varrho_{\sigma,\beta} |u_\sigma| [\varrho]_\sigma^2 \leq C_2,$$

where $\varrho_{\sigma,\beta}$ is defined in (6.2). In particular, since $\gamma > 3$, we get by taking $\beta = 2$ in (6.8):

$$(6.9) \quad \sum_{\sigma \in \mathcal{E}_{\text{int}}} |\sigma| |u_\sigma| [\varrho]_\sigma^2 \leq C_2.$$

Note that if, in Proposition 1, we choose a fixed value of C_s , for instance $C_s = 1$, There exists $\bar{h} > 0$, depending of the data, such that the conclusions of Proposition 1 are true for $h_{\mathcal{M}} \leq \bar{h}$. This is easy to see with (6.11).

Proof. To prove Proposition 1, we proceed in several steps. We will follow the proof established in the continuous case to obtain uniform bounds of the approximate solutions.

Step 1 : Estimates on $\|\mathbf{u}\|_{1,\mathcal{E},0}$ and inequality (6.8).

Taking \mathbf{u} as a test function in (3.30), using the Hölder's inequality and thanks to the fact that the discrete H^1 norm controls the L^2 norm (see Theorem 2), we have:

$$(6.10) \quad \frac{\mu}{2} \|\mathbf{u}\|_{1,\mathcal{E},0}^2 + (\mu + \lambda) \|\text{div}_{\mathcal{M}} \mathbf{u}\|_{L^2(\Omega)}^2 - \int_{\Omega} p \text{div}_{\mathcal{M}} \mathbf{u} \, d\mathbf{x} \\ + \sum_{i=1}^3 \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ \epsilon = D_\sigma | D_{\sigma'}}} \frac{1}{2} F_{\sigma,\epsilon} (u_\sigma + u_{\sigma'}) (u_\sigma - u_{\sigma'}) \leq C$$

where C depends only on \mathbf{f} and Ω . Moreover, by virtue of (3.14),

$$\sum_{i=1}^3 \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ \epsilon = D_\sigma | D_{\sigma'}}} \frac{1}{2} F_{\sigma,\epsilon} (u_\sigma + u_{\sigma'}) (u_\sigma - u_{\sigma'}) = \sum_{i=1}^3 \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ \epsilon = D_\sigma | D_{\sigma'}}} \frac{1}{2} F_{\sigma,\epsilon} ((u_\sigma)^2 - (u_{\sigma'})^2) \\ = \sum_{i=1}^3 \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \frac{(u_\sigma)^2}{2} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} = -\frac{1}{2} C_s h_{\mathcal{M}}^\alpha \left(\int_{\Omega} \varrho \|\mathbf{u}\|^2 \, d\mathbf{x} - \varrho^* \int_{\Omega} \|\mathbf{u}\|^2 \, d\mathbf{x} \right)$$

Lemma 7 with $\beta = \gamma$ yields, since $p = \varrho^\gamma$:

$$\int_{\Omega} p \text{div}_{\mathcal{M}}(\mathbf{u}) \, d\mathbf{x} + \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\text{int}}} \gamma |\sigma| \varrho_{\sigma,\gamma} |u_\sigma| [\varrho]_\sigma^2 \leq C,$$

where C depends only on $M, \gamma, \alpha, \mu, \Omega$ and η .
Consequently

$$\frac{\mu}{2} \|\mathbf{u}\|_{1,\varepsilon,0}^2 + \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\text{int}}} \gamma |\sigma| \rho_{\sigma,\gamma} |u_\sigma| [\rho]_\sigma^2 \leq \frac{1}{2} C_s M h^\alpha \|\mathbf{u}\|_{L^\infty(\Omega)}^2 + C$$

By virtue of Theorem 2 we have $h_{\mathcal{M}}^3 \|\mathbf{u}\|_{L^\infty(\Omega)}^6 \leq C(\eta) \|\mathbf{u}\|_{L^6(\Omega)}^6 \leq C(\eta) \|\mathbf{u}\|_{1,\varepsilon,0}^6$ and therefore

$$\|\mathbf{u}\|_{L^\infty(\Omega)} \leq C(\eta) \frac{1}{\sqrt{h_{\mathcal{M}}}} \|\mathbf{u}\|_{1,\varepsilon,0}.$$

Summing these two relations, we thus obtain:

$$(6.11) \quad \frac{\mu}{2} \|\mathbf{u}\|_{1,\varepsilon,0}^2 + \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\text{int}}} \gamma |\sigma| \varrho_{\sigma,\gamma} |u_\sigma| [\varrho]_\sigma^2 \leq C + \frac{1}{2} C(\eta) C_s M h_{\mathcal{M}}^{\alpha-1} \|\mathbf{u}\|_{1,\varepsilon,0}^2$$

and consequently, since $\alpha > 1$,

$$\frac{1}{2} (\mu - C(\eta) C_s M \text{diam}(\Omega)^{\alpha-1}) \|\mathbf{u}\|_{1,\varepsilon,0}^2 + \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\text{int}}} \gamma |\sigma| \varrho_{\sigma,\gamma} |u_\sigma| [\varrho]_\sigma^2 \leq C.$$

Taking C_s such that $0 < C_s < \frac{\mu}{C(\eta) M \text{diam}(\Omega)^{\alpha-1}}$ (see (6.1) for a possible explicit choice), we infer that

$$\|\mathbf{u}\|_{1,\varepsilon,0} + \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\text{int}}} \gamma |\sigma| \varrho_{\sigma,\gamma} |u_\sigma| [\varrho]_\sigma^2 \leq C.$$

Step 2: Estimate on $\|p\|_{L^2(\Omega)}$.

Let $m(p)$ stand for the mean value of p . Let $\mathbf{v} = (v_1, v_2, v_3) \in H_0^1(\Omega)^3$ such that

$$\begin{cases} \text{div} \mathbf{v} = p - m(p), \\ \|\mathbf{v}\|_{H_0^1(\Omega)^3} \leq C(\Omega) \|p - m(p)\|_{L^2(\Omega)}, \end{cases}$$

(the existence of \mathbf{v} coming from Lemma 1) Multiplying (3.3b) by $\tilde{\mathcal{P}}_\varepsilon \mathbf{v}$ (where $\tilde{\mathcal{P}}_\varepsilon$ is defined in Lemma 8) and integrating over Ω we have:

$$\|p - m(p)\|_{L^2(\Omega)}^2 \leq C \|p - m(p)\|_{L^2(\Omega)} + \sum_{i=1}^3 \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ \epsilon = D_\sigma | D_{\sigma'}}} F_{\sigma,\epsilon} \frac{1}{2} (u_\sigma + u_{\sigma'}) ((\tilde{\mathcal{P}}_\varepsilon^{(i)} v_i)_\sigma - (\tilde{\mathcal{P}}_\varepsilon^{(i)} v_i)_{\sigma'})$$

where C depends on $\mathbf{f}, \Omega, \eta, \mu, \alpha, \gamma, M$. Now keeping in mind the definition of the dual fluxes (see (3.11) and (3.12)) and the definition of $\|\cdot\|_{1,\varepsilon,0}$, a technical but straightforward computation gives

$$\begin{aligned} \left| \sum_{i=1}^3 \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ \epsilon = D_\sigma | D_{\sigma'}}} F_{\sigma,\epsilon} \frac{1}{2} (u_\sigma + u_{\sigma'}) ((\tilde{\mathcal{P}}_\varepsilon^{(i)} v_i)_\sigma - (\tilde{\mathcal{P}}_\varepsilon^{(i)} v_i)_{\sigma'}) \right| &\leq C \|\varrho\|_{L^6(\Omega)} \|\mathbf{u}\|_{L^6(\Omega)}^2 \|\tilde{\mathcal{P}}_\varepsilon \mathbf{v}\|_{1,\varepsilon,0} \\ &\leq C \|p\|_{L^2(\Omega)}^{\frac{1}{7}} \|p - m(p)\|_{L^2(\Omega)}, \end{aligned}$$

where the constant C depends only on the data. To obtain the last inequality we have used the energy inequality (6.7) to get a bound on $\|\mathbf{u}\|_{L^6(\Omega)}$ (thanks to Theorem 2) and the Hölder's inequality since $2\gamma \geq 6$ and $p = \varrho^\gamma$. Consequently

$$\|p - m(p)\|_{L^2(\Omega)} \leq C (\|p\|_{L^2(\Omega)}^{\frac{1}{7}} + 1)$$

where C depends on \mathbf{f} , μ , M , Ω , γ , α and on η . Since $\int_{\Omega} p^{\frac{1}{\gamma}} \, d\mathbf{x} = \int_{\Omega} \varrho \, d\mathbf{x} = M$, Lemma 2 gives an L^2 bound for p depending only on the data. To conclude, we obtain a $L^{2\gamma}$ bound for the density since $p = \varrho^\gamma$.

