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Characterizations of Wishart Processes and Wishart Distributions

Piotr Graczyk ^{*}, Jacek Małeckı [†] and Eberhard Mayerhofer [‡]

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Abstract

We characterize the existence of non-central Wishart distributions (with shape and non-centrality parameter) as well as the existence of solutions to Wishart stochastic differential equations (with initial data and drift parameter) in terms of their exact parameter domains. In particular, we show that the exact parameter domain of Wishart distributions equals the so-called non-central Gindikin set, which links the rank of the non-centrality parameter (resp. initial value) to the size of the shape parameter (resp. drift parameter). Our novel approach utilizes the action of Wishart semigroups on symmetric polynomials. Also, we prove a conjecture by Damir Filipovic (2009) on the existence of such semigroups on the cones of lower rank matrices.

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1 Introduction and Preliminaries

The aim of this paper is to characterize the parameter domain of non-central Wishart distributions (with shape, scale and non-centrality parameters) and that of Wishart processes, a class of continuous-time Markov processes with positive semi-definite state space (with constant drift parameter).

Denote by \mathcal{S}_p the space of symmetric $p \times p$ matrices and let \mathcal{S}_p^+ be the open cone of positive definite matrices, with topological closure $\bar{\mathcal{S}}_p^+$, the positive semi-definite matrices. The classical Gindikin¹ set W_0 is defined as the set of admissible $\beta \in \mathbb{R}$ such that there exists a random matrix X with values in \mathcal{S}_p^+ (equivalently a measure with support in $\bar{\mathcal{S}}_p^+$) such that its Laplace transform is of the form

$$\mathbb{E}e^{-\text{tr}(uX)} = (\det(I + \Sigma u))^{-\beta}, \quad u \in \bar{\mathcal{S}}_p^+,$$

where $\Sigma \in \mathcal{S}_p^+$. It is well-known that

$$W_0 = \frac{1}{2}B \cup \left[\frac{p-1}{2}, \infty \right),$$

where $B = \{0, 1, \dots, p-2\}$ (cf. [6], pp. 137, 349). A more intricate question concerns the existence of non-central Wishart distributions, which in addition involves a parameter of non-centrality:

Definition 1.1. *The general non-central Wishart distribution $\Gamma_p(\beta, \omega; \Sigma)$ on $\bar{\mathcal{S}}_p^+$ is defined (whenever it exists) by its Laplace transform*

$$\mathcal{L}(\Gamma(\beta, \omega; \sigma))(u) = (\det(I + \Sigma u))^{-\beta} e^{-\text{tr}(u(I + \Sigma u)^{-1}\omega)}, \quad u \in \mathcal{S}_p^+, \quad (1.1)$$

where $\beta > 0$ denotes its shape parameter, $\Sigma \in \mathcal{S}_p^+$ is the scale parameter and the parameter of non-centrality equals $\omega \in \bar{\mathcal{S}}_p^+$.

Random matrices X verifying (1.1) arise in statistics as estimators of the covariance matrix parameter Σ of a normal population. In fact, for the random matrix

$$X = \xi_1 \xi_1^T + \dots + \xi_n \xi_n^T =: q(\xi), \quad \xi = (\xi_1, \dots, \xi_p),$$

where $\xi_i \sim N_p(m_i, \Sigma/2)$ are independent normal vectors in \mathbb{R}^p , the Laplace transform of X is given by the right side of (1.1) with $\beta = n/2$ and $\omega = q(m_1, \dots, m_n)$.

Accordingly, we shall say that the pair (ω, β) belongs to the non-central Gindikin set W if there exists a random matrix X with values in $\bar{\mathcal{S}}_p^+$ having the Laplace transform (1.1) for a matrix $\Sigma \in \mathcal{S}_p^+$ ².

Note the following:

¹The name of this set originates from Gindikin's [8] work in a general multivariate setting .

²As long as Σ is of maximal rank, this definition is indeed independent of Σ , see Lemma 3.5.

- Whenever $(\omega, \beta) \in W$ then we have $\beta \geq 0$, otherwise $\mathbb{E}e^{-\text{tr}(uX)}$ would be unbounded; and clearly, $(0, \beta) \in W$ if and only if $\beta \in W_0$.
- In the case $\text{rank}(\omega) = 1$, $\beta \neq 0$, the characterization of the non-central Gindikin set W was given in [16]: one then has $(\omega, \beta) \in W$ if and only if $\beta \in W_0$.

The general problem of existence and non-existence of non-central Wishart distribution has been next studied by Letac and Massam [12]³. A first hint on the non-triviality of this problem is given by Mayerhofer [15] who reveals that there is an interplay between the rank of the non-centrality parameter ω and the magnitude of β in the discrete part of the classical Gindikin ensemble. The method used therein involves the construction of affine Markov processes on $\bar{\mathcal{S}}_p^+$ whose positivity requires a certain magnitude of their constant drift parameter. The results in [15] allow to conjecture⁴ the following:

NCGS Conjecture. *The non-central Gindikin set is characterized by*

$$(\omega, \beta) \in W \Leftrightarrow \left(\beta \in \left[\frac{p-1}{2}, \infty \right), \omega \in \bar{\mathcal{S}}_p^+ \right) \text{ or } (2\beta \in B, \text{rank}(\omega) \leq 2\beta).$$

Sufficiency of these conditions was shown by Bru in [2], except the case $2\beta = p - 1$, that may be found in [9] or [15]. What concerns the necessity, [15] proved that if $(\omega, \beta) \in W$ and $2\beta \in B$, then $\text{rank}(\omega) \leq 2\beta + 1$. A proof of the necessity in the NCGS Conjecture has been put forward by the preprint [13].

