



**HAL**  
open science

# On the regularity of a generalized diffusion problem arising in population dynamics set in a cylindrical domain

R Labbas, S Maingot, D Manceau, A Thorel

► **To cite this version:**

R Labbas, S Maingot, D Manceau, A Thorel. On the regularity of a generalized diffusion problem arising in population dynamics set in a cylindrical domain. 2016. hal-01342758

**HAL Id: hal-01342758**

**<https://hal.science/hal-01342758>**

Preprint submitted on 6 Jul 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the regularity of a generalized diffusion problem arising in population dynamics set in a cylindrical domain

R. Labbas, S. Maingot, D. Manceau & A. Thorel\*

Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, 76600 Le Havre, France

## Abstract

In this paper, we consider a generalized diffusion problem arising in population dynamics. To this end, we study a fourth order operational equation of elliptic type, with various boundary conditions. We show existence, uniqueness and regularity of a classical solution on a cylindrical domain under some necessary and sufficient conditions on the data. This elliptic problem is solved in  $L^p(a, b; X)$ ,  $p \in (1, +\infty)$ , where  $(a, b) \subset \mathbb{R}$  and  $X$  is a UMD Banach space. Our techniques use essentially the functional calculus and the semigroup theory.

2010 Mathematics Subject Classification. 35A09, 35B65, 35C15, 35J30, 35R20, 47D06.

Keywords: Population dynamics, Abstract differential equations, UMD spaces, Bounded imaginary powers, Interpolation spaces, Maximal regularity.

\* This research is supported by CIFRE contract 2014/1307 with Qualiom Eco company.

## 1 Introduction

Many problems in biology may be described by partial differential equations. It means that the model is constructed by averaging the density of a population and keeping only time and space variables. This leads to study the population density, so-called population dynamics, governed by reaction-diffusion equations. The model considered in general is the following:

$$\partial_t u = \alpha \Delta u + f(u), \quad \text{in } \mathbb{R}_+ \times \Omega, \quad (1.1)$$

where  $u(t, x)$  is the density of population at time  $t$  and position  $x$ ,  $\Omega$  is an open set of  $\mathbb{R}^d$ ,  $d \geq 1$ , and  $f$  is a non-linear growth interaction. The diffusion operator term  $\Delta u$  is obtained from Fick's law. In 1981, D.S. Cohen and J.D. Murray [6] derived a more complete model with a biharmonic term added to the harmonic one. This model has been obtained by studying the motility of cells, for which they noticed that the classical diffusion model was not sufficient. This model is:

$$\partial_t u = k_1 \Delta u - k_2 \Delta^2 u + f(u) \quad \text{in } \mathbb{R}_+ \times \Omega, \quad (1.2)$$

where  $k_1, k_2 \in \mathbb{R} \setminus \{0\}$ . The biharmonic term  $\Delta^2 u$  represents the long range diffusion, whereas the harmonic term  $\Delta u$  represents the short range diffusion. Another derivation of this model has been obtained in 1984 by F.L. Ochoa [20] who studied the dynamics of (1.2) when  $f$  is a cubic function. His work brings out the importance of the biharmonic term in the diffusion of the population.

The classical study of nonlinear problems like (1.1) or (1.2) needs first to consider the associated linearized steady problem. This first step is essential to deduce maximal regularity results for the linearized unsteady problem. From these maximal regularity results, one can obtain the existence and uniqueness of the nonlinear problem by a fixed point theorem. In the case where  $\Omega$  is an open set with  $C^4$ -boundary, the result is well-known. For problems (1.1), we can find a detailed study in [22], see section 4, p. 13 for instance. For problems (1.2), results of existence and regularity are obtained in Hilbertian case in [19] and some other results were recently obtained in [5].

In this paper, we study problem (1.2) in the linear steady case on a bounded cylindrical open set  $\Omega := (a, b) \times \omega$  of  $\mathbb{R}^d$ , with  $\omega$  an open bounded set of  $\mathbb{R}^{d-1}$  with  $C^2$ -boundary and  $f \in L^p(\Omega)$ ,  $p \in (1, +\infty)$ , *i.e.*

$$k_2 \Delta^2 u - k_1 \Delta u = f \quad \text{on } \Omega. \quad (1.3)$$

We follow the work of [11] for second order problems that we adapt to fourth order problems. For this, we set  $k := k_1/k_2$  and we consider the Laplace operator  $A_0$  on the Banach space  $L^q(\omega)$ ,  $q \in (1, +\infty)$ , with homogeneous Dirichlet boundary condition on  $\omega$ . Thus, equation (1.3) can be formulated as the fourth order operational differential equation

$$u^{(4)}(x) + (2A_0 - kI)u''(x) + (A_0^2 - kA_0)u(x) = f(x), \quad x \in (a, b),$$

where  $f \in L^p(a, b; L^q(\omega))$ ,  $p \in (1, +\infty)$ , with  $u(x) := u(x, \cdot)$  and  $f(x) := f(x, \cdot)$ . Then, we will consider a generalization of this problem, that is

$$u^{(4)}(x) + (2A - kI)u''(x) + (A^2 - kA)u(x) = f(x), \quad x \in (a, b), \quad (1.4)$$

under various boundary conditions, with  $(A, D(A))$  a BIP operator of angle  $\theta \in (0, \pi)$  on a UMD space  $X$  (see section 2 below for the definitions of BIP operator and UMD spaces) and  $f \in L^p(a, b; X)$ . More precisely, we study (1.4) under one of the following boundary conditions:

$$\begin{cases} u(a) = \varphi_1, & u(b) = \varphi_2, \\ u''(a) = \varphi_3, & u''(b) = \varphi_4, \end{cases} \quad (\text{BC1})$$

$$\begin{cases} u'(a) = \varphi_1, & u'(b) = \varphi_2, \\ u''(a) + Au(a) = \varphi_3, & u''(b) + Au(b) = \varphi_4, \end{cases} \quad (\text{BC2})$$

$$\begin{cases} u(a) = \varphi_1, & u(b) = \varphi_2, \\ u'(a) = \varphi_3, & u'(b) = \varphi_4, \end{cases} \quad (\text{BC3})$$

or

$$\begin{cases} u'(a) = \varphi_1, & u'(b) = \varphi_2, \\ u''(a) = \varphi_3, & u''(b) = \varphi_4, \end{cases} \quad (\text{BC4})$$

where  $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in X$ . We obtain (see Theorem 2.5) the existence and uniqueness of a classical solution  $u$  of (1.4)-(BC*i*),  $i = 1, 2, 3, 4$ , if and only if  $\varphi_1, \varphi_2, \varphi_3, \varphi_4$  are in some real interpolation spaces.

Let  $B_0, \dots, B_{n-1}$  be linear operators in  $X$ , we recall that  $u$  is a classical solution (in  $L^p(a, b; X)$ ) of a differential equation

$$u^{(n)}(x) + \sum_{j=0}^{n-1} B_j u^{(j)}(x) = f(x), \quad x \in (a, b), \quad (1.5)$$

if  $u$  is a solution of (1.5) with the following regularity

$$u \in W^{n,p}(a, b; X) \quad \text{and} \quad x \mapsto B_j u^{(j)}(x) \in L^p(a, b; X), \quad j = 0, \dots, n-1. \quad (1.6)$$

Similarly,  $u$  is a classical solution of problem (1.5)-(BC), where (BC) are some boundary conditions if  $u$  is a classical solution of (1.5) and satisfies (BC).

The paper is organised as follows. In section 2, first we detail our assumptions on space  $X$  and operator  $A$  and we describe our results. First, we state the existence and the uniqueness of the solution of (1.4)-(BC2) in Theorem 2.2. This allows us to establish in Proposition 2.3, a general representation formula for classical solutions of (1.4). Then, we state a result about the regularity of a difference of two analytic semigroups (Theorem 2.4) which is an interesting result in itself and constitutes an important tool to prove the main result (Theorem 2.5). We then present Theorem 2.5, generalizing Theorem 2.2, which gives existence and uniqueness of a classical solution  $u$  of (1.4) under one of the boundary conditions (BC1), (BC2), (BC3) or (BC4). As a consequence of this theorem, we obtain the Corollary 2.7 which states existence and uniqueness of a solution  $u$  in  $W^{4,p}(\Omega)$  of (1.3). In section 3, we prove Theorem 2.2 and Proposition 2.3. Section 4 is devoted to the proof of Theorem 2.4. Finally, in the last section, we prove Theorem 2.5.

## 2 Assumptions and statement of results

In all the paper,  $(X, \|\cdot\|)$  is a complex Banach space and  $\mathcal{L}(X)$  stands for the space of bounded linear operators in  $X$ . For any linear operator  $T$  in  $X$ , we denote by  $D(T), R(T)$  the domain and the range of  $T$  while

$$\rho(T) := \{\lambda \in \mathbb{C} : (\lambda I - T)^{-1} \in \mathcal{L}(X)\} \quad \text{and} \quad \sigma(T) := \mathbb{C} \setminus \rho(T),$$

denote the resolvent set and the spectrum of  $T$ . Recall (see, for instance, [14]) that a closed linear operator  $T$  is called sectorial of angle  $\theta \in (0, \pi)$  if

- i)  $\sigma(T) \subset \overline{S_\theta}$ ,
- ii)  $\forall \theta' \in (\theta, \pi), \quad M(T, \theta') := \sup \{ \|\lambda(\lambda I - T)^{-1}\|_{\mathcal{L}(X)}, \lambda \in \mathbb{C} \setminus \overline{S_{\theta'}} \} < \infty,$

where

$$S_\theta := \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \theta\}. \quad (2.1)$$

It is known that any injective sectorial operator  $T$  admits imaginary powers  $T^{is}$ ,  $s \in \mathbb{R}$ , but, in general,  $T^{is}$  is not bounded (see [14], p. 70). We say that a linear operator  $T$  in  $X$  has bounded imaginary powers, denoted by  $T \in \text{BIP}(X)$ , if  $T$  is an injective sectorial operator such that

$$\overline{D(T) \cap R(T)} = X \quad \text{and} \quad \forall s \in \mathbb{R}, \quad T^{is} \in \mathcal{L}(X).$$

Moreover, if  $T \in \text{BIP}(X)$ , we set

$$\theta_T := \inf \left\{ \theta \geq 0 : \exists C > 0, \|T^{is}\|_{\mathcal{L}(X)} \leq Ce^{|s|\theta}, \forall s \in \mathbb{R} \right\},$$

and we write  $T \in \text{BIP}(X, \theta)$  if  $T \in \text{BIP}(X)$  and  $\theta \geq \theta_T$ . All the results stated in this paper used the well-known Dore-Venni Theorem (see [8] and its generalization [23]), which needs to consider a UMD space  $X$ . Recall that a Banach space  $X$  is a UMD space if and only if for some  $p \in (1, +\infty)$  (and thus for all  $p$ ) the Hilbert transform is bounded from  $L^p(\mathbb{R}, X)$  into itself ([1, 3, 4]).

In all the sequel,  $k \in \mathbb{R} \setminus \{0\}$ ,  $A$  denotes a linear operator in  $X$  and we assume:

- (H<sub>1</sub>)  $X$  is a UMD space,
- (H<sub>2</sub>)  $0 \in \rho(A)$ ,
- (H<sub>3</sub>)  $-A \in \text{BIP}(X, \theta)$  for some  $0 < \theta < \pi$ ,
- (H<sub>4</sub>)  $[k, +\infty) \subset \rho(A)$ .

For some of our results, we need a supplementary assumption:

$$(H_5) \quad \sigma(A) \subset (-\infty, 0) \quad \text{and} \quad \forall \theta \in (0, \pi), \sup_{\lambda \in S_\theta} \|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} < +\infty,$$

which means that is a sectorial operator of any angle  $\theta \in (0, \pi)$ . Let us give some remarks on our assumptions.