In order to prove (6.8) for $1 \leq \beta \leq \gamma$, let us use once again Lemma 7, to obtain:

$$\frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\text{int}}} \beta |\sigma| \varrho_{\sigma, \beta} |u_\sigma| [\varrho]_\sigma^2 \leq - \int_{\Omega} \varrho^\beta \operatorname{div}_{\mathcal{M}} \mathbf{u} \, d\mathbf{x} + C,$$

where C depends on M , β , μ , α , Ω , η . Since ϱ is bounded in $L^{2\beta}(\Omega)$ and $\|\operatorname{div}_{\mathcal{M}} \mathbf{u}\|_{L^2(\Omega)}$ is controlled by $\|\mathbf{u}\|_{1, \varepsilon, 0}$, this concludes the proof. \square

7. CONVERGENCE ANALYSIS

The aim of this section is to pass to the limit in the discrete equations (3.3a)–(3.3c). As in the continuous case, thanks to the estimates established in the previous section, taking a sequence of meshes, we can assume the convergence, up to a subsequence, of the discrete solution to some (\mathbf{u}, p, ϱ) , in a convenient sense. We will first prove that (\mathbf{u}, p, ϱ) satisfies the weak form of Problem (1.1)–(1.2). We then prove that $p = \varrho^\gamma$. The first difficulty is the convergence of the discrete convective term (the second consists in passing to the limit in the equation of state). Indeed it is not easy to manipulate the discrete convective operator defined with the dual fluxes. We then introduce velocity interpolators in order to transform the discrete convective operator. It relies on the reconstruction of each velocity component on all faces (or edges in 2D) of the mesh. Similar results are used in [16] for the incompressible case.

7.1. Passing to the limit in the mass and momentum balance equations.

Lemma 9 (Velocity interpolators). *For a given MAC grid $\mathcal{D} = (\mathcal{M}, \mathcal{E})$, we define, for $i, j = 1, 2, 3$, the full grid velocity reconstruction operator with respect to (i, j) by*

$$(7.1) \quad \begin{aligned} \mathcal{R}_{\mathcal{E}}^{(i,j)} : H_{\mathcal{E},0}^{(i)} &\rightarrow H_{\mathcal{E},0}^{(j)} \\ v &\mapsto \mathcal{R}_{\mathcal{E}}^{(i,j)} v = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(j)}} (\mathcal{R}_{\mathcal{E}}^{(i,j)} v)_\sigma \chi_{D_\sigma}, \end{aligned}$$

where

$$(7.2) \quad \begin{aligned} (\mathcal{R}_{\mathcal{E}}^{(i,i)} v)_\sigma &= v_\sigma \text{ for } \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \\ \text{and, for } \sigma = K|L \in \mathcal{E}_{\text{int}}^{(j)}, j \neq i, \\ (7.3) \quad (\mathcal{R}_{\mathcal{E}}^{(i,j)} v)_\sigma &= \frac{1}{4} \sum_{\sigma' \in \mathcal{N}_\sigma} v_{\sigma'}, \quad \mathcal{N}_\sigma = \{\sigma' \in \mathcal{E}^{(i)}, \sigma' \in \mathcal{E}(K) \cup \mathcal{E}(L)\}. \end{aligned}$$

For any $i = 1, 2, 3$, we also define a projector from $H_{\mathcal{E}}^{(i)}$ into $L_{\mathcal{M}}$ by

$$(7.4) \quad \begin{aligned} \mathcal{R}_{\mathcal{M}}^{(i)} : H_{\mathcal{E}}^{(i)} &\rightarrow L_{\mathcal{M}} \\ v &\mapsto \mathcal{R}_{\mathcal{M}}^{(i)} v = \sum_{K \in \mathcal{M}} (\mathcal{R}_{\mathcal{M}}^{(i)} v)_K \chi_K, \end{aligned}$$

where

$$(7.5) \quad (\mathcal{R}_{\mathcal{M}}^{(i)}v)_K = \frac{1}{2} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} v_\sigma.$$

Then there exists $C \geq 0$, depending only on the regularity of the mesh (defined by (3.2)) in a decreasing way, such that, for any $1 \leq q < \infty$ and for any $i, j = 1, 2, 3$,

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j)}v\|_{L^q(\Omega)} \leq C\|v\|_{L^q(\Omega)} \text{ for any } v \in H_{\mathcal{E},0}^{(i)},$$

$$\|\mathcal{R}_{\mathcal{M}}^{(i)}v\|_{L^q(\Omega)} \leq C\|v\|_{L^q(\Omega)} \text{ for any } v \in H_{\mathcal{E}}^{(i)}.$$

Proof. Let us prove the bound on $\|\mathcal{R}_{\mathcal{E}}^{(i,j)}v\|_{L^q(\Omega)}$ for $d = 2$, $i = 1$ and $j = 2$. The other cases are similar. In this case, for a given $\sigma = K|L \in \mathcal{E}_{\text{int}}^{(i)}$, the edge σ belongs to $\mathcal{N}_{\sigma'}$ for $\sigma' \in \{\sigma_K^t, \sigma_K^b, \sigma_L^t, \sigma_L^b\}$ where σ_K^t (resp. σ_K^b) denotes the the top (resp. bottom) edge of K , as depicted in Figure 4. Let $v \in H_{\mathcal{E},0}^{(i)}$; by definition of $\mathcal{R}_{\mathcal{E}}^{(i,j)}v$, noting that $[\frac{1}{4}(a+b+c+d)]^q \leq a^q + b^q + c^q + d^q$, we have:

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j)}v\|_{L^q(\Omega)}^q \leq \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(i)} \\ \sigma = K|L}} v_\sigma^q (|D_{\sigma_K^t}| + |D_{\sigma_K^b}| + |D_{\sigma_L^t}| + |D_{\sigma_L^b}|) \leq 4\eta^{-2} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(i)} \\ \sigma = K|L}} v_\sigma^q |D_\sigma|,$$

which concludes the proof. \square

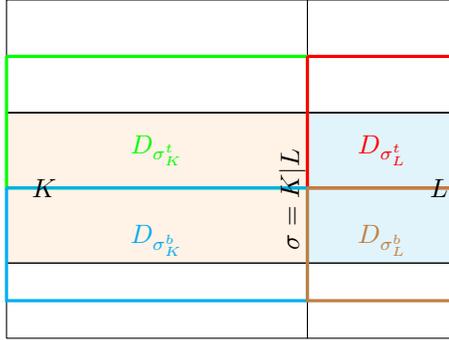


FIGURE 4. Full grid velocity interpolate.

Lemma 10 (Convergence of the full grid velocity interpolate). *Let $(\mathcal{M}_n, \mathcal{E}_n)_{n \in \mathbb{N}}$ be a sequence MAC meshes such that $h_{\mathcal{M}_n} \rightarrow 0$ as $n \rightarrow +\infty$, and, for all n , $\eta_{\mathcal{M}_n} \geq \eta > 0$. Let $1 \leq q < \infty$.*

Let $v \in L^q(\Omega)$, and let $(v_n)_{n \in \mathbb{N}}$ be such that $v_n \in H_{\mathcal{E}_n,0}^{(i)}$ and v_n converges to v as $n \rightarrow +\infty$ in $L^q(\Omega)$. Let $i, j = 1, 2, 3$ and $\mathcal{R}_{\mathcal{E}_n}^{(i,j)}$ be the full grid velocity reconstruction operator defined by (7.1).

Then $\mathcal{R}_{\mathcal{E}_n}^{(i,j)}v_n \rightarrow v$ in $L^q(\Omega)$ as $n \rightarrow +\infty$.

Similarly, if $(v_n)_{n \in \mathbb{N}}$ is such that $v_n \in H_{\mathcal{E}}^{(i)}$ and v_n converges to v as $n \rightarrow +\infty$ in $L^q(\Omega)$, then, for all i , $\mathcal{R}_{\mathcal{M}_n}^{(i)}v_n \rightarrow v$ in $L^q(\Omega)$ as $n \rightarrow +\infty$, where $\mathcal{R}_{\mathcal{M}_n}^{(i)}v$ is defined by (7.4).

