From Static Output Feedback to Structured Robust Static Output Feedback: A Survey

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Abstract

This paper reviews the vast literature on static output feedback design for linear time-invariant systems including classical results and recent developments. In particular, we focus on static output feedback synthesis with performance specifications, structured static output feedback, and robustness. The paper provides a comprehensive review on existing design approaches including iterative linear matrix inequalities heuristics, linear matrix inequalities with rank constraints, methods with decoupled Lyapunov matrices, and non-Lyapunov-based approaches. We describe the main difficulties of dealing with static output feedback design and summarize the main features, advantages, and limitations of existing design methods.

Keywords: Static output feedback (SOF), structured static output feedback, robustness, convex optimization, linear matrix inequality (LMI), bilinear matrix inequality (BMI), non-smooth non-convex optimization.

1 Introduction

Static output feedback design is a theoretically challenging issue in control theory and it has attracted considerable attention due to its great importance in practice. However, so far, there has been no exact solution to this prominent problem which can guarantee the design of static output feedback or determine that such a feedback does not exist. The fact is that the problem is intrinsically a Bilinear Matrix Inequality (BMI) problem which is generally NP-hard [116]; furthermore, it becomes non-smooth in the case of problem formulation in the space of the controller parameters [117].

To solve the static output feedback design problem, well-known bilinear matrix inequality (BMI) solvers such as the commercial software package PENBMI [64,79] and the free open-source MATLAB toolbox PENLAB [45] can be applied. The algorithms behind these solvers combines the ideas of the (exterior) penalty and (interior) barrier methods with the augmented Lagrangian approach [78]. These solvers can locally solve all kinds of BMI problems, including static output feedback. Since our aim is to survey dedicated static output feedback design methods, we no longer discuss these general BMI approaches. Note however that BMI solvers most often fail to provide a solution for the static output feedback BMI problems, and the choice of an initial guess is very crucial for these solvers.

The only survey dedicated to static output feedback has been conducted in [115]. Since then, the past two decades have witnessed much theoretical progress on static output feedback design which has not been covered in that survey. A large amount of research has been carried out on the development of the static output feedback controllers according to Lyapunov theory via linear matrix inequality based (LMI-based) approaches, e.g. [3, 9, 13, 14, 18, 22, 23, 27, 32, 33, 35, 38, 49, 53–56, 60, 68, 72, 75, 77, 80, 82, 83, 89–91, 94, 105–107, 109, 118]. Most of these methods present an iterative algorithm in which a set of LMIs are iteratively repeated until some certain termination criteria are met. In addition to the Lyapunov-based approaches, there exist non-Lyapunov-based static output feedback control strategies, see, e.g. [4, 6, 8, 12, 20, 24, 57, 58, 101].

The objective of this paper is to provide a comprehensive review on the existing static output feedback design methods. The main focus is on pure stabilizing static output feedback design with no other specification. But the paper also addresses the problem of structured feedback, simultaneous stabilization, multi-performance and robust control design. All methods and approaches described in the survey are gathered in order to provide a comprehensive classification. All results have been reinterpreted and rewritten so as to fit a common notation/framework. The notation uniformization allows a simplified overview on the differences and ressemblances of the results. It allows as well to provide direct extensions of the existing results for example using system duality. Due to the fact that fixed-order dynamic output-feedback can be equivalently transformed into static output feedback by introducing an augmented plant [53], this survey paper can be also used for fixed/low-order control design problem.

The paper is organized as follows. Section 2 presents the problem statement

and main difficulties associated with stabilizing static output feedback design and its extensions to structured feedback, simultaneous stabilization, multi-performance and robust control.

The five sections that follow provide our classification of SOF design methods. Section 3 focuses on special cases where under specific structures of the open-loop system, the SOF problem becomes convex. Section 4 reviews the available literature on iterative LMI heuristics for the intrinsically MI nature of SOF design. Section 5 covers the heuristics related to a reformulation of the SOF design as LMIs with rank constraints. While all the previous sections describe results build out of classical Lyapunov conditions, Section 6 is devoted to methods with decoupled Lyapunov matrices that have better characteristics with respect to robustness. Section 7 exposes alternative approaches which are non-Lyapunov-based. All the classes of results are analyzed in terms of their known or claimed numerical characteristics, well as in terms of their ability to address the structured feedback, simultaneous multi-performance, and robustness issues.

The paper ends with global concluding remarks in Section 8.

The notation used in this paper is standard. In particular, matrices I and 0 are the identity matrix and the zero matrix of appropriate dimensions, respectively. The symbol $\star$ denotes symmetric blocks in block matrices. The symbols A^T, $\{A\}^{\mathcal{S}}$, $A^{\perp}$, $\|A\|_F$, and $A^{\frac{1}{2}}$ are respectively notations for the transpose of A, $\{A\}^{\mathcal{S}} = A + A^T$, the maximal rank perpendicularity such that $A^{\perp}A = 0$, Frobenius norm of A, and the unique nonnegative-definite square root of positive-definite matrix A. For symmetric matrices, $P > 0$ ($P < 0$) indicates the positive-definiteness (the negative-definiteness).

2 Problem Formulation and Main Difficulties

2.1 Main SOF stabilization problem

Consider a linear time-invariant (LTI) continuous-time system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

and a static output feedback controller

$$u(t) = Ky(t) \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^{n_i}$ in the control input, and $y \in \mathbb{R}^{n_o}$ is the output of the system. The state-space matrices A, B, C, and the control gain K are of appropriate dimensions. The closed-loop system is described as follows:

$$\dot{x}(t) = (A + BKC)x(t) \tag{3}$$

and its stability is equivalent to that of the dual system

$$\dot{x}_d(t) = (A + BKC)^T x_d(t). \tag{4}$$

Theorem 1 *The following statements are equivalent and prove that the static output feedback (2) stabilizes the system (1).*

(a) The eigenvalues of $A + BKC$ are all in the left-half plane.

(b) There exists a symmetric matrix P satisfying the following matrix inequalities (Lyapunov inequalities for the primal system):

$$P > 0 \ , \quad \{P(A + BKC)\}^{\mathcal{S}} < 0 \tag{5}$$

(c) There exists a symmetric matrix Q satisfying the following matrix inequalities (Lyapunov inequalities for the dual system):

$$Q > 0 \ , \quad \{(A + BKC)Q\}^{\mathcal{S}} < 0 \tag{6}$$

Moreover, $Q = P^{-1}$ holds to prove equivalence of the two last conditions.

The main difficulties associated with static output feedback design are as follows [62]:

- Non-differentiability: The performance objective related to the first statement (maximal real part of all eigenvalues) is a non-differential function of K. The spectral abscissa of the closed-loop state matrix $A + BKC$ is a continuous but non-Lipschitz function of K; thus, its gradient can be locally unbounded.
- Non-convexity: The stability conditions (5) or (6) are not convex in the unknowns due to the terms containing products of P and K and products of Q and K respectively.

For concrete control system design, the problem formulation is scarcely limited to proving stability of the closed-loop. The actual problems to be solved include multi-objective and robustness specifications as well as structure constraints on the control gains. In this survey we shall not enter in all the details of how these specifications are formulated for each considered method, and most often they are not. We will rather give a general appreciation of the ability of the methods to address these specifications. To help the understanding we for a start briefly formalize the specifications.

2.2 Structured SOF

Constraints on the control structure are mainly rooted in different sources. The first source comes from the well-known Internal Model Principle (IMP) [46] which states that for tracking and disturbance rejection, the dynamics of persistently exciting references and/or disturbances must be replicated in the structure of the controller. Furthermore, the well-known proportional-integral (PI) and proportional-integral-derivative (PID) controllers, widely used in industrial control systems, inherently have a fixed structure. Finally, the last main source

results from a need for decentralized or distributed control of large-scale interconnected systems due to cost, reliability issues, and limitations on communication links among the local controllers [126]. All these reasons highlight the paramount importance of structured control design.

Mathematically these structural constraints usually boil down to imposing that some coefficients are zero in the K matrix and/or that some others are linearly dependent. More general non linear constraints may also occur but for the present survey we shall assume that structure constraints are linear equality constraints of the type $L_s K R_s = C_s$ where there may be several triples of given matrices (L_s, R_s, C_s).

2.3 Simultaneous stability and multi-performance

Stability is in general only one of the expected features of a closed-loop system. On top of stability, classical performance specifications are in terms of input/output performances such as H_∞ or H_2 performances (see [1, 3, 8, 9, 36, 58, 61, 73, 83, 102, 105, 106, 123, 124] and [12, 83, 94, 105, 106, 108, 109] respectively for papers addressing these issues in the SOF framework). Moreover, the actual requirements are most often in terms of a tradeoff between several such specifications. This is the multi-performance problem.

To formulate the problem let us define a system with performance inputs w and outputs z:

$$H^{[i]}(s): \begin{array}{l} \dot{x}(t) = A^{[i]}x(t) + B_w^{[i]}w(t) + B^{[i]}u(t) \\ z(t) = C_z^{[i]}x(t) + D_{zw}^{[i]}w(t) + D_{zu}^{[i]}u(t) \\ y(t) = C^{[i]}x(t) + D_{yw}^{[i]}w(t) \end{array} \tag{7}$$

and denote $H^{[i]}(s, K)$ the closed-loop with SOF $u = Ky$. For that system define a performance Π_i which could be among H_∞, H_2, stability (or others). The multi-performance problem is, given some collection of open-loop systems $(H^{[1]}, \dots H^{[\bar{\imath}]})$, associated performances $(\Pi_1, \dots \Pi_{\bar{\imath}})$ and performance levels $(\gamma_1 \dots \gamma_{\bar{\imath}})$ to find a common to all gain K guaranteeing that each closed-loop satisfies its prescribed performance level:

$$\Pi_i(H^{[i]}(s, K)) \leq \gamma_i, \quad \forall i = 1 \dots \bar{\imath}. \tag{8}$$

In case all 'performances' Π_i are stability specifications and the models are all different, this problem corresponds to simultaneous stabilization of the collection of plants by the same SOF control (see [20,22,57,65,66]). In case of two identical plant models and the specifications are H_∞ and H_2, the design problem amounts to finding a mixed H_2/H_∞ controller (see [14, 41, 59, 83]). A variant of this problem is the multi-objective control problem where one aims at minimizing a linear combination of the performance levels γ_i.