**Remark 2.1.**

1. Assumption (H<sub>4</sub>) is relevant only if  $k < 0$ , since for  $k > 0$ , (H<sub>2</sub>) and (H<sub>3</sub>) imply (H<sub>4</sub>).
2. The case  $k = 0$  has been stated (with particular boundary conditions) in [10]. The study is different from our and requires a particular representation formula for the solution.
3. To solve (1.4) in the scalar case (with  $-A > 0$ ), it is necessary to introduce the roots  $\pm\sqrt{-A+k}$ ,  $\pm\sqrt{-A}$  of the characteristic equation

$$r^4 + (2A - k)r^2 + (A^2 - kA)r = 0,$$

this is why, in our operational case, we consider the operators

$$L := -\sqrt{-A+k}I \quad \text{and} \quad M := -\sqrt{-A}. \quad (2.2)$$

Due to (H<sub>3</sub>) and (H<sub>4</sub>),  $-A$  and  $-A+kI$  are sectorial operators, so the existence of  $L$  and  $M$  is ensured (see for instance [14]).

4. Applying Proposition 3.19 in [14], we have  $D(L) = D(M)$ . Thus, for  $k, l \in \mathbb{N}$  and  $l \leq k$

$$D(L^k) = D(M^k) = D(L^l M^{k-l}).$$

5. Due to (H<sub>3</sub>) and (H<sub>4</sub>), we get that  $-A+kI$  is sectorial. So  $-A+kI \in \text{BIP}(X, \theta)$  (see [2], Theorem 2.3, p. 69), from which we deduce

$$-L, -M \in \text{BIP}(X, \theta/2),$$

(see [14], Proposition 3.2.1, e), p. 71). Since  $0 < \theta/2 < \pi/2$ ,  $L$  and  $M$  generate analytic semigroups  $(e^{xL})_{x \geq 0}$  and  $(e^{xM})_{x \geq 0}$  (see [23], Theorem 2, p. 437).

6. From  $(H_2)$  and  $(H_4)$ , we deduce that  $0 \in \rho(M) \cap \rho(L)$  (see [14], Proposition 3.3.1, e), p. 62). Thus, assumptions  $(H_1)$ ,  $(H_2)$  and  $(H_3)$  lead us to apply Dore-Venni Theorem [8] to obtain  $0 \in \rho(L + M)$ .
7. Definitions (2.2) imply that

$$\forall \psi \in D(L^2) = D(M^2), \quad (L^2 - M^2)\psi = k\psi. \quad (2.3)$$

Moreover  $L - M = (L - M)(L + M)(L + M)^{-1}$ , so

$$\forall \psi \in D(L), \quad (L - M)\psi = k(L + M)^{-1}\psi. \quad (2.4)$$

To determine the conditions on  $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in X$  characterizing the existence of a unique classical solution of (1.4)-(BC1), (1.4)-(BC2), (1.4)-(BC3) and (1.4)-(BC4), we need to introduce interpolation spaces. To this end, we denote  $(Y, X)_{\theta, q}$ ,  $0 < \theta < 1$ ,  $1 \leq q \leq \infty$ , the real interpolation spaces (see, for instance, [13, 16]) between a subspace  $Y \subset X$  and  $X$ . Moreover, we will use the following notation:

$$(D(A), X)_{k+\theta, q} := \{\psi \in D(A^k) : A^k\psi \in (D(A), X)_{\theta, q}\}.$$

First, we solve problem (1.4)-(BC1) whose boundary conditions are well adapted to the equation and which can be easily treated by considering two second order problems. We obtain:

**Theorem 2.2.** *Let  $f \in L^p(a, b; X)$  with  $a, b \in \mathbb{R}$ ,  $a < b$  and  $p \in (1, \infty)$ . Assume that  $(H_1)$ ,  $(H_2)$ ,  $(H_3)$  and  $(H_4)$  hold. Then, there exists a unique classical solution of (1.4)-(BC1) if and only if*

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}.$$

*This unique classical solution is denoted by  $F_{\Phi, f}$  with  $\Phi := (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$  and is explicitly described by*

$$\begin{aligned} F_{\Phi, f}(x) &= e^{(x-a)M} Z \varphi_1 + e^{(b-x)M} Z \varphi_2 \\ &\quad - \frac{1}{2} e^{(x-a)M} Z M^{-1} \int_a^b e^{(s-a)M} v_0(s) ds - \frac{1}{2} e^{(b-x)M} Z M^{-1} \int_a^b e^{(b-s)M} v_0(s) ds \\ &\quad + \frac{1}{2} M^{-1} \int_a^x e^{(x-s)M} v_0(s) ds + \frac{1}{2} M^{-1} \int_x^b e^{(s-x)M} v_0(s) ds \\ &\quad - e^{(b-x)M} e^{(b-a)M} \varphi_1 - e^{(x-a)M} e^{(b-a)M} \varphi_2 + \frac{1}{2} e^{(x-a)M} Z e^{(b-a)M} M^{-1} \int_a^b e^{(b-s)M} v_0(s) ds \\ &\quad + \frac{1}{2} e^{(b-x)M} Z e^{(b-a)M} M^{-1} \int_a^b e^{(s-a)M} v_0(s) ds, \quad x \in [a, b], \end{aligned}$$

where

$$\begin{aligned} v_0(x) &:= e^{(x-a)L} W (\varphi_3 + A\varphi_1) + e^{(b-x)L} W (\varphi_4 + A\varphi_2) \\ &\quad - \frac{1}{2} e^{(x-a)L} W L^{-1} \int_a^b e^{(s-a)L} f(s) ds - \frac{1}{2} e^{(b-x)L} W L^{-1} \int_a^b e^{(b-s)L} f(s) ds \\ &\quad + \frac{1}{2} L^{-1} \int_a^x e^{(x-s)L} f(s) ds + \frac{1}{2} L^{-1} \int_x^b e^{(s-x)L} f(s) ds \\ &\quad - e^{(b-x)L} e^{(b-a)L} (\varphi_3 + A\varphi_1) - e^{(x-a)L} e^{(b-a)L} (\varphi_4 + A\varphi_2) \\ &\quad + \frac{1}{2} e^{(x-a)L} W e^{(b-a)L} L^{-1} \int_a^b e^{(b-s)L} f(s) ds \\ &\quad + \frac{1}{2} e^{(b-x)L} W e^{(b-a)L} L^{-1} \int_a^b e^{(s-a)L} f(s) ds, \quad x \in [a, b], \end{aligned}$$

$$Z := (I - e^{2(b-a)M})^{-1} \quad \text{and} \quad W := (I - e^{2(b-a)L})^{-1}.$$

The existence of  $Z$  and  $W$  are proved in virtue of (5.9) in subsection 5.1.

Under other boundary conditions, equation (1.4) is more complicated to study. We need two results which are important by themselves. The first one is a representation formula for the classical solution of equation (1.4).

**Proposition 2.3.** *If  $u$  is a classical solution of (1.4), then there exist  $K_i \in X$ ,  $i = 1, 2, 3, 4$ , such that for any  $x \in [a, b]$*

$$u(x) = e^{(x-a)M}K_1 + e^{(b-x)M}K_2 + e^{(x-a)L}K_3 + e^{(b-x)L}K_4 + F_{0,f}(x). \quad (2.5)$$

where  $F_{0,f}$  is defined in Theorem 2.2.

The second result concerns the regularity of the difference of two analytic semigroups stated below.

**Theorem 2.4.** *Assume that  $(H_1)$ ,  $(H_2)$ ,  $(H_3)$  and  $(H_4)$  hold. For any  $\psi \in X$  and  $x \in [a, b]$ , we set*

$$u_\psi(x) := \left( e^{(x-a)L} - e^{(x-a)M} \right) \psi \quad \text{and} \quad v_\psi(x) := \left( e^{(b-x)L} - e^{(b-x)M} \right) \psi.$$

Then, we have

1.  $u_\psi \in W^{2,p}(a, b; X) \cap L^p(a, b; D(M^2))$  and  $u'_\psi \in L^p(a, b; D(M))$ .
2.  $u_\psi \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4))$  if and only if  $\psi \in (D(M), X)_{1+\frac{1}{p}, p}$ .  
Furthermore, in this case,  $u''_\psi \in L^p(a, b; D(M^2))$ .
3. Statements 1. and 2. hold true if we replace  $u_\psi$  by  $v_\psi$  (it suffices to write  $v_\psi(x) = u_\psi(b+a-x)$ ).

Note that if  $\psi \in (D(M), X)_{3+\frac{1}{p}, p}$  then

$$e^{(\cdot-a)L}\psi, e^{(\cdot-a)M}\psi \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4)),$$

(see Remark 4.2 below), from which we deduce that

$$u_\psi \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4)),$$

but our result is more precise since  $(D(M), X)_{3+\frac{1}{p}, p} \subset (D(M), X)_{1+\frac{1}{p}, p}$ .

We are now in position to state our main result which generalizes Theorem 2.2:

**Theorem 2.5.** *Let  $f \in L^p(a, b; X)$  with  $a < b$ ,  $a, b \in \mathbb{R}$  and  $p \in (1, \infty)$ . Assume that  $(H_1)$ ,  $(H_2)$ ,  $(H_3)$  and  $(H_4)$  hold. Then*

1. *There exists a unique classical solution  $u$  of (1.4)-(BC1) if and only if*

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}.$$

2. *There exists a unique classical solution  $u$  of (1.4)-(BC2) if and only if*

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}.$$

If in addition, we assume  $(H_5)$ , then

3. *There exists a unique classical solution  $u$  of (1.4)-(BC3) if and only if*

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}.$$

4. *There exists a unique classical solution  $u$  of (1.4)-(BC4) if and only if*

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}.$$

**Remark 2.6.**

1. The result for the boundary conditions (BC1) can be directly generalized to the following boundary conditions:

$$\begin{cases} u(a) = \varphi_1, & u(b) = \varphi_2, \\ u''(a) + Au(a) = \varphi_3, & u''(b) + Au(b) = \varphi_4. \end{cases} \quad (\text{BC5})$$

2. In the same way, one can obtain similar results for mixed boundary conditions.

From Theorem 2.5, we deduce regularity results for equation (1.3) under some particular boundary conditions (other conditions can be considered). Let us explicit Theorem 2.5, statement 1 for instance, when we consider problem (1.3)-(BC1) with  $A := A_0$ .

**Corollary 2.7.** *Consider a cylindrical domain  $\Omega := (a, b) \times \omega$  of  $\mathbb{R}^d$ , where  $a, b \in \mathbb{R}$ ,  $a < b$  and  $\omega$  is a bounded domain of  $\mathbb{R}^{d-1}$  with  $C^2$ -boundary. Let  $f \in L^p(\Omega)$ ,  $p \in (1, +\infty)$ ; let  $k_1, k_2 \in \mathbb{R} \setminus \{0\}$  such that  $k_1 > k_2 C_\omega$ , where  $C_\omega > 0$  is the Poincaré constant in  $\omega$ . Then, there exists a unique solution  $u \in W^{4,p}(\Omega)$  of*

$$\left\{ \begin{array}{ll} -k_1 \Delta u(x, y) + k_2 \Delta^2 u(x, y) = f(x, y), & (x, y) \in \Omega, \\ u(x, y) = \Delta_y u(x, y) = 0, & (x, y) \in (a, b) \times \partial\omega, \\ u(a, y) = \varphi_1, & y \in \omega, \\ u(b, y) = \varphi_2, & y \in \omega, \\ \Delta_y u(a, y) = \varphi_3, & y \in \omega, \\ \Delta_y u(b, y) = \varphi_4, & y \in \omega, \end{array} \right. \quad (2.6)$$

if and only if

$$\varphi_1, \varphi_2 \in W^{2,p}(\omega) \cap W_0^{1,p}(\omega) \quad \text{and} \quad \Delta\varphi_1, \Delta\varphi_2, \varphi_3, \varphi_4 \in \left( W^{2,p}(\omega) \cap W_0^{1,p}(\omega), L^p(\omega) \right)_{\frac{1}{2p}, p}. \quad (2.7)$$

In view to make explicit the above result, let us consider  $\omega := (c, d) \subset \mathbb{R}$ . Then  $X := L^p(c, d)$  and

$$\left\{ \begin{array}{ll} D(A_0) & := W^{2,p}(c, d) \cap W_0^{1,p}(c, d), \\ (A_0\psi)(y) & := \psi''(y). \end{array} \right.$$

With these notations, problem (2.6) becomes (1.4) with the boundary conditions (BC1). From [25] (Proposition 23),  $X$  satisfies  $(H_1)$  and from [12] (Theorem 19.15 and Lemma 9.17),  $A_0$  satisfies  $(H_2)$ . Theorem C of [24] ensured that  $(H_3)$  is satisfied and, from [15], we have

$$\sigma(A_0) = \left\{ -\frac{n^2 \pi^2}{(d-c)^2} : n \in \mathbb{N} \setminus \{0\} \right\}.$$

So  $\sigma(A_0) \subset (-\infty, C_\omega]$ , where  $C_\omega := -\frac{\pi^2}{(d-c)^2}$ . Since  $k > C_\omega$ , we obtain  $[k, +\infty) \subset \rho(A_0)$ . Thus, assumption  $(H_4)$  is satisfied. Finally, all the assumptions of Theorem 2.5 are satisfied. We have

$$(D(A_0), X)_{\frac{1}{2p}, p} = \left( W^{2,p}(\omega) \cap W_0^{1,p}(\omega), L^p(\omega) \right)_{\frac{1}{2p}, p} = \left\{ \psi \in B_p^{2-\frac{1}{p}}(c, d) : \psi(c) = \psi(d) = 0 \right\},$$

where  $B_p^{2-\frac{1}{p}}(0, 1)$  is a Besov space described in [13], p. 680.