Proof. We give the proof for $\mathcal{R}_{\mathcal{E}_n}^{(i,j)}$ (the proof is similar for $\mathcal{R}_{\mathcal{M}_n}^{(i)}$).

Let $\varphi \in C_c^\infty(\Omega)$. Denoting $\mathcal{R}_{\mathcal{E}_n}^{(i,j)}$ by \mathcal{R}_n and $\mathcal{P}_{\mathcal{E}_n}^{(i)}$ (defined by (3.4)) by \mathcal{P}_n for short, we have:

$$\begin{aligned} \|\mathcal{R}_n v_n - v\|_{L^q(\Omega)} &\leq \|\mathcal{R}_n v_n - \mathcal{R}_n \circ \mathcal{P}_n v\|_{L^q(\Omega)} + \|\mathcal{R}_n \circ \mathcal{P}_n v - \mathcal{R}_n \circ \mathcal{P}_n \varphi\|_{L^q(\Omega)} + \\ &\quad \|\mathcal{R}_n \circ \mathcal{P}_n \varphi - \varphi\|_{L^q(\Omega)} + \|\varphi - v\|_{L^q(\Omega)}. \end{aligned}$$

Since $\mathcal{R}_n v_n = \mathcal{R}_n \circ \mathcal{P}_n v_n$, and thanks to the fact that $\|\mathcal{R}_n w\|_{L^q(\Omega)} \leq C\|w\|_{L^q(\Omega)}$ (for some $C > 0$, see Lemma 9) and that $\|\mathcal{P}_n w\|_{L^q(\Omega)} \leq \|w\|_{L^q(\Omega)}$, we get

$$\|\mathcal{R}_n v_n - v\|_{L^q(\Omega)} \leq C\|v_n - v\|_{L^q(\Omega)} + C\|v - \varphi\|_{L^q(\Omega)} + \|\mathcal{R}_n \circ \mathcal{P}_n \varphi - \varphi\|_{L^q(\Omega)} + \|\varphi - v\|_{L^q(\Omega)}.$$

Let $\varepsilon > 0$. Let us choose $\varphi \in C_c^\infty(\Omega)$ such that $\|\varphi - v\|_{L^q(\Omega)} \leq \frac{\varepsilon}{C+1}$. There exists n_1 such that $C\|v_n - v\|_{L^q(\Omega)} \leq \varepsilon$ for all $n \geq n_1$, and there exists n_2 such that $\|\mathcal{R}_n \circ \mathcal{P}_n \varphi - \varphi\|_{L^q(\Omega)} \leq \varepsilon$, for all $n \geq n_2$. Therefore $\|\mathcal{R}_n v_n - v\|_{L^q(\Omega)} \leq 3\varepsilon$ for $n \geq \max(n_1, n_2)$, which concludes the proof. \square

With the above definitions the following algebraic identity holds (a similar identity is in [21]):

Lemma 11. *Let $\varrho \in L_{\mathcal{M}}$ and $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{H}_{\mathcal{E},0}$. Let $i \in \{1, 2, 3\}$ and $\varphi = (\varphi_\sigma)_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \in H_{\mathcal{E},0}^{(i)}$ be a discrete scalar function. Let the primal fluxes $F_{K,\sigma}$ be given by (3.5) and let the dual fluxes $F_{\sigma,\varepsilon}$ be given by (3.11) or (3.12). Then we have*

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\varepsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\varepsilon} u_\varepsilon \varphi_\sigma = \sum_{j=1}^3 S_j,$$

where

$$S_i = \sum_{\substack{K=[\sigma\sigma'] \\ \sigma, \sigma' \in \mathcal{E}^{(i)}}} (\varrho_\sigma^{up} u_\sigma |D_{K,\sigma}| + \varrho_{\sigma'}^{up} u_{\sigma'} |D_{K,\sigma'}|) (\mathcal{R}_{\mathcal{M}}^{(i)} u_i)_K \frac{\varphi_\sigma - \varphi_{\sigma'}}{d(x_\sigma, x_{\sigma'})},$$

and, for $j \neq i$,

$$S_j = \sum_{\tau \in \mathcal{E}_{\text{int}}^{(j)}} |D_\tau| \frac{\varrho_\tau^{up} u_\tau}{4} \left[(u_{\sigma_3} + u_{\sigma_1}) \frac{\varphi_{\sigma_1} - \varphi_{\sigma_3}}{d(x_{\sigma_1}, x_{\sigma_3})} + (u_{\sigma_4} + u_{\sigma_2}) \frac{\varphi_{\sigma_2} - \varphi_{\sigma_4}}{d(x_{\sigma_2}, x_{\sigma_4})} \right]$$

where $(\sigma_k)_{k=1,\dots,4}$ are the four neighbouring faces (or edges) of τ belonging to $\mathcal{E}^{(i)}$, with $x_{\sigma_3} x_{\sigma_1} \cdot e_j > 0$ and $x_{\sigma_4} x_{\sigma_2} \cdot e_j > 0$.

Proof. We write $\sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\varepsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\varepsilon} u_\varepsilon \varphi_\sigma = \sum_{j=1}^3 S_j$ with, using (3.11), (3.12) and the centred choice for u_ε ,

$$\begin{aligned} S_i &= \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\substack{\varepsilon = \overrightarrow{\sigma\sigma'} \in \tilde{\mathcal{E}}_{\text{int}}^{(i)} \\ \varepsilon \perp e_i, \varepsilon \subset K}} \frac{1}{2} [F_{K,\sigma} \mathbf{n}_{K,\sigma} + F_{K,\sigma'} \mathbf{n}_{K,\sigma'}] \cdot \mathbf{n}_{D_\sigma, \varepsilon} \frac{u_\sigma + u_{\sigma'}}{2} \varphi_\sigma, \\ S_j &= \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\substack{\varepsilon = \overrightarrow{\sigma\sigma'} \in \tilde{\mathcal{E}}_{\text{int}}^{(i)} \\ \varepsilon \not\perp e_i, \varepsilon \subset \tau \cup \tau'}} \frac{1}{2} [F_{K,\tau} + F_{L,\tau'}] \frac{u_\sigma + u_{\sigma'}}{2} \varphi_\sigma, \text{ for } j \neq i, \end{aligned}$$

where τ and τ' are the faces of $\mathcal{E}^{(j)}$ such that $\varepsilon \subset \tau \cup \tau'$.

For S_i , a reordering of the summation and the fact that $(u_\sigma + u_{\sigma'})/2 = (\mathcal{R}_{\mathcal{M}}^{(i)} u_i)_K$ yield

$$S_i = \sum_{\substack{K=\overrightarrow{[\sigma\sigma']} \\ \sigma, \sigma' \in \mathcal{E}^{(i)}}} \frac{1}{2} [F_{K,\sigma} - F_{K,\sigma'}] (\mathcal{R}_{\mathcal{M}}^{(i)} u_i)_K (\varphi_\sigma - \varphi_{\sigma'}).$$

Since $F_{K,\sigma} = |\sigma| \varrho_\sigma^{up} u_\sigma$, this gives

$$S_i = \sum_{\substack{K=\overrightarrow{[\sigma\sigma']} \\ \sigma, \sigma' \in \mathcal{E}^{(i)}}} (\varrho_\sigma^{up} u_\sigma |D_{K,\sigma}| + \varrho_{\sigma'}^{up} u_{\sigma'} |D_{K,\sigma'}|) (\mathcal{R}_{\mathcal{M}}^{(i)} u_i)_K \frac{\varphi_\sigma - \varphi_{\sigma'}}{d(x_\sigma, x_{\sigma'})}.$$

For S_j , $j \neq i$, we have

$$S_j = \sum_{\tau \in \mathcal{E}_{\text{int}}^{(j)}} |\tau| \frac{\varrho_\tau^{up} u_\tau}{4} [(u_{\sigma_3} + u_{\sigma_1}) \varphi_{\sigma_1} + (u_{\sigma_4} + u_{\sigma_2}) \varphi_{\sigma_2} - (u_{\sigma_1} + u_{\sigma_3}) \varphi_{\sigma_3} - (u_{\sigma_2} + u_{\sigma_4}) \varphi_{\sigma_4}]$$

where $(\sigma_k)_{k=1,\dots,4}$ are the four neighbouring faces (or edges) of τ belonging to $\mathcal{E}^{(i)}$, *i.e.* such that $\bar{\tau} \cap \bar{\sigma}_k \neq \emptyset$, see figure 5.

