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Reliability Assessment of Systems Subject to Dependent Degradation Processes and Random Shocks

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Abstract –System failures can be induced by internal degradation mechanisms or by external causes. In this paper, we consider the reliability of systems experiencing both degradation and random shock processes. The dependencies between degradation processes and random shocks, and among degradation processes are explicitly modelled. The degradation processes of system components are modeled by multi-state models (MSMs) and physics-based models (PBMs). The piecewise-deterministic Markov process modeling framework is employed to combine MSMs and PBMs, and for incorporating degradation and random shocks dependencies. The Monte Carlo simulation and finite-volume methods are used to compute the system reliability. A subsystem of a residual heat removal system in a nuclear power plant is considered as illustrative case.

Key Words –multi-state system, system reliability assessment, degradation, random shocks, dependency, piecewise-deterministic Markov process, Monte Carlo simulation, residual heat removal system.

Acronyms

PBMs	Physics-based models
MSMs	Multi-state models
PDMP	Piecewise-deterministic Markov process
MCS	Monte Carlo simulation

FV	Finite-volume
RHRS	Residual heat removal system

Notations

K	Group of degradation processes modeled by MSMs
L	Group of degradation processes modeled by PBMs
S_{K_n}	Finite state set of degradation process K_n
$Y_{K_n}(t)$	State variable of degradation process K_n
$\lambda_i(j \theta_{K_n})$	Transition rate from state i to j
$\mathcal{F}_{K_n}$	Set of failure states of degradation process K_n
$X_{L_m}(t)$	Time-dependent continuous variables of degradation process L_m
$X_{L_m}^D(t)$	Non-decreasing degradation variables vector
$X_{L_m}^P(t)$	Physical variables vector
$\mathcal{F}_{L_m}$	Set of failure states of degradation process L_m
$N(t)$	Number of random shocks occurred until time t
μ	Arrival rate of random shock process
$Y'_{K_n}(t)$	Degradation level of K_n considering random shocks
$\mathcal{F}'_{K_n}$	Set of failure states of $Y'_{K_n}(t)$
W_i	Shock load of the i -th shock
D	Maximal material strength
H_i	Instantaneous random increase caused by the i -th cumulative shock
$N^c(t)$	Number of cumulative shocks occurred until time t
D_{L_m}	Degradation level of L_m considering random shocks

$Y(t)$	Degradation state of the processes of set K
$X(t)$	Degradation state of the processes of set L
θ_K	Environmental and operational factors in K
θ_L	Environmental and operational factors in L
$Z(t)$	Degradation process of the system
$Y'(t)$	Degradation state of the processes of set K considering random shocks
$X'(t)$	Degradation state of the processes of set L considering random shocks
T_k	k -th jump time in $Y'(t)$
$Z_k = (X'_k, Y'_k)$	State of $Z(t), X'(t), Y'(t)$ after k -th jump of $Y'(t)$
$f_L^{Y'(t)}(X(t) \theta_L)$	Deterministic physics equations of $X(t)$
$\lambda_{y_i, y_j}(x \theta_K)$	Transition rate of $Y'(t)$ from state y_i to y_j
$N(i, (dx, y_j), dt)$	Semi-Markov kernel of $\{Z_n, T_n\}_{n \geq 0}$
$dF_i(t)$	Probability distribution of holding time given $Z_k = i$

1. INTRODUCTION

The evaluation of the system reliability over time is an important and critical task. For example, the reliability of safety systems in nuclear power plants, such as reactor shutdown, emergency core cooling systems and other safety multi-component systems in nuclear industry, need to meet safety requirements imposed by regulation to ensure their operational safety (Marseguerra, *et al.*, 2004). The instances when the requirements are not satisfied can be identified by reliability analysis. Reliability improvement actions can, then, be performed by design on maintenance, to avoid possible human and economic losses. In this paper, we investigate the reliability assessment of multi-component systems subject to dependent degradation processes influenced by random shocks. The dependencies pose challenging issues in system reliability modeling and assessment (Schöttl, 1996) (e.g. for micro-electromechanical systems, which are complex systems experiencing dependent component failure processes and multiple dependent competing failure processes for each component (Song, *et al.*, 2014b)). System failures can be induced by internal degradation mechanisms (e.g. wear, fatigue and erosion) or by external causes (e.g. thermal and mechanical shocks) (Jiang, *et al.*, 2015). The

interactions between these factors need to be considered under certain circumstances, e.g. when degradation processes and random shocks are s-dependent (e.g. single-event overloads with safe shock magnitudes can influence the fatigue crack growth of stents by causing instantaneous increase on the crack propagation (Keedy and Feng, 2013)), or the degradation state of some components in one system can influence the degradation dynamics or the remaining useful life of the others (e.g. the degradation of the pre-filtration stations leading to a lower performance level of the sand filter in a water treatment plant (Rasmekomen and Parlikad, 2013)). Neglecting these aspects may result in overestimation of system reliability (Wang and Pham, 2012).