Note that since  $2 - \frac{1}{p} > \frac{1}{p}$ , we have  $C([c, d]) \hookrightarrow B_p^{2-\frac{1}{p}}(c, d)$  with continuous embedding and, since  $2 - \frac{1}{p} > \frac{1}{p}$  is never integer,  $B_p^{2-\frac{1}{p}}(c, d) = W^{2-\frac{1}{p}, p}(c, d)$ . Therefore,

$$(D(A_0), X)_{\frac{1}{2p}, p} = \left\{ \psi \in W^{2-\frac{1}{p}, p}(c, d) : \psi(c) = \psi(d) = 0 \right\},$$

and

$$\begin{aligned} (D(A_0), X)_{1+\frac{1}{2p}, p} &= \left\{ \psi \in D(A_0) : A_0\psi \in (D(A_0), X)_{\frac{1}{2p}, p} \right\} \\ &= \left\{ \psi \in W^{2,p}(c, d) \cap W_0^{1,p}(c, d) : \psi'' \in (D(A_0), X)_{\frac{1}{2p}, p} \right\} \\ &= \left\{ \psi \in W^{2,p}(c, d) \cap W_0^{1,p}(c, d) : \psi''(c) = \psi''(d) = 0 \quad \text{and} \quad \psi'' \in W^{2-\frac{1}{p}, p}(c, d) \right\} \\ &= \left\{ \psi \in W^{2,p}(c, d) : \psi'' \in W^{2-\frac{1}{p}, p}(c, d) \quad \text{and} \quad \psi(c) = \psi(d) = \psi''(c) = \psi''(d) = 0 \right\}. \end{aligned}$$

Now, the classical solution  $u$  of

$$u^{(4)}(x) + (2A_0 - kI)u''(x) + (A_0^2 - kA_0)u(x) = f(x), \quad \text{a.e. } x \in (a, b),$$

satisfies

$$\begin{cases} u \in W^{4,p}(a, b; L^p(c, d)) \\ x \mapsto A_0 u''(x) \in L^p(a, b; L^p(c, d)) \\ x \mapsto A_0^2 u(x) \in L^p(a, b; L^p(c, d)), \end{cases}$$

and

$$\begin{cases} u(a, y) = \varphi_1(y), & y \in (c, d) \\ u(b, y) = \varphi_2(y), & y \in (c, d) \\ \frac{\partial^2 u}{\partial y^2}(a, y) = \varphi_3(y), & y \in (c, d) \\ \frac{\partial^2 u}{\partial y^2}(b, y) = \varphi_4(y), & y \in (c, d) \\ u(x, c) = \frac{\partial^2 u}{\partial y^2}(x, c) = 0, & x \in (a, b) \\ u(x, d) = \frac{\partial^2 u}{\partial y^2}(x, d) = 0, & x \in (a, b). \end{cases}$$

Since

$$\begin{cases} u \in W^{2,p}(a, b; L^p(c, d)) \\ u \in L^p(a, b; W^{2,p}(c, d) \cap W_0^{1,p}(c, d)) \\ u(x, c) = u(x, d) = 0 \\ \varphi_1(c) = \varphi_1(d) = 0, \end{cases}$$

it is possible to use the Sobolev extension Theorem to  $\mathbb{R}^2$  and the Mihlin Theorem (see [18]). We deduce that

$$u \in W^{2,p}((a, b) \times (c, d)) = W^{2,p}(\Omega).$$

By reiterating the same arguments to the other regularities, we obtain

$$u \in W^{4,p}((a, b) \times (c, d)) = W^{4,p}(\Omega).$$

In the same way, we can obtain this result with  $\omega$  in  $\mathbb{R}^{d-1}$ . Equation (1.3) can be studied with other boundary conditions taking into account the results of Theorem 2.5. We can also obtain anisotropic results by considering  $f \in L^p(a, b; L^q(\omega))$ ,  $p, q \in (1, +\infty)$ .

### 3 Proofs of preliminary results

First, we recall some classical trace results:

**Remark 3.1.**

1. Let  $T$  be a closed linear operator in  $X$  and  $k \in \mathbb{N} \setminus \{0\}$ . From [13] p. 677 and [16] p. 39, if

$$u \in W^{n,p}(a, b; X) \cap L^p(a, b; D(T^k)),$$

where  $n \in \mathbb{N} \setminus \{0\}$ , then for any  $j \in \mathbb{N}$  satisfying the Poulsen condition  $0 < \frac{1}{p} + j < n$  and  $c \in \{a, b\}$ , we have

$$u^{(j)}(c) \in (D(T^k), X)_{\frac{j}{n} + \frac{1}{np}, p}. \quad (3.1)$$

Moreover, recall that, if we denote  $(X, D(T^k))_{1-\theta, p} := (D(T^k), X)_{\theta, p}$ , where  $k \in \mathbb{N} \setminus \{0\}$  and  $\theta \in (0, 1)$  such that  $k\theta \notin \mathbb{N}$ , from reiteration theorem [13, 16], we have

$$(D(T^k), X)_{\theta, p} = (X, D(T^k))_{1-\theta, p} = (X, D(T))_{k-k\theta, p}. \quad (3.2)$$



2. Let  $u \in W^{4,p}(a, b; X) \cap L^p(a, b; D(A^2))$ . Then, from (3.1), for any  $c \in \{a, b\}$ , we obtain

$$u(c) \in (D(A^2), X)_{\frac{1}{4p}, p}, \quad u'(c) \in (D(A^2), X)_{\frac{1}{4} + \frac{1}{4p}, p} \quad \text{and} \quad u''(c) \in (D(A^2), X)_{\frac{1}{2} + \frac{1}{4p}, p}.$$

From (3.2), we deduce, for any  $c \in \{a, b\}$ ,

$$u(c) \in (D(A), X)_{1 + \frac{1}{2p}, p}, \quad u'(c) \in (D(A), X)_{1 + \frac{1}{2} + \frac{1}{2p}, p} \quad \text{and} \quad u''(c) \in (D(A), X)_{\frac{1}{2p}, p}. \quad (3.3)$$

### 3.1 Proof of Theorem 2.2

Let  $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in X$  be such that

$$\varphi_1, \varphi_2 \in (D(A), X)_{1 + \frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}. \quad (3.4)$$

From (3.4), there exists (see e.g. [11], Theorem 4, p. 200) a unique classical solution

$$v_0 \in W^{2,p}(a, b; X) \cap L^p(a, b; D(A)),$$

of

$$\begin{cases} v''(x) + (A - kI)v(x) = f(x), & a. e. x \in (a, b) \\ v(a) = \varphi_3 + A\varphi_1, \\ v(b) = \varphi_4 + A\varphi_2. \end{cases} \quad (3.5)$$

Then, since  $\varphi_1, \varphi_2 \in (D(A), X)_{1 + \frac{1}{2p}, p} \subset (D(A), X)_{\frac{1}{2p}, p}$ , there exists a unique classical solution

$$u_0 \in W^{2,p}(a, b; X) \cap L^p(a, b; D(A)),$$

of

$$\begin{cases} u''(x) + Au(x) = v_0(x), & a. e. x \in (a, b) \\ u(a) = \varphi_1, \\ u(b) = \varphi_2. \end{cases} \quad (3.6)$$

A simple computation shows that  $u_0$  satisfies (1.4)-(BC1). It remains to show that  $u_0$  satisfies (1.6). Let  $w_0 \in W^{2,p}(a, b; X) \cap L^p(a, b; D(A))$  be the unique classical solution of

$$\begin{cases} w''(x) + Aw(x) = Av_0(x), & a. e. x \in (a, b) \\ w(a) = A\varphi_1, \\ w(b) = A\varphi_2. \end{cases}$$

Then  $A^{-1}w_0$  is a classical solution of (3.6). So, by uniqueness of the solution, we obtain

$$u_0 = A^{-1}w_0 \in L^p(a, b; D(A^2)).$$

Moreover, from (3.6), we have

$$u_0'' = v_0 - Au_0 = v_0 - w_0 \in W^{2,p}(a, b; X) \cap L^p(a, b; D(A)).$$

Hence  $u_0$  is a classical solution of (1.4)-(BC1). Uniqueness is straightforward since if  $u$  is a classical solution of (1.4)-(BC1) then  $v := u'' + Au$  is a classical solution of (3.5). By uniqueness,  $v = v_0$ , so  $u$  is a classical solution of (3.6) and again, by uniqueness,  $u = u_0$ .

Conversely, if there exists a classical solution of (1.4)-(BC1), then (3.4) holds (see (3.3)).

### 3.2 Proof of Proposition 2.3 (Representation formula)

Let  $u$  be a classical solution of (1.4). By Theorem 2.2, we can consider the classical solution  $F_{0,f}$  of (1.4)-(BC1); *i.e.* which satisfies

$$F_{0,f}(a) = F_{0,f}(b) = F_{0,f}''(a) = F_{0,f}''(b) = 0. \quad (3.7)$$

We set  $u_{hom} := u - F_{0,f}$ . Then,  $u_{hom}$  is a classical solution of

$$u^{(4)}(x) + (2A - kI)u''(x) + (A^2 - kA)u(x) = 0, \quad a. e. x \in (a, b).$$

So it remains to show the existence of constants  $K_i$ , for  $i = 1, 2, 3, 4$  such that

$$u_{hom}(x) = e^{(x-a)M}K_1 + e^{(b-x)M}K_2 + e^{(x-a)L}K_3 + e^{(b-x)L}K_4, \quad a. e. x \in (a, b). \quad (3.8)$$

It implies that

$$u^{(4)}(x) = (L^2 + M^2)u''(x) - L^2M^2u(x), \quad a. e. x \in (a, b). \quad (3.9)$$

Since  $u$  and  $F_{0,f}$  are classical solutions of (1.4),  $u_{hom}$  is also a classical solution, so

$$u_{hom} \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4)) \quad \text{and} \quad u''_{hom} \in L^p(a, b; D(M^2)).$$

Thus, for almost every  $x \in (a, b)$ , we can set

$$\begin{cases} v(x) & := k^{-1}L^2u_{hom}(x) - k^{-1}u''_{hom}(x) & \in L^p(a, b; D(M^2)) \\ w(x) & := -k^{-1}M^2u_{hom}(x) + k^{-1}u''_{hom}(x) & \in L^p(a, b; D(L^2)). \end{cases} \quad (3.10)$$

From (2.3), we obtain

$$v(x) + w(x) = k^{-1}(L^2 - M^2)u_{hom}(x) = u_{hom}(x), \quad a. e. x \in (a, b). \quad (3.11)$$

Moreover, from (3.9), for almost every  $x \in (a, b)$ , we deduce that

$$\begin{aligned} L^{-2}v''(x) &= k^{-1}u''_{hom}(x) - k^{-1}L^{-2}u^{(4)}_{hom}(x) \\ &= k^{-1}u''_{hom}(x) - k^{-1}L^{-2}((L^2 + M^2)u''(x) - L^2M^2u(x)) \\ &= M^2L^{-2}v(x), \end{aligned}$$

and in the same way

$$M^{-2}w''(x) = -k^{-1}u''_{hom}(x) + k^{-1}M^{-2}u^{(4)}_{hom}(x) = L^2M^{-2}w(x).$$

Thus,  $v$  and  $w$  are, respectively, solutions of

$$v''(x) - M^2v(x) = 0 \quad \text{and} \quad w''(x) - L^2w(x) = 0, \quad a. e. x \in (a, b).$$

The previous functional equalities are set in  $L^p(a, b; X)$  and make sense due to (3.9). From [11], we obtain

$$v(x) = e^{(x-a)M}K_1 + e^{(b-x)M}K_2 \quad \text{and} \quad w(x) = e^{(x-a)L}K_3 + e^{(b-x)L}K_4, \quad x \in (a, b). \quad (3.12)$$

Then, from (3.11) and (3.12), we deduce (3.8).