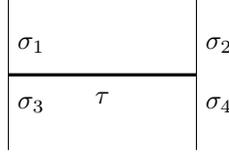


FIGURE 5. neighbouring faces of τ

Thus,

$$\begin{aligned} S_j &= \sum_{\tau \in \mathcal{E}_{\text{int}}^{(j)}} |\tau| \frac{\varrho_\tau^{up} u_\tau}{4} [(u_{\sigma_3} + u_{\sigma_1}) (\varphi_{\sigma_1} - \varphi_{\sigma_3}) + (u_{\sigma_4} + u_{\sigma_2}) (\varphi_{\sigma_2} - \varphi_{\sigma_4})] \\ &= \sum_{\tau \in \mathcal{E}_{\text{int}}^{(j)}} |D_\tau| \frac{\varrho_\tau^{up} u_\tau}{4} \left[(u_{\sigma_3} + u_{\sigma_1}) \frac{\varphi_{\sigma_1} - \varphi_{\sigma_3}}{d(x_{\sigma_1}, x_{\sigma_3})} + (u_{\sigma_4} + u_{\sigma_2}) \frac{\varphi_{\sigma_2} - \varphi_{\sigma_4}}{d(x_{\sigma_2}, x_{\sigma_4})} \right] \end{aligned}$$

□

With the uniform estimates stated in Proposition 1 and the material introduced above we are able to pass to the limit in the discrete equations (3.3a)–(3.3b).

Proposition 2. *Let $\eta > 0$ and $(\mathcal{D}_n = (\mathcal{M}_n, \mathcal{E}_n))_{n \in \mathbb{N}}$ be a sequence of MAC grids with step size $h_{\mathcal{M}_n}$ tending to zero as $n \rightarrow +\infty$. Assume that $\eta \leq \eta_{\mathcal{M}_n}$, $\forall n \in \mathbb{N}$ where $h_{\mathcal{M}_n}$ is defined by (3.2). Let $(\mathbf{u}_n)_{n \in \mathbb{N}}$, $(p_n)_{n \in \mathbb{N}}$ and $(\varrho_n)_{n \in \mathbb{N}}$ be the corresponding sequence of solutions to (3.3). Then, up to the extraction of a subsequence:*

- (1) *the sequence $(\mathbf{u}_n)_{n \in \mathbb{N}}$ converges in $(L^q(\Omega))^3$ where $q \in [1, 6)$ to a function $\mathbf{u} \in (\mathbf{H}_0^1(\Omega))^3$ and $(\nabla_{\mathcal{E}_n} \mathbf{u}_n)_{n \in \mathbb{N}}$ converges weakly in $L^2(\Omega)^{3 \times 3}$ to $\nabla \mathbf{u}$.*
- (2) *the sequence $(\varrho_n)_{n \in \mathbb{N}}$ weakly converges to a function ϱ in $L^{2\gamma}(\Omega)$,*
- (3) *the sequence $(p_n)_{n \in \mathbb{N}}$ weakly converges to a function p in $L^2(\Omega)$,*

- (4) \mathbf{u} and ϱ satisfy the continuous mass equation (1.1a).
 (5) \mathbf{u} , p and ϱ satisfy momentum balance equation (1.1b).
 (6) $\varrho \geq 0$ a.e. and $\int_{\Omega} \varrho \, d\mathbf{x} = M$.

Proof. The stated convergences (i.e. points 1. to 3.) are straightforward consequences of the uniform bounds for the sequence of solutions, together, for the velocity, with the compactness theorem 3 and the Sobolev inequalities stated in Theorem 2. Point 6. is an easy consequence of point 2.. We refer the reader to [8] for the proof of point 4. Let us then prove point 5. i.e. that \mathbf{u} , p and ϱ satisfy (1.1b) in the weak sense that is (2.2).

Let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ be a function of $C_c^\infty(\Omega)^3$. Taking $\tilde{\mathcal{P}}_{\varepsilon_n} \varphi \in \mathbf{H}_{\varepsilon_n, 0}$ as a test function in (3.30) we infer:

$$\begin{aligned} & \int_{\Omega} \operatorname{div}_{\varepsilon_n}(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) \cdot \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x} + \mu \int_{\Omega} \nabla_{\varepsilon_n} \mathbf{u}_n : \nabla_{\varepsilon_n} \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x} \\ & + (\mu + \lambda) \int_{\Omega} \operatorname{div}_{\mathcal{M}_n} \mathbf{u}_n \operatorname{div}_{\mathcal{M}_n} \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x} - \int_{\Omega} p_n \operatorname{div}_{\mathcal{M}_n} \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x} = \int_{\Omega} \mathcal{P}_{\varepsilon_n} \mathbf{f} \cdot \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x}. \end{aligned}$$

The convergence of the second term may be proven by slight modifications of a classical result [6, Chapter III]:

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \nabla_{\varepsilon_n} \mathbf{u}_n : \nabla_{\varepsilon_n} \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x} = \int_{\Omega} \nabla \mathbf{u} : \nabla \varphi \, d\mathbf{x}.$$

From the definition of $\tilde{\mathcal{P}}_{\varepsilon_n} \varphi$ and thanks to the L^2 weak convergence of the pressure, we have:

$$\int_{\Omega} p_n \operatorname{div}_{\mathcal{M}_n} \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x} = \int_{\Omega} p_n \operatorname{div} \varphi \, d\mathbf{x}$$

and therefore

$$\lim_{n \rightarrow +\infty} \int_{\Omega} p_n \operatorname{div}_{\mathcal{M}_n} \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x} = \int_{\Omega} p \operatorname{div} \varphi \, d\mathbf{x}.$$

By virtue of the L^2 weak convergence of $\operatorname{div}_{\mathcal{M}_n} \mathbf{u}_n$, we have also:

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \operatorname{div}_{\mathcal{M}_n} \mathbf{u}_n \operatorname{div}_{\mathcal{M}_n} \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x} = \int_{\Omega} \operatorname{div} \mathbf{u} \operatorname{div} \varphi \, d\mathbf{x}.$$

From (6.4) and the strong convergence of $\mathcal{P}_{\varepsilon_n} \mathbf{f}$ towards \mathbf{f} we infer that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \mathcal{P}_{\varepsilon_n} \mathbf{f} \cdot \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \varphi \, d\mathbf{x}.$$

Now it remains to treat the convective term. Here again the dependency of the mesh on n will be omitted for short. First of all we have

$$\int_{\Omega} \operatorname{div}_{\varepsilon_n}(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) \cdot \tilde{\mathcal{P}}_{\varepsilon_n} \varphi \, d\mathbf{x} = \sum_{i=1}^3 \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma, \epsilon} u_\epsilon (\tilde{\mathcal{P}}_{\varepsilon_n}^{(i)} \varphi_i)_\sigma.$$

Let $1 \leq i \leq 3$. Using Lemma 11 we can write, setting $(\tilde{\mathcal{P}}_{\varepsilon_n}^{(i)} \varphi_i)_\sigma = \psi_\sigma$ and using the notation of Lemma 11,

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma, \epsilon} u_\epsilon (\tilde{\mathcal{P}}_{\varepsilon_n}^{(i)} \varphi_i)_\sigma = \sum_{j=1}^3 S_j,$$

where

$$S_i = \sum_{\substack{K=\overrightarrow{[\sigma\sigma']} \\ \sigma,\sigma' \in \mathcal{E}^{(i)}}} (\varrho_\sigma^{up} u_\sigma |D_{K,\sigma}| + \varrho_{\sigma'}^{up} u_{\sigma'} |D_{K,\sigma'}|) (\mathcal{R}_{\mathcal{M}}^{(i)} u_i)_K \frac{\psi_\sigma - \psi_{\sigma'}}{d(x_\sigma, x_{\sigma'})},$$

and, for $j \neq i$,

$$S_j = \sum_{\tau \in \mathcal{E}_{\text{int}}^{(j)}} |D_\tau| \frac{\varrho_\tau^{up} u_\tau}{4} \left[(u_{\sigma_3} + u_{\sigma_1}) \frac{\varphi_{\sigma_1} - \varphi_{\sigma_3}}{d(x_{\sigma_1}, x_{\sigma_3})} + (u_{\sigma_4} + u_{\sigma_2}) \frac{\varphi_{\sigma_2} - \varphi_{\sigma_4}}{d(x_{\sigma_2}, x_{\sigma_4})} \right].$$