All mentioned performances happen to be associated to formulations similar to Theorem 1. That is: (a) some possibility to compute numerically the performance knowing $H^{[i]}$ and K; (b) a matrix inequality formulation involving some

matrices $P^{[i]}$ and the system representation $H^{[i]}$; (c) a dual matrix inequality formulation involving some $Q^{[i]}$. Similar properties hold for these individual performance problems as for the stabilization problem. The multi-performance problem is more involved due to the several constraints to satisfy simultaneously.

2.4 Robustness

The robustness problem is the extension of simultaneous stabilisation to an infinite number of plant models defined by a set of membership constraint. The problem is trivially more involved since one cannot simply concatenate the infinitely many constraints.

Among these robustness problems, one is simpler than others. In case, the uncertain system is defined as a rational function $A(\Delta) = A + B_\Delta \Delta (I - D_\Delta \Delta)^{-1} C_\Delta$ of some unstructured matrix Δ constrained by a convex quadratic constraint, the problem of static output feedback design can be recast via linear fractional transformation (LFT) as a system in feedback-loop with respect to a norm-bounded uncertainty. Then, by means of the small gain theorem, the robust stabilization problem is translated into controller design for a given plant under H_∞ performance constraint [127]. It falls under the previously discussed category.

More general robustness problems are when Δ is structured (block diagonal). In that case, even robust analysis issues are complex to solve for example via μ-analysis tools [15,125], internal quadratic constraints (IQCs) results [88], and parameter-dependent Lyapunov functions [95,113]. Rather than going into the details of these, we shall rather concentrate our attention at formally simpler case, but that has in the end similar features in SOF design as when considering uncertain systems with structured LFT-type uncertainty.

We focus our attention on affine polytopic uncertain systems modeled as the convex combination of a finite number of given vertices. Polytopic uncertainty can cover interval, linear parameters, and multi-model uncertainties [74]. Although these models may seem limited to affine dependence in the uncertain parameters, papers such as [25,37,87,121] show that at the expense of extensions to descriptor-type representations, affine polytopic models include rationally dependent plants. For the present survey we will concentrate on the simplest case without descriptor representations. Moreover, we shall consider only systems with uncertainties on the A and B matrices that have the advantage to keep the affine dependence when considering the closed-loop.

Polytopic uncertain systems will have the following notation:

$$\begin{aligned} \dot{x}(t) &= A(\xi)x(t) + B(\xi)u(t) \\ y(t) &= Cx(t) \end{aligned} \tag{9}$$

where the uncertain matrices are defined as the convex linear combinations of a finite number of vertices

$$\begin{bmatrix} A(\xi) & B(\xi) \end{bmatrix} = \sum_{j=1}^{\bar{j}} \xi_j \begin{bmatrix} A^{[j]} & B^{[j]} \end{bmatrix} \tag{10}$$

where ξ is the vector of barycentric coordinates defined as in the following simplex

$$\xi \in \Xi = \{\xi = (\xi_1, \ldots, \xi_{\bar{j}}),\, \xi_j \geq 0,\, \sum_{j=1}^{\bar{j}} \xi_j = 1\}. \tag{11}$$

The problem of robust static output feedback synthesis is to design a control law $u(t) = Ky(t)$ for the polytopic system (9) such that every model inside the polytope is stable. The problem of robust static output feedback is more challenging compared to simultaneous stabilization as the controller must ensure the stability and performances for every model in the uncertainty domain.

2.5 SOF results and their potential extensions

In the following, we shall consider the existing static output feedback design results from the literature that deal either with the non-differentiability issue via non-smooth optimization or with the non-convexity issue via iterative LMI approaches. In all cases we shall expose the results assuming the main SOF design problem (stabilisation of one plant without any structure constraint). This will allow us, without entering into all the details, to question these results with respect to their potentialities of extension for structured SOF, multi-performance and robustness issues. But before that we recall some special cases when the static output feedback design can be recast as a convex optimization problem.

3 Convex Cases

From our study of the literature we have been able to extract five types of results in which a particular structure of the data enables to convert the non convex static output feedback design to a convex optimization problem. The first of these results is the well known state-feedback case (and its dual case related to observer design).

Proposition 1 *In the two following cases a change of variables involving the Lyapunov matrix makes the problem convex:*

1. *In the full-actuation case, $B = I$, if there exist $P = P^T$ and L solution to the following LMIs:*

$$P > 0\ ,\quad \{PA + LC\}^{\mathcal{S}} < 0 \tag{12}$$

then $K = P^{-1}L$ is a stabilizing feedback gain.

2. *In the full-information case, $C = I$, if there exist $Q = Q^T$ and F solution to the following LMIs:*

$$Q > 0\ ,\quad \{AQ + BF\}^{\mathcal{S}} < 0 \tag{13}$$

then $K = FQ^{-1}$ is a stabilizing feedback gain.

The full-actuation case is generally used in the literature for output filtering [50] or for observer gain design. The result is indeed such that the following Luenberger-type observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

guarantees that the error signal $x(t) - \hat{x}(t)$ converges asymptotically to zero. Due to separation principle, the combination of state-feedback and observer feedback allows to build full-order dynamic controllers. It is not the only convex method for such full-order design (see for example [39, 70, 110]). Since these methods are out of scope of the present survey, we shall not comment further on these. Nevertheless, note that all these methods fail to provide results for the robust case when systems are with uncertainties (see for example [52] for a discussion about robust full-order control and [100] for a discussion on robust observers).

Note that conditions in Proposition 1 allow to describe all possible controllers as soon as the assumptions hold. All stabilizing state-feedback gains can be rewritten as $K = P^{-1}L$ for some pair (P, L) solution to (12). This feature does not hold for the following results by [26].

Proposition 2 *In the two following cases, a change of variables involving an auxiliary matrix, that is constrained linearly to the Lyapunov matrix, makes the problem convex:*

1. *If there exists $P = P^T$, L and square matrix $\hat{P}$ solution to:*

$$P > 0 \ , \quad \{PA + BLC\}^{\mathcal{S}} < 0 \ , \quad B\hat{P} = PB \tag{14}$$

   then $K = \hat{P}^{-1}L$ is a stabilizing feedback gain.

2. *If there exists $Q = Q^T$, F and a square square matrix $\hat{Q}$ solution to:*

$$Q > 0 \ , \quad \{AQ + BFC\}^{\mathcal{S}} < 0 \ , \quad \hat{Q}C = CQ \tag{15}$$

   then $K = F\hat{Q}^{-1}$ is a stabilizing feedback gain.

These conditions hold only in special cases and can be understood as constraints on BP or CQ of being close to commute. This feature is trivial when $B = I$ or $C = I$, respectively. In that sense, the approach is a generalization of Proposition 1. The property on BP or CQ of being close to commute is not possible in general. Moreover, if the conditions hold, these describe only a subset of all stabilizing gains.

In terms of simultaneous stabilization, results of Propositions 1 and 2 impose inevitably to search for a common Lyapunov matrix P or Q for all systems. This is the classical "Lyapunov Shaping Paradigm" as formalized in [110]. Inevitably, this same conclusion applies to the multi-objective problems and to robustness issues. For these conditions, the simultaneous stabilization of several systems and the robust stability of the polytope having these same systems

as vertices coincide. The robust counterpart of (15) for systems (9) reads as finding common Q, $\hat{Q}$, and F solution to the following conditions:

$$Q > 0 \ , \quad \{A^{[j]}Q + B^{[j]}FC\}^{\mathcal{S}} < 0 \ , \quad \hat{Q}C = CQ \tag{16}$$

for all vertices $j = 1 \dots \bar{j}$. These conditions for state-feedback are well-known since [16, 17, 76] and have been intensively applied.

Another convex sub-case of the SOF design problem proposed in [102] is when some structural constraints hold.

Proposition 3 *Consider the following assumptions:*

i) (1) is a minimal realization and the plant is square ($n_i = n_o$).

ii) CB is full row-rank.

iii) Let T_C be an orthogonal basis of the null-space of C, let $T = \begin{bmatrix} C^T & T_C^T \end{bmatrix}^T$ and let

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = TAT^{-1}, \ \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} = TB$$

with dimensions such that $\tilde{A}_{11} \in \mathbb{R}^{n_o \times n_o}$, $\tilde{B}_1 \in \mathbb{R}^{n_o \times n_i}$. The matrix $\tilde{A}_{22} - \tilde{B}_2\tilde{B}_1^T(\tilde{B}_1\tilde{B}_1^T)^{-1}\tilde{A}_{12}$ is Hurwitz stable.

If the three assumptions hold, then there always exist a solution $Q_1 = Q_1^T \in \mathbb{R}^{n_o \times n_o}$, $Q_2 = Q_2^T$, and Y to the following LMIs:

$$\begin{aligned} &Q_d = diag\big(Q_1, Q_2\big) > 0 \\ &\Big\{(T_N^{-1}\tilde{A}T_N)Q_d + (T_N^{-1}\tilde{B})Y \begin{bmatrix} I_{n_o} & 0 \end{bmatrix}\Big\}^{\mathcal{S}} < 0 \end{aligned} \tag{17}$$

where $T_N = \begin{bmatrix} I_{n_o} & 0 \\ (\tilde{B}_1\tilde{B}_1^T)^{-T}\tilde{B}_1\tilde{B}_2^T & I_{n-n_o} \end{bmatrix}$. The solution is such that $K = YQ_1^{-1}$ stabilizes the system.

This result has the advantage to describe all possible stabilizing gains as soon as the assumptions hold. Of course satisfaction of the assumptions is a strong limit to the method. As for results of Propositions 1 and 2, the result does not allow to incorporate any structural constraint on the control gains, because of the change of variables of the type $K = YQ_1^{-1}$. Simultaneous stabilization, and further robustness, are hardly achievable because of the assumption iii) to be verified on several systems.