In industrial systems, many critical components (e.g. valves and pumps in the nuclear and aerospace industries) are designed to be highly reliable, for which statistical degradation/failure data are often limited. In this case, multi-state models (MSMs) (Black, *et al.*, 2005, Giorgio, *et al.*, 2011, Moghaddass and Zuo, 2014) and physics-based models (PBMs) (Chookah, *et al.*, 2011, Daigle and Goebel, 2011, Wang, *et al.*, 2014) can be used to describe the evolution of degradation in components and systems. A MSM describes the degradation process in a discrete way, supported by material science knowledge (Lin, *et al.*, 2015b) and/or available but limited degradation/failure historical data from field collection or degradation tests (Giorgio, *et al.*, 2011). On the contrary, a PBM gives an integrated mechanistic description of the component life consistent with the underlying real degradation mechanisms under operating conditions (Hu, *et al.*, 2014), by using physics knowledge modeled by corresponding mathematical equations (Chookah, *et al.*, 2011). In practice, degradation models of different nature have to be applied depending on the available information of the degradation processes. Recently, the piecewise-deterministic Markov process (PDMP) modeling framework has been employed to incorporate PBMs and MSMs, and to treat the dependencies among degradation processes but without considering the influences of random shocks (Lin, *et al.*, 2015a). On the other hand, random shocks can accelerate the degradation processes (e.g. internal thermal shocks and water hammers onto power plant components (Salonen, *et al.*, 2007)).

The reliability of systems experiencing both degradation and random shocks is a problem that has been widely studied (Becker, *et al.*, 2002, Jiang, *et al.*, 2012, Lin, *et al.*, 2015b, Peng, *et al.*, 2010, Rafiee, *et al.*, 2014, Song, *et al.*, 2014a, Song, *et al.*, 2014b, Wang and Pham, 2012, Yang, *et al.*, 2011). The dependency among these processes leading to failure has posed some challenges to reliability modeling. A literature review is presented below, to position our contributions within the existing works. Previous research has focused on the dependency between one type of degradation processes (continuous or multi-state) and random shocks. For continuous degradation processes, Peng, *et al.* (2010) considered systems with one linear degradation path where shocks can bring additional abrupt degradation damage if the shock loads do not exceed the maximum strength of the material; multi-component systems subject to multiple linear degradation paths have been further considered by Song, *et al.* (2014b); Jiang, *et al.* (2012) considered changes in the maximal strength of the material when systems are deteriorating under different situations; Becker, *et al.* (2002) extended

the theory of dynamic reliability to incorporate random changes of the degradation variables due to random shocks; Ye, *et al.* (2011) considered the destructive power of a shock depending not only on the shock magnitude but also on the state of the system; Wang, *et al.* (2011) considered two types of the effects of shocks: a sudden increase in the failure rate after a shock and a direct random change in the degradation after the occurrence of a shock; Rafiee, *et al.* (2014) proposed reliability models for systems for which the degradation path has a changing degradation rate according to particular random shock patterns; Song, *et al.* (2014a) studied random shocks with specific sizes or functions, which can selectively affect the degradation processes of one or more components (not necessarily all components) in one system. For multi-state degradation processes, Yang, *et al.* (2011) combined random shocks with Markov degradation models where shocks can lead the systems to further degraded states; Lin, *et al.* (2015b) integrated random shocks into multi-state physics models of degradation processes where the influences of shocks are dependent on the current degradation condition; Ruiz-Castro (2016) considered external shocks which could produce several effects; extreme failure, cumulative damage and when the damage reaches a threshold state, a non-repairable failure occurs, and changes in the internal performance of the device. Note that no work has considered systems with both continuous and multi-state degradation processes and subject to random shocks, and few studies have explicitly considered both the dependencies between degradation processes and random shocks, and among degradation processes themselves. Wang and Pham (2012) employed copula to handle these two types of dependencies; however, sufficient degradation/failure data is required to determine the copula function through statistical inference.