The attempt of [13] is very technical and lengthy and as such makes it difficult to understand the key reasons for the particular parametric restrictions of shape and non-centrality parameter.

The present paper gives a first complete proof of the NCGS conjecture, which reveals, and builds on, the intimate connection between the existence of non-central Wishart distributions and that of Wishart processes. These are positive semi-definite solutions $(X_t)_{t \geq 0}$ of stochastic differential equations of the form

$$dX_t = \sqrt{X_t} dW_t + dW_t^T \sqrt{X_t} + \alpha I dt, \quad X_t \in \bar{\mathcal{S}}_p^+, t \geq 0; \quad X_0 = x_0 \in \bar{\mathcal{S}}_p^+, \quad (1.2)$$

where $\sqrt{X_t}$ is the unique positive square root of a X_t , W is a $d \times d$ matrix of standard Brownian motions, and $\alpha \geq 0$ is a single drift parameter.

At the same time, we formulate and prove here as a novelty sufficient and necessary condition for the existence of Wishart processes.

³However, the statement in [12] is incomplete. In fact, the absence of an additional condition implies existence of certain affine Markov processes, which has however been ruled out by [3]. This fact has been pointed out by [14] and [15]. Accordingly, the proof in section 5 of [12] has a gap.

⁴In [13] and a previous versions of this paper, the name *Mayerhofer Conjecture* is used. The conjecture has been formally presented at the CIMPA Workshop in Hammamet in 2011 (http://www.cimpa-icpam.org/IMG/pdf/Cours_Mayerhofer_Hammamet.pdf).

Already Bru [2], who introduced Wishart processes, realized that Wishart processes are Wishart distributed⁵:

Proposition 1.2. *Bru([2, Theorem 3]) If the stochastic differential equation (1.2) with $x_0 \in \bar{\mathcal{S}}_p^+$ has a global weak solution in $\bar{\mathcal{S}}_p^+$, then X_t is Wishart distributed for each $t \geq 0$. In particular, we have*

$$\mathbb{E}^{x_0}[\exp(-\operatorname{tr}(uX_t))] = (\det(I + 2tu))^{-\alpha/2} \exp[-\operatorname{tr}(x_0(I + 2tu)^{-1}u)], \quad u \in \mathcal{S}_p^+. \quad (1.3)$$

In the present paper, we shall also show how to construct from individual Wishart distributions full-fledged Wishart processes. Our main result is thus a three-fold characterization:

Theorem 1.3. *Let $x_0 \in \bar{\mathcal{S}}_p^+$ and $\alpha \geq 0$. The following are equivalent:*

- (i) *The SDE (1.2) has a global weak solution with $X_0 = x_0$.*
- (ii) *Either $\alpha \geq p - 1$, or $\alpha \in B$ and $\operatorname{rank}(x_0) \leq \alpha$.*
- (iii) *$(x_0, \alpha/2) \in W$.*

Our proof of the NCGS Conjecture (that is, Theorem 1.3 (ii) \Leftrightarrow (iii)) is based on analysis of affine Wishart semigroups. We use as new tool a class of symmetric polynomials which arise as coefficients of the characteristic polynomial of a symmetric matrix. A full characterization of Wishart processes is provided by (Theorem 1.3 (i) \Leftrightarrow (ii)).

For convenience of the reader, but at the expense of proving an additional implication, Theorem 1.3 is split into two independent theorems in the following two chapters. They require different mathematical tools and therefore can be read independently. Chapter 2 is concerned with the existence of solutions to Wishart stochastic differential equations using elementary stochastic analysis with symmetric polynomials (Theorem 2.3 comprises the equivalence (i) \Leftrightarrow (ii) of Theorem 1.3). Chapter 3 concerns the existence of Wishart distributions (the NCGS conjecture, which comprises (ii) \Leftrightarrow (iii) of Theorem 1.3). Here we use the Markovian viewpoint, in particular the fact that Wishart semigroups are affine Feller semigroups

2 Gindikin sets for Wishart Processes

We consider the question of solutions in $\bar{\mathcal{S}}_p^+$ of the so-called Wishart SDE (1.2). In particular, using the dynamics of symmetric polynomial functionals of these solutions, we characterize drift parameters and the ranks of initial values that allow solutions of this SDE.

If X is a symmetric $p \times p$ matrix, we define the polynomials $e_n(X)$ as basic symmetric polynomials

$$e_n(X) = \sum_{i_1 < \dots < i_n} \lambda_{i_1}(X) \lambda_{i_2}(X) \dots \lambda_{i_n}(X), \quad n = 1, \dots, p; \quad (2.1)$$

⁵Bru did not introduce the notion of *non-central* Wishart distribution, but she realized the explicit formula for the Laplace transform.

in the eigenvalues $\lambda_1(X) \leq \dots \leq \lambda_p(X)$ of X . Moreover, we use the convention that $e_0(X) \equiv 1$. Up to the sign change, the polynomials e_n are the coefficients of the characteristic polynomial of X , i.e.

$$\det(X - uI) = (-1)^p u^p + (-1)^{p-1} e_1(X) u^{p-1} + \dots - e_{p-1}(X) u + e_p(X)$$

and are polynomial functions of the entries of the matrix X . In particular, $e_p(X) = \det X$.