A last sub-case when the static output feedback happens to be convex is as follows [36].

Proposition 4 *Consider two given matrices $C_z \in \mathbb{R}^{p \times n}$ and $D_z \in \mathbb{R}^{p \times n_i}$ the last one satisfying $D_z^T D_z = I$, then the solutions P and K to the following LMIs:*

$$\begin{aligned} &P > 0, \\ &\begin{bmatrix} \{PA + C_z^T D_z KC\}^{\mathcal{S}} + C_z C_z & PB + C^T K^T \\ B^T P + KC & -I \end{bmatrix} < 0 \end{aligned}$$

describe exactly the set of controllers such that the plant

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Bw(t)\\ z(t) &= C_z x(t) + D_z u(t)\\ y(t) &= Cx(t)\end{aligned}$$

in closed-loop with $u(t) = Ky(t)$ is stable and has an H_∞ norm smaller than 1 for the transfer function from w to z.

At the difference of the previous results this one has the advantage that the conditions are directly convex in the control parameters K, thus allowing to add structural constraints. Moreover, this same feature may allow to address the simultaneous stabilization issue by searching for one Lyapunov function per system. The simultaneous stabilization counterpart of the inequality from Proposition 4 reads as finding $P^{[i]}$ matrices and a common K solution to

$$\begin{array}{l} P^{[i]} > 0,\\ \left[\begin{array}{cc} \{P^{[i]}A^{[i]} + C_z^T D_z K C^{[i]}\}^{\mathcal{S}} + C_z C_z & \star \\ B^{[i]T}P^{[i]} + KC^{[i]} & -I \end{array}\right] < 0 \end{array} \tag{18}$$

for all systems $i = 1 \ldots \bar{\imath}$. Multi-performance problems are unfortunately not achievable since there is no such formulation for performance criteria (even the H_∞ is only partly handled). Robustness issues can be handled but with the same "Lyapunov Shaping Paradigm" as for results of Propositions 1 and 2. The other advantage is that it provides an exact description of a subset of stabilizing controllers (those that make the H_∞ norm smaller than one). The disadvantages are that such set may be empty and depends on the choice of matrices C_z and D_z. Moreover, the constraint $D_z^T D_z = I$ is strongly limiting.

To these cases when the static output feedback design may be solved via one convex LMI, we add the following result from [18] that is also purely LMI, but based on three steps. These are summarized in the following proposition:

Proposition 5 *Let K_s be a stabilizing state-feedback (for example obtained applying Proposition 1, or by solving a Riccati equation as proposed in [18]). Then, there exists $P > 0$ that proves stability of the closed-loop under state-feedback and a scalar σ solution to the following LMIs:*

$$\{P(A + BK_s)\}^{\mathcal{S}} < 0, \quad \{PA\}^{\mathcal{S}} < \sigma C^T C.$$

Let P be a solution to this problem. If the LMI (5) in K holds, then K is a stabilizing static output-feedback gain.

The success of this three steps method depends on the choices made at the two first steps and is hence heuristic. If the last step fails (LMIs are infeasible), no conclusion can be made. It potentially describes all existing stabilizing gains and allows at the last step to add structural constraints on the gains K. Simultaneous stabilization, multi-objective, and robustness issues have the same answers as for Proposition 4, but with increased complexity for the state-feedback initialization step.

4 Iterative LMI Heuristics

As seen in the previous section, only special cases provide convex LMI conditions for the design of SOF gains. Meaning that only subsets of all possible SOF gains can be designed with these methods. Some of these subsets can be empty, even when there exists some stabilizing gain. To go further, one can address the original problem that happens to be bilinear in the variables. A natural approach, largely explored in the literature, is to proceed as in Proposition 5 by freezing some of the variables to render the problem LMI. For BMI problems one can freeze the variables alternatively, providing a heuristic algorithm. With appropriate design, the algorithms have monotonically non-increasing criteria during iterations. Such iterative LMI heuristics are gathered in this section.

4.1 P-K iteration

The Lyapunov equation (5) or its dual version in (6) are not convex with respect to K and the Lyapunov matrix P or Q. However, once P or Q is specified, it becomes linear (hence convex) in K. Conversely, when K is specified, it is linear in P or Q. This is also the idea behind the well-known $D-K$ iteration for μ-synthesis [86, 93]. Based on this idea, several approaches have been developed to fix the Lyapunov matrix P or Q in different ways. We recall the simplest one proposed in [54]. The iterations are based on (5) and aim at minimizing the largest eigenvalue of the matrix involved in the LMI, until it becomes negative. A similar result can trivially be achieved for the dual formulation (6).

Proposition 6 *In the following algorithm, $\{\delta_{k=1\ldots\bar{k}}\}$ is a monotonically non-increasing sequence and if it concludes with a negative value of $\delta_{\bar{k}}$, then $K_{\bar{k}}$ is a stabilizing SOF gain.*
[Init.] *Set $k=1$ and choose $P_0=I$.*
[Step k,1] *Take $P=P_{k-1}$ and solve the following LMI optimization with respect to K and δ.*

$$\min \delta \quad : \quad \{P(A+BKC)\}^{\mathcal{S}} < \delta I \tag{19}$$

Set $K_k=K$.
[Step k,2] *Take $K=K_k$ and solve the LMI optimization (19) with respect to P and δ. Set $P_k=P$ and $\delta_k=\delta$.*
[Term.] *Let ϵ be some predefined threshold. If $\delta_{k-1}-\delta_k<\epsilon$ or $\delta_k<0$, stop and set $\bar{k}=k$. Otherwise, go to* [Step k,1] *with $k \leftarrow k+1$.*

The main advantage of this method is that [Step k,2] is convex in the SOF gain. One can hence impose at this stage some structural constraints on the control gain. Unfortunately, in practice the objective function δ_k often gets, after very few iterations, to a local plateau or a local minimum, and stops, which does not allow to conclude.

4.2 Path-following method

A disadvantage of the upper described algorithm is that it optimizes over a scalar δ that has no direct control oriented interpretation. The following variant proposed in [60] allows to optimize over the maximal decay rate of all modes based on the following matrix inequality:

$$\{P(A + BKC - \alpha I)\}^{\mathcal{S}} < 0 \tag{20}$$

which proves, when it holds, that the real part of the poles of the closed-loop plant are smaller than α. Trivially if α is negative, the condition proves stability. In this variant the BMI problem is not linearized by freezing alternatively some decision variables, but by first-order perturbation approximation and under the assumption of small search steps. It is a path-following approach parameterized by two tuning parameters $\epsilon > 0$ and $\delta_\alpha > 0$, both chosen small.

Proposition 7 *In the following algorithm, $\{\alpha_{k=1\ldots\bar{k}}\}$ is a monotonically non increasing sequence and if it concludes with a negative value of $\alpha_{\bar{k}}$, then $K_{\bar{k}}$ is a stabilizing SOF gain.*
[Init.] *Set $k = 1$ and choose $K_0 = 0$.*
[Step k,1] *Let $\alpha = \max(Real(\lambda(A + BK_{k-1}C)))$ be the maximal real part of the eigenvalues of $A + BK_{k-1}C$ and solve the following LMI optimization problem with respect to P and κ:*

$$\begin{array}{l} \min \kappa : \quad I < P < \kappa I, \\ \{P(A + BK_{k-1}C - (\alpha + \epsilon)I)\}^{\mathcal{S}} < 0. \end{array}$$

[Step k,2] *Fix P and solve the following LMI problem with respect to matrices $\tilde{P}$ and $\tilde{K}$*

$$\begin{array}{l} P + \tilde{P} > 0, \quad \|\tilde{P}\| < 0.2\|P\| \\ \{(P + \tilde{P})(A + BK_{k-1}C - (\alpha + \delta_\alpha)I) + PB\tilde{K}C\}^{\mathcal{S}} < 0 \end{array} \tag{21}$$

Set $\alpha_k = \alpha + \delta_\alpha$ and $K_k = K_{k-1} + \tilde{K}$.
[Term.] *If the LMI problem of* [Step k,2] *is infeasible, stop, the algorithm fails. If $\alpha_k < 0$, stop and set $\bar{k} = k$. Otherwise, go to* [Step k,1] *with $k \leftarrow k + 1$.*

The convergence is unfortunately with no better properties than for Proposition 6. Moreover, it may be slow if choosing small values of the tuning parameters ϵ and δ_α, or fail for too large steps δ_α.

4.3 Linearized convex-concave decomposition

The idea behind this approach proposed in [118] is to decompose the BMI condition in (20) as the difference of two positive semidefinite convex mappings. The convex-concave decomposition is as follows:

$$2\{P(A + BKC - \alpha I)\}^{\mathcal{S}} = G(P, K, \alpha) - H(P, K, \alpha)$$

where

$$G(P,K,\alpha) = \\ (P + A + BKC - \alpha I)^T (P + A + BKC - \alpha I) \\ H(P,K,\alpha) = \\ (P - A - BKC + \alpha I)^T (P - A - BKC + \alpha I).$$

Based on this representation, the convex part $G(P,K,\alpha)$ has an LMI representation thanks to a Schur complement argument, while only the concave term $H(P,K,\alpha)$ is linearized at each iteration. This is hence an improvement compared to Proposition 7 where the whole constraint is linearized. The following notation stands for the linearization around the point (P_k, K_k, α_k):

$$H_k(P,K,\alpha) = \\ \{(P_k - A - BK_kC + \alpha_k I)^T (P - A - BKC + \alpha I)\}^{\mathcal{S}}.$$

Proposition 8 *In the following algorithm, $\{\alpha_{k=1\ldots\bar{k}}\}$ is a monotonically non increasing sequence and if it concludes with a negative value of $\alpha_{\bar{k}}$, then $K_{\bar{k}}$ is a stabilizing SOF gain.*

[Init.] *Set $k = 1$, choose $K = 0$, $\alpha = \max(Real(\lambda(A)))$ be the maximal real part of the eigenvalues of the A matrix and solve the LMI (20) with respect to P. Set $P_0 = P$, $K_0 = K$, and $\alpha_0 = \alpha$.*

[Step k] *Solve the following LMI optimization with respect to P, K, and α.*

$$\begin{array}{l} \min \alpha \quad : \quad P > 0, \\ \left[\begin{array}{cc} H_{k-1}(P,K,\alpha) & (P + A + BKC - \alpha I)^T \\ P + A + BKC - \alpha I & I \end{array} \right] > 0. \end{array} \tag{22}$$

Set $P_k = P$, $K_k = K$, and $\alpha_k = \alpha$.