Replacing, in S_i , ϱ_σ^{up} by ϱ_K , the term S_i can be written as $S_i = \bar{S}_i + R_i$ with

$$\bar{S}_i = \sum_{\substack{K=\overrightarrow{[\sigma\sigma']} \\ \sigma,\sigma' \in \mathcal{E}^{(i)}}} (\varrho_K u_\sigma |D_{K,\sigma}| + \varrho_K u_{\sigma'} |D_{K,\sigma'}|) (\mathcal{R}_{\mathcal{M}}^{(i)} u_i)_K \frac{\psi_\sigma - \psi_{\sigma'}}{d(x_\sigma, x_{\sigma'})}.$$

Thanks to the weak convergence ϱ in $L^2(\Omega)$, the convergence of \mathbf{u} in $L^4(\Omega)^3$, Lemma 10 and the uniform convergence of the term $\frac{\psi_\sigma - \psi_{\sigma'}}{d(x_\sigma, x_{\sigma'})}$ to $-\partial_i \varphi_i$, we obtain

$$\lim_{n \rightarrow +\infty} \bar{S}_i = - \int_{\Omega} \varrho u_i u_i \partial_i \varphi_i dx.$$

Furthermore, using Hölder inequality and Inequality (6.9) one has $|R_i| \leq C \sqrt{h_{\mathcal{M}_n}}$ and then

$$\lim_{n \rightarrow +\infty} S_i = - \int_{\Omega} \varrho u_i u_i \partial_i \varphi_i dx.$$

For $j \neq i$ we can write $S_j = \bar{S}_j + R_j$ with

$$\begin{aligned} \bar{S}_j &= \sum_{\tau \in \mathcal{E}_{\text{int}}^{(j)}} |D_\tau| \frac{\varrho_\tau^{up} u_\tau}{4} [(u_{\sigma_3} + u_{\sigma_1}) \partial_j \varphi_i(x_\tau) + (u_{\sigma_4} + u_{\sigma_2}) \partial_j \varphi_i(x_\tau)] \\ &= \sum_{\tau \in \mathcal{E}_{\text{int}}^{(j)}} |D_\tau| \varrho_\tau^{up} u_\tau (\mathcal{R}_{\mathcal{E}_n}^{(i,j)} u_i)_\tau \partial_j \varphi_i(x_\tau), \end{aligned}$$

and $|R_j| \leq Ch_{\mathcal{M}_n}$ thanks to the L^2 -bound for ϱ , the L^4 -bound for \mathbf{u} , Lemma 9 and the regularity of φ_i .

Now, as for S_i , we replace ϱ_τ^{up} by ϱ_K or ϱ_L (for $\tau = K|L$), the term \bar{S}_i can be written as $\bar{S}_i = \tilde{S}_i + \tilde{R}_i$ with

$$\tilde{S}_j = \sum_{\tau \in \mathcal{E}_{\text{int}}^{(j)}} (|D_{K,\tau}| \varrho_K + |D_{L,\tau}| \varrho_L) u_\tau (\mathcal{R}_{\mathcal{E}_n}^{(i,j)} u_i)_\tau \partial_j \varphi_i(x_\tau).$$

As for \tilde{S}_i (weak convergence ϱ in $L^2(\Omega)$, convergence of \mathbf{u} in $L^4(\Omega)^3$, Lemma 10 and regularity of φ_i), we obtain

$$\lim_{n \rightarrow +\infty} \tilde{S}_j = - \int_{\Omega} \varrho u_i u_j \partial_j \varphi_i dx.$$

Furthermore, using Hölder inequality and Inequality (6.9) one has $|\tilde{R}_i| \leq C \sqrt{h_{\mathcal{M}_n}}$ and then

$$\lim_{n \rightarrow +\infty} S_j = - \int_{\Omega} \varrho u_i u_j \partial_j \varphi_i dx.$$

Summing the limit of S_j for $j = 1, 2, 3$, we obtain

$$\lim_{n \rightarrow +\infty} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma, \epsilon} u_\epsilon (\tilde{\mathcal{P}}_{\mathcal{E}_n}^{(i)} \varphi_i)_\sigma = - \int_{\Omega} u_i \rho \mathbf{u} \cdot \nabla \varphi_i dx.$$

Now, summing for $i \in \{1, 2, 3\}$ we obtain

$$\int_{\Omega} \text{div}_{\mathcal{E}_n} (\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) \cdot \boldsymbol{\varphi} \, d\mathbf{x} \rightarrow - \int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \, d\mathbf{x} \text{ as } n \rightarrow +\infty.$$

Finally \mathbf{u}, p, ϱ satisfy point 5. and the proof of Proposition 2 is complete. \square

7.2. Passing to the limit in the equation of the state. The goal of this part is to pass to the limit in the nonlinear equation (3.3c). As in the continuous case, the main idea is to prove the a.e convergence of ϱ_n towards ϱ (up to a subsequence).

7.2.1. The effective viscous flux. To overtake this difficulty in the continuous case we have proved that the sequence of approximate solution satisfy (2.14). The following proposition is nothing than the discrete version of this identity.

Proposition 3. *Under the assumptions of Proposition 2 we have for all $\varphi \in C_c^\infty(\Omega)$,*

$$(7.6) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} (p_n - (\lambda + 2\mu) \text{div}_{\mathcal{M}_n} \mathbf{u}_n) \varrho_n \varphi \, d\mathbf{x} = \int_{\Omega} (p - (\lambda + 2\mu) \text{div} \mathbf{u}) \varrho \varphi \, d\mathbf{x},$$

passing to subsequences if necessary.

Remark 5. *The quantity $p(\varrho) - (\lambda + 2\mu) \text{div} \mathbf{u}$ is called usually the effective viscous flux. This quantity enjoys many remarkable properties for which we refer to Hoff [22], Lions [28], or Serre [31]. Note that this quantity is nothing other than the amplitude of the normal viscous stress augmented by the hydrostatic pressure p , that is, the “real” pressure acting on a volume element of the fluid.*

Proof. The following proof can be seen as a discrete version of Step 3 of the proof of Theorem 1. Let $\varphi \in C_c^\infty(\Omega)$. For a MAC grid \mathcal{M} , we define $\varphi_{\mathcal{M}} \in L_{\mathcal{M}}$, $\varphi_{\mathcal{E}}^{(i)} \in H_{\mathcal{E},0}^{(i)}$ by:

$$\begin{cases} \varphi_{\mathcal{M}}(\mathbf{x}) = \varphi(\mathbf{x}_K), \forall \mathbf{x} \in K, \forall K \in \mathcal{M}, \\ \varphi_{\mathcal{E}}^{(i)}(\mathbf{x}) = \varphi(\mathbf{x}_\sigma), \forall \mathbf{x} \in D_\sigma, \forall \sigma \in \mathcal{E}^{(i)}. \end{cases}$$

For a sequence of grids \mathcal{M}_n , for short we shall denote $\varphi_n = \varphi_{\mathcal{M}_n}$. We define w_n with (4.8) (with \mathcal{M}_n and ϱ_n instead of \mathcal{M} and ϱ) and \mathbf{v}_n with $\mathbf{v}_n = -\overline{\nabla}_{\mathcal{E}_n} w_n$. We set $\mathbf{V}_n = (V_{n,1}, V_{n,2}, V_{n,3}) = (v_{n,1} \varphi_{\mathcal{E}_n}^{(1)}, v_{n,2} \varphi_{\mathcal{E}_n}^{(2)}, v_{n,3} \varphi_{\mathcal{E}_n}^{(3)})$.

Thanks to Lemma 6, since ϱ_n is bounded in $L^2(\Omega)$, the compactness theorem 3 gives that, up to a subsequence, as $n \rightarrow \infty$, \mathbf{v}_n converges to some $\mathbf{v} = (v_1, v_2, v_3)$ in $L_{loc}^2(\Omega)^3$ and that $\mathbf{v} \in H_{loc}^1(\Omega)^3$. As a consequence, and using Theorem 2, the sequence $(\mathbf{V}_n)_{n \in \mathbb{N}}$ converges to $\boldsymbol{\varphi} \mathbf{v}$ in $L^q(\Omega)^3$ for any $q \in [1, 6)$. As a consequence of the compactness theorem 3 we also have that $\text{div}_{\mathcal{M}_n} \mathbf{u}_n$ and $\text{curl}_{\mathcal{M}_n} \mathbf{u}_n$ converge weakly in $L^2(\Omega)$ towards $\text{div} \mathbf{u}$ and $\text{curl} \mathbf{u}$.