In [10], symmetric polynomials related to general class of non-colliding particle systems were studied in details. Here we present similar results adapted to the matrix SDE

$$dX_t = g(X_t) dW_t h(X_t) + h(X_t) dW_t^T g(X_t) + b(X_t) dt, \quad (2.2)$$

where the continuous functions g, h, b act spectrally⁶ on S_p and W_t is a Brownian $p \times p$ matrix. Henceforth we abbreviate $\sigma = 2gh$ and $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$. We use the natural bijection (2.1) between the eigenvalues $\Lambda = (\lambda_1 \dots \lambda_p)$ and the polynomials $e = (e_1, \dots, e_p)$, extended to the closed Weyl chamber $\bar{C}_+ = \{(x_1, \dots, x_p) \in \mathbb{R}^p : x_1 \leq x_2 \leq \dots \leq x_p\}$, see [10], p.6. We write $\Lambda = \Lambda(e)$ for the inverse bijection on the set $e(\bar{C}_+)$. We use the notation $e_n^{\bar{i}}$ for the incomplete polynomial of order n , not containing the variable $\lambda_i(e)$; the notation $e_n^{i, \bar{j}}$ is analogous.

Proposition 2.1. *Let X be a solution of (2.2). Then the symmetric polynomials $e_n = e_n(X)$, $n = 1, \dots, p$, are continuous semimartingales described by the system of SDEs ($n = 1, \dots, p$)*

$$de_n = \left(\sum_{i=1}^p \sigma^2(\lambda_i(e)) (e_{n-1}^{\bar{i}})^2 \right)^{1/2} dV_n + \left(\sum_{i=1}^p b(\lambda_i(e)) e_{n-1}^{\bar{i}} - \sum_{i < j} G(\lambda_i(e), \lambda_j(e)) e_{n-2}^{\bar{i}, \bar{j}} \right) dt \quad (2.3)$$

where V_n are Brownian motions on \mathbb{R} such that $d\langle e_n, e_m \rangle = \sum_{i=1}^p \sigma^2(\lambda_i(e)) e_{n-1}^{\bar{i}} e_{m-1}^{\bar{i}} dt$.

Proof. The symmetric polynomials (e_1, \dots, e_n) are given by an analytic function (polynomial of the coefficients) of the matrix X . Thus Itô's formula, applied to the SDE for the matrix process X_t , gives a system of the SDEs for (e_1, \dots, e_n) . We determine these SDEs like in Propositions 3.1 and 3.2 in [10], using the SDE for the eigenvalues

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)dB_i + \left(b(\lambda_i) + \sum_{j \neq i} \frac{G(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt, \quad i = 1, \dots, p, \quad (2.4)$$

which are available, according to Theorem 3 from [9], when eigenvalues $\lambda_i(0)$ of x_0 are all distinct and before their eventual collision.

However, the form of the SDEs (2.3) does not depend on the starting point x_0 or on the non-collision of the eigenvalues, i.e. it does not change if we remove the conditions that eigenvalues of the initial point are all different and that they are non-colliding for $t > 0$. \square

⁶Recall that if $g : \mathbb{R} \mapsto \mathbb{R}$ then $g(X)$ is defined spectrally, i.e. $g(U \operatorname{diag}(\lambda_i) U^T) = U \operatorname{diag}(g(\lambda_i)) U^T$, where $U \in SO(p)$.

Using Proposition 2.1 we get the following characterization of the symmetric polynomials related to Wishart processes.

Proposition 2.2. *Let X_t be a Wishart process, i.e. a solution of the matrix SDE (1.2). Then the symmetric polynomials $e_n = e_n(X)$, $n = 1, \dots, p$ are semimartingales satisfying the following system of SDEs*

$$de_n = M_n(e_1, \dots, e_p)dV_n + (p - n + 1)(\alpha - n + 1)e_{n-1}dt, \quad n = 1, \dots, p - 1, \quad (2.5)$$

$$de_p = 2\sqrt{e_{p-1}e_p}dV_p + (\alpha - p + 1)e_{p-1}dt, \quad (2.6)$$

where V_i , $i = 1, \dots, p$ are one-dimensional Brownian motions and the functions M_n are continuous on \mathbb{R}^p .

Note that by Proposition 2.1, the explicit forms of the martingale parts $M_n(e_1, \dots, e_p)dV_n$ as well as their brackets $d\langle e_n, e_m \rangle$ are known for every $n, m = 1, \dots, p$.

Proof. Applying Proposition 2.1 to the SDE (1.2), we find that $M_n = 2 \left(\sum_{i=1}^p \lambda_i (e_{n-1}^i)^2 \right)^{1/2}$. Moreover, we have the following expressions for the drift parts of de_n :

$$\sum_{i=1}^p \alpha e_{n-1}^i - \sum_{i < j} (\lambda_i + \lambda_j) e_{n-2}^{i,j} = (p - n + 1)(\alpha - n + 1)e_{n-1}.$$

□

Since a Wishart process is $\bar{\mathcal{S}}_p^+$ valued by construction, so $e_k \geq 0$, for all $k = 1, \dots, p$. The SDEs describing the symmetric polynomials e_1, \dots, e_p can be used to show that the e_p would become negative if $e_p(0) = 0$ and $\alpha = 2\beta$ was small enough, a mere impossibility. This idea is used in the proof of the next Theorem to find the precise range for $\alpha \geq 0$, where Wishart SDEs have global weak solutions.

2.1 Solving the Wishart stochastic differential equations

This section gives a full characterization of the existence of solutions to Wishart SDEs (1.2).