## 4 Proof of Theorem 2.4

The proof is essentially based on analytic semigroups results (see [17]) and interpolation spaces (see, for instance, [7] p. 381-386 and [26], Theorem p. 96) and on a corollary of the well-known Dore-Venni Theorem [8]. We recall below these results as lemmas for the readers convenience.

**Lemma 4.1** ([7, 26]). *Let  $\psi \in X$  and  $T$  be a generator of an analytic semigroup in  $X$ . Then, for any  $n \in \mathbb{N} \setminus \{0\}$ , the two next properties are equivalent:*

1.  $x \mapsto T^n e^{(x-a)T} \psi \in L^p(a, b; X)$ ,
2.  $\psi \in (D(T^n), X)_{\frac{1}{np}, p}$ .

**Remark 4.2.** Under the assumptions of Lemma 4.1, from (3.2) it follows that, for any  $n \in \mathbb{N} \setminus \{0\}$ , the two next properties are equivalent:

1.  $x \mapsto e^{(x-a)T} \psi \in W^{n,p}(a, b; X) \cap L^p(a, b; D(T^n))$ ,
2.  $\psi \in (D(T), X)_{n-1+\frac{1}{p}, p}$ .

**Lemma 4.3** ([8]). *Let  $T \in BIP(X, \theta)$  with  $\theta \in (0, \pi/2)$ , and  $g \in L^p(a, b; X)$ . Then, for almost every  $x \in (a, b)$ , we have*

$$\int_a^x e^{(x-s)T} g(s) dt \in D(T) \quad \text{and} \quad \int_x^b e^{(s-x)T} g(s) dt \in D(T).$$

Moreover,

$$x \mapsto T \int_a^x e^{(x-s)T} g(s) dt \in L^p(a, b; X) \quad \text{and} \quad x \mapsto T \int_x^b e^{(s-x)T} g(s) dt \in L^p(a, b; X).$$

**Remark 4.4.** The assumption that  $X$  is a UMD space is not necessary in Lemma 4.1, but is essential in Lemma 4.3.

Now, we state the proof of Theorem 2.4.

1. Let  $\psi \in X$ . Recall that

$$\forall x > a, \quad u_\psi(x) := \left( e^{(x-a)L} - e^{(x-a)M} \right) \psi.$$

Since, for  $T := L$  or  $M$ ,  $(e^{xT})_{x \geq 0}$  is an analytic semigroup, we have (see, for instance, [17], Proposition 2.1.1, p. 35)

$$x \mapsto e^{(x-a)T} \in C^\infty((a, b]; X) \cap C^0([a, b]; X), \quad (4.1)$$

and for any  $x > a$ ,

$$e^{(x-a)L} \psi \in D(L^\infty) = D(M^\infty) \quad \text{and} \quad e^{(x-a)M} \psi \in D(M^\infty), \quad (4.2)$$

where

$$D(M^\infty) = D(L^\infty) := \bigcap_{k \geq 0} D(M^k) = \bigcap_{k \geq 0} D(L^k). \quad (4.3)$$

From (4.1) and (4.2), we deduce

$$\forall x > a, \quad u_\psi \in C^\infty((a, b]; X) \cap C^0([a, b]; X) \quad \text{and} \quad u(x) \in D(M^\infty).$$

Then, from (2.4), we obtain

$$\begin{aligned} \forall x > a, \quad u'_\psi(x) &= \left( L e^{(x-a)L} - M e^{(x-a)M} \right) \psi \\ &= M \left( e^{(x-a)L} - e^{(x-a)M} \right) \psi + (L - M) e^{(x-a)L} \psi \\ &= M u_\psi(x) + k (L + M)^{-1} e^{(x-a)L} \psi. \end{aligned}$$

Thus,  $u_\psi \in C^1((a, b]; X) \cap C^0([a, b]; X)$  is a solution of the Cauchy problem

$$\begin{cases} u'(x) = M u(x) + k (L + M)^{-1} e^{(x-a)L} \psi, & a.e. x \in (a, b], \\ u(a) = 0. \end{cases}$$

Hence, (see e.g. [21], Corollary 2.2 p. 106),  $u$  is given by the variation of constant formula:

$$\begin{aligned} \forall x \in (a, b], \quad u_\psi(x) &= \int_a^x e^{(x-s)M} k (L + M)^{-1} e^{(s-a)L} \psi ds \\ &= k (L + M)^{-1} M^{-1} \left[ M \int_a^x e^{(x-s)M} e^{(s-a)L} \psi ds \right]. \end{aligned} \quad (4.4)$$

Since

$$s \mapsto e^{(s-a)L} \psi \in C^0([a, b]; X) \subset L^p(a, b; X),$$

from Lemma 4.3, we have

$$g_\psi : x \mapsto M \int_a^x e^{(x-s)M} e^{(s-a)L} \psi ds \in L^p(a, b; X).$$

Then, since  $M^2(L + M)^{-1}M^{-1} \in \mathcal{L}(X)$ , we deduce that

$$x \mapsto M^2 u_\psi(x) = k M^2 (L + M)^{-1} M^{-1} g_\psi(x) \in L^p(a, b; X).$$

We also have  $u_\psi \in W^{1,p}(a, b; X)$  because  $u_\psi \in C^1((a, b]; X) \cap C^0([a, b]; X)$  and

$$u'_\psi(\cdot) = M u_\psi(\cdot) + k (L + M)^{-1} e^{(\cdot-a)L} \psi \in L^p(a, b; X).$$

Furthermore,

$$x \mapsto M u'_\psi(x) = M^2 u_\psi(x) + k M (L + M)^{-1} e^{(x-a)L} \psi \in L^p(a, b; X).$$

so  $u'_\psi \in L^p(a, b; D(M))$ . For *a. e.*  $x \in (a, b]$ , since  $e^{(x-a)L} \psi \in D(M^2)$ , from (2.3) we obtain

$$\begin{aligned} u''_\psi(x) &= (L^2 e^{(x-a)L} - M^2 e^{(x-a)L}) \psi \\ &= M^2 (e^{(x-a)L} - e^{(x-a)L}) \psi + (L^2 - M^2) e^{(x-a)L} \psi \\ &= M^2 u_\psi(x) + k e^{(x-a)L} \psi. \end{aligned} \quad (4.5)$$

Thus  $u''_\psi \in L^p(a, b; X)$ , hence  $u_\psi \in W^{2,p}(a, b; X)$ .

2. *First step.* Assume that  $\psi \in (D(M), X)_{1+\frac{1}{p}, p}$ . We set  $\varphi := M\psi \in (D(M), X)_{\frac{1}{p}, p}$ . Then, we have  $M u_\psi = u_\varphi$ . Thus, equality (4.4), for *a. e.*  $x \in (a, b)$ , yields

$$\begin{aligned} M^3 u_\varphi(x) &= k M (L + M)^{-1} M^2 M^{-1} \left[ M \int_a^x e^{(x-s)L} e^{(s-a)L} \varphi ds \right] \\ &= k M (L + M)^{-1} M L^{-1} \left[ M \int_a^x e^{(x-s)L} L e^{(s-a)L} \varphi ds \right]. \end{aligned} \quad (4.6)$$

Since  $\varphi = M\psi \in (D(M), X)_{\frac{1}{p}, p}$ , from Lemma 4.1, we have

$$s \mapsto L e^{(s-a)L} \varphi \in L^p(a, b; X).$$

Then, from Lemma 4.3, we deduce

$$x \mapsto M \int_a^x e^{(x-s)L} L e^{(s-a)L} \varphi ds \in L^p(a, b; X).$$

Hence, from (4.6) and the fact that  $M(L + M)^{-1} M L^{-1} \in \mathcal{L}(X)$ , we obtain

$$x \mapsto M^4 u_\psi(x) = M^3 u_\varphi \in L^p(a, b; X). \quad (4.7)$$

It follows

$$u_\psi \in L^p(a, b; D(M^4)) \quad \text{and} \quad u_\varphi \in L^p(a, b; D(M^3)). \quad (4.8)$$

Moreover, from (4.5) and (2.4), for  $x > a$ , we have

$$\begin{aligned} u'''_\psi(x) &= M^2 u'_\psi(x) + k L e^{(x-a)L} \psi \\ &= M^2 (L e^{(x-a)L} - M e^{(x-a)L}) \psi + k L e^{(x-a)L} \psi \\ &= M^3 (e^{(x-a)L} - e^{(x-a)L}) \psi + M^2 (L - M) e^{(x-a)L} \psi + k L e^{(x-a)L} \psi \\ &= M^3 u_\psi(x) + k M^2 (L + M)^{-1} e^{(x-a)L} \psi + k L e^{(x-a)L} \psi \\ &= M^3 u_\psi(x) + k (M^2 (L + M)^{-1} + L) e^{(x-a)L} \psi \\ &= M^2 u_\varphi(x) + k (M^2 (L + M)^{-1} + L) M^{-1} e^{(x-a)L} \varphi. \end{aligned}$$

Since  $(M^2 (L + M)^{-1} + L) M^{-1} \in \mathcal{L}(X)$ , from (4.8) and (4.1), we deduce

$$u'''_\psi \in L^p(a, b; X). \quad (4.9)$$

Furthermore, for  $x > a$ , we have

$$\begin{aligned} u_\psi^{(4)}(x) &= M^2 u_\varphi'(x) + k (M^2(L+M)^{-1} + L) M^{-1} L e^{(x-a)L} \varphi \\ &= M^3 u_\varphi(x) + k M^2(L+M)^{-1} e^{(x-a)L} \varphi + k (M^2(L+M)^{-1} + L) M^{-1} L e^{(x-a)L} \varphi \\ &= M^3 u_\varphi(x) + k (M^2(L+M)^{-1} L^{-1} + M^2(L+M)^{-1} M^{-1} + L M^{-1}) L e^{(x-a)L} \varphi. \end{aligned}$$

From Lemma 4.1, since  $\varphi = M\psi \in (D(M), X)_{\frac{1}{p}, p} = (D(L), X)_{\frac{1}{p}, p}$ , we obtain

$$x \mapsto L e^{(x-a)L} \varphi \in L^p(a, b; X).$$

Then, since  $M^2(L+M)^{-1} L^{-1} + M^2(L+M)^{-1} M^{-1} + L M^{-1} \in \mathcal{L}(X)$  and from (4.8), we deduce

$$u_\psi^{(4)} \in L^p(a, b; X). \quad (4.10)$$

From statement 1. (see above), (4.8), (4.9) and (4.10), we deduce the result.

*Second step.* Assume that  $u_\psi \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4))$ . From (3.1), with  $k = n = 4$  and  $j = 2$ , we have  $u_\psi''(a) \in (D(M^4), X)_{\frac{1}{4p} + \frac{1}{2}, p}$ , then from (3.2), we deduce  $u_\psi''(a) \in (D(M), X)_{1 + \frac{1}{p}, p}$ . From (4.5), for  $x > a$ , we have

$$M^{-2} u_\psi''(x) = u_\psi(x) + k M^{-2} e^{(x-a)L} u_\psi.$$

Passing to the limit when  $x$  tends to  $a$ , we obtain  $u_\psi''(a) = k\psi$  and thus

$$\psi = k^{-1} u_\psi''(a) \in (D(M), X)_{1 + \frac{1}{p}, p}.$$

Finally, since  $u_\psi(x) = M^{-1} u_\varphi(x)$ , from (4.5), we obtain

$$\forall x > a, \quad M^2 u_\psi''(x) = M^3 u_\varphi(x) + k M L^{-1} L e^{(x-a)L} \varphi.$$

Applying Lemma 4.1 with  $\varphi := M\psi \in (D(M), X)_{\frac{1}{p}, p}$ , the fact that  $M L^{-1} \in \mathcal{L}(X)$  and (4.8), the previous equality yields  $u_\psi'' \in L^p(a, b; D(M^2))$ .