[Term.] *Let ϵ be some predefined threshold. If $\alpha_{k-1} - \alpha_k < \epsilon$ or $\alpha_k < 0$, stop and set $\bar{k} = k$. Otherwise, go to* [Step k] *with $k \leftarrow k + 1$.*

This method has the same disadvantage of rapid convergence to a local plateau or a local minimum that may not allow to conclude. Compared to Proposition 6, the optimization is done with respect to the exponential decay rate α which has a control interpretation. Compared to Proposition 7, there is no need for tuning a priori some small step parameter δ_α, the condition (22) being convex in α.

4.4 Riccati related approach

In [22, 23] another iterative LMI approach is proposed (with the explicit usage of the term ILMI to denote the iterative LMI algorithm). The approach relies on the following alternative condition for stabilizability of the plant in which the additional matrix X is directly related to the solution of a Riccati equation.

Lemma 1 *The system in (1) is stabilizable via a static output feedback if and only if there exist a Lyapunov matrix $P > 0$, $X > 0$, and a matrix K satisfying the following condition:*

$$\begin{array}{l} \{P(A - BB^T X)\}^{\mathcal{S}} + XBB^T X \\ +(B^T P + KC)^T (B^T P + KC) < 0. \end{array} \tag{23}$$

The following algorithm uses the fact that for given X the condition is convex in both P and K. Moreover, $X = P$ is an admissible solution.

Proposition 9 *In the following algorithm, $\{\alpha_{k=1\ldots\bar{k}}\}$ is a monotonically non-increasing sequence and if it concludes with a negative value of $\alpha_{\bar{k}}$, then $K_{\bar{k}}$ is a stabilizing SOF gain.*
[Init.] *Set $k = 1$, select some $R > 0$ and solve the following algebraic Riccati equation: $A^T X + XA - XBB^T X + R = 0$ and set $X_k = X$.*
[Step k] *For fixed X_k, solve the following optimization with respect to P, K, and α:*

$$\begin{array}{l} \min \alpha \quad : \quad P > 0, \\ \left[\begin{array}{cc} \{PA(X_k,\alpha)\}^{\mathcal{S}} + X_k BB^T X_k & PB + C^T K^T \\ B^T P + KC & -I \end{array} \right] < 0 \end{array} \tag{24}$$

where $A(X_k, \alpha) = (A - BB^T X_k - \alpha I)$. Set $K_k = K$ and $\alpha_k = \alpha$.
[Term.] *If $\alpha_k < 0$, stop and set $\bar{k} = k$. Else, solve the following LMI problem in P and K:*

$$\begin{array}{l} \min \mathrm{Tr}(P) \quad : \quad P > 0, \\ \left[\begin{array}{cc} \{PA(X_k,\alpha_k)\}^{\mathcal{S}} + X_k BB^T X_k & PB + C^T K^T \\ B^T P + KC & -I \end{array} \right] < 0. \end{array}$$

If $\|X_k - P\| < \epsilon$, where ϵ is a pre-specified tolerance, stop and set $\bar{k} = k$. Otherwise, set $X_{k+1} = P$ and go to [Step k] *with $k \leftarrow k + 1$.*

Similar comments as for the previous results hold. There is nevertheless an additional complexity related to the quasi convex search for minimal α in (24). This search can be done with solvers that handle the generalized eigenvalue minimization problem, or by performing a bisection search over α (for fixed α the conditions are LMI).

4.5 Dual iteration approach [68]

All methods described until now in this section are variations on the condition (5). They may also be derived following the exposed methodologies for the dual version (6). The results proposed in [68], which we describe now, take advantage of both conditions (5) and (6). More precisely they rely on the following lemma that states that the plant is stabilizable with an SOF gain K if and only if there exist a state feedback F and an observer gain L such that two inequalities are satisfied with a common Lyapunov matrix Q or P.

Lemma 2 *The following statements hold:*
(i) There exist a pair (K, P) solution to (20), if and only if, for that same P, there exist matrices K_s and K_o such that:

$$\{P(A + BK_s - \alpha I)\}^{\mathcal{S}} < 0, \quad \{P(A + K_o C - \alpha I)\}^{\mathcal{S}} < 0. \tag{25}$$

(i) There exist a pair (K, P) solution to (20), if and only if, for $Q = P^{-1}$, there exist matrices K_s and K_o such that:

$$\{(A + BK_s - \alpha I)Q\}^{\mathcal{S}} < 0, \quad \{(A + K_o C - \alpha I)Q\}^{\mathcal{S}} < 0. \tag{26}$$

The inequalities given in (26) and (25) are BMI with respect to the unknown matrices K_s, K_o, and Q or P. However, by keeping K_s or K_o fixed, the BMI problems in (26) or (25) become convex in the other variable, and vice versa. The proposed algorithm iterates between these two stages.

Proposition 10 *In the following algorithm, $\{\alpha_{s,k=1\ldots\bar{k}}\}$ and $\{\alpha_{o,k=1\ldots\bar{k}}\}$ are monotonically non-increasing sequences such that $\alpha_{o,k} \leq \alpha_{s,k}$. If it concludes with a negative value of $\alpha_{s,\bar{k}}$ or $\alpha_{o,\bar{k}}$, then $K_{\bar{k}}$ is a stabilizing SOF gain.*
[Init.] *Set $k = 1$ and design a stabilizing state-feedback gain $K_{s,0}$.*
[Step k,1] *For fixed $K_s = K_{s,k-1}$, solve the following optimization problem with respect to α, P, and L:*

$$\begin{array}{l} \min \alpha \quad : \quad P > 0, \\ \{P(A + BK_s - \alpha I)\}^{\mathcal{S}} < 0, \quad \{PA + LC - \alpha I\}^{\mathcal{S}} < 0. \end{array} \tag{27}$$

Set $\alpha_{s,k} = \alpha$, $P_k = P$, and $K_{o,k} = P^{-1}L$.
[Step k,2] *For fixed $K_o = K_{o,k}$, solve the following optimization problem with respect to α, Q, and F:*

$$\begin{array}{l} \min \alpha \quad : \quad Q > 0, \\ \{(A + K_oC - \alpha I)Q\}^{\mathcal{S}} < 0, \quad \{AQ + BF - \alpha I\}^{\mathcal{S}} < 0. \end{array} \tag{28}$$

Set $\alpha_{o,k} = \alpha_o$, $Q_k = Q$, and $K_{s,k} = FQ^{-1}$.
[Term.] *If $\alpha_{s,k} < 0$, solve (5) with respect to K for fixed $P = P_k$ and stop. Else, if $\alpha_{o,k} < 0$, solve (6) with respect to K for fixed $Q = Q_k$ and stop. Otherwise, if either $\alpha_{s,k}$ or $\alpha_{o,k}$ have decreased more than some predefined threshold, go to* [Step k,1] *with $k \leftarrow k + 1$.*

The main drawback is that it requires at each step to solve a quasi convex optimization problem due to the bilinear terms αP and αQ. Meanwhile, the main advantage compared to all previous methods, is that at each step the conditions involve an optimization over the Lyapunov matrices P or Q. It provides many more degrees of freedom. The other advantage is that the initialization relies on a state-feedback gain (as for Proposition 5). Such initialization is more relevant than when choosing some arbitrary SOF gain ($K = 0$ in Proposition 8) or some arbitrary Lyapunov matrix ($P = I$ in Proposition 6). This initialization based on a stabilizing state-feedback is also what is done in Proposition 9, at least implicitly, via the Riccati equation at initialization step.

4.6 Conclusions about iterative LMI heuristics

All these iterative heuristics are variants to address the original BMI problem by iteratively freezing some of the decision variables. They all have in common that results are much dependent on the initialization step. Replace for example in the algorithm of Proposition 7, the choice $K_0 = 0$ by any random choice of K_0 and results will inevitably be much different for every run of the algorithm. On examples one often notices that the objective function gets to a local plateau

or a local minimum after very few iterations and does not allow to conclude. On the other hand, the coding of the iterations is very simple and provides an easy to implement local method that does succeed more than occasionally.

In all the exposed iterative LMI methods, except the one of Proposition 10, the SOF gains are decision variables. These therefore allow without additional complexity to deal with linear structure constraints. For the results given in Proposition 10, structural constraints may also be imposed at the termination step but with no guarantee whatsoever of having a feasible structured solution.

In terms of simultaneous stabilization and robustness, the results of Propositions 6, 7, 8, 9 share same properties as reported for Proposition 4. Simultaneous stabilisation can be done with individual Lyapunov matrices for each system. Robustness can be achieved applying the "Lyapunov Shaping Paradigm", that is with a Lyapunov matrix common to all realizations of the uncertain plant. The multi-performance problem is more involved since it implies to revisit the optimization criteria used in the algorithms. It may need to start from a stabilizing SOF and perform iterates to improve closed-loop performances (see [118] for example). Concerning Proposition 10, due to the change of variables $K_{o,k} = P^{-1}L$ and $K_{s,k} = FQ^{-1}$ performed at each iteration, the simultaneous stabilization and robustness problems can only be solved for common to all Lyapunov matrices. The properties are as for the Proposition 2 results.

5 Heuristics for Solving LMIs with Rank Constraints

As seen in the last proposition, there is a high potential of algorithms that would be based on the two dual constraints (5) and (6). It is the case of all the algorithms that follow which are based on the following reformulation of the problem, where the first two conditions are noting but (6) and (5) after applying the elimination lemma [70,71,114] and the last two guarantee that $QP = I$ and hence the SOF problem has a solution.