In this paper, we extend the PDMP modeling framework for system reliability assessment, considering not only the dependencies among degradation processes but also the impacts of random shocks. To the best knowledge of the authors, this is the first work investigating systems with both continuous and multi-state degradation processes, subject to random shocks and considering the dependencies between degradation processes and random shocks, and among degradation processes. Since the analytical solution is difficult to obtain due to the complexity of the system being considered, we employ two numerical approaches to assess system reliability: the Monte Carlo (MC) simulation (Marseguerra and Zio, 1996) and the finite-volume (FV) (Coccozza-Thivent, *et al.*, 2006) methods.

The remainder of this article is organized as follows. Section 2 provides the assumptions and descriptions of the degradation processes and random shocks. Section 3 presents the extended model for systems with degradation and random shock processes, considering their dependencies. The proposed MC simulation and FV methods are presented in Section 4. Section 5 presents an illustrative study taken from a real residual heat removal system (RHRS) operated by Électricité de France (EDF). The RHRS is used for cooling the reactor during and following shutdown, contributing to safety by removing heat from the core and transferring it to the environment. Specifically, we consider an important subsystem consisting of a pneumatic valve and a centrifugal pump in series. This system

setting is widely used, under different conditions, in a variety of domains for fluid delivery (from water supply to spacecraft fueling systems) (Daigle and Goebel, 2011, Daigle and Goebel, 2013). Numerical results and analysis are presented in Section 6. Section 7 concludes the work.

2. ASSUMPTIONS AND MODEL DESCRIPTIONS

We consider a multi-component system. Each component may be affected by multiple degradation mechanisms or processes, possibly dependent. The degradation processes can be separated into two groups: (1) $\mathbf{K}$ consists of processes fit to be modeled by MSMs; (2) $\mathbf{L}$ consists of processes fit to be modeled by PBMs.

2.1. Degradation models

2.1.1. MSMs

We follow the assumptions on MSMs made in Lin, *et al.* (2014):

- A degradation process, $Y_{K_n}(t)$, $K_n \in \mathbf{K}$ of group (1), takes values from a finite state set denoted by $\mathcal{S}_{K_n} = \{0, 1, \dots, d\}$, where d is the perfect functioning state and 0 is the complete failure state. The component is functioning or partially functioning in the intermediate degradation states. The transition rates $\lambda_i(j | \boldsymbol{\theta}_{K_n})$, $\forall i, j \in \mathcal{S}_{K_n}, i > j$ characterize the degradation transition probabilities from state i to state j , where $\boldsymbol{\theta}_{K_n}$ represents the environmental factors relevant to K_n and the related parameters of λ_{K_n} . We follow the assumption of Markov property made in Giorgio, *et al.* (2011), Yeh (1997) and Zuo, *et al.* (2000); Markov processes are widely used to describe components degradation processes. The transition rates between different degradation states are estimated from the degradation and/or failure data from historical field collection. The failure state set of the process K_n is denoted by $\mathcal{F}_{K_n} = \{0\}$.

2.1.2. PBMs

We follow the assumptions on PBMs made in Lin, *et al.* (2014):

- A degradation process $\mathbf{X}_{L_m}(t)$, $L_m \in \mathbf{L}$ of group (2), has d_{L_m} time-dependent continuous variables, whose evolution is characterized by a system of first-order differential equations $\dot{\mathbf{X}}_{L_m}(t) = \mathbf{f}_{L_m}(\mathbf{X}_{L_m}(t), t | \boldsymbol{\theta}_{L_m})$, i.e. physics equations, where $\boldsymbol{\theta}_{L_m}$ represents the environmental factors influential to L_m (e.g. temperature and pressure) and the parameters used in $\mathbf{f}_{L_m}$. This assumption is made in Lorton, *et al.* (2013) and widely used (Daigle and

Goebel, 2011, Daigle and Goebel, 2013). Note that higher-order differential equations can be converted into a system of first-order differential equations by introducing extra variables(Zwillinger, 1998).