## 5 Proof of Theorem 2.5

Statement 1 has been treated in section 2.2. We give here the proofs for the others boundary conditions. Note that from Remark 3.1, in Theorem 2.5, we only have to prove the reverse implications.

For each kind of boundary conditions, the proof of Theorem 2.5 is divided in two steps. First, from the representation formula obtained in section 2.3, we show uniqueness of a classical solution. Then, we state the existence of this solution. At this end, we establish the following technical result.

**Lemma 5.1.** *Let  $V \in \mathcal{L}(X)$  such that  $I + V$  is invertible in  $\mathcal{L}(X)$ . Then, there exists  $W \in \mathcal{L}(X)$  such that*

$$(I + V)^{-1} = I - W$$

*and  $W(X) \subset V(X)$ . Moreover, if  $T$  is a linear operator in  $X$  such that  $V(X) \subset D(T)$  and for any  $\psi \in D(T)$ ,  $TV\psi = VT\psi$ , then*

$$\forall \psi \in D(T), \quad WT\psi = TW\psi.$$

*Proof.* We set

$$W := V(I + V)^{-1} \in \mathcal{L}(X).$$

Then, obviously  $(I + V)^{-1} = I - W$  and  $W(X) \subset V(X) \subset D(T)$ . Moreover, for  $\psi \in D(T)$

$$(I + V)WT\psi = VT\psi = TV\psi = T(I + V)W\psi = (I + V)TW\psi,$$

thus  $WT\psi = TW\psi$ . □

## 5.1 Proof of statement 2. (Boundary conditions (BC2))

Assume  $(H_1) \sim (H_4)$  and

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}. \quad (5.1)$$

First we prove the uniqueness of the solution of (1.4)-(BC2) by determining the constants  $K_i$ ,  $i = 1, 2, 3, 4$ , of the representation formula (2.5). Then, we show that the formula obtained constitutes a classical solution.

*First step:* Uniqueness.

If  $u$  is a classical solution of (1.4)-(BC2), then from Proposition 2.3,  $u$  satisfies (2.5). In order to obtain a simplified model, we set

$$\alpha_1 := \frac{K_1 - K_2}{2}, \quad \alpha_2 := \frac{K_3 - K_4}{2}, \quad \alpha_3 := \frac{K_1 + K_2}{2} \quad \text{and} \quad \alpha_4 := \frac{K_3 + K_4}{2}. \quad (5.2)$$

Then for *a.e.*  $x \in (a, b)$ ,  $u$  is given by

$$\begin{aligned} u(x) &= \left( e^{(x-a)M} - e^{(b-x)M} \right) \alpha_1 + \left( e^{(x-a)L} - e^{(b-x)L} \right) \alpha_2 + \left( e^{(x-a)M} + e^{(b-x)M} \right) \alpha_3 \\ &\quad + \left( e^{(x-a)L} + e^{(b-x)L} \right) \alpha_4 + F_{0,f}(x). \end{aligned} \quad (5.3)$$

Since  $F_{0,f}$  satisfies (3.7), the boundary conditions (BC2) applied to  $M^{-2}u(x)$  for *a.e.*  $x \in (a, b)$ , imply the following relations

$$\begin{aligned} &\left( I + e^{(b-a)M} \right) M^{-1} \alpha_1 + \left( I + e^{(b-a)L} \right) LM^{-2} \alpha_2 \\ &+ \left( I - e^{(b-a)M} \right) M^{-1} \alpha_3 + \left( I - e^{(b-a)L} \right) LM^{-2} \alpha_4 = M^{-2} \left( \varphi_1 - F'_{0,f}(a) \right), \end{aligned} \quad (5.4)$$

$$\begin{aligned} &\left( e^{(b-a)M} + I \right) M^{-1} \alpha_1 + \left( e^{(b-a)L} + I \right) LM^{-2} \alpha_2 \\ &+ \left( e^{(b-a)M} - I \right) M^{-1} \alpha_3 + \left( e^{(b-a)L} - I \right) LM^{-2} \alpha_4 = M^{-2} \left( \varphi_2 - F'_{0,f}(b) \right), \end{aligned} \quad (5.5)$$

$$\left( I - e^{(b-a)L} \right) (L^2 - M^2) M^{-2} \alpha_2 + \left( I + e^{(b-a)L} \right) (L^2 - M^2) M^{-2} \alpha_4 = M^{-2} \varphi_3, \quad (5.6)$$

$$\left( e^{(b-a)L} - I \right) (L^2 - M^2) M^{-2} \alpha_2 + \left( e^{(b-a)L} + I \right) (L^2 - M^2) M^{-2} \alpha_4 = M^{-2} \varphi_4. \quad (5.7)$$

Note that we have considered  $M^{-2}u(x)$  for *a.e.*  $x \in (a, b)$ , because we do not know if  $\alpha_i \in D(M)$ ,  $i = 1, 2, 3, 4$ . Using (2.3) and summing (5.4) with (5.5) and (5.6) with (5.7), we obtain the abstract system

$$\left\{ \begin{array}{l} \left( I + e^{(b-a)M} \right) M^{-1} \alpha_1 + \left( I + e^{(b-a)L} \right) LM^{-2} \alpha_2 \\ + \left( I - e^{(b-a)M} \right) M^{-1} \alpha_3 + \left( I - e^{(b-a)L} \right) LM^{-2} \alpha_4 = M^{-2} \left( \varphi_1 - F'_{0,f}(a) \right), \\ 2 \left( I + e^{(b-a)M} \right) M^{-1} \alpha_1 + 2 \left( I + e^{(b-a)L} \right) LM^{-2} \alpha_2 = M^{-2} \left( \varphi_1 + \varphi_2 - F'_{0,f}(a) - F'_{0,f}(b) \right), \\ k \left( I - e^{(b-a)L} \right) M^{-2} \alpha_2 + k \left( I + e^{(b-a)L} \right) M^{-2} \alpha_4 = M^{-2} \varphi_3, \\ \qquad \qquad \qquad 2k \left( I + e^{(b-a)L} \right) M^{-2} \alpha_4 = M^{-2} \left( \varphi_3 + \varphi_4 \right). \end{array} \right.$$

This leads to the system

$$\left\{ \begin{array}{l} M \left( I - e^{(b-a)M} \right) M^{-2} \alpha_3 + L \left( I - e^{(b-a)L} \right) M^{-2} \alpha_4 = M^{-2} \tilde{\varphi}_2 \\ M \left( I + e^{(b-a)M} \right) M^{-2} \alpha_1 + L \left( I + e^{(b-a)L} \right) M^{-2} \alpha_2 = M^{-2} \tilde{\varphi}_1, \\ \qquad \qquad \qquad k \left( I - e^{(b-a)L} \right) \alpha_2 = \frac{\varphi_3 - \varphi_4}{2}, \\ \qquad \qquad \qquad k \left( I + e^{(b-a)L} \right) \alpha_4 = \frac{\varphi_3 + \varphi_4}{2}, \end{array} \right.$$

where

$$\tilde{\varphi}_1 := \frac{\varphi_1 + \varphi_2 - F'_{0,f}(a) - F'_{0,f}(b)}{2} \quad \text{and} \quad \tilde{\varphi}_2 := \frac{\varphi_1 - \varphi_2 - F'_{0,f}(a) + F'_{0,f}(b)}{2}. \quad (5.8)$$

For  $T$  invertible with bounded inverse, from [21] p. 70, for any  $n \in \mathbb{N}$ , we have:

$$\exists \delta > 0, \exists M > 1 : \left\| e^{n(b-a)T} \right\|_{\mathcal{L}(X)} \leq M e^{-n(b-a)\delta}. \quad (5.9)$$

Therefore, there exists  $n_0 \in \mathbb{N} \setminus \{0\}$  such that  $\|e^{n_0(b-a)T}\|_{\mathcal{L}(X)} < 1$ . So  $I \pm e^{n_0(b-a)T}$  is invertible with bounded inverse. Then, for  $T := L$  or  $M$ , we deduce

$$\begin{cases} \alpha_1 &= \left( I + e^{(b-a)M} \right)^{-1} \left( M^{-1} \tilde{\varphi}_1 - LM^{-1} \left( I + e^{(b-a)L} \right) \alpha_2 \right) \\ \alpha_2 &= k^{-1} \left( I - e^{(b-a)L} \right)^{-1} \left( \frac{\varphi_3 - \varphi_4}{2} \right), \\ \alpha_3 &= \left( I - e^{(b-a)M} \right)^{-1} \left( M^{-1} \tilde{\varphi}_2 - LM^{-1} \left( I - e^{(b-a)L} \right) \alpha_4 \right), \\ \alpha_4 &= k^{-1} \left( I + e^{(b-a)L} \right)^{-1} \left( \frac{\varphi_3 + \varphi_4}{2} \right). \end{cases} \quad (5.10)$$

It follows that  $u$  is uniquely determined by (5.3), (5.8) and (5.10).

*Second step: Existence.*

Consider this uniquely determined  $u$  and set

$$\tilde{K}_1 := K_1 + K_3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \quad \text{and} \quad \tilde{K}_2 := K_2 + K_4 = \alpha_3 + \alpha_4 - \alpha_1 - \alpha_2,$$

thus, for any  $x \in (a, b)$ ,

$$\begin{aligned} u(x) &= e^{(x-a)M} \tilde{K}_1 + e^{(b-x)M} \tilde{K}_2 + \left( e^{(x-a)L} - e^{(x-a)M} \right) K_3 \\ &\quad + \left( e^{(b-x)L} - e^{(b-x)M} \right) K_4 + F_{0,f}(x). \end{aligned} \quad (5.11)$$

Then, from Remark 4.2 and Theorem 2.4, it suffices to show that  $\tilde{K}_1, \tilde{K}_2 \in (D(M), X)_{3+\frac{1}{p}, p}$  and  $K_3, K_4 \in (D(M), X)_{1+\frac{1}{p}, p}$ . From Lemma 5.1, for  $T := L$  or  $M$ , we have

$$\left( I \pm e^{2(b-a)T} \right)^{-1} = I + R_{\pm}, \quad \text{where } R_{\pm}(X) \subset D(M^{\infty}) \quad \text{and} \quad R_{\pm}T = TR_{\pm}.$$

Hence, from (5.10), there exists  $R_i \in D(M^{\infty})$ ,  $i = 1, 2, 3, 4$ , such that

$$\begin{cases} \alpha_1 &= \frac{1}{2} M^{-1} (2\tilde{\varphi}_1 - k^{-1}L(\varphi_3 - \varphi_4)) + R_1 \\ \alpha_2 &= k^{-1} \frac{\varphi_3 - \varphi_4}{2} + R_2, \\ \alpha_3 &= \frac{1}{2} M^{-1} (2\tilde{\varphi}_2 - k^{-1}L(\varphi_3 + \varphi_4)) + R_3, \\ \alpha_4 &= k^{-1} \frac{\varphi_3 + \varphi_4}{2} + R_4. \end{cases}$$

From (5.2) and (5.8), we obtain

$$\begin{cases} K_1 &= M^{-1} (\varphi_1 - F'_{0,f}(a)) - k^{-1}LM^{-1}\varphi_3 + \tilde{R}_1 \\ K_2 &= -M^{-1} (\varphi_2 - F'_{0,f}(b)) - k^{-1}LM^{-1}\varphi_4 + \tilde{R}_2, \\ K_3 &= k^{-1}\varphi_3 + \tilde{R}_3, \\ K_4 &= k^{-1}\varphi_4 + \tilde{R}_4, \end{cases}$$

where  $\tilde{R}_i \in D(M^\infty)$ ,  $i = 1, 2, 3, 4$ . From (3.2), we have

$$(D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} = (D(M^2), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} = (D(M), X)_{2+\frac{1}{p}, p} \quad (5.12)$$

and

$$(D(A), X)_{\frac{1}{2p}, p} = (X, D(M))_{1+1-\frac{1}{p}, p} = (D(M), X)_{1+\frac{1}{p}, p}. \quad (5.13)$$

Since  $F'_{0,f}(a) \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}$ , from (2.4), (5.1), (5.12) and (5.13), we have

$$\begin{aligned} \tilde{K}_1 &= M^{-1}(\varphi_1 - F'_{0,f}(a)) + k^{-1}(I - LM^{-1})\varphi_3 + \tilde{R}_1 + \tilde{R}_3 \\ &= M^{-1}(\varphi_1 - F'_{0,f}(a)) - (L + M)^{-1}M^{-1}\varphi_3 + \tilde{R}_1 + \tilde{R}_3 \in (D(M), X)_{3+\frac{1}{p}, p}, \end{aligned}$$

and, in the same way,  $\tilde{K}_2 \in (D(M), X)_{3+\frac{1}{p}, p}$ . Moreover,

$$K_3 = k^{-1}\varphi_3 + \tilde{R}_3 \in (D(M), X)_{1+\frac{1}{p}, p} \quad \text{and} \quad K_4 = k^{-1}\varphi_4 + \tilde{R}_4 \in (D(M), X)_{1+\frac{1}{p}, p},$$

which gives the result.