Since $\mathbf{V}_n \in \mathbf{H}_{\mathcal{E}_n,0}$, it is possible to take \mathbf{V}_n in (3.30):

$$(7.7) \quad \int_{\Omega} \text{div}_{\mathcal{E}_n} (\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) \cdot \mathbf{V}_n \, d\mathbf{x} + (\lambda + 2\mu) \int_{\Omega} \text{div}_{\mathcal{M}_n} \mathbf{u}_n \, \text{div}_{\mathcal{M}_n} \mathbf{V}_n \, d\mathbf{x} \\ + \mu \int_{\Omega} \text{curl}_{\mathcal{M}_n} \mathbf{u}_n \cdot \text{curl}_{\mathcal{M}_n} \mathbf{V}_n \, d\mathbf{x} - \int_{\Omega} p_n \, \text{div}_{\mathcal{M}_n} \mathbf{V}_n \, d\mathbf{x} = \int_{\Omega} \mathcal{P}_{\mathcal{E}_n} \mathbf{f} \cdot \mathbf{V}_n \, d\mathbf{x}.$$

where we have used formula (4.4). We now mimick the proof given in the continuous case for the proof of (2.15). Since $\operatorname{div}_{\mathcal{M}_n} \mathbf{v}_n = \varrho_n$, we first remark that:

$$(7.8) \quad \int_{\Omega} \operatorname{div}_{\mathcal{M}_n} \mathbf{u}_n \operatorname{div}_{\mathcal{M}_n} \mathbf{V}_n \, d\mathbf{x} = \int_{\Omega} (\operatorname{div}_{\mathcal{M}_n} \mathbf{u}_n) \varrho_n \varphi \, d\mathbf{x} + \int_{\Omega} (\operatorname{div}_{\mathcal{M}_n} \mathbf{u}_n) \mathbf{v}_n \cdot \nabla \varphi \, d\mathbf{x} + R_{1,n},$$

where $\lim_{n \rightarrow +\infty} R_{1,n} = 0$, thanks to the discrete $H^1(\Omega)$ -estimate on \mathbf{u}_n and the $L^2_{loc}(\Omega)$ estimate on \mathbf{v}_n . Replacing $\operatorname{div}_{\mathcal{M}_n} \mathbf{u}_n$ by p_n , the same computation gives:

$$(7.9) \quad \int_{\Omega} p_n \operatorname{div}_{\mathcal{M}_n} \mathbf{V}_n \, d\mathbf{x} = \int_{\Omega} p_n \varrho_n \varphi \, d\mathbf{x} + \int_{\Omega} p_n \mathbf{v}_n \cdot \nabla \varphi \, d\mathbf{x} + R_{2,n},$$

where $\lim_{n \rightarrow +\infty} R_{2,n} = 0$. In accordance with [8], the second term of (7.7) can be transformed as follows:

$$(7.10) \quad \int_{\Omega} \operatorname{curl}_{\mathcal{M}_n} \mathbf{u}_n \operatorname{curl}_{\mathcal{M}_n} \mathbf{V}_n \, d\mathbf{x} = \int_{\Omega} \operatorname{curl}_{\mathcal{M}_n} \mathbf{u}_n (\operatorname{curl}_{\mathcal{M}_n} \mathbf{v}_n) \varphi \, d\mathbf{x} + \int_{\Omega} \operatorname{curl}_{\mathcal{M}_n} \mathbf{u}_n \cdot L(\varphi) \bar{\mathbf{v}}_n \, d\mathbf{x} + R_{3,n},$$

where $\lim_{n \rightarrow +\infty} R_{3,n} = 0$ (for the same reasons as $R_{1,n}$), $L(\varphi)$ is the same matrix involving the first order derivatives of φ as in the proof of (2.15) and $\bar{\mathbf{v}}_n$ satisfy:

$$(7.11) \quad \bar{\mathbf{v}}_n \rightarrow \mathbf{v} \text{ in } L^2_{loc}(\Omega)^3 \text{ as } n \rightarrow +\infty.$$

We refer the interested reader to [8] for an explicit expression of $\bar{\mathbf{v}}_n$ and for a proof of (7.11). Since $\operatorname{curl}_{\mathcal{M}_n} \mathbf{v}_n = 0$, (7.10) leads to:

$$(7.12) \quad \int_{\Omega} \operatorname{curl}_{\mathcal{M}_n} \mathbf{u}_n \operatorname{curl}_{\mathcal{M}_n} \mathbf{V}_n \, d\mathbf{x} = \int_{\Omega} \operatorname{curl}_{\mathcal{M}_n} \mathbf{u}_n \cdot L(\varphi) \bar{\mathbf{v}}_n \, d\mathbf{x} + R_{3,n}.$$

Let us turn our attention to the convective term. For the readability, the dependency of some terms with respect to n will be omitted when there are indices related to the mesh (such as σ, ϵ, τ).

One has

$$\int_{\Omega} \operatorname{div}_{\mathcal{E}_n} (\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) \cdot \mathbf{V}_n \, d\mathbf{x} = \sum_{i=1}^3 \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_{\sigma})} F_{\sigma, \epsilon} u_{\epsilon} V_{\sigma},$$

where V_{σ} is the value of $V_{n,i}$ in D_{σ} .

Let $i \in \{1, 2, 3\}$.

Setting $Q_n = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} Q_{\sigma} \chi_{D_{\sigma}}$ with $Q_{\sigma} = (1/|D_{\sigma}|) \sum_{\epsilon \in \tilde{\mathcal{E}}(D_{\sigma})} F_{\sigma, \epsilon} u_{\epsilon}$, one has

$$(7.13) \quad \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_{\sigma})} F_{\sigma, \epsilon} u_{\epsilon} V_{\sigma} = \int_{\Omega} Q_n V_{n,i} \, dx.$$

We recall that the sequence $V_{n,i} \rightarrow \varphi v_i$ in $L^q(\Omega)$ for $q < 6$ (as $n \rightarrow +\infty$). In a first step, we prove that the sequence $(Q_n)_{n \in \mathbb{N}}$ is bounded in $L^p(\Omega)$ for some $p > 6/5$ (indeed we will have p such that $1/p = 1/(2\gamma) + 1/2 + 1/6$ and then $p > 6/5$ since $\gamma > 3$). Then, up to subsequence, $Q_n \rightarrow Q$ weakly in $L^p(\Omega)$. In a second step we identify Q , proving that $Q = \rho \sum_{j=1}^3 u_j \partial_j u_i$.

Estimate on Q_n

For $\sigma \in \mathcal{E}_{\text{int}}^{(i)}$, we use (3.14). It gives

$$(7.14) \quad Q_\sigma = \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} (u_\epsilon - u_\sigma) - C_s h_{\mathcal{M}}^\alpha (\varrho_{D_\sigma} - \varrho^*) u_\sigma.$$

Let $\epsilon \in \tilde{\mathcal{E}}(D_\sigma)$.

If $\epsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}$, $\epsilon \perp \mathbf{e}_i$, $\epsilon \subset K$, then

$$|F_{\sigma,\epsilon}| \leq \frac{1}{2} (|F_{K,\sigma}| + |F_{K,\sigma'}|) \leq \frac{1}{2} (|\sigma| \varrho_\sigma^{up} |u_\sigma| + |\sigma'| \varrho_{\sigma'}^{up} |u_{\sigma'}|).$$

If $\epsilon = \overrightarrow{\sigma|\sigma'} \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}$, $\epsilon \not\perp \mathbf{e}_i$, $\epsilon \subset \tau \cup \tau'$, where τ and τ' are the faces of $\mathcal{E}^{(j)}$ such that $\epsilon \subset \tau \cup \tau'$, then, with $\sigma = K|L$,

$$|F_{\sigma,\epsilon}| \leq \frac{1}{2} (|F_{K,\tau}| + |F_{\tau',L}|) \leq \frac{1}{2} (|\tau| \varrho_\tau^{up} |u_\tau| + |\tau'| \varrho_{\tau'}^{up} |u_{\tau'}|).$$

Using the estimates on ϱ in $L^{2\gamma}(\Omega)$, \mathbf{u} in $L^6(\Omega)$, $\nabla \varepsilon u_i$ in $L^2(\Omega)$ and the fact that $\eta_n \geq \eta$ for all n , the part of Q given by the first term of (7.14) is bounded in $L^p(\Omega)$ with p such that $1/p = 1/(2\gamma) + 1/2 + 1/6$. The part of Q given by the second term of (7.14) tends to 0 in $L^{3/2}(\Omega)$ for instance (since ϱ is bounded in $L^2(\Omega)$ and \mathbf{u} in $L^6(\Omega)$) and then also in $L^p(\Omega)$.