- $\mathbf{X}_{L_m}(t) = (\mathbf{X}_{L_m}^D(t), \mathbf{X}_{L_m}^P(t))$ contains: (1) the non-decreasing degradation variables $\mathbf{X}_{L_m}^D(t)$ (e.g. leak area) describing the degradation process, where $\mathbf{D}$ is the set of degradation variables indices (the same assumption has been already widely used(Daigle and Goebel, 2011, Daigle and Goebel, 2013, Keedy and Feng, 2013)); (2) the physical variables $\mathbf{X}_{L_m}^P(t)$ (e.g. velocity and force), which influence $\mathbf{X}_{L_m}^D(t)$, where $\mathbf{P}$ is the set of physical variable indices. The generic degradation process L_m reaches failure when one $x_{L_m}^i(t) \in \mathbf{X}_{L_m}^D(t)$ reaches or exceeds its corresponding failure threshold denoted by $x_{L_m}^{i*}$. The failure state set of the process L_m is denoted by $\mathcal{F}_{L_m}$.

2.2. Random shocks

Random shocks can influence the degradation processes of the components. The following assumptions are made, similarly to various previous works(Jiang, *et al.*, 2012, Peng, *et al.*, 2010, Rafiee, *et al.*, 2014, Song, *et al.*, 2014a, Yang, *et al.*, 2011).

- Random shocks occur in time according to a homogeneous Poisson process $\{N(t), t \geq 0\}$ with constant arrival rate μ (Fig. 1), where the random variable $N(t)$ denotes the number of random shocks occurred until time t .
- The damages of random shocks are divided into two types: extreme and cumulative.
- Extreme and cumulative shocks are mutually exclusive.
- Extreme shocks immediately lead the components to failure, whereas cumulative shocks gradually deteriorate the components.

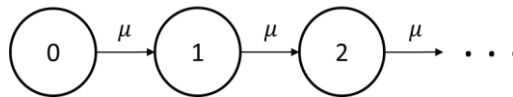


Fig. 1. Random shock process

3. DEPENDENT DEGRADATION PROCESSES AND RANDOM SHOCKS

3.1. Dependency between degradation processes and random shocks

Due to the different nature of PBMs and MSMs, the impacts of random shocks on the two groups

of components are characterized in different ways.

3.1.1. Impacts on MSMs

In the generic degradation process $K_n \in \mathbf{K}$, random shocks can cause the process variable $Y_{K_n}(t)$ to step from state i to a further degraded state j with probability $p_{ij}, i > j$ (Yang, *et al.*, 2011), with p_{i0} denoting the probability that the random shock is extreme, i.e. leading to failure state 0 upon occurrence from state $Y_{K_n}(t) = i$. By combining the original degradation and the random shock processes, the resulting process $\mathbf{Y}'_{K_n}(t)$ is a homogeneous continuous-time Markov chain of the type depicted in Fig. 2. Each layer indicates one degradation state of $Y_{K_n}(t)$, and the numbers in each layer indicate the numbers of shocks experienced up to time t in the process K_n , denoted by k . The state of $\mathbf{Y}'_{K_n}(t)$ is, then, represented by the pair $(Y_{K_n}(t), k)$. The transitions represented by solid lines are due to the original degradation process, characterized by the original transition rates, which do not influence the value of k . The transitions represented by dotted lines are due to random shocks, which cause k to be increased by one. $\mu p_{ij}, i > j$ is the rate of occurrence of a shock which will cause the process stepping to the j -th layer from the i -th layer. Note that K_n fails whenever $Y_{K_n}(t)$ reaches the degradation state 0, no matter how many shocks it has experienced. Therefore, the space of the failure states of $\mathbf{Y}'_{K_n}(t)$ is denoted by $\mathcal{F}'_{K_n} = \{(0, b), \forall b \in \mathbb{N}\}$. The state space of $\mathbf{Y}'_{K_n}(t)$ is denoted by $\mathcal{S}'_{K_n} = \{(a, b), \forall a \in \mathcal{S}_{K_n}, b \in \mathbb{N}\}$.

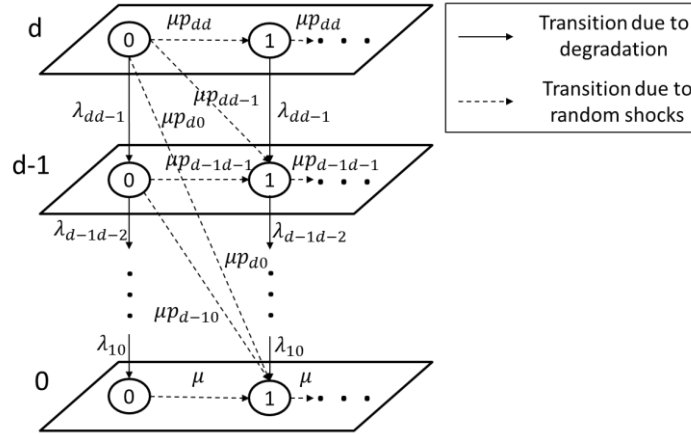


Fig. 2. Degradation process K_n and random shocks.