## 5.2 Proof of statement 3. (Boundary conditions (BC3))

In all the sequel, we assume  $(H_1) \sim (H_5)$ . For statements 1. and 2., the representation formula is easily obtained by taking into account the boundary conditions. But for statement 3. (and also statement 4.), to build the representation, we need the invertibility of some determinant operators.

More precisely, we have to prove that  $U$  and  $V$  given by

$$\begin{cases} U & := I - e^{(b-a)(L+M)} - k^{-1}(L+M)^2 \left( e^{(b-a)M} - e^{(b-a)L} \right), \\ V & := I - e^{(b-a)(L+M)} + k^{-1}(L+M)^2 \left( e^{(b-a)M} - e^{(b-a)L} \right), \end{cases} \quad (5.14)$$

are invertible with bounded inverse. This will be a consequence of a functional calculus result. To state this result, we need some notations and technical lemmas. For  $\theta \in (0, \pi)$ , we denote by  $H(S_\theta)$  the space of holomorphic functions on  $S_\theta$  (defined by (2.1)) with values in  $\mathbb{C}$ . Moreover, we consider the following subspace of  $H(S_\theta)$ :

$$\mathcal{E}_\infty(S_\theta) := \{f \in H(S_\theta) : f = O(|z|^{-s}) \text{ } (|z| \rightarrow +\infty) \text{ for some } s > 0\}.$$

In other words,  $\mathcal{E}_\infty(S_\theta)$  is the space of polynomial decreasing holomorphic functions at  $\infty$ . Let  $T$  be an invertible sectorial operator of angle  $\theta_T \in (0, \pi)$ . If  $f \in \mathcal{E}_\infty(S_\theta)$ , with  $\theta \in (\theta_T, \pi)$ , then we can define, by functional calculus,  $f(T) \in \mathcal{L}(X)$ , see [14], p. 45.

Set, for  $z \in \mathbb{C}$

$$f_k(z) := 1 - e^{-c(\sqrt{z} + \sqrt{z+k})} - k^{-1}(\sqrt{z} + \sqrt{z+k})^2 \left( e^{-c\sqrt{z}} - e^{-c\sqrt{z+k}} \right),$$

and

$$g_k(z) := 1 - e^{-c(\sqrt{z} + \sqrt{z+k})} + k^{-1}(\sqrt{z} + \sqrt{z+k})^2 \left( e^{-c\sqrt{z}} - e^{-c\sqrt{z+k}} \right),$$

where  $c := b - a > 0$ . Note that, we can write formally  $U = f_k(-A)$  and  $V = g_k(-A)$ .

**Lemma 5.2.** *For any  $k > 0$ ,  $f_k$  and  $g_k$  do not vanish in  $\mathbb{R}^+ \setminus \{0\}$ .*

*Proof.* First, we consider for  $\tau > 0$ ,  $\varphi(\tau) := \tau \frac{1 + e^{-\tau}}{1 - e^{-\tau}}$ , then  $\varphi$  is a strictly increasing function on  $\mathbb{R}^+ \setminus \{0\}$  since

$$\varphi'(\tau) = 2 \frac{\sinh(\tau) - \tau}{e^{-\tau} (e^\tau - 1)^2} > 0, \quad \tau > 0.$$

Let  $x > 0$ . Since  $f_k(x) \leq g_k(x)$ , it suffices to show that  $f_k(x) > 0$ . Set  $s := c\sqrt{x}$  and  $t := c\sqrt{x+k}$ , then  $0 < s < t$  and

$$\begin{aligned} f_k(x) &= 1 - e^{-(s+t)} - \frac{t+s}{t-s} (e^{-s} - e^{-t}) \\ &= \frac{1}{t-s} \left[ (t-s) \left( 1 - e^{-(s+t)} \right) - (t+s) (e^{-s} - e^{-t}) \right] \\ &= \frac{1}{t-s} \left[ t(1 - e^{-s})(1 + e^{-t}) - s(1 - e^{-t})(1 + e^{-s}) \right] \\ &= \frac{(1 - e^{-s})(1 - e^{-t})}{t-s} [\varphi(t) - \varphi(s)] > 0, \end{aligned}$$



which leads to the result.  $\square$

**Lemma 5.3.** *Let  $P$  be a sectorial operator in  $X$  of angle  $\theta$ , for all  $\theta \in (0, \pi)$ . Let  $f \in H(S_\theta)$ , for some  $\theta \in (0, \pi)$ , be such that*

- (i)  $1 - f \in \mathcal{E}_\infty(S_\theta)$ ,
- (ii)  $f \neq 0$  on  $\mathbb{R}^+ \setminus \{0\}$ .

*Then,  $f(P) \in \mathcal{L}(X)$ , is invertible with bounded inverse.*

*Proof.* Since we have  $\frac{1}{f} = 1 + h$ , where  $h := \frac{1-f}{f}$ , it suffices to show that  $h \in \mathcal{E}_\infty(S_{\theta_0})$ , for some  $\theta_0 \in (0, \pi)$ , to obtain the result. Indeed, since  $\sigma(P) \subset S_{\theta_0}$ , from [14], pp. 28 and 45, in this case we can set:

$$\frac{1}{f}(P) = I + h(P) \in \mathcal{L}(X),$$

and the invertibility of  $f(P)$  is obtained by writing

$$\frac{1}{f}(P)f(P) = \left(\frac{1}{f} \times f\right)(P) = 1(P) = I,$$

and similarly,  $f(P)\frac{1}{f}(P) = I$ .

From (i), there exists  $R > 0$  such that, for all  $z \in S_\theta$  with  $|z| > R$ , we have  $|1 - f(z)| < 1/2$ . Then  $f(z)$  is never equal to 0 if  $z \in S_\theta$  and  $|z| > R$ , so

$$h \in H\left(S_\theta \cap \left(\mathbb{C} \setminus \overline{B(0, R)}\right)\right). \quad (5.15)$$

Let  $\theta_1 \in (0, \theta)$ . In the compact set  $\overline{S_{\theta_1} \cap B(0, R)}$ , the holomorphic function  $f$  has only a finite number of zeros, which from (ii), are not in  $(0, R]$ . So there exists  $\theta_0 \in (0, \theta_1)$  such that  $f \neq 0$  in  $S_{\theta_0} \cap \overline{B(0, R)}$ . Thus  $h \in H\left(S_{\theta_0} \cap \overline{B(0, R)}\right)$  and from (5.15), we deduce  $h \in H(S_{\theta_0})$ .

Since  $1 - f \in \mathcal{E}_\infty(S_{\theta_0})$ , there exists  $R_0 > 0$  such that, if  $z \in S_{\theta_0}$  and  $|z| > R_0$ , then  $|1 - f| < C|z|^{-\alpha}$  where  $\alpha > 0$ ,  $C > 0$ .

Moreover, since for all  $z \in S_{\theta_0}$  with  $|z| > \max(R, R_0)$  we have  $|1 - f(z)| < 1/2$ , we deduce that  $|f(z)| > 1/2$ . So, for all  $z \in S_{\theta_0}$  with  $|z| > \max(R, R_0)$  we obtain  $|h(z)| \leq 2C|z|^{-\alpha}$ . It follows that  $h \in \mathcal{E}_\infty(S_{\theta_0})$ .  $\square$

**Proposition 5.4.**  *$U$  and  $V$ , defined by (5.14), are invertible with bounded inverse.*

*Proof.* We first consider the case  $k > 0$ . Due to  $(H_5)$ ,  $P := -A$  satisfies the assumptions of Lemma 5.3. Moreover, it is clear that for a given  $\theta \in (0, \pi)$ ,  $f_k \in H(S_\theta)$  and  $1 - f_k \in \mathcal{E}_\infty(S_\theta)$ . Furthermore, from Lemma 5.2,  $f_k$  does not vanish on  $\mathbb{R}^+ \setminus \{0\}$ . So applying Lemma 5.3, we get that  $U = f_k(-A)$  is invertible. The invertibility of  $V = g_k(-A)$  is obtained in the same way.

Now, assume that  $k < 0$ . The linear operator  $-A + kI$  is a sectorial operator from Remark 2.1 5. and as in the first case, setting  $l := -k > 0$ , we deduce that  $U = f_l(-A + kI)$  and  $V = g_l(-A + kI)$  are invertible with bounded inverse.  $\square$

Now, we prove Theorem 2.5 for the boundary conditions (BC3). The proof is divided in two steps. First we prove the uniqueness of the solution of (1.4)-(BC3) by determining the constants  $K_i$ ,  $i = 1, 2, 3, 4$ , of the representation formula (2.5). Then, we show that the formula obtained is a classical solution.

*First step: Uniqueness.*

Assume that

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}. \quad (5.16)$$

If  $u$  is a classical solution of (1.4)-(BC3) then, as in section 5.1,  $u$  satisfies (5.3). Since  $F_{0,f}$  satisfies (3.7), the boundary conditions (BC3) applied to  $M^{-1}u(\cdot)$  imply the following relations

$$\begin{aligned} & \left(I - e^{(b-a)M}\right) M^{-1}\alpha_1 + \left(I - e^{(b-a)L}\right) M^{-1}\alpha_2 \\ & + \left(I + e^{(b-a)M}\right) M^{-1}\alpha_3 + \left(I + e^{(b-a)L}\right) M^{-1}\alpha_4 = M^{-1}\varphi_1, \end{aligned} \quad (5.17)$$

$$\begin{aligned} & \left( e^{(b-a)M} - I \right) M^{-1} \alpha_1 + \left( e^{(b-a)L} - I \right) M^{-1} \alpha_2 \\ & + \left( e^{(b-a)M} + I \right) M^{-1} \alpha_3 + \left( e^{(b-a)L} + I \right) M^{-1} \alpha_4 = M^{-1} \varphi_2, \end{aligned} \quad (5.18)$$

$$\begin{aligned} & M \left( I + e^{(b-a)M} \right) M^{-1} \alpha_1 + L \left( I + e^{(b-a)L} \right) M^{-1} \alpha_2 \\ & + M \left( I - e^{(b-a)M} \right) M^{-1} \alpha_3 + L \left( I - e^{(b-a)L} \right) M^{-1} \alpha_4 = M^{-1} \left( \varphi_3 - F'_{0,f}(a) \right), \end{aligned} \quad (5.19)$$

$$\begin{aligned} & M \left( e^{(b-a)M} + I \right) M^{-1} \alpha_1 + L \left( e^{(b-a)L} + I \right) M^{-1} \alpha_2 \\ & + M \left( e^{(b-a)M} - I \right) M^{-1} \alpha_3 + L \left( e^{(b-a)L} - I \right) M^{-1} \alpha_4 = M^{-1} \left( \varphi_4 - F'_{0,f}(b) \right). \end{aligned} \quad (5.20)$$