Theorem 2.3. *Let $\alpha \geq 0$, and $x_0 \in \bar{\mathcal{S}}_p^+$. The following are equivalent.*

- (i) *The SDE (1.2) has a global weak solution with $X_0 = x_0$.*
- (ii) *$\alpha \geq p - 1$, or $\alpha \in \{0, 1, \dots, p - 2\}$ and $\text{rank}(x_0) \leq \alpha$.*

Proof. Assume first (i). If $\alpha \geq p - 1$, nothing has to be shown. Suppose, therefore, $\alpha < p - 1$. Recall equations (2.5) –(2.6) from Proposition 2.2 .

We can compute explicitly the expected value of the polynomials starting from the first one.

$$\mathbb{E}e_1(t) = e_1(0) + p\alpha \int_0^t ds = e_1(0) + pat$$

Thus we have

$$\begin{aligned}\mathbb{E}e_2(t) &= e_2(0) + (p-1)(\alpha-1) \int_0^t \mathbb{E}e_1(s) ds \\ &= e_2(0) + (p-1)(\alpha-1)e_1(0)t + p(p-1)\alpha(\alpha-1)\frac{t^2}{2}\end{aligned}$$

and so on. Consequently $\mathbb{E}e_n(t)$ is a polynomial of degree not greater than n . In particular, the coefficient of t^n is

$$\frac{p(p-1) \cdots (p-n+1) \cdot \alpha(\alpha-1) \cdots (\alpha-n+1)}{n!}$$

If $\alpha \notin B$, we take $n = \lceil \alpha \rceil + 1$ and get that $\mathbb{E}e_n(t)$ is a polynomial of degree n such that the leading coefficient is negative. Consequently, it can not stay positive for every $t > 0$. Contradiction.

If $\alpha = m \in B$ we look at $\mathbb{E}e_n(t)$ where $n = m + 1$. Then

$$\mathbb{E}e_n(t) = e_n(0) + (p-n+1)(\alpha-n+1) \int_0^t \mathbb{E}e_{n-1}(s) ds = e_n(0)$$

If $e_n(0) > 0$, then

$$\mathbb{E}e_{n+1}(t) = e_{n+1}(0) + (p-n)(\alpha-n)e_n(0)t$$

i.e. the leading term is negative and thus $\mathbb{E}e_{n+1}(t) < 0$ for large t . It implies $e_n(0) = 0$, i.e. $\text{rank}(x_0) \leq n-1 = m = \alpha$.

Proof of (ii) \Rightarrow (i): Several proofs of this fact are known: see, e.g. Bru [2], where she does however not cover the case $\alpha = p-1$. Another proof can be obtained by using results from [9] which show solvability (even in the strong sense) of the SDEs for the eigenvalues of a Wishart process. Here we provide similar, but more elaborate arguments than Bru's, using the theory of affine processes:

- If $\alpha \geq p-1$, then existence of an affine Markov process $(X_t)_{t \geq 0}$ with state-space $\bar{\mathcal{S}}_p^+$ and Laplace transform (1.3) is provided by the general theory of [4]. This implies a solution to the corresponding martingale problem, that is, for any $f \in C_c^2(\bar{\mathcal{S}}_p^+)$, and any initial value x_0 , we have that

$$f(X_t) - f(x_0) - \int_0^t \mathcal{A}f(X_s) ds$$

is a martingale, where \mathcal{A} is the infinitesimal generator of X , see eq. (3.3) below.

- If $\alpha \in \{0, 1, \dots, p-2\}$, then existence of a solution to the corresponding martingale problem is shown in [2] by an explicit construction (squaring matrix valued standard Brownian motions), see also [14, Example 2.3].

In any case, by [14, Lemma 2.5] (see also [4, Proof of Theorem 6.2, p.53, last paragraph]), the solution of the Wishart martingale problem implies the existence of a weak solution to the Wishart SDE (1.2). \square

Remark 2.4. *Necessity of (ii)* can be also proved, if we assume the validity of the NCGS Conjecture (a fact that is proven in Section 3, and which we have not used above to keep the section self-contained). Suppose the existence of a weak solution. Then by Proposition 1.2, the solution is Wishart distributed, that is, for each $t \geq 0$, $X_t \sim \Gamma_p(\alpha/2, x_0; 2tI)$. By the NCGS Conjecture, we therefore must have $\alpha/2 \in W_0$ and, in addition, if $\alpha < p - 1$ then $\text{rank}(x_0) \leq \alpha$.

3 The NCGS Conjecture and Wishart Semigroups

In this section we introduce Wishart semigroups, which are the main tool for the proof of the NCGS Conjecture in Section 3.2 below. In the final subsection 3.3 we characterize all Wishart semigroups on lower rank matrices, which are non-convex state spaces.

3.1 Wishart semigroups

For $p \geq 1$, let $D_p(k) \subset \bar{\mathcal{S}}_p^+$ be the subcones of rank $\leq k$ matrices, $0 \leq k \leq p$, where clearly $D_p(0) = \{0\}$ and $D_p(p) = \bar{\mathcal{S}}_p^+$. Denote by $f_u(x) = \exp(\text{tr}(-ux))$, where $u, x \in \bar{\mathcal{S}}_p^+$.