3.1.2. Impacts on PBMs

In the generic degradation process $L_m \in \mathbf{L}$, the i -th shock becomes extreme if the shock load W_i exceeds the maximal material strength D , otherwise, it can bring an instantaneous random increase H_i

to $\mathbf{X}_{L_m}(t)$ (Song, *et al.*, 2014b). The overall degradation level of L_m is expressed as follows:

$$\mathbf{D}_{L_m}(t) = \begin{cases} \mathbf{X}_{L_m}(t) + \sum_{i=1}^{N^c(t)} \mathbf{H}_i, & \text{if } N'(t) \neq 0 \\ \mathbf{X}_{L_m}(t), & \text{if } N'(t) = 0 \end{cases} \quad (1)$$

where $N^c(t)$ is the number of cumulative shocks occurred in the developing L_m process before the extreme shock occurs until time t . The process L_m leads to failure if $\mathbf{D}_{L_m}(t)$ reaches the predefined failure state set $\mathcal{F}_{L_m}$ or a shock with load larger than D occurs. An example of degradation process L_m considering random shocks is shown in Fig. 3, where W_i is the shock load of the i -th shock occurred at time t_i , $i = 1, 2, 3$. The center figure in Fig. 3 represents the evolution of the physical variable (e.g. velocity and force), which can influence the degradation variable (top figure) and may also be influenced by random shocks (bottom figure).

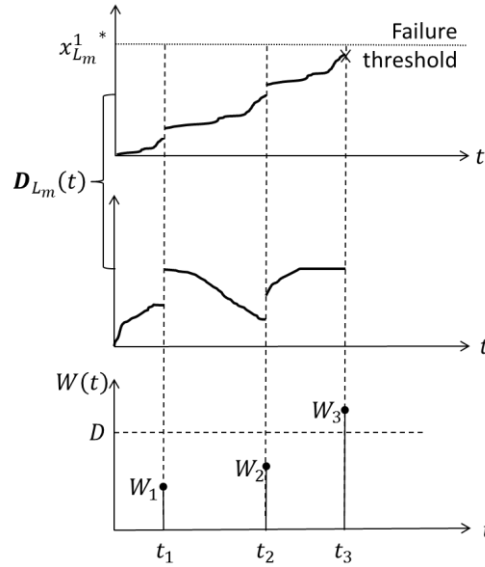


Fig. 3. An example of degradation process L_m with random shocks. Top Figure: degradation variable; Center Figure: physical variable; Bottom Figure: random shock process.

3.2. Dependency among degradation processes

Dependencies may exist among degradation processes within each group and between the two groups. The degradation states of the processes of set $\mathbf{K}$ may influence the evolution of the continuous variables of the degradation processes of set $\mathbf{L}$, and the degradation levels of the latter may influence the transition times and transition directions of the former (the detailed formulations are shown in eqs. (2) and (3)) (Lin, *et al.*, 2015a).

Let $\mathbf{Y}(t) = (Y_{K_1}(t), \dots, Y_{K_N}(t)) \in \mathcal{S} = \{0, 1, \dots, d_S\}$ and $\mathbf{X}(t) = (\mathbf{X}_{L_1}(t), \dots, \mathbf{X}_{L_M}(t)) \in \mathbb{R}^{d_L}$. The evolution of $\mathbf{Y}(t)$ is governed by the transition rates which depend on the states of the degradation processes in the first group $\mathbf{X}(t)$ and also in the second group $\mathbf{Y}(t)$, as follows:

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} P \left(Y(t + \Delta t) = j | X(t), Y(t) = i, \theta_K = \bigcup_{n=1}^N \theta_{K_n} \right) / \Delta t \\ = \lambda_i(j | X(t), \theta_K), \forall t \geq 0, i, j \in S, i \neq j \end{aligned} \quad (2)$$