Then, up to a subsequence, we can assume that $Q_n \rightarrow Q$ weakly in $L^p(\Omega)$ and this gives

$$(7.15) \quad \lim_{n \rightarrow +\infty} \int_\Omega Q_n V_{n,i} dx = \int_\Omega Q \varphi v_i dx.$$

Identification of Q

Let $\varphi \in C_c^\infty(\Omega)$. For $\sigma \in \mathcal{E}_{\text{int}}^{(i)}$, let $\varphi_\sigma = (\tilde{\mathcal{P}}_{\mathcal{E}_n}^{(i)} \varphi_i)_\sigma$.

Then, one has (at least for h_n small enough)

$$\int_\Omega Q_n \varphi dx = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} u_\epsilon \varphi_\sigma.$$

We already pass to the limit on this term in Proposition 2 where we obtain

$$\lim_{n \rightarrow +\infty} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} u_\epsilon (\tilde{\mathcal{P}}_{\mathcal{E}_n}^{(i)} \varphi_i)_\sigma = - \int_\Omega u_i \rho \mathbf{u} \cdot \nabla \varphi_i dx.$$

Then $\int_\Omega Q \varphi dx = - \int_\Omega u_i \rho \mathbf{u} \cdot \nabla \varphi_i dx$. Since we already know that $\text{div}(\varrho \mathbf{u}) = 0$ we obtain (using $u_i \in H^1(\Omega)$ and $\varrho \mathbf{u} \in L^2(\Omega)^3$)

$$Q = \sum_{j=1}^3 \varrho u_j \partial_j u_i.$$

Finally, we have the limit of the convection term:

$$(7.16) \quad \lim_{n \rightarrow +\infty} \int_\Omega \text{div}_{\mathcal{E}_n} (\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) \cdot \mathbf{V}_n dx = \int_\Omega \sum_{i=1}^3 \sum_{j=1}^3 \varrho u_j (\partial_j u_i) \varphi v_i dx.$$

We recall now that $(\mathbf{V}_n)_{n \in \mathbb{N}}$ converges to $\varphi \mathbf{v}$ in $L^q(\Omega)^3$ for any $q \in [1, 6)$ and that we also have that $\text{div}_{\mathcal{M}_n} \mathbf{u}_n$, $\text{curl}_{\mathcal{M}_n} \mathbf{u}_n$ and p_n weakly converge in $L^2(\Omega)$ towards $\text{div} \mathbf{u}$, $\text{curl} \mathbf{u}$ and p .

Then, using (7.8)–(7.12), we deduce from (7.7) and (7.16):

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\Omega} \left((\lambda + 2\mu) \operatorname{div}_{\mathcal{M}} \mathbf{u}_n - p_n \right) \varrho_n \varphi \, d\mathbf{x} &= \int_{\Omega} \left(p - (\lambda + 2\mu) \operatorname{div} \mathbf{u} \right) \mathbf{v} \cdot \nabla \varphi \, d\mathbf{x} \\ &\quad - \mu \int_{\Omega} (\operatorname{curl} \mathbf{u}) L(\varphi) \mathbf{v} \, d\mathbf{x} - \int_{\Omega} \varrho ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \varphi \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \varphi \, d\mathbf{x}. \end{aligned}$$

Finally, since p_n and \mathbf{u}_n are solution of the discrete momentum balance equations, we already know thanks to the estimates on p_n and ρ_n that the limits p and \mathbf{u} are solution of the momentum balance equation; hence, since $\mathbf{v} \in \mathbf{H}_{loc}^1(\Omega)^3$ and in accordance with the continuous case:

$$\begin{aligned} \int_{\Omega} \left((2\mu + \lambda) \operatorname{div} \mathbf{u} - p \right) (\operatorname{div} \mathbf{v}) \varphi \, d\mathbf{x} - \int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla(\varphi \mathbf{v}) \, d\mathbf{x} = \\ \int_{\Omega} \left((p - (2\mu + \lambda) \operatorname{div} \mathbf{u}) \mathbf{v} \cdot \nabla \varphi - \mu (\operatorname{curl} \mathbf{u}) \cdot L(\varphi) \mathbf{v} - \mu \operatorname{curl} \mathbf{u} \cdot (\operatorname{curl} \mathbf{v}) \varphi + \mathbf{f} \cdot \mathbf{v} \varphi \right) d\mathbf{x}. \end{aligned}$$

Moreover we know that $\operatorname{div}(\varrho \mathbf{u}) = 0$ and $(\varrho, \mathbf{u}) \in L^6(\Omega) \times H_0^1(\Omega)^3$ and consequently $\int_{\Omega} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla(\varphi \mathbf{v}) \, d\mathbf{x} = - \int_{\Omega} \varrho (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \varphi \mathbf{v} \, d\mathbf{x}$. Since $\operatorname{div}_{\mathcal{M}_n} \mathbf{v}_n$ and $\operatorname{curl}_{\mathcal{M}_n} \mathbf{v}_n$ converge weakly in $L_{loc}^2(\Omega)$ towards $\operatorname{div} \mathbf{v}$ and $\operatorname{curl} \mathbf{v}$, one has $\operatorname{div} \mathbf{v} = \varrho$ and $\operatorname{curl} \mathbf{v} = 0$ and therefore:

$$\begin{aligned} \int_{\Omega} \left((2\mu + \lambda) \operatorname{div} \mathbf{u} - p \right) \varrho \varphi \, d\mathbf{x} &= \int_{\Omega} \left((p - (2\mu + \lambda) \operatorname{div} \mathbf{u}) \mathbf{v} \cdot \nabla \varphi - \mu (\operatorname{curl} \mathbf{u}) \cdot L(\varphi) \mathbf{v} \right) d\mathbf{x} \\ &\quad - \int_{\Omega} \varrho (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \varphi \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \varphi \, d\mathbf{x}. \end{aligned}$$

Then, we obtain the desired result, that is:

$$(7.17) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} (p_n - (\lambda + 2\mu) \operatorname{div}_{\mathcal{M}_n} \mathbf{u}_n) \varrho_n \varphi \, d\mathbf{x} = \int_{\Omega} (p - (\lambda + 2\mu) \operatorname{div} \mathbf{u}) \varrho \varphi \, d\mathbf{x}.$$

□

7.2.2. *A.e. and strong convergence of ϱ_n and p_n .* Let us now prove the a.e. convergence of ϱ_n and p_n . Using [9, Lemma 2.1], one can take $\varphi = 1$ in (7.6), which gives:

$$\lim_{n \rightarrow +\infty} \int_{\Omega} (p_n - (2\mu + \lambda) \operatorname{div}_{\mathcal{M}_n} \mathbf{u}_n) \varrho_n \, d\mathbf{x} = \int_{\Omega} (p - (2\mu + \lambda) \operatorname{div}_{\mathcal{M}} \mathbf{u}) \varrho \, d\mathbf{x}$$

Now using Lemma 7 and (2.5) we obtain the discrete version of (2.15) that is

$$(7.18) \quad \limsup_{n \rightarrow +\infty} \int_{\Omega} p_n \varrho_n \, d\mathbf{x} \leq \int_{\Omega} p \varrho \, d\mathbf{x}.$$

Let $G_n = (\varrho_n^\gamma - \varrho^\gamma)(\varrho_n - \varrho)$. One has $G_n \in L^1(\Omega)$ and $G_n \geq 0$ a.e. in Ω . Furthermore:

$$\int_{\Omega} G_n \, d\mathbf{x} = \int_{\Omega} p_n \varrho_n \, d\mathbf{x} - \int_{\Omega} p_n \varrho \, d\mathbf{x} - \int_{\Omega} \varrho^\gamma \varrho_n \, d\mathbf{x} + \int_{\Omega} \varrho^\gamma \varrho \, d\mathbf{x}.$$

Using the weak convergence in $L^2(\Omega)$ of p_n and ϱ_n , and (7.18), we obtain:

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} G_n \, d\mathbf{x} \leq 0.$$

Then (up to a subsequence), $G_n \rightarrow 0$ a.e. and then $\varrho_n \rightarrow \varrho$ a.e. (since $y \mapsto y^\gamma$ is an increasing function on \mathbb{R}_+). Finally, $\varrho_n \rightarrow \varrho$ in $L^q(\Omega)$ for all $1 \leq q < 2\gamma$, $p_n = \varrho_n^\gamma \rightarrow \varrho^\gamma$ in $L^q(\Omega)$ for all $1 \leq q < 2$, and $p = \varrho^\gamma$. We have thus proved the convergence of the approximate pressure and density, which, together with Proposition 2 concludes the proof of Theorem 4.