Summing (5.17) with (5.18) and (5.19) with (5.20), we obtain the system

$$\left\{ \begin{array}{l} \left( I - e^{(b-a)M} \right) M^{-1} \alpha_1 + \left( I - e^{(b-a)L} \right) M^{-1} \alpha_2 \\ + \left( I + e^{(b-a)M} \right) M^{-1} \alpha_3 + \left( I + e^{(b-a)L} \right) M^{-1} \alpha_4 = M^{-1} \varphi_1, \\ 2 \left( I + e^{(b-a)M} \right) M^{-1} \alpha_3 + 2 \left( I + e^{(b-a)L} \right) M^{-1} \alpha_4 = M^{-1} \left( \varphi_1 + \varphi_2 \right), \\ M \left( I + e^{(b-a)M} \right) M^{-1} \alpha_1 + L \left( I + e^{(b-a)L} \right) M^{-1} \alpha_2 \\ + M \left( I - e^{(b-a)M} \right) M^{-1} \alpha_3 + L \left( I - e^{(b-a)L} \right) M^{-1} \alpha_4 = M^{-1} \left( \varphi_3 - F'_{0,f}(a) \right), \\ 2M \left( I + e^{(b-a)M} \right) M^{-1} \alpha_1 + 2L \left( I + e^{(b-a)L} \right) M^{-1} \alpha_2 = M^{-1} \left( \varphi_3 + \varphi_4 - F'_{0,f}(a) - F'_{0,f}(b) \right). \end{array} \right.$$

From which we deduce the two second order systems

$$\left\{ \begin{array}{l} \left( I - e^{(b-a)M} \right) M^{-1} \alpha_1 + \left( I - e^{(b-a)L} \right) M^{-1} \alpha_2 = M^{-1} \left( \frac{\varphi_1 - \varphi_2}{2} \right) \\ M \left( I + e^{(b-a)M} \right) M^{-1} \alpha_1 + L \left( I + e^{(b-a)L} \right) M^{-1} \alpha_2 = M^{-1} \tilde{\varphi}_1 \end{array} \right. \quad (5.21)$$

and

$$\left\{ \begin{array}{l} \left( I + e^{(b-a)M} \right) M^{-1} \alpha_3 + \left( I + e^{(b-a)L} \right) M^{-1} \alpha_4 = M^{-1} \left( \frac{\varphi_1 + \varphi_2}{2} \right) \\ M \left( I - e^{(b-a)M} \right) M^{-1} \alpha_3 + L \left( I - e^{(b-a)L} \right) M^{-1} \alpha_4 = M^{-1} \tilde{\varphi}_2, \end{array} \right. \quad (5.22)$$

where

$$\tilde{\varphi}_1 := \frac{\varphi_3 + \varphi_4 - F'_{0,f}(a) - F'_{0,f}(b)}{2} \quad \text{and} \quad \tilde{\varphi}_2 := \frac{\varphi_3 - \varphi_4 - F'_{0,f}(a) + F'_{0,f}(b)}{2}. \quad (5.23)$$

Then, the unique determination of the constants  $\alpha_i$ ,  $i = 1, 2, 3, 4$ , is equivalent to show that the matrices  $\Lambda_1$  and  $\Lambda_2$  of the systems (5.21) and (5.22) have an invertible determinant. From (2.4), we have

$$\begin{aligned} \det(\Lambda_1) &= \begin{vmatrix} \left( I - e^{(b-a)M} \right) M^{-1} & \left( I - e^{(b-a)L} \right) M^{-1} \\ M \left( I + e^{(b-a)M} \right) M^{-1} & L \left( I + e^{(b-a)L} \right) M^{-1} \end{vmatrix} \\ &= \left( \left( I - e^{(b-a)M} \right) L \left( I + e^{(b-a)L} \right) - M \left( I + e^{(b-a)M} \right) \left( I - e^{(b-a)L} \right) \right) M^{-2} \\ &= \left( L - M - (L - M) e^{(b-a)(L+M)} - (L + M) \left( e^{(b-a)M} - e^{(b-a)L} \right) \right) M^{-2} \\ &= kM^{-2} (L + M)^{-1} U, \end{aligned}$$

and in the same way

$$\begin{aligned} \det(\Lambda_2) &= \begin{vmatrix} \left( I + e^{(b-a)M} \right) M^{-1} & \left( I + e^{(b-a)L} \right) M^{-1} \\ M \left( I - e^{(b-a)M} \right) M^{-1} & L \left( I - e^{(b-a)L} \right) M^{-1} \end{vmatrix} \\ &= kM^{-2} (L + M)^{-1} V, \end{aligned}$$

where  $U$  and  $V$  are given by (5.14). Since  $U$  and  $V$  are invertible with bounded inverse (see Proposition 5.4), from (5.21) and (5.22), we deduce

$$\begin{cases} \alpha_1 &= \frac{1}{2k}(L+M)U^{-1} \left[ L(I+e^{(b-a)L})(\varphi_1-\varphi_2) - 2(I-e^{(b-a)L})\tilde{\varphi}_1 \right] \\ \alpha_2 &= -\frac{1}{2k}(L+M)U^{-1} \left[ M(I+e^{(b-a)M})(\varphi_1-\varphi_2) - 2(I-e^{(b-a)M})\tilde{\varphi}_1 \right] \\ \alpha_3 &= \frac{1}{2k}(L+M)V^{-1} \left[ L(I-e^{(b-a)L})(\varphi_1+\varphi_2) - 2(I+e^{(b-a)L})\tilde{\varphi}_2 \right] \\ \alpha_4 &= -\frac{1}{2k}(L+M)V^{-1} \left[ M(I-e^{(b-a)M})(\varphi_1+\varphi_2) - 2(I+e^{(b-a)M})\tilde{\varphi}_2 \right]. \end{cases} \quad (5.24)$$

Note that, since  $F_{0,f} \in W^{4,p}(a,b;X) \cap L^p(a,b;D(M^2))$ , from Remark 3.1 we have

$$F'_{0,f}(a) \in (D(M^2), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \quad \text{and} \quad F'_{0,f}(b) \in (D(M^2), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}. \quad (5.25)$$

This combined with (5.16) implies that  $F'_{0,f}(a), F'_{0,f}(b), \varphi_i \in D(M^2)$ ,  $i = 1, 2, 3, 4$ . Thus all the previous equalities are well defined. This shows that the eventual classical solution of (1.4)-(BC3) is characterized by (5.3), (5.23) and (5.24).

*Second step: Existence.*

It suffices to prove that  $u$ , determined in the first step, satisfies (1.6). From Lemma 5.1, we have

$$U^{-1} = I + R_U \quad \text{and} \quad V^{-1} = I + R_V.$$

Set  $T := L$  or  $M$ . Then, from (4.2), since  $Te^{xT} = e^{xT}T$  on  $D(T)$ , for any  $x \geq 0$ , Lemma 5.1 implies

$$R_U(X), R_V(X) \subset D(M^\infty), \quad R_U T = T R_U \quad \text{and} \quad R_V T = T R_V.$$

This combined with (4.2) implies that there exist  $R_i \in D(M^\infty)$ ,  $i = 1, 2, 3, 4$ , such that

$$\begin{cases} \alpha_1 &= \frac{1}{2k}(L+M) [L(\varphi_1-\varphi_2) - 2\tilde{\varphi}_1] + R_1 \\ \alpha_2 &= -\frac{1}{2k}(L+M) [M(\varphi_1-\varphi_2) - 2\tilde{\varphi}_1] + R_2 \\ \alpha_3 &= \frac{1}{2k}(L+M) [L(\varphi_1+\varphi_2) - 2\tilde{\varphi}_2] + R_3 \\ \alpha_4 &= -\frac{1}{2k}(L+M) [M(\varphi_1+\varphi_2) - 2\tilde{\varphi}_2] + R_4. \end{cases}$$

Then, from (5.2) and (5.23), we obtain

$$\begin{cases} K_1 &= \frac{1}{k}(L+M) [L\varphi_1 - \varphi_3 + F'_{0,f}(a)] + \tilde{R}_1 \\ K_2 &= \frac{1}{k}(L+M) [L\varphi_2 + \varphi_4 - F'_{0,f}(b)] + \tilde{R}_3 \\ K_3 &= -\frac{1}{k}(L+M) [M\varphi_1 - \varphi_3 + F'_{0,f}(a)] + \tilde{R}_2 \\ K_4 &= -\frac{1}{k}(L+M) [M\varphi_2 + \varphi_4 - F'_{0,f}(b)] + \tilde{R}_4, \end{cases} \quad (5.26)$$

with  $\tilde{R}_i \in D(M^\infty)$ ,  $i = 1, 2, 3, 4$ . We set

$$\tilde{K}_1 := K_1 + K_3 \quad \text{and} \quad \tilde{K}_2 := K_2 + K_4,$$

thus, from (5.11), for any  $x \in (a, b)$ ,

$$u(x) = e^{(x-a)M}\tilde{K}_1 + e^{(b-x)M}\tilde{K}_2 + \left( e^{(x-a)L} - e^{(x-a)M} \right) K_3 + \left( e^{(b-x)L} - e^{(b-x)M} \right) K_4 + F_{0,f}(x).$$

From (3.2), we have

$$(D(A), X)_{1+\frac{1}{2p}, p} = (D(M), X)_{3+\frac{1}{p}, p}. \quad (5.27)$$

Thus, since  $\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p}$ , from (2.3), we have

$$\tilde{K}_1 = K_1 + K_3 = \frac{1}{k}(L + M)(L - M)\varphi_1 + \tilde{R}_1 + \tilde{R}_3 = \varphi_1 + \tilde{R}_1 + \tilde{R}_3 \in (D(M), X)_{3+\frac{1}{p}, p}$$

and, in the same way,

$$\tilde{K}_2 = K_2 + K_4 = \varphi_2 + \tilde{R}_2 + \tilde{R}_4 \in (D(M), X)_{3+\frac{1}{p}, p}.$$

Then, from Lemma 4.1 and Remark 4.2, we obtain that

$$u_M : x \mapsto e^{(x-a)M} \tilde{K}_1 + e^{(b-x)M} \tilde{K}_2,$$

satisfies

$$u_M \in W^{4,p}(a, b; X) \cap L^p(a, b; D(A^2)) \quad \text{and} \quad u_M'' \in L^p(a, b; D(A)). \quad (5.28)$$

Thus, from (5.25), (5.16) and (5.27),  $M\varphi_1, \varphi_3, F'_{0,f}(a) \in (D(M), X)_{2+\frac{1}{p}, p}$ . From (5.26), we deduce

$$K_3 \in (D(M), X)_{1+\frac{1}{p}, p}.$$

So, from Theorem 2.4, we obtain that  $v_{K_3} : x \mapsto (e^{(x-a)L} - e^{(x-a)M}) K_3$  satisfies

$$v_{K_3} \in W^{4,p}(a, b; X) \cap L^p(a, b; D(A^2)) \quad \text{and} \quad v_{K_3}'' \in L^p(a, b; D(A)). \quad (5.29)$$

By the same arguments,  $v_{K_4} : x \mapsto (e^{(x-a)L} - e^{(x-a)M}) K_4$  satisfies

$$v_{K_4} \in W^{4,p}(a, b; X) \cap L^p(a, b; D(A^2)) \quad \text{and} \quad v_{K_4}'' \in L^p(a, b; D(A)). \quad (5.30)$$

Since  $F_{0,f}$  satisfies (1.6), from (5.11), (5.28), (5.29) and (5.30), we deduce that  $u$  satisfies (1.6) and so is a classical solution of (1.4)-(BC3).

### 5.3 Proof of statement 4. (Boundary conditions (BC4))

We proceed as in the proof of the previous statement. We only point out the differences between the two proofs.

*First step:* Uniqueness.