The evolution of $X(t)$ is described by mathematical equations representing the underlying physics and depends on the states of the degradation processes in the second group $Y(t)$ and also in the first group $X(t)$, as follows:

$$\begin{aligned} \dot{X}(t) = (\dot{X}_{L_1}(t), \dots, \dot{X}_{L_M}(t)) = (f_{L_1}^{Y(t)}(X(t), t | \theta_{L_1}), \dots, f_{L_M}^{Y(t)}(X(t), t | \theta_{L_M})) \\ = f_L^{Y(t)}(X(t), t | \theta_L = \bigcup_{m=1}^M \theta_{L_m}) \end{aligned} \quad (3)$$

3.3. PDMPs for systems subject to degradation dependency and random shocks

Let $Z(t)$ denote the overall degradation process of the system:

$$Z(t) = (X'(t) = (D_{L_1}(t), \dots, D_{L_M}(t)), Y'(t) = (Y(t), N(t))) \in E = \mathbb{R}^{d_L} \times S' \quad (4)$$

where E is a space combining $\mathbb{R}^{d_L}$ and $S' = S \times \mathbb{N}$. Let $T_k, k \in \mathbb{N}$ denote the k -th jump time in $Y'(t)$ and $Z_k = Z(T_k) = (X'(T_k), Y'(T_k)) = (X'_k, Y'_k)$. The evolution of $Z(t)$ between two consecutive jumps of $Y'(t)$, between which no shock occurs to the system and the degradation state does not change, can be written as follows:

$$\begin{aligned} \dot{Z}(t) = (\dot{X}'(t), \dot{Y}'(t)) \\ = (f_L^{Y'(t)}(X(t) | \theta_L), (0, 0)), \text{ for } t \in [T_k, T_{k+1}[\end{aligned} \quad (5)$$

According to the definition in Coccozza-Thivent (2011), $Z(t)$ is a PDMP since (1) it can be written as $Z(t) = \varphi(Z_k, t - T_k)$, for $t \in [T_k, T_{k+1}[$ and φ satisfies $\varphi(y, t + s) = \varphi(\varphi(y, t), s)$, $\forall t, s \geq 0, y \in E$, and $t \rightarrow \varphi(y, t)$, $\forall t \geq 0, y \in E$ is right continuous with left limits and (2) $\{Z_n, T_n\}_{n \geq 0}$ is a Markov renewal process defined on the space $E \times \mathbb{R}^+$. The probability that $Z(t)$ will step to state j from state Z_k in the time interval $[T_k, T_k + t]$, given $\{Z_i, T_i\}_{i \leq k}$ is as follows:

$$\begin{aligned} P[Z_{k+1} = j, T_{k+1} \in [T_k, T_k + t] | \{Z_i, T_i\}_{i \leq k}] = P[Z_{k+1} = j, T_{k+1} \in [T_k, T_k + t] | Z_k], \\ \forall k \in \mathbb{N}, j \in E, j \neq Z_k \end{aligned} \quad (6)$$

$\{Z_n, T_n\}_{n \geq 0}$ is characterized by the semi-Markov kernel $N(i = (x_i, y_i), (dx, y_j), dt) = P[X'_{k+1} \in [x, x + dx], Y'_{k+1} = y_j, T_{k+1} - T_k \in [t, t + dt] | Z_k = i], \forall k \in \mathbb{N}, y_i, y_j \in S', x_i, dx \in \mathbb{R}^{d_L}, dx \rightarrow 0, dt \rightarrow 0$, which can be reformulated as follows:

$$\begin{aligned} N(i = (x_i, y_i), (dx, y_j), dt) \\ = P[X'_{k+1} \in [x, x + dx], Y'_{k+1} = y_j | T_{k+1} - T_k \in [t, t + dt], Z_k = i] \\ \cdot P[T_{k+1} - T_k \in [t, t + dt] | Z_k = i] \\ = Q(\varphi(i, t), (dx, y_j)) dF_i(t) \end{aligned} \quad (7)$$

where $Q(\varphi(i, t), (dx, y_j))$ is the probability distribution of state Z_{k+1} given $T_{k+1} - T_k = t$ and