Definition 3.1. *Let $D \subset \bar{\mathcal{S}}_p^+$. A Wishart semigroup $(P_t)_{t \geq 0}$ on D is a positive, strongly continuous $C_0(D)$ contraction semigroup which for any $u \in \bar{\mathcal{S}}_p^+$ acts as*

$$P_t f_u(x) = \det(I + 2tu)^{-\alpha/2} e^{-\text{tr}(x(u^{-1} + 2tI)^{-1})}, \quad x \in \bar{\mathcal{S}}_p^+. \quad (3.1)$$

Here $\alpha \geq 0$ is called the constant drift parameter of $(P_t)_{t \geq 0}$.

We summarize a few essential facts in the following

Remark 3.2. *Let $(P_t)_{t \geq 0}$ be a Wishart semigroup with drift parameter α .*

- (i) *(Markovian representation) In view of the Riesz representation theorem for positive functionals [17, Chapter 2.14], for each $t > 0$, $x \in D$ there exists a positive measure $p_t(x, d\xi)$ such that*

$$P_t f(x) = \int_D f(\xi) p_t(x, d\xi). \quad (3.2)$$

Furthermore, the semigroup property of $(P_t)_{t \geq 0}$ implies, that $p_t(x, d\xi)$ satisfies the Chapman-Kolmogorov equations, thus $p_t(x, d\xi)$ is a Markov transition function. Hence, the semigroup has a stochastic representation as a Markov process $(\mathbb{P}^x)_{x \in D}$, where for each $x \in D$, \mathbb{P}^x denotes the resulting probability on the canonical path space $D^{\mathbb{R}^+}$ with initial law $X_0 = x$, and $X_t(\omega) := \omega(t)$, where $\omega \in D^{\mathbb{R}^+}$.

- (ii) (*Càdlàg Paths*) It is a well established fact, that any Feller process (that is, a Markov process with strongly continuous C_0 semigroup), has a càdlàg modification.
- (iii) (*Affine Property*) By definition, Wishart semigroups are affine semigroups (see [3]), that is, the Laplace transform of its transition function is of the form

$$\mathbb{E}[e^{-\text{tr}(uX_t)} \mid X_0 = x] = e^{-\phi(t,u) - \text{tr}(\psi(t,u)x)},$$

where

$$\phi(t, u) = \frac{\alpha}{2} \log(\det(I + 2tu)), \quad \psi(t, u) = (u^{-1} + 2tI)^{-1}.$$

- (iv) (*Wishart transition function*) By construction, the Markovian transition function $p_t(x, d\xi)$ is $\Gamma_p(\alpha/2, x; 2It)$ distributed, for each $t \geq 0$ and for all $x \in D$.
- (v) (*Non-Explosion*) $(P_t)_{t \geq 0}$ is conservative: Let $u_n \in \mathcal{S}_p^+$ such that $u_n \rightarrow 0$ as $n \rightarrow \infty$. By (3.2) we thus have

$$P_t 1 = \lim_{n \rightarrow \infty} P_t f_{u_n}(x) = 1.$$

- (vi) (*Continuity*) If, in addition, we assume that the linear span of D has non-empty interior, (X, \mathbb{P}_x) for each x is (by [5]) an affine semimartingale, that is, a semimartingale with characteristics which are affine functions in the state. Since the associated jump measure vanishes, continuity of the sample paths follows.
- (vii) (*Strong Maximum Principle*) For a strongly continuous C_0 semigroup $(P_t)_{t \geq 0}$ with infinitesimal generator \mathcal{A} , the following are equivalent

- \mathcal{A} satisfies the strong maximum principle, that is, $\mathcal{A}f(x_0) \geq 0$, for any $f \in C_0$ that satisfies $f(x) \geq f(x_0)$.
- $(P_t)_{t \geq 0}$ is positive (hence a Feller semigroup).

The second implication is simple. A proof of the non-trivial implication employs the positivity of the Yoshida approximations of \mathcal{A} ([11, Corollary 2.8]).

Wishart semigroups on $D = \bar{\mathcal{S}}_p^+$ are well understood; they are affine diffusion processes. By [3] the following are equivalent:

- The Wishart semigroup with drift parameter α exists with state space $D = \bar{\mathcal{S}}_p^+$.
- $\alpha \geq p - 1$.

However, for state-spaces D which are strict subsets of $D = \bar{\mathcal{S}}_p^+$, less is known about Wishart semigroups. We shall state and prove a particular result when D is the set of rank $k \leq p - 1$ matrices (Theorem 3.9).

We consider operators acting on functions on \mathcal{S}_p (as a $p \times (p + 1)/2$ -dimensional vector space), and the nabla operator as the symmetric matrix operator $(\partial_{ij} f)_{ij}$ (for this notation and results related to the following, see [2, section 2.2] or [1, Proposition 3]).

For a subset $D \subseteq \bar{\mathcal{S}}_p^+$ we let $S_p^*(D)$ be the restriction of the space of rapidly decreasing smooth functions on \mathcal{S}_p to D .

Proposition 3.3. *Suppose $(P_t)_{t \geq 0}$ is a Wishart semigroup on D , and that $D_p(1) \subseteq D$. Then its infinitesimal generator \mathcal{A} is given by*

$$\mathcal{A} = \text{tr}(2x\nabla^2 + \alpha\nabla). \quad (3.3)$$

and $S_p^*(D) \subset \mathcal{D}(\mathcal{A})$ is a core.