Lemma 3 *There exists a static output feedback controller which stabilizes a plant of order n, if and only if there exist matrices P and Q such that*

$$B^{\perp}\{AQ\}^{\mathcal{S}}B^{\perp^T} < 0 \tag{29}$$

$$C^{\perp^T}\{PA\}^{\mathcal{S}}C^{\perp} < 0 \tag{30}$$

$$W(P,Q) = \begin{bmatrix} Q & I \\ I & P \end{bmatrix} \geq 0 \tag{31}$$

$$\mathit{rank}(W(P,Q)) = n \tag{32}$$

When removing the rank constraint (32), the problem becomes LMI and corresponds exactly to the case of full-order output feedback control design [39, 70, 110]. Moreover, if replacing (32) by $\mathrm{rank}(W(P,Q)) = n + m$, the conditions

correspond to the design of dynamic controllers of order m. Hence, solving this problem combining LMIs with a rank constraint allows to solve the general problem of fixed-order control design.

Unfortunately, the rank constraint in (32) renders the static output feedback synthesis problem algorithmically complex and numerically difficult to solve. Nevertheless, due to the practical importance of the problem, many approaches have been developed. Several are summarized in the following.

5.1 Min/Max algorithm

The Min/Max algorithm exposed in [49] mimics the type of iterations on the primal and dual formulations of the SOF problem. The rank constraint is handled via the fact that (32) holds if and only if $PQ = I$.

Proposition 11 *In the following algorithm $\{\lambda_{p,k=1\ldots\bar{k}}\}$ is monotonically non-increasing and $\{\lambda_{q,k=1\ldots\bar{k}}\}$ is monotonically non-decreasing. Moreover, $\lambda_{p,k} \geq 1$, $\lambda_{q,k} \leq 1$, $P_k \leq P_{k+1}$, and $Q_k \geq Q_{k+1}$ hold. If it concludes with rank$(W(P_k, Q_k)) = n$, then K is a stabilizing SOF gain.*
[Init.] *Set $k = 1$, solve the LMI problem composed of $Q > 0$, (29) and set $Q_0 = Q$.*
[Step k,1] *Solve the following LMI optimization problem with respect to P*

$$\min \lambda_p \quad : \quad (30), \quad Q_{k-1}^{-1} \leq P \leq \lambda_p Q_{k-1}^{-1}$$

Set $\tilde{P}_k = P$ and $\lambda_{p,k} = \lambda_p$.
[Step k,2] *Solve the following LMI optimization problem with respect to Q*

$$\min \lambda_q \quad : \quad (29), \quad \lambda_q P_k^{-1} \leq Q \leq P_k^{-1}$$

Set $\tilde{Q}_k = P$ and $\lambda_{q,k} = \lambda_q$.
[Term.] *If rank$(W(P_k, Q_k)) = n$, solve (5) with respect to K for fixed $P = P_k$, set $\bar{k} = k$ and stop. Otherwise, if either $\lambda_{p,k}$ or $\lambda_{q,k}$ have not evolved more than some predefined threshold, go to [Step k,1] with $k \leftarrow k + 1$.*

5.2 XY-centring algorithm

An improved version of the Min/Max algorithm is proposed in [72]. It has the major advantage of providing (at least theoretically) strictly decreasing criteria during the iterations.

Proposition 12 *In the following algorithm $\{\delta_{p,k=1\ldots\bar{k}}\}$ and $\{\delta_{q,k=1\ldots\bar{k}}\}$ are strictly decreasing sequences such that $\delta_{p,k-1} > \delta_{q,k} > \delta_{p,k} > 1$ holds. If it stops before the maximal number of iterations is reached, then K is a stabilizing SOF gain.*
[Init.] *Set $k = 1$, solve the LMI problem composed of (29), (30), (31) and set $Q_0 = Q$. Take some values $\theta \in [0\,1]$ and $\delta_{p,0} > \lambda_{\max}(PQ)$.*
[Step k,1] *Find the analytic center of the following LMI problem with respect to P*

$$(30), \quad I < Q_{k-1}^{\frac{1}{2}} P Q_{k-1}^{\frac{1}{2}} < \delta_{p,k-1} I$$

Set $P_k = P$ *and* $\delta_{q,k} = (1-\theta)\lambda_{\max}(P_k Q_{k-1}) + \theta\delta_{p,k}$.
[Step k,2] *Find the analytic center of the following LMI problem with respect to* Q

$$(29), \quad I < P_k^{\frac{1}{2}} Q P_k^{\frac{1}{2}} < \delta_{q,k} I$$

Set $Q_k = Q$ *and* $\delta_{p,k} = (1-\theta)\lambda_{\max}(P_k Q_k) + \theta\delta_{q,k}$.
[Term.] *If* $Q = P_k^{-1}$ *satisfies (29), then solve (5) with respect to* K *for fixed* $P = P_k$, *set* $\bar{k} = k$ *and stop. If* $P = Q_k^{-1}$ *satisfies (30), then solve (5) with respect to* K *for fixed* $P = Q_k^{-1}$, *set* $\bar{k} = k$ *and stop. Otherwise, if a maximal number of iterations is not reached, go to* [Step k,1] *with* $k \leftarrow k+1$.

Note that at the difference of Proposition 11 the stopping criterion is not on the fact that (32) holds. There is no actual need to converge exactly to such solution to have a solution to the SOF problem. This remark holds as well for the further exposed methods. For these we shall formulate the following positive stoping criterion:

[CT] If $\|P_k Q_k - I\|_F$ is close to zero, solve (5) with respect to K for fixed $P = P_k$. If it is feasible, stop the algorithm.

5.3 Alternating projection method

In the methods that follow from now on the specificity is that the iterations allow optimization over both P and Q matrices at each step.

The alternating projection algorithm proposed in [56] searches a pair (P, Q) in the intersection of the non-convex set $\mathcal{Z}_{rank}(0)$ and convex set $\mathcal{Z}_{LMI}$ by taking orthogonal projections on each set alternately, where

$$\begin{aligned} \mathcal{Z}_{rank}(m) &:= \left\{(Q,P) \middle| \quad \text{rank}(W(P,Q)) \leq n+m\right\}, \\ \mathcal{Z}_{LMI} &:= \left\{(Q,P) \middle| \quad (29),\ (30),\ (31) \text{ hold}\right\}. \end{aligned}$$

The projection on the set $\mathcal{Z}_{LMI}$ is denoted $(P,Q) = \mathcal{P}_{\mathcal{Z}_{LMI}}(\tilde{P},\tilde{Q})$ and is the solution of the following LMI optimization problem with respect to P and Q:

$$\begin{aligned} &\min \quad \|P-\tilde{P}\|_F^2 + \|Q-\tilde{Q}\|_F^2 \\ &\text{subject to } (29),\ (30),\ (31) \end{aligned} \tag{33}$$

The projection on the set $\mathcal{Z}_{rank}(0)$ is denoted $(P,Q) = \mathcal{P}_{\mathcal{Z}_{rank}(0)}(\tilde{P},\tilde{Q})$ and is obtained by computing the eigenvalue decomposition of the positive semi-definite matrix $W(\tilde{P},\tilde{Q}) = V\Sigma V^T$, taking $V^T = \left[\begin{array}{cc} V_q^T & V_p^T \end{array}\right]$ where V_q and V_p have same number or rows, and choosing $Q = V_q \Sigma V_q^T$, $P = V_p \Sigma V_p^T$.

Proposition 13 *If the following algorithm concludes a positive test* [CT], *then* K *is a stabilizing SOF gain.*
[Init.] *Set* $k = 1$ *and take any* $P_0 > 0$, $Q_0 > 0$.
[Step k,1] *Compute* $(\tilde{P},\tilde{Q}) = \mathcal{P}_{\mathcal{Z}_{LMI}}(P_{k-1}, Q_{k-1})$.
[Step k,2] *Compute* $(P_k, Q_k) = \mathcal{P}_{\mathcal{Z}_{rank}(0)}(\tilde{P},\tilde{Q})$.
[Term.] *Test* [CT]. *If it fails and the number of iterations has not reached a predefined upper limit, go to* [Step k,1] *with* $k \leftarrow k+1$.

The main drawback of the alternating projection is that its convergence cannot be guaranteed [29]. An improved version can be found in [92] and is associated with a software package named LMIrank.

5.4 Directional alternating projection method

Directional alternating projection algorithm developed in [55] accelerates the convergence of the alternating projection algorithm. In fact, it uses information about the geometry of the sets of the constraints to improve convergence by choosing a clever direction.

Proposition 14 *If the following algorithm concludes a positive test* [CT]*, then K is a stabilizing SOF gain.*
[Init.] *Set $k = 1$ and solve the following LMI optimization problem with respect to P and Q*

$$\min \quad \mathrm{Tr}(P+Q) \quad : \quad (29), (30), (31)$$

Set $P_0 = P$ and $Q_0 = Q$.
[Step k] *Compute the following sequence of projections*

$$\begin{aligned} (Q_k^a, P_k^a) &= \mathcal{P}_{\mathcal{Z}_{rank}(0)}(Q_{k-1}, P_{k-1}) \\ (Q_k^b, P_k^b) &= \mathcal{P}_{\mathcal{Z}_{LMI}}(Q_k^a, P_k^a) \\ (Q_k^c, P_k^c) &= \mathcal{P}_{\mathcal{Z}_{rank}(0)}(Q_k^b, P_k^b) \end{aligned} \tag{34}$$

and set

$$Q_k = Q_k^a + \frac{\|Q_k^a - Q_k^b\|_F^2}{\mathrm{Tr}\left((Q_k^a - Q_k^c)^T(Q_k^a - Q_k^b)\right)}(Q_k^c - Q_k^a)$$

$$P_k = P_k^a + \frac{\|P_k^a - P_k^b\|_F^2}{\mathrm{Tr}\left((P_k^a - P_k^c)^T(P_k^a - P_k^b)\right)}(P_k^c - P_k^a).$$

[Term.] *Test* [CT]*. If it fails and the number of iterations has not reached a predefined upper limit, go to* [Step k] *with $k \leftarrow k+1$.*

Note that we have exposed here a simplified version of the algorithm in [55]. The original version contains two imbricated loops. The outer-loop is with respect to the order of the controller m and aims at reducing it by one at each step by projections on the set $\mathcal{Z}_{rank}(m)$. The inner-loop being the one described in [Step k] of Proposition 14.