$\mathbf{Z}_k = \mathbf{i}$ and $dF_i(t)$ is the probability distribution of $T_{k+1} - T_k$ given $\mathbf{Z}_k = \mathbf{i}$. $Q(\varphi(\mathbf{i}, t), (d\mathbf{x}, \mathbf{y}_j))$ can be reformulated as follows:

$$\begin{aligned} & Q(\varphi(\mathbf{i} = (\mathbf{x}_i, \mathbf{y}_i), t), (d\mathbf{x}, \mathbf{y}_j)) \\ &= P[\mathbf{X}'_{k+1} \in [\mathbf{x}, \mathbf{x} + d\mathbf{x}], \mathbf{Y}'_{k+1} = \mathbf{y}_j \mid T_{k+1} - T_k \in [t, t + dt], \mathbf{Z}_k = \mathbf{i}] \\ &= P[\mathbf{X}'_{k+1} \in [\mathbf{x}, \mathbf{x} + d\mathbf{x}] \mid \mathbf{Y}'_{k+1} = \mathbf{y}_j, T_{k+1} - T_k \in [t, t + dt], \mathbf{Z}_k = \mathbf{i}] \\ &\quad \cdot P[\mathbf{Y}'_{k+1} = \mathbf{y}_j \mid T_{k+1} - T_k \in [t, t + dt], \mathbf{Z}_k = \mathbf{i}] \end{aligned} \quad (8)$$

Let $p_t(d\mathbf{z} = (d\mathbf{x}, \mathbf{y}_i))$ denote the probability distribution of $\mathbf{Z}(t)$, which obeys the Chapman-Kolmogorov equation (Davis, 1993) as follows:

$$\begin{aligned} & \int_0^t \sum_{\mathbf{y}_i \in \mathcal{S}'} \int_{\mathbb{R}^{d_L}} \sum_{\mathbf{y}_j \in \mathcal{S}'} \lambda_{\mathbf{y}_i, \mathbf{y}_j}(\mathbf{x} \mid \boldsymbol{\theta}_K) \left(\int_{\mathbb{R}^{d_L}} \psi(\mathbf{y}_j, \mathbf{y}) \mu(\mathbf{y}_i, \mathbf{y}_j, \mathbf{x})(d\mathbf{y}) - \psi(\mathbf{y}_i, \mathbf{x}) \right) p_s(d\mathbf{x}, \mathbf{y}_i) ds + \\ & \int_0^t \sum_{\mathbf{y}_i \in \mathcal{S}'} \int_{\mathbb{R}^{d_L}} \mathbf{f}_L^{\mathbf{y}_i}(\mathbf{x} \mid \boldsymbol{\theta}_L) \text{div}(\psi(\mathbf{y}_i, \mathbf{x})) p_s(d\mathbf{x}, \mathbf{y}_i) ds - \\ & \sum_{\mathbf{y}_i \in \mathcal{S}'} \int_{\mathbb{R}^{d_L}} \psi(\mathbf{y}_i, \mathbf{x}) p_t(d\mathbf{x}, \mathbf{y}_i) + \sum_{\mathbf{y}_i' \in \mathcal{S}'} \int_{\mathbb{R}^{d_L}} \psi(\mathbf{y}_i, \mathbf{x}) p_0(d\mathbf{x}, \mathbf{y}_i) = 0 \end{aligned} \quad (9)$$

where $\lambda_{\mathbf{y}_i, \mathbf{y}_j}(\mathbf{x} \mid \boldsymbol{\theta}_K)$ is the transition rate of $\mathbf{Y}'(t)$ from state $\mathbf{y}_i$ to $\mathbf{y}_j$, $\psi(\cdot, \cdot)$ is any continuously differentiable function from $\mathcal{S}' \times \mathbb{R}^{d_L}$ to $\mathbb{R}$ with a compact support and $\mu(\mathbf{y}_i, \mathbf{y}_j, \mathbf{x})(d\mathbf{y})$ is the probability of $\mathbf{X}'(t) \in [\mathbf{y}, \mathbf{y} + d\mathbf{y}]$ after jumping from $\mathbf{x}$ when $\mathbf{Y}'(t)$ steps to state $\mathbf{y}_j$ from state $\mathbf{y}_i$.

The reliability of the system at time t is defined as follows:

$$R(t) = P[\mathbf{Z}(s) \notin \mathcal{F}, \forall s \leq t] = \int_{\mathbf{z} \notin \mathcal{F}} p_t(d\mathbf{z}) \quad (10)$$

where $\mathcal{F}$ is the space of the failure states of the system.

In the general modeling framework, the temporal variability is considered as follows: (1) the randomness in MSMs could imply the temporal variability of the degradation processes modeled by PBMs, as shown in Figs. (3) and (8); (2) in PBMs, the temporal variability of the degradation variables can be attributed to the time-varying physical variables associated with the underlying degradation mechanisms.