Proof. By the definition of the affine property, we have

$$\mathcal{A}f_u(x) = (F(u) + \text{tr}(R(u)x))f_u(x) \quad (3.4)$$

for $f_u(x) = \exp(-\text{tr}(ux))$ and $u \in \bar{\mathcal{S}}_p^+$, and thus $f_u(x) \in \mathcal{D}(\mathcal{A})$. Here $F(u) = \alpha \text{tr}(u) = \frac{\partial \phi(t,u)}{\partial t}|_{t=0}$ and $R(u) = -2u^2 = \frac{\partial \psi(t,u)}{\partial t}|_{t=0}$. The assumption that D contains rank one matrices implies that the convex hull of D equals $\bar{\mathcal{S}}_p^+$, and thus F and R are uniquely defined. It is readily seen that the action of (3.3) on $f_u(x)$ coincides with (3.4).

According to the density argument [3, Theorem B.3], the linear hull of such exponentials for strictly positive definite u is dense in the space of rapidly decreasing functions on $\bar{\mathcal{S}}_p^+$, and thus, its restriction $S_p^*(D)$ is a valid choice for a core of \mathcal{A} . □

Recall that a Feller process is polynomial if the action of its semigroup can be extended to linear polynomials of any order ([4]).

Proposition 3.4. *Suppose X is a Wishart semigroup supported on $D \subset \bar{\mathcal{S}}_p^+$ with drift $\alpha \geq 0$. X is polynomial and its infinitesimal generator acts on symmetric polynomials as follows*

$$\mathcal{A}e_k(x) = (p - k + 1)(\alpha - k + 1)e_{k-1}(x), \quad 1 \leq k \leq p. \quad (3.5)$$

Proof. Since the domain of the moment generating function of $X_t | X_0 = x$ for each t contains an open neighborhood of zero, the process is polynomial [4] and thus the semigroup action of X can be extended to polynomials of any order. Equation (3.5) thus can be inferred from Proposition 2.2. □

We further require three Lemmas.

Lemma 3.5. *If $\Gamma_p(\beta, \omega; \sigma)$ is a positive measure on $\bar{\mathcal{S}}_p^+$, where σ is invertible, then $\Gamma_p(\beta, \omega'; \sigma')$ exists for any ω' satisfying $\text{rank}(\omega') \leq \text{rank}(\omega)$ and for any invertible σ' .*

Proof. The statement is a mild improvement of Proposition 3.1 (iii)(b) in [15]) (where the case $\text{rank}(\omega) = p$ is discussed). However, it applies also in this situation of lower rank matrices. In fact, $\bar{\mathcal{S}}_p^+$ is a symmetric cone, as its linear automorphism group is transitive. Hence, any ω' with $\text{rank } k = \text{rank}(\omega)$ can be mapped by a linear automorphism on ω : there exists a $d \times d$ matrix A such that $\omega' = A\omega A^\top$. Hence, by the proof of Proposition 3.1 (iii)(b) in [15]) we obtain a distribution $\Gamma(p, \omega'; \sigma)$. Finally, let $\text{rank}(\omega') \leq \text{rank}(\omega)$. Then there exists a sequence $\omega_n \rightarrow \omega'$, where $\text{rank}(\omega_n) = k$ for each n . A glance at the characteristic function⁷ of $\Gamma(p, \omega_n; \sigma)$ suffices to realize that Lévy's continuity theorem ensures also the existence of a distribution $\Gamma(p; \omega'; \sigma)$. □

⁷Lévy's continuity theorem in its original form applies to characteristic functions only, but it can be extended to the Laplace transform for cone-valued measures.

Lemma 3.6. *Let Ξ be a positive semi-definite random matrix supported on $D_p(r-1)$ and $\text{rank}(\Xi) = r-1$ with nonzero probability. Let further $\eta \sim \mathcal{N}(\mu, \Sigma)$ with $\mu \in \mathbb{R}^p$ and with covariance matrix $\Sigma \in \mathcal{S}_p^+$. If Ξ and η are independent, then $\text{rank}(\Xi + \eta\eta^\top) = r$ with nonzero probability.*

Proof. Assume first the constant case $\Xi = \Xi_0 \in \bar{\mathcal{S}}_p^+$. Without loss of generality, we may assume $\Xi_0 = \text{diag}(I_{r-1}, 0)$, where I_k is the $k \times k$ unit matrix. Define

$$V = \begin{pmatrix} I_{r-1} & -\Omega \\ 0 & I_{p-r+1} \end{pmatrix}$$

with a $(r-1) \times (p-r+1)$ matrix $\Omega_{ij} = \delta_{ij} \frac{\eta_i}{\eta_{r+1-j}}$. Then

$$V(\Xi_0 + \eta\eta^\top)V^\top = \text{diag}(I_{r-1}, (\eta\eta^\top)_{r \leq i, j \leq d})$$

and since $(\eta_k)_{r \leq k \leq d} \sim \mathcal{N}((\mu_k)_{r \leq k \leq d}, (\Sigma_{ij})_{r \leq i, j \leq d})$, we have that $\eta\eta^\top$ has rank 1 almost surely. Thus $\text{rank}(V(\Xi_0 + \eta\eta^\top)V^\top) = r-1 + 1 = r$ almost surely.