The convergence of the algorithm is expected to be faster than that of Proposition 13. Yet it cannot be guaranteed and there is no clear indication whether the iterations bring any improvement.

5.5 Penalty function method

Another variant to alternating projection algorithm presented in [77] proposes to minimize iteratively the following objective function:

$$\mathrm{Tr}(P+Q) + \mu\,\mathrm{Tr}\left(V^T W(P,Q) V\right)$$

where V is obtained from the eigenvalue decomposition of $W(P,Q)$ and μ is a penalty parameter. It can be seen as a variation to the alternating projection algorithm since the objective contains implicitly the projection onto the the rank constraint condition.

Proposition 15 *If the following algorithm concludes a positive test* [CT]*, then K is a stabilizing SOF gain.*
[Init.] *Set $k = 1$, some values $\mu_1 > 1$, $\alpha \in [0\,1]$, $\tau > 1$. Solve the LMI problem composed of (29), (30), (31) with respect to P and Q. Set $P_0 = P$, $Q_0 = Q$ and V_0 such that $V_0^T V_0 = I_n$ and it contains the eigenvectors corresponding to the n smallest eigenvalues of $W(P_0, Q_0)$.*
[Step k,1] *Solve the following LMI optimization problem with respect to P and Q:*

$$\begin{array}{ll} \min & \mathrm{Tr}(P+Q) + \mu_k \,\mathrm{Tr}(V_{k-1}^T W(P,Q) V_{k-1}) \\ & \textit{subject to (29), (30), (31)} \end{array} \tag{35}$$

Set $P_k = P$, $Q_k = Q$, and V_k such that $V_k^T V_k = I_n$ and it contains the eigenvectors corresponding to the n smallest eigenvalues of $W(P_k, Q_k)$.
[Step k,2] *If the following condition holds*

$$\mathrm{Tr}(V_k^T W(P_k, Q_k) V_k) > \alpha \,\mathrm{Tr}(V_{k-1}^T W(P_{k-1}, Q_{k-1}) V_{k-1})$$

Update the penalty parameter $\mu_{k+1} = \tau \mu_k$, else keep $\mu_{k+1} = \mu_k$.
[Term.] *Test* [CT]*. If it fails and the number of iterations has not reached a predefined upper limit, go to* [Step k,1] *with $k \leftarrow k+1$.*

5.6 Cone complementarity linearization approach

The cone complementarity linearization approach proposed in [53] considers the problem of static output feedback design by iteratively minimizing $\mathrm{Tr}(PQ)$, which, with cone complementarity arguments, is minimal when (32) holds ($\mathrm{Tr}(PQ) = 2n$ if and only if $PQ = I$). The criterion is nonlinear, but is tackled via a Frank and Wolfe linearization procedure.

Proposition 16 *In the following algorithm $\{f_{k=1\ldots\bar{k}}\}$ is monotonically non-increasing. If the following algorithm concludes a positive test* [CT]*, then K is a stabilizing SOF gain.*
[Init.] *Set $k = 1$, $P_0 = 0$, and $Q_0 = 0$.*
[Step k] *Solve the following LMI optimization problem with respect to P and Q*

$$\min \quad \mathrm{Tr}(P_{k-1}Q + PQ_{k-1}) \quad : \quad (29), (30), (31)$$

Set $P_k = P$, $Q_k = Q$ and $f_k = \mathrm{Tr}\big(P_{k-1}Q_k + P_k Q_{k-1}\big)$.
[Term.] *Test* [CT]*. If it fails and $f_{k-1} - f_k$ is greater than some predefined threshold, go to* [Step k] *with $k \leftarrow k+1$.*

5.7 Sequential linear programming matrix method

The sequential linear programming matrix method proposed in [83] is a variation on the cone complementarity linearization approach, where there is an added line search in the Frank and Wolfe algorithm. The convergence is improved.

Proposition 17 *In the following algorithm* $\{f_{k=1\ldots\bar{k}}\}$ *is monotonically non-increasing. If the following algorithm concludes a positive test* [CT]*, then* K *is a stabilizing SOF gain.*
[Init.] *Set* $k = 1$*, solve the LMI problem composed of (29), (30), (31) and set* $P_0 = P$ *and* $Q_0 = Q$*.*
[Step k,1] *Solve the following LMI optimization problem with respect to* P *and* Q

$$\min \quad \mathrm{Tr}(P_{k-1}Q + PQ_{k-1}) \quad : \quad (29), (30), (31).$$

Set $\tilde{P}_k = P$*,* $\tilde{Q}_k = Q$*.*
[Step k,2] *Perform a line search over* $\alpha \in [0\,1]$ *to find the value that minimizes*

$$\mathrm{Tr}\big((P_{k-1} + \alpha(\tilde{P}_k - P_{k-1}))(Q_{k-1} + \alpha(\tilde{Q}_k - Q_{k-1}))\big).$$

Set $\alpha_k = \alpha$*,* $P_k = (1-\alpha_k)P_{k-1} + \alpha_k\tilde{P}_k$*,* $Q_k = (1-\alpha_k)Q_{k-1} + \alpha_k\tilde{Q}_k$*, and* $f_k = \mathrm{Tr}(P_kQ_k)$*.*
[Term.] *Test* [CT]*. If it fails and* α_k *is greater than some predefined threshold, go to* [Step k,1] *with* $k \leftarrow k+1$*.*

5.8 Concave minimization approach

In [10, 11] the problem is addressed with the same type of Frank and Wolfe algorithm. The difference compared to the cone complementarity approach is in the criterion to be minimized which is replaced by $\mathrm{Tr}(P - Q^{-1})$. As for the previous results, and as reported in [42], the algorithm has no guarantee of convergence, even if starting in the neighbourhood of a local solution.

5.9 Augmented Lagrangian method

One last variant for the rank constraint problem is proposed in [9, 91]. This time the rank constraint is tackled by minimization of the following criterion:

$$\Phi = \mathrm{Tr}\left(\Lambda^T(PQ - I)\right) + \frac{c}{2}\|PQ - I\|_F^2$$

where Λ is a Lagrange multiplier matrix and $c > 0$ is a penalty parameter. More precisely, the algorithm minimizes iteratively its tangent version $\Phi_k(P, Q)$ around a point $(P_k, Q_k, \Lambda_k, c_k)$. This tangent version includes both first order and convex second order information.

Proposition 18 *If the following algorithm concludes a positive test* [CT]*, then* K *is a stabilizing SOF gain.*
[Init.] *Set* $k = 1$*, solve the LMI problem composed of (29), (30), (31) and set*

$P_0 = P$ and $Q_0 = Q$. Choose some parameters $c_0 > 0$, $\Lambda_0 > 0$, $\rho < 1$, $\mu \in [0\,1]$.
[Step k,1] *Solve the following LMI optimization problem with respect to P and Q*

$$\min \quad \Phi_{k-1}(P,Q) \quad : \quad (29),(30),(31).$$

Set $\tilde{P}_k = P$, $\tilde{Q}_k = Q$.
[Step k,2] *Perform a line search similar to that in [Step 2] of Proposition 17 to find the updates P_k and Q_k. Update the Lagrange multiplier as*

$$\Lambda_k = \Lambda_{k-1} + c_{k-1}(P_k Q_k - I)$$

If $\|P_k Q_k - I\|_F > \mu \|P_{k-1} Q_{k-1} - I\|_F$, update the penalty parameter as $c_k = \rho c_{k-1}$, else keep it constant $c_k = c_{k-1}$.
[Term.] *Test* [CT]. *If it fails and the number of iterations has not reached a predefined upper limit, go to* [Step k,1] *with $k \leftarrow k+1$.*

As illustrated on examples in [9], the augmented Lagrangian method brings numerical improvements to the basic cone complementarity approach. Yet as we show here, it can be considered as of the same type of result.

5.10 Conclusions about LMIs with rank constraints

As said in the introduction of this section, the SOF design based on LMIs with rank constraints is a major approach that has the advantage to cope in a unified framework with static or reduced-order dynamic feedback design. Moreover, compared to iterative methods described in Section 4, there are many algorithms that have been proposed and these optimize at each step over P or Q matrices. Most algorithms even optimize over both P and Q at each step.

In terms of drawbacks, the common one to all methods is that the algorithms are heuristic and have no guarantee to converge to a solution. Yet, the experiments show that the latest versions of the algorithms do perform quite well on some examples. Another drawback is that there is no good manner to initialize the algorithms in terms of knowledge of some controller, not even with the knowledge of a state-feedback gain. Finally the optimization being done on constraints that are independent of the control gains, there is no possibility to add structural constraints on the control. At the final termination step, the LMIs (5) are solved with respect to K, and structural constraints could be added at this point, but if the LMIs are infeasible, there is no other alternative than going back to iterative methods of Section 4.

With respect to simultaneous stabilization, and hence also for robustness issues, the algorithms will almost inevitably fail. Assuming some "Lyapunov Shaping Paradigm" the algorithms may be applied for the search of common Lyapunov matrices for several systems. Unfortunately, the fact that conditions (29), (30), (31), and (32) hold for several matrices $A^{[i]}$, $B^{[i]}$, $C^{[i]}$ does not imply that a common to all systems matrix K is solution to conditions (5).

6 Methods with Decoupled Lyapunov Matrices

In the previous sections we have recalled many heuristic techniques based on the formulations (5) and (6) that in general do not allow during iterations to optimize simultaneously over the control gain and the Lyapunov matrices. The reason for this difficulty is in the products between these decision matrices in the matrix inequalities. In the following, we expose two alternative formulations that allow decoupling of the control gains (or some matrices which in turn allow to build the SOF gains) with the Lyapunov matrices. These conditions hence offer more design freedom but do not render the problem convex. Yet, similar algorithms as the ones exposed up to this point can apply, with some noticeable advantages.