Assume that

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}. \quad (5.31)$$

If  $u$  is a classical solution of (1.4)-(BC4), then  $u$  is given by (5.3). Since  $F_{0,f}$  satisfies (3.7), the boundary conditions (BC4) applied to  $M^{-1}u'(a), M^{-1}u'(b), M^{-2}u''(a), M^{-2}u''(b)$  and the same computations as in the proof of subsection 5.2 lead to the two second order systems

$$\begin{cases} (I + e^{(b-a)M}) \alpha_1 + LM^{-1} (I + e^{(b-a)L}) \alpha_2 = M^{-1} \tilde{\varphi}_1, \\ (I - e^{(b-a)M}) \alpha_1 + L^2 M^{-2} (I - e^{(b-a)L}) \alpha_2 = M^{-2} \frac{\varphi_3 - \varphi_4}{2}, \end{cases} \quad (5.32)$$

and

$$\begin{cases} (I - e^{(b-a)M}) \alpha_3 + LM^{-1} (I - e^{(b-a)L}) \alpha_4 = M^{-1} \tilde{\varphi}_2, \\ (I + e^{(b-a)M}) \alpha_3 + L^2 M^{-2} (I + e^{(b-a)L}) \alpha_4 = M^{-2} \frac{\varphi_3 + \varphi_4}{2}, \end{cases} \quad (5.33)$$

where

$$\tilde{\varphi}_1 := \frac{\varphi_1 + \varphi_2 - F'_{0,f}(a) - F'_{0,f}(b)}{2} \quad \text{and} \quad \tilde{\varphi}_2 := \frac{\varphi_1 - \varphi_2 - F'_{0,f}(a) + F'_{0,f}(b)}{2}. \quad (5.34)$$

Let  $\Lambda_1$  and  $\Lambda_2$  be their corresponding matrices. From (2.4), we have

$$\begin{aligned} \det(\Lambda_1) &= \begin{vmatrix} I + e^{(b-a)M} & LM^{-1} (I + e^{(b-a)L}) \\ I - e^{(b-a)M} & L^2 M^{-2} (I - e^{(b-a)L}) \end{vmatrix} \\ &= L^2 M^{-2} (I - e^{(b-a)L}) (I + e^{(b-a)M}) - LM^{-1} (I + e^{(b-a)L}) (I - e^{(b-a)M}) \\ &= kLM^{-2} V(L + M)^{-1}, \end{aligned}$$

and

$$\begin{aligned}
\det(\Lambda_2) &= \begin{vmatrix} I - e^{(b-a)M} & LM^{-1}(I - e^{(b-a)L}) \\ I + e^{(b-a)M} & L^2M^{-2}(I + e^{(b-a)L}) \end{vmatrix} \\
&= L^2M^{-2}(I - e^{(b-a)M})(I + e^{(b-a)L}) - LM^{-1}(I - e^{(b-a)L})(I + e^{(b-a)M}) \\
&= kLM^{-2}(L + M)^{-1}U,
\end{aligned}$$

where  $U$  and  $V$  are given by (5.14). Since  $U$  and  $V$  are invertible in  $\mathcal{L}(X)$  (see Proposition 5.4), from (5.32) and (5.33), we deduce

$$\begin{cases} \alpha_1 &= \frac{1}{2k}(L + M)V^{-1} \left[ 2(I - e^{(b-a)L})LM^{-1}\tilde{\varphi}_1 - (I + e^{(b-a)L})M^{-1}(\varphi_3 - \varphi_4) \right] \\ \alpha_2 &= -\frac{1}{2k}(L + M)V^{-1} \left[ 2(I - e^{(b-a)M})ML^{-1}\tilde{\varphi}_1 - (I + e^{(b-a)M})L^{-1}(\varphi_3 - \varphi_4) \right] \\ \alpha_3 &= \frac{1}{2k}(L + M)U^{-1} \left[ 2(I + e^{(b-a)L})LM^{-1}\tilde{\varphi}_2 - (I - e^{(b-a)L})M^{-1}(\varphi_3 + \varphi_4) \right] \\ \alpha_4 &= -\frac{1}{2k}(L + M)U^{-1} \left[ 2(I + e^{(b-a)M})ML^{-1}\tilde{\varphi}_2 - (I - e^{(b-a)M})L^{-1}(\varphi_3 + \varphi_4) \right]. \end{cases} \quad (5.35)$$

Then, from (5.2) and (5.35) we obtain the uniqueness of the solution of (1.4)-(BC4).

*Second step: Existence.*

Let  $u$ , determined in the first step, then  $u$  is also given by (5.11). Then, it suffices to show that  $\tilde{K}_1, \tilde{K}_2 \in (D(M), X)_{3+\frac{1}{p}, p}$  and  $K_3, K_4 \in (D(M), X)_{1+\frac{1}{p}, p}$ . By the same arguments as in the proof of subsection 5.2, there exist  $R_i \in D(M^\infty)$ ,  $i = 1, 2, 3, 4$ , such that

$$\begin{cases} \alpha_1 &= \frac{1}{2k}(L + M) [2LM^{-1}\tilde{\varphi}_1 - M^{-1}(\varphi_3 - \varphi_4)] + R_1 \\ \alpha_2 &= -\frac{1}{2k}(L + M) [2ML^{-1}\tilde{\varphi}_1 - L^{-1}(\varphi_3 - \varphi_4)] + R_2 \\ \alpha_3 &= \frac{1}{2k}(L + M) [2LM^{-1}\tilde{\varphi}_2 - M^{-1}(\varphi_3 + \varphi_4)] + R_3 \\ \alpha_4 &= -\frac{1}{2k}(L + M) [2ML^{-1}\tilde{\varphi}_2 - L^{-1}(\varphi_3 + \varphi_4)] + R_4. \end{cases}$$

Then, from (5.2) and (5.34), we obtain

$$\begin{cases} K_1 &= \frac{1}{k}(L + M) [LM^{-1}(\varphi_1 - F'_{0,f}(a)) - M^{-1}\varphi_3] + \tilde{R}_1 \\ K_2 &= \frac{1}{k}(L + M) [LM^{-1}(-\varphi_2 + F'_{0,f}(b)) - M^{-1}\varphi_4] + \tilde{R}_2 \\ K_3 &= -\frac{1}{k}(L + M) [ML^{-1}(\varphi_1 - F'_{0,f}(a)) - L^{-1}\varphi_3] + \tilde{R}_3 \\ K_4 &= -\frac{1}{k}(L + M) [ML^{-1}(-\varphi_2 + F'_{0,f}(b)) - L^{-1}\varphi_4] + \tilde{R}_4, \end{cases} \quad (5.36)$$

with  $\tilde{R}_i \in D(M^\infty)$ ,  $i = 1, 2, 3, 4$ . Then, combining (5.12) and (5.13), since  $\varphi_1, F'_{0,f}(a) \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}$  and  $\varphi_3 \in (D(A), X)_{\frac{1}{2p}, p}$ , from (2.3), we have

$$\begin{aligned}
\tilde{K}_1 &= K_1 + K_3 = \frac{1}{k}(L + M) [(LM^{-1} - ML^{-1})(\varphi_1 - F'_{0,f}(a)) + (L^{-1} - M^{-1})\varphi_3] + \tilde{R}_1 + \tilde{R}_2 \\
&= \frac{1}{k}(L + M) [(L^2 - M^2)M^{-1}L^{-1}(\varphi_1 - F'_{0,f}(a)) - (L - M)M^{-1}L^{-1}\varphi_3] + \tilde{R}_1 + \tilde{R}_2 \\
&= (L + M)M^{-1}L^{-1}(\varphi_1 - F'_{0,f}(a)) - M^{-1}L^{-1}\varphi_3 + \tilde{R}_1 + \tilde{R}_2 \in (D(M), X)_{3+\frac{1}{p}, p}
\end{aligned}$$

and, in the same way,

$$\tilde{K}_2 = K_2 + K_4 = (L + M)M^{-1}L^{-1}(F'_{0,f}(b) - \varphi_2) - M^{-1}L^{-1}\varphi_4 + \tilde{R}_2 + \tilde{R}_4 \in (D(M), X)_{3+\frac{1}{p}, p}.$$

Moreover, since  $(L + M)\varphi_1, (L + M)F'_{0,f}(a), \varphi_3 \in (D(M), X)_{2+\frac{1}{p}, p}$ , from (5.36), we deduce

$$K_3 \in (D(M), X)_{1+\frac{1}{p}, p},$$

and, in the same way,  $K_4 \in (D(M), X)_{1+\frac{1}{p}, p}$ , which leads to the result.

## References

- [1] W. ARENDT, C.J.K. BATTY, M. HIEBER & F. NEUBRANDER, *Vector-valued Laplace transforms and Cauchy problems*, second Edition, Monographs in Mathematics, 96, Birkhauser, 2011.
- [2] W. ARENDT, S. BU, M. HAASE, “Functional calculus, variational methods and Liapunov’s theorem”, *Arch. Math.*, 77, 2001, pp. 65-75.
- [3] J. BOURGAIN, “Some remarks on Banach spaces in which martingale difference sequences are unconditional”, *Ark. Mat.*, vol. 21, 1983, pp. 163-168.
- [4] D.L. BURKHOLDER, “A geometrical characterisation of Banach spaces in which martingale difference sequences are unconditional”, *Ann. Probab.*, vol. 9, 1981, pp. 997-1011.
- [5] L. CHERFILS, A. MIRANVILLE & S. ZELIK, “On a generalized Cahn-Hilliard equation with biological applications”, *Discrete and continuous dynamical systems series B*, vol. 19, No 7, 2014, pp. 2013-2026.
- [6] D.S. COHEN & J.D. MURRAY, “A generalized diffusion model for growth and dispersal in population”, *Journal of Mathematical Biology*, 12, Springer-Verlag, 1981, pp. 237-249.
- [7] G. DA PRATO & P. GRISVARD, “Somme d’opérateurs linéaires et équations différentielles opérationnelles”, *J. Math. Pures et Appl.*, 54, 1975, pp. 305-387.
- [8] G. DORE & A. VENNI, “On the closedness of the sum of two closed operators”, *Math. Z.*, 196, 1987, pp. 189-201.
- [9] G. DORE & A. VENNI, “ $H^\infty$  functional calculus for sectorial and bisectorial operators”, *Studia Math.*, 166, no. 3, 2005, pp. 221-241.
- [10] A. FAVINI, R. LABBAS, K. LEMRABET, S. MAINGOT & H. SIDIBÉ, *Transmission Problem for an Abstract Fourth-order Differential Equation of Elliptic Type in UMD Spaces*, Advances in Differential Equations, Volume 15, Numbers 1-2, 2010, pp. 43-72.
- [11] A. FAVINI, R. LABBAS, S. MAINGOT, H. TANABE & A. YAGI, “A simplified approach in the study of elliptic differential equations in UMD spaces and new applications”, *Funkc. Ekv.*, 451, 2008, pp. 165-187.
- [12] D. GILBARG & N. TRUDINGER, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [13] P. GRISVARD, “Spazi di tracce e applicazioni”, *Rendiconti di Matematica*, (4) Vol.5, Serie VI, 1972, pp. 657-729.
- [14] M. HAASE, *The functional calculus for sectorial Operators*, Birkhauser, 2006.
- [15] T. KATO, *Perturbation theory for linear operators*, Springer, 1980.
- [16] J.-L. LIONS & J. PEETRE, “Sur une classe d’espaces d’interpolation”, *Publications mathématiques de l’I.H.É.S.*, tome 19, 1964, pp. 5-68.
- [17] A. LUNARDI, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhauser, Basel, Boston, Berlin, 1995.
- [18] S.G. MIHLIN, “On the multipliers of Fourier integrals”, *Dokl. Akad. Nauk SSSR*, N.S., 109, 1956, pp. 701-703.
- [19] A. NOVICK-COHEN, “On Cahn-Hilliard type equations”, *Nonlinear Analysis, Theory, Methods and Applications*, vol. 15, No 9, 1990, pp. 797-814.
- [20] F.L. OCHOA, “A generalized reaction-diffusion model for spatial structures formed by motile cells”, *BioSystems*, 17, 1984, pp. 35-50.
- [21] A. PAZY, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

- [22] J. PRÜSS, “Maximal regularity for evolution equations in  $L^p$ -spaces”, *Conf. Semin. Mat. Univ. Bari*, No. 285, 2002, pp. 1-39.
- [23] J. PRÜSS & H. SOHR, “On operators with bounded imaginary powers in Banach spaces”, *Mathematische Zeitschrift, Springer-Verlag, Math. Z.*, 203, 1990, pp. 429-452.
- [24] J. PRÜSS & H. SOHR, “Imaginary powers of elliptic second order differential operators in  $L^p$ -spaces”, *Hiroshima Math. J.*, 23, no. 1, 1993, pp. 161-192.
- [25] J.L. RUBIO DE FRANCIA, *Martingale and integral transforms of Banach space valued functions*, in: J. Bastero, M. San Miguel (Eds.), *Probability and Banach Spaces*, Zaragoza, 1985, in: *Lecture Notes in Math.*, vol. 1221, Springer-Verlag, Berlin, 1986, pp. 195-222.
- [26] H. TRIEBEL, *Interpolation theory, function Spaces, differential Operators*, North-Holland publishing company Amsterdam New York Oxford, 1978.