Now consider a random matrix Ξ . Clearly, $\text{rank}(\Xi + \eta\eta^\top) \leq r$. The set $\Omega_\Xi := \{\omega \in \Omega \mid \text{rank}(\Xi(\omega)) = r-1\}$ is Borel, since it is given by the inverse image of the measurable function⁸ $\text{rank}(\xi) : \bar{\mathcal{S}}_p^+ \rightarrow \{0, 1, \dots, p\}$. By assumption $\mathbb{P}[\Omega_\Xi] > 0$, thus the first part of the proof implies

$$\mathbb{E}[\text{rank}(\Xi + \eta\eta^\top) \mid \text{rank}(\Xi) = r-1] = r$$

and thus we conclude $\text{rank}(\Xi + \eta\eta^\top) = r$ on all of Ω_Ξ . \square

Lemma 3.7. *Suppose $\Xi_0 \in \bar{\mathcal{S}}_p^+$ with $\text{rank}(\Xi_0) = p-1$, and let $\Xi \sim \Gamma_p((p-1)/2, \Xi_0; \sigma)$, where σ is non-degenerate. Then $\text{rank}(\Xi) \leq p-1$, almost surely, and equals $p-1$ with positive probability.*

Proof. By Lemma 3.5 we can without loss of generality assume $\sigma = I$. Let \mathcal{A} be the infinitesimal generator of an Wishart semigroup $(P_t)_t$ with drift $\alpha = p-1$ and state space $\bar{\mathcal{S}}_p^+$, and denote by Ξ its canonical realization. Then $\Xi_{1/2} \mid \Xi_0 \sim \Gamma_p((p-1)/2, \Xi_0; I)$. By Proposition 3.4

$$\mathcal{A}e_p(x) = 0$$

and hence for any $x \in \bar{\mathcal{S}}_p^+$,

$$(P_t \det)(x) = \mathbb{E}[\det(\Xi_t) \mid \Xi_0 = x] = e_p(x_0) + \mathbb{E}\left[\int_0^t \mathcal{A}f(\Xi_s) ds\right] = 0 + 0 = 0,$$

⁸This map is measurable, since it can be written as

$$\text{rank}(\xi) = \min\{r \in \{0, 1, \dots, p-1\} \mid e_r(\xi) \neq 0\}.$$

where we recall the convention $e_0 = 1$ (to identify the 0).

and thus $\text{rank}(\Xi_{1/2}) \leq p - 1$ almost surely. Furthermore, by Proposition 3.4 we have $\mathcal{A}e_{p-1}(x) = 2e_{p-2}(x)$ and thus for each $t > 0$

$$(P_t e_{p-1})(x) = \mathbb{E}[e_{p-1}(\Xi_t) \mid \Xi_0 = x] = e_{p-1}(x_0) + \int_0^t \mathbb{E}[e_{p-2}(X_s) ds] \geq e_{p-1}(x_0) > 0$$

We conclude that whenever $\text{rank}(x_0) = p - 1$, we have for each $t > 0$, $\text{rank}(\Xi_t \mid \Xi_0 = x) = p - 1$ with non-zero probability and thus also $\text{rank}(\Xi) = \text{rank}(\Xi_{1/2} \mid \Xi_0 = \omega) = p - 1$ with non-zero probability. \square

The following statement concerns the support of Wishart distributions with general shape parameter.

Proposition 3.8. *Suppose $\beta \in \{0, 1/2, \dots, (p - 1)/2\}$. Suppose $\text{rank}(\omega) = 2\beta + k$, where $1 \leq k \leq p - (2\beta + 1)$. Then $\Gamma_p(\beta, \omega; \sigma)$, if exists, is supported in $D_p(2\beta + k)$. In other words, almost surely,*

$$\text{rank}(\Xi) \leq 2\beta + k \tag{3.6}$$

for any $\Xi \sim \Gamma_p(\beta, \omega; \sigma)$.

Proof. The case $\beta = (p - 1)/2$ is trivial.

Suppose next, $\beta = 0$, and $\text{rank}(\omega) \geq 1$. Then, also $\Gamma_p(\beta = 0, \omega; 2tI)$ exists, with $\text{rank}(\omega) = 1$, see Lemma 3.5. Let $x \in \bar{\mathcal{S}}_p^+$, then we can write

$$x = \sum_{i=1}^p \mu_i \mu_i^\top, \quad \mu_i \in \mathbb{R}^p$$

Let $\Xi_i \sim \Gamma_p(\beta = 0, \mu_i \mu_i^\top; 2tI)$, for $i = 1, \dots, p$, then by p -fold convolution

$$\Xi = \Xi_1 + \dots + \Xi_p \sim \Gamma_p(0, x; 2tI),$$

and thus we have constructed a transition function of a Wishart semigroup with zero drift, thus violating the drift condition for affine Markov processes on $\bar{\mathcal{S}}_p^+$ [3] (which rules out drifts strictly below $(p - 1)/2$). Thus $\omega = 0$ whenever $\beta = 0$.

Let now $\beta \in \{1/2, \dots, (p - 2)/2\}$, then, since $2\beta + k \geq 2\beta + 1 \geq 2$, there is nothing to show when $p \leq 2$. Set therefore $p \geq 3$. We have

- $\beta' := (p - 1)/2 - \beta$ satisfies $1/2 \leq \beta' \leq (p - 2)/2$.

- Since

$$2 \leq \text{rank}(\omega) = 2\beta + k \leq 2\beta + (p - (2\beta + 1)) = d - 1$$

there exists $\omega' \in \bar{\mathcal{S}}_p^+$ with $\text{rank}(\omega') = (p - 1) - \text{rank}(\omega) = (p - 1) - (2\beta + k)$ and such that $\omega_* := \omega + \omega'$ satisfies $\text{rank}(\omega_*) = p - 1$. Furthermore, since

$$\text{rank}(\omega') = \beta - 1 - (2\beta + k) = 2\beta' - k \leq 2\beta''$$

we conclude that a random variable $Y \sim \Gamma_p(\beta', \omega'; \sigma)$ exists, independent of Ξ .