6.1 Resilient approach

The resilient method proposed in [43, 44, 96, 98] interprets the SOF problem as if a robustness problem with respect to a feedback uncertainty. The obtained formulation reassembles those LMIs build for proving robustness with respect to bounded uncertainties in LFT form [40, 69, 111, 112]. For robustness analysis the set in which the uncertainties are known a priori, for the resilient design approach the aim is to find such non empty set. The significant feature is that the design provides not a single SOF gain, but a whole convex set of stabilizing gains. Thus allowing imprecisions in the control implementation, i.e. robustness to uncertainties on the control gain which is known as the resilience problem (or non-fragility).

Theorem 2 *The LTI system in (1) is stabilizable by a static output feedback K if and only if there exist matrix Y and symmetric matrices P, X, and Z such that the following constraints are satisfied:*

$$\begin{gathered} P > 0, \quad Z > 0, \quad X \leq YZ^{-1}Y^T, \\ M_A^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} M_A < M_C^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} M_C \end{gathered} \tag{36}$$

where $M_A = \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}$, $M_C = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$. *Moreover, solutions to this problem are such that any gain K in the ellipsoidal set defined by a center $K_0 = -Z^{-1}Y^T$, a radius $R = K_0^T Z K_0 - X$, and the convex quadratic matrix inequality $(K - K_0)^T Z (K - K_0) \leq R$ stabilizes the system (1).*

The conditions in the theorem are all purely LMI except for the constraint $X \leq YZ^{-1}Y^T$. Equivalently, this constraint reads as $X \leq \hat{X}$ with $\hat{X} = YZ^{-1}Y^T$, or also as

$$X \leq \hat{X}, \quad \text{rank} \begin{bmatrix} \hat{X} & Y \\ Y^T & Z \end{bmatrix} = n_i$$

where n_i is the size of the input vector $u \in \mathbb{R}^{n_i}$ of the plant (1). All algorithms described in Section 5 can apply. For example, the cone complementarity algorithm is applied in [96]. The main differences with results in Section 5 are:

- The rank constraint is of smaller size, dependent only on the size of the SOF gain which is in general much smaller than the order of the plant.
- Variables related to the control gain enter explicitly in the constraints thus allowing to add structure constraints. To this end, one can choose without conservatism Z proportional to the identity matrix and impose the structure constrains on the Y matrix.
- The stopping criterion [CT] may be replaced by testing whether $X \leq YZ^{-1}Y^T$ holds or not.
- A whole convex set of controllers is obtained.
- Simultaneous stabilization and multi-performance can be achieved without additional complexity. It only imposes to concatenate the LMIs of the type (36) for each pair of system/performance in the same vein as done in (18). For each of these a different Lyapunov variable may be searched for. The constraints are all coupled by the common matrices X, Y, Z that parameterize the controller.
- Robust stabilization is also easily achievable, at least assuming the "Lyapunov Shaping Paradigm", that is searching for common Lyapunov matrices for all plants. The LMIs involved for stabilization of polytopic models of the type (9) are copies of (36) for each vertex with common P matrices, in the same way as it is done in (16). Robustness with respect to uncertainties entering the model as LFTs can be dealt with as well without difficulty by applying the tools developed for the analysis of such uncertain systems [40, 69, 99, 111, 112].

6.2 S-variable methods

The S-variable approach initiated in [28, 30, 31, 34, 51, 97, 119, 122] has from the beginning been used for robustness analysis issues. Results with lower conservatism where obtained in that framework thanks to the decoupling between Lyapunov matrices and the state-space matrices of the system, decoupling enabled by the introduction of additional variables in an S-procedure like manner (also understood as the Finsler Lemma result, or the reverse use of the elimination lemma). The potential of this decoupling has also been understood quite rapidly for static output feedback design with results such as [94] that corresponds to condition (38) in the next theorem, [63] that correspond to (37), or [38] that corresponds to (39). See Chapter 7 of [37] for a complete description of these formulations.

Theorem 3 *The following statements are equivalent.*

i) There exists a static output feedback $u = Ky$ that stabilizes the system (1).

ii) There exist $P > 0$, K, S_1, and S_2 such that:

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} < \left\{ \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \begin{bmatrix} A + BKC & -I \end{bmatrix} \right\}^{\mathcal{S}}. \tag{37}$$

The solutions are such that S_2 is non singular, $A_o^T = -S_1 S_2^{-1}$ is Hurwitz stable, and P proves the stability of both $A + BKC$ and A_o.

iii) There exist $P > 0$, K_s, W, and S such that:

$$M_A^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} M_A < \left\{ \begin{bmatrix} K_s^T \\ -I \end{bmatrix} \begin{bmatrix} -WC & S \end{bmatrix} \right\}^{\mathcal{S}}. \tag{38}$$

The solutions are such that S is non singular, $K = -S^{-1}W$ is a stabilizing output-feedback gain, K_s is a stabilizing state-feedback gain, and P proves the stability of both $A + BKC$ and $A + BK_s$.

iv) There exist $P > 0$, $\hat{S}_1$, $\hat{S}_2$, L, and S such that

$$\begin{bmatrix} M_A^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} M_A & \begin{bmatrix} C^T W^T \\ -S^T \end{bmatrix} \\ \begin{bmatrix} WC & -S \end{bmatrix} & 0 \end{bmatrix} < \left\{ \begin{bmatrix} \hat{S}_1 \\ -\hat{S}_2 \end{bmatrix} \right\}^{\mathcal{S}} \tag{39}$$
$$rank \begin{bmatrix} \hat{S}_1^T & \hat{S}_2^T \end{bmatrix} \leq n_i.$$

The rank constraint ensures that there exists K_s solution to $\hat{S}_1 = K_s^T \hat{S}_2$. $P > 0$, K_s, W, and S are solutions to (38).

The dual versions of the above conditions can be easily obtained by replacing the matrices (A, B, C, P, K, K_s) with $(A^T, C^T, B^T, Q, K^T, K_s^T)$. In the dual case, K_s is a full-actuation feedback.

The three formulations proposed in the theorem have many variants in the literature, some are listed here [3, 13, 14, 27, 32, 33, 35, 75, 80–82, 89, 90, 105–107, 109, 120, 123]. For example the condition proposed in [32] reads after some slight change of notation as

$$\begin{bmatrix} AQ - \frac{1}{\tau} BLTC & -BL \\ CQ + \frac{1}{\tau} STC & S \end{bmatrix}^{\mathcal{S}} < 0 \tag{40}$$

which corresponds to the dual version of (38) for the choice of $K_s = \frac{1}{\tau} TC$. Another example is the condition in [107] which reads as

$$\begin{bmatrix} P(A + BK_s) & PB \\ -WC + SK_s & S \end{bmatrix}^{\mathcal{S}} < 0 \tag{41}$$

and can be obtained by post-multiplying (38) by $\left[\begin{array}{cc} I & K_s \\ 0 & I \end{array}\right]$ and pre-multiplying it by its transpose. [3] provides another version of this last result, with additional S-variables (it corresponds to condition (45) exposed in the following).

The first two conditions of Theorem 3 are bilinear matrix inequalities to which one can apply methods as those described in section 4. The main difference is that the matrix P (or Q in the dual formulations) may be optimized at each step.

The other main difference is in the initialization. In condition (37) it is non conservative to impose $S_2 = -I$, hence an appropriate initialization is to choose some stable matrix $A_o^T = S_1$ and to set $S_2 = -I$. The condition is then linear in P and K and allows to design structured SOF gains. This strategy has been adopted in [63]. An example of algorithm that can be produced is as follows.

Proposition 19 *In the following algorithm, $\{\alpha_{k=1\ldots\bar{k}}\}$ is a monotonically non increasing sequence. If it concludes with a negative value of $\alpha_{\bar{k}}$, then $K_{\bar{k}}$ is a stabilizing SOF gain.*
[Init.] *Set $k = 1$, and choose some stable matrix A_0.*
[Step k] *For fixed A_{k-1} solve the following optimization problem with respect to α, P and K*

$$\begin{array}{l} \min \alpha \quad : \quad P > 0, \\ \left[\begin{array}{cc} 0 & P \\ P & 0 \end{array}\right] < \left\{ \left[\begin{array}{c} A_{k-1}^T \\ -I \end{array}\right] \left[\begin{array}{cc} A + BKC - \alpha I & -I \end{array}\right] \right\}^{\mathcal{S}}. \end{array} \tag{42}$$

Set $\alpha_k = \alpha$, $K_k = K$, and $A_k = A + BK_kC - \alpha_k I$.
[Term.] *If $\alpha_k < 0$, set $\bar{k} = k$ and stop. Otherwise, if α_k has decreased more than some predefined threshold, go to [Step k] with $k \leftarrow k + 1$.*

Similar properties hold for condition (38) except that the initialization is with a state-feedback (as in Propositions 5 and 10). An example of algorithm that can be produced is the following (see also [94] for an algorithm that minimizes the H_∞ cost). Without loss of generality, it is possible to search for diagonal matrices S, hence structural constraints on K can be imposed via constraints on W.

Proposition 20 *In the following algorithm, $\{\alpha_{k=1\ldots\bar{k}}\}$ is a monotonically non increasing sequence. If it concludes with a negative value of $\alpha_{\bar{k}}$, then $K_{\bar{k}}$ is a stabilizing SOF gain.*
[Init.] *Set $k = 1$, and choose some stabilizing state feedback gain $K_{s,0}$.*
[Step k] *For fixed $K_{s,k-1}$, solve the following optimization problem with respect to α, P, W, and S*

$$\begin{array}{l} \min \alpha \quad : \quad P > 0, \\ M_A^T \left[\begin{array}{cc} -2\alpha P & P \\ P & 0 \end{array}\right] M_A < \left\{ \left[\begin{array}{c} K_{s,k-1}^T \\ -I \end{array}\right] \left[\begin{array}{cc} -WC & S \end{array}\right] \right\}^{\mathcal{S}}. \end{array} \tag{43}$$

Set $\alpha_k = \alpha$, $K_k = -S^{-1}W$, *and* $K_{s,k} = K_k C$.
[Term.] *If* $\alpha_k < 0$, *set* $\bar{k} = k$ *and stop. Otherwise, if* α_k *has decreased more than some predefined threshold, go to [Step k] with* $k \leftarrow k+1$.