8. CONCLUSION

In this paper, we considered the MAC scheme for the stationary barotropic compressible Navier-Stokes equations. This scheme, which is very popular in the computational fluid dynamics community, is also proved to be quite adapted to a convergence analysis. Ongoing work concerns the extension to the non stationary Navier-Stokes equations in two or three space dimensions.

APPENDIX A. EXISTENCE OF A DISCRETE SOLUTION

This section is devoted to the proof of Theorem 5. We now state the abstract theorem which will be used hereafter.

Theorem 6. *Let N and M be two positive integers and V be defined as follows:*

$$V = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^M, y > 0\},$$

where, for any real number c , the notation $y > c$ means that each component of y is greater than c . Let F be a continuous function from $V \times [0, 1]$ to $\mathbb{R}^N \times \mathbb{R}^M$ satisfying:

(1) $\forall \zeta \in [0, 1]$, if $v \in V$ is such that $F(v, \zeta) = 0$ then $v \in W$ where W is defined as follows:

$$W = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^M, \|x\| < C_1, \text{ and } \varepsilon < y < C_2\},$$

with C_1, C_2 and ε three positive constants and $\|\cdot\|$ a norm defined over \mathbb{R}^N ;

(2) the topological degree of $F(\cdot, 0)$ with respect to 0 and W is equal to $d_0 \neq 0$.

Then the topological degree of $F(\cdot, 1)$ with respect to 0 and W is also equal to $d_0 \neq 0$; consequently, there exists at least a solution $v \in W$ such that $F(v, 1) = 0$.

Let us now prove the existence of a solution to (3.3). Let us define

$$V = \{(\mathbf{u}, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}, \varrho_K > 0 \forall K \in \mathcal{M}\}.$$

and consider the continuous mapping

$$\begin{aligned} F : \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}} \times [0, 1] &\longrightarrow \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}} \\ (\mathbf{u}, \varrho, \zeta) &\mapsto F(\mathbf{u}, \varrho, \zeta) = (\hat{\mathbf{u}}, \hat{\varrho}) \end{aligned}$$

where $(\hat{\mathbf{u}}, \hat{\varrho})$ is the unique element of $\mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}$ such that

$$\begin{aligned} \int_{\Omega} \hat{\mathbf{u}} \cdot \mathbf{v} \, d\mathbf{x} &= \mu[\mathbf{u}, \mathbf{v}]_{1,\mathcal{E},0} + (\mu + \lambda) \int_{\Omega} \operatorname{div}_{\mathcal{M}} \mathbf{u} \operatorname{div}_{\mathcal{M}} \mathbf{v} \, d\mathbf{x} \\ \text{(A.1)} \quad &+ \zeta \int_{\Omega} \operatorname{div}_{\mathcal{E}}(\varrho \mathbf{u} \otimes \mathbf{u}) \cdot \mathbf{v} \, d\mathbf{x} - \zeta \int_{\Omega} \varrho^\gamma \operatorname{div}_{\mathcal{M}} \mathbf{v} - \int_{\Omega} \mathcal{P}_{\mathcal{E}} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{H}_{\mathcal{E},0}, \end{aligned}$$

$$\text{(A.2)} \quad \int_{\Omega} \hat{\varrho} q \, d\mathbf{x} = \zeta \int_{\Omega} \operatorname{div}_{\mathcal{M}}^{\text{up}}(\varrho \mathbf{u}) q \, d\mathbf{x} + \int_{\Omega} C_s h_{\mathcal{M}}^{\alpha}(\varrho - \varrho^*) q \, d\mathbf{x}, \quad \forall q \in L_{\mathcal{M}}.$$

Note that the values of $\hat{u}_i; i = 1, \dots, d$, and $\hat{\varrho}$ are readily obtained by setting in this system $v_i = 1_{D_\sigma}$, $v_j = 0, j \neq i$ in (A.1) and $q = 1_K$ in (A.2).

Any solution of $F(\mathbf{u}, \varrho, 1) = 0$ is a solution of Problem 3.3 where $p = \varrho^\gamma$.

The mapping F is continuous.

Let $(\mathbf{u}, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}$ and $\zeta \in [0, 1]$ such that $F(\mathbf{u}, \varrho, \zeta) = (0, 0)$ (in particular $\varrho > 0$). Then for any $(\mathbf{v}, q) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}$,

$$(A.3a) \quad \zeta \int_{\Omega} \operatorname{div}_{\mathcal{E}}(\varrho \mathbf{u} \otimes \mathbf{u}) \, d\mathbf{x} + \mu[\mathbf{u}, \mathbf{v}]_{1,\mathcal{E},0} + (\mu + \lambda) \int_{\Omega} \operatorname{div}_{\mathcal{M}} \mathbf{u} \operatorname{div}_{\mathcal{M}} \mathbf{v} \, d\mathbf{x} - \zeta \int_{\Omega} \varrho^{\gamma} \operatorname{div}_{\mathcal{M}} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathcal{P}_{\mathcal{E}} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x},$$

$$(A.3b) \quad \zeta \int_{\Omega} \operatorname{div}_{\mathcal{M}}^{\text{up}}(\varrho \mathbf{u}) q \, d\mathbf{x} + \int_{\Omega} C_s h_{\mathcal{M}}^{\alpha} (\varrho - \varrho^*) q \, d\mathbf{x} = 0.$$

Taking $q = 1$ as a test function in (A.3b), and using the conservativity of the fluxes we obtain

$$(A.4) \quad \int_{\Omega} \varrho \, d\mathbf{x} = \|\varrho\|_{L^1(\Omega)} = M > 0.$$

This relation provides a bound for ϱ in the L^1 norm, and therefore in all norms since the problem is of finite dimension.

Taking \mathbf{u} as a test function in (A.3a) and following Step 1 of the proof of Proposition 1 gives

$$(A.5) \quad \|\mathbf{u}\|_{1,\mathcal{E},0} < C_1$$

where the constant C_1 depends only on the data of the problem. Now a straightforward computation gives

$$\varrho_K \geq \frac{C_s \min_{L \in \mathcal{M}} |L| h_{\mathcal{M}}^{\alpha} \varrho^*}{C_s h_{\mathcal{M}}^{\alpha} |\Omega| + \sum_{\sigma \in \mathcal{E}_{\text{int}}, \sigma=K|L} |\sigma| |u_{K,\sigma}|}$$

Consequently by virtue of (A.5) there exists $\varepsilon > 0$ such that

$$(A.6) \quad \varrho_K > \varepsilon, \quad \forall K \in \mathcal{M},$$

where the constant ε depends only on the data of the problem. Clearly from (A.4) one has also

$$(A.7) \quad \varrho_K \leq \frac{M}{\min_{K \in \mathcal{M}} |K|} = C_2 - 1, \quad \forall K \in \mathcal{M}.$$

Moreover $\zeta = 0$ the system $F(\mathbf{u}, \varrho, 0) = 0$ reads:

$$(A.8a) \quad \mu[\mathbf{u}, \mathbf{v}]_{1,\mathcal{E},0} + (\mu + \lambda) \int_{\Omega} \operatorname{div}_{\mathcal{M}} \mathbf{u} \operatorname{div}_{\mathcal{M}} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathcal{P}_{\mathcal{E}} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{H}_{\mathcal{E},0},$$

$$(A.8b) \quad \varrho_K = \varrho^*, \quad \forall K \in \mathcal{M}.$$

which has clearly one and only one solution. Let W defined by

$$W = \{(\mathbf{u}, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}} \text{ such that } \|\mathbf{u}\| < C_1, \varepsilon < \varrho_K < C_2\}$$

Since $F(\mathbf{u}, \varrho, 0) = 0$ is a linear system which has one and only one solution belonging to W , the topological degree d_0 of $F(\cdot, \cdot, 0)$ with respect to 0 and W is not zero. Then, using the inequalities (A.5), (A.6), (A.7), Theorem 6 applies, which concludes the proof.

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