By convolution, we define $\Xi' = \Xi + Y$, which is $\Gamma_p((p - 1)/2, \omega_*, \sigma)$ distributed. Since $\text{rank}(\omega_*) = p - 1$, Lemma 3.7 applies and states that $\text{rank}(\Xi') = p - 1$, and thus, by Lemma 3.6 (applied exactly $2\beta'$ times, since Y can be constructed by a sum of $2\beta'$ squares of normally distributed vectors) we must have $\text{rank}(\Xi) \leq 2\beta$ almost surely, done. \square

3.2 Proof of the NCGS Conjecture.

The existence of non-central Wishart distribution satisfying the stated rank condition is proved in [14] by quadratic construction (for shape parameters less than p) and by their explicit densities (for shape parameters greater or equals $p - 1$). Conversely, suppose the existence of a single distribution $\Gamma_p(\beta, \omega; I)$. Then by Lemma 3.5, also $\Gamma_p(\beta, 0; I)$ exists. Since the latter is a classical Wishart distribution with non-degenerate scale parameter, we infer $\beta \in W_0$, the classical Gindikin set. Let us assume $\beta \in \{0, 1/2, \dots, (p - 2)/2\}$ and suppose that $\text{rank}(\omega) = 2\beta + k$ and $1 \leq k \leq p - 2\beta$. By Lemma 3.5 we can obtain non-central Wishart distributions for $\Gamma_p(\beta, \omega'; \sigma)$ with any $\text{rank}(\omega') \leq 2\beta + k$ and any invertible σ .

Using, in addition, the support information of Proposition 3.8, we thus obtain a Wishart semigroup $(P_t)_{t \geq 0}$ with state space $D_p(2\beta + k)$ and with drift 2β , by creating $\Gamma_p(\beta, x; 2tI)$, for each $t > 0$, and for each x with $\text{rank}(x) \leq 2\beta + k$. Denote by \mathcal{A} the infinitesimal generator of $(P_t)_{t \geq 0}$.

We distinguish now the following two cases.

- (i) $k < p - 2\beta$. We know that for all $x \in D_p(2\beta + k)$, $e_{2\beta+k+1} \equiv 0$, thus by Proposition 3.4, equation (3.5),

$$0 = \mathcal{A}0 = \mathcal{A}e_{2\beta+k+1}(x) = (p - (2\beta + k))(-\beta - k)e_{2\beta+k}(x) \neq 0, \quad \text{for all } x \text{ with } \text{rank}(x) = 2\beta + k,$$

a contradiction.

- (ii) $k = p - 2\beta$, then $\text{rank}(\omega) = p$. Then $(P_t)_{t \geq 0}$ acts on $C_0(\bar{\mathcal{S}}_p^+)$. The positivity of the Feller semigroup implies that its infinitesimal generator \mathcal{A} satisfies the positive maximum principle. Applied to $e_p(x) = \det(x)$ this implies that

$$\mathcal{A} \det(x_0) \geq 0$$

for any x_0 with $\text{rank}(x_0) < p$. In particular, for rank $p - 1$ matrices x_0 , this yields by Proposition 3.4, equation (3.5) (using $k = p$),

$$\mathcal{A} \det(x_0) = (2\beta - p + 1)e_{p-1}(x_0) \geq 0$$

and since $e_{p-1}(x_0) > 0$, we must have $2\beta \geq p - 1$, done.

3.3 Non-convex states

We conclude the paper with the following characterization of Wishart semigroups on $p \times p$ symmetric positive semi-definite matrices of rank $\leq k$.⁹ The statement has been conjectured by Damir Filipovic [7] in fall 2009.

⁹Note that for $k < p$ these are non-convex domains, but the semigroups on $D_k(p)$ cannot be extended to the convex hull $\bar{\mathcal{S}}_p^+$.

Theorem 3.9. *Let $k \in \{1, \dots, p\}$ and let $\alpha \geq 0$. The following are equivalent:*

- (i) *The Wishart semigroup with state-space $D = D_k(p)$ exists.*
- (ii) *If $k \in \{1, \dots, p-1\}$, then $\alpha = k$, and if $k = p$, then $\alpha \geq p-1$.*

Proof. When $k = p$, that is $D = \bar{\mathcal{S}}_p^+$, we have $\alpha \geq p-1$ due to [3]. We thus confine ourselves to $k < p$.

Proof of (ii) \Rightarrow (i): The existence is shown by construction, using squares. See, for instance, the proof of Theorem 2.3, or [14, Example 2.2 and Example 2.3].

Proof of (i) \Rightarrow (ii): Assume the existence of a Wishart semigroup on $D_k(p)$ ¹⁰. Since e_{k+1} vanishes on $D_k(p)$, we obtain using Proposition 3.4 that

$$0 = (\mathcal{A}e_{k+1})(x) = (p-k)(\alpha-k)e_k(x).$$

Since $k < p$, and $e_k(x) > 0$ for $\text{rank}(x) = k$ matrices, we conclude that $\alpha = k$. □

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¹⁰Using the NCGS conjecture, the following, weaker, conclusion may be made. Assume the existence of a Wishart semigroup on $D_k(p)$. Then $\Gamma_p(\alpha, x_0, I)$ exists with $\text{rank}(x_0) = k$. By the NCGS Conjecture, we must have $\alpha/2 \in W_0$ and, if $\alpha < p-1$, then $\text{rank}(x_0) \leq \alpha$. This implies $\alpha \geq k$.

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