All such algorithms are highly sensitive to initial conditions. For this reason, a randomized approach is proposed in [13] and [37] to generate using Hit-and-Run methods many initial guesses of state-feedback matrices K_s. This is shown to be quite efficient to get feasible solutions to LMIs (38).

The last condition of Theorem 3 is an LMI with a rank constraint. Any methods of Section 5 may be applied. The (directional) alternating projection approach is for example applied in [35, 38].

Among variants of these algorithms, contributions such as [67, 80] address the multi-performance problem and show that similarly to the resilient approach there is no difficulty in searching for independent Lyapunov matrices for each system/performance pair. This is thanks to the decoupling property between the Lyapunov matrices and the variables used to parameterize the controller gains. [67] also adresses the case of reduced order control with an original method, starting from a dynamical controller, that searches for controllers of lower order.

In terms of extensions to robustness issues, the S-variables approach has the other characteristic to allow for the search of parameter-dependent Lyapunov matrices, thus having reduced conservatism compared to all results based on formulations (5) and (6). The robust counterpart for polytopic systems (9) of condition (37) given in [37] reads as searching for $P^{[j]} > 0$ for each vertex and common SOF gain K and S-variables S_1, S_2 such that the following matrix inequalities hold for all vertices $j = 1 \dots \bar{j}$

$$\left[\begin{array}{cc} 0 & P^{[j]} \\ P^{[j]} & 0 \end{array}\right] < \left\{ \left[\begin{array}{c} S_1 \\ S_2 \end{array}\right] \left[\begin{array}{cc} A^{[j]} + B^{[j]}KC & -I \end{array}\right] \right\}^{\mathcal{S}}. \tag{44}$$

The numerical characteristics of this problem is similar to the non-robust case (except for the increase of the number of decision variables and the number of constraints). The same algorithm as in Proposition 19 can be applied, having in mind that $A_o^T = -S_1 S_2^{-1}$ needs to be Hurwitz stable, and each $P^{[j]}$ proves the stability of both $A^{[j]} + B^{[j]}KC$ and A_o. The resulting parameter-dependent Lyapunov matrix is of the type $P(\xi) = \sum_{j=1}^{\bar{j}} \xi_j P^{[j]}$.

The robust counterpart for polytopic systems (9) of condition (38) is obtained by introducing an additional S-variable, see [37]. It reads as searching for $P^{[j]} > 0$ and $K_s^{[j]}$ for each vertex and common SOF gain W and S-variables S, S_1, S_2 such that the following matrix inequalities hold for all vertices $j = 1 \dots \bar{j}$

$$\left[\begin{array}{ccc} 0 & 0 & P^{[j]} \\ 0 & 0 & 0 \\ P^{[j]} & 0 & 0 \end{array}\right] < \left\{ \left[\begin{array}{c} S_1 \\ 0 \\ S_2 \end{array}\right] N^{[j]} + \left[\begin{array}{c} K_s^{[j]T} \\ -I \\ 0 \end{array}\right] N(W,S) \right\}^{\mathcal{S}} \tag{45}$$

where $N^{[j]} = \left[\begin{array}{ccc} A^{[j]} & B^{[j]} & -I \end{array}\right]$ and $N(W,S) = \left[\begin{array}{ccc} -WC & S & 0 \end{array}\right]$. The same algorithm as in Proposition 20 can be applied, having in mind that each $P^{[j]}$

proves the stability of both $A^{[j]} + B^{[j]}KC$ and $A^{[j]} + B^{[j]}K_s^{[j]}$. The parameter-dependent Lyapunov matrix is as in the former case. Further extensions are proposed in [3, 107] for searching for homogeneous parameter-dependent Lyapunov matrices of arbitrary degree.

7 Non-Lyapunov-based Approaches

In addition to Lyapunov-based approaches, there exist non-Lyapunov-based methods. The first two we shall discuss focus on solving the following optimization problem:

$$\min_K \quad \alpha(A + BKC) \tag{46}$$

where α is the spectral abscissa of the closed loop system, i.e. the maximum real part of its eigenvalues. The above optimization problem is non-convex and non-smooth. In fact, the lack of convexity and smoothness of the spectral abscissa and other similar performance criteria make the above optimization problem difficult to solve [19].

Free package HIFOO (H_∞-H_2 Fixed Order Optimization) [12,20,57,58] and recent MATLAB functions, available in the Robust Control Toolbox, can cope with the above non-smooth non-convex optimization problem.

HIFOO is a public-domain MATLAB package for static output feedback and fixed-order stabilizing control design in state space setting with several performance objectives, e.g. H_∞, H_2, multiobjective optimization, simultaneous stabilization, spectral abscissa, and complex stability radius optimization. HIFOO relies on quasi-Newton updating and gradient sampling algorithm proposed in [21] and [19].

MATLAB commands *hinfstruct*, *looptune*, *systune* [4, 6, 8, 48], available in the Robust Control Toolbox since release R2010b, address the problem of fixed-structure and fixed-order H_∞ control synthesis in both state space and transfer function framework. *looptune* tunes fixed-structure and fixed-order feedback loops while satisfying the common engineering requirements including performance bandwidth, set-point tracking, roll-off, and multi-loop gain and phase margins [5]. The MATLAB routine *systune* deals with the fixed-structure and fixed-order control synthesis with time-domain, frequency domain, open-loop shape, stability margin, and closed-loop pole requirements [5]. *systune* can also handle multiple requirements as well as multiple models.

The main properties of HIFOO, *hinfstruct*, *looptune*, and *systune* are that:

- They are purely optimization-based, i.e. the static output feedback problem is formulated as a solution to a non-smooth non-convex optimization.

- As compared to Lyapunov-based methods, they are quite fast in terms of execution time due to the absence of the Lyapunov matrix or other variables such as the S-variables. For instance, given a 55×55 matrix A, a 55×2 matrix B, and a 2×55 matrix C, we are looking for a 2×2 stabilizing static output feedback K. The BMI problem given in (5) or its

dual version in (6) needs 1544 decision variables, most of them, i.e. 1540, for Lyapunov matrix P or Q whereas the optimization problem in (46) requires just 4 tunable controller parameters.

- They can address structured SOF design for the case described in subsection 2.2.
- They can address simultaneous stabilization and other multi-performance problems as described in subsection 2.3.
- The optimization, although very efficient compared to other Lyapunov based techniques, remains local. For different initializations the results may differ. Yet, usually for several runs for randomized initializations, the methods almost always provide different but appropriate results.

For all these reasons, and the fact that the solvers are available to users, these tools have gained very rapidly a large reputation and are intensively used on applications. See [2, 47, 85, 103, 104] to cite just a few in different application fields.

The main drawback of these methods is that these cannot cope with robustness issues, at least not in the same guaranteed way as the Lyapunov based methods. Yet robustness may be addressed via design/analysis iterations as proposed in [7] where the design steps are simultaneous stabilization type for scenarios (finite number of systems among possible ones in the uncertainty set) and where the analysis step allows to exhibit "worst-case" samples of the uncertainties to be integrated in the scenarios.

Among non-Lyapunov type methods, one can also cite the recently published paper [101] that proposes a randomized approximation algorithm. On examples it claims to be competitive with HIFOO and *hinfstruct* in terms of computation time. But it fails on some of the most difficult examples from the $COMPl_eib$ library of SOF problems [84] where HIFOO and *hinfstruct* succeed in providing SOF gains. At this stage the package is not available for independent testing. It is limited to stabilization problems (no performances except pole placement in regions) and does not handle structure issues on the control gains. Moreover, it does not apply to simultaneous stabilization problems (and hence neither for robustness issues).

Finally we shall mention the non-Lyapunov method proposed in [24]. It relies on an old standard tool: Routh-Hurwitz table. The rationale is the following. Assume the closed-loop system uncertain matrix $A(\xi) + B(\xi)KC$. Its coefficient are affine in both the uncertainties and the control gain coefficients. Take the characteristic polynomial of that matrix, its coefficients are polynomials in the uncertainties and the control gains. The Routh-Hurwitz table coefficients are rational in the uncertainties and the control gains. Positivity of these coefficients may be recast as a set of polynomial constraints in the uncertainties and the control gains. Using sum-of-squares methodology, the problem can hence be solved using convex optimization. The method applies also for systems that depend polynomially on the uncertainties. The sum-of-squares relaxations are

moreover guaranteed to provide a necessary and sufficient stabilization test for finite degree of relaxation as soon as one imposes the control gains to be searched in bounded sets. Although highly appealing theory, the method is expected to be impracticable for real-world systems. This is because the degree of polynomials involved in the Routh-Hurwitz table coefficients and the number of polynomial constraints grow extremely fast with respect to the order of the plant (and the number of uncertain parameters). The sum-of-squares relaxations are therefore in practice of huge dimensions.

8 Conclusions

This paper reviews the existing methods for the design of a static output feedback. It does not claim to provide an exhaustive list of the many contributions on the topic, but we believe it covers all main methods. The survey proposes a comprehensive classification of these methods: convex cases; iterative LMIs for the BMI problem; iterative LMIs for the rank constraint formulation; achievable improvements with decoupled Lyapunov matrices; non-Lyapunov approaches. The global conclusion is that the latest non-Lyapunov based optimization tools are most effective for SOF design, even when structure and multi-performance issues are considered. Iterative LMI approaches for decoupled Lyapunov matrix condition may as well have competitive features, in particular in terms of robustness. Purely convex results for systems with specific structure should also not be neglected, and there may be more such cases to be discovered. Globally, although there has been a very high effort of the control community to address the static output feedback issue, we would say that it is still open, in particular when dealing with structured robust multi-performance design.

The efforts for this survey were focused on building the classification, providing a uniform notation and discussing the flexibilities of each method for addressing structure and robustness issues. A remaining work would be to conduct a detailed numerical comparison of all these methods (or at least of the main ones). We believe that such comparison can be fair, and hence satisfactory, only if all methods are given the same attention in terms of accurate and open-source coding. We believe that it is not achievable at this stage.

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