The parameters in the proposed model are mainly divided into three groups: (1) transition rates in multi-state models; (2) parameters in physics equations of physics-based models and (3) parameters characterizing random shock processes. The values of the first group can be estimated, using degradation and/or failure data from historical field collection or degradation tests, through maximum likelihood estimation for complete or incomplete data (Lisnianski, *et al.*, 2010, Ogurtsova, 2014); they can also be estimated by domain experts using physics knowledge (e.g. the values of the transition rates in multi-state physics model (Lin, *et al.*, 2015b)) are described by physics equations). For the second group, the laws of physics are used to build the equations describing the development of the underlying degradation mechanisms (e.g. fatigue, wear, corrosion, etc.) (Daigle and Goebel, 2011); the related parameter values can be estimated through regression models using degradation and/or failure data. For example, the physics equations of the fatigue cracking of the seal are built according

to Paris-Erdogan law (Newby, 1991), which relates the stress intensity factor range to the crack growth under a fatigue stress regime; the values of the parameters are estimated through least squares regression methods by using data on crack length and cycles. The values of the third group can be estimated using related degradation and/or failure data obtained from historical field collection or shock tests (Chan and Englert, 2001), using likelihood-based inference or regression models (Ye, *et al.*, 2011). For example, the Brown-Proshan model is employed to model wear and shock processes of tire treads (Ye, *et al.*, 2011), whereby the likelihood function can be derived based on cumulative hazard function and the parameter values are estimated through maximum likelihood estimation.

4. SYSTEM RELIABILITY ASSESSMENT UNDER DEPENDENT DEGRADATION AND RANDOM SHOCK PROCESSES

The analytical solution of $R(t)$ is difficult to obtain mainly due to the complex PDMPs used to model the dependent degradation and random shock processes (Labeau, 1996). Therefore, we consider the following two approximate methods: the MC simulation method (Marseguerra and Zio, 1996) based on the semi-Markov kernel of $\{\mathbf{Z}_n, T_n\}_{n \geq 0}$ (eq. (7)) and the FV method (Cocozza-Thivent, *et al.*, 2006) based on the Chapman-Kolmogorov equation (eq. (9)). They are two widely used approaches for solving PDMPs to evaluate reliability quantities. The FV method approximates the probability density function of PDMPs by discretizing the state space of the continuous variables and the time space. It is a method that can lead to comparable results as the MC simulation method, using less computing time for low dimensional problems (Eymard and Mercier, 2008). However, it is typically unsuited for high-dimensional problems or problems with complex equations describing the deterministic evolution. Besides, it is relatively more difficult to implement than the MC simulation method.

4.1. MC simulation method

The MC simulation method to compute the system reliability at time t consists of replicating several times the lifeprocess of the system by repeatedly sampling its holding time and arrival state from the corresponding probability distributions. Each replication continues until the time of system evolution reaches t or until the system enters a state in the failure set $\mathcal{F}$. The procedure of the MC simulation method is as follows:

Set N_{max} (the maximum number of replications) and $k = 0$ (index of replication)

Set $k' = 0$ (number of replication that end in a system failure state)

While $k < N_{max}$

Initialize the system by setting $\mathbf{Z} = (\mathbf{X}'(0), \mathbf{Y}'(0))$ (initial system state), and the time $T = 0$ (initial system time)

Set $t' = 0$ (state holding time)

While $T < t$

Sample a t' by using the probability distribution $dF_Z(t)$

Sample an arrival state y for stochastic process $Y'(t)$ and an arrival state x for process $X'(t)$ by using eq. (8)

Set $T = T + t'$

If $T \leq t$

Set $Z = (x, y)$

If $Z \in \mathcal{F}$

Set $k' = k' + 1$

Break

End if

Else(when $T > t$)

If $\varphi(Z, t + t' - T) \in \mathcal{F}$

Set $k' = k' + 1$

Break

End if

End if

End While

Set $k = k + 1$

End While □

The estimated system reliability at time t can be obtained by

$$\widehat{R}_{MC}(t) = 1 - k' / N_{max} \quad (11)$$

where k' represents the number of trials that end in the failure state of the system, and the sample variance (Lewis and Böhm, 1984) is:

$$var_{\widehat{R}_{MC}(t)} = \widehat{R}_{MC}(t)(1 - \widehat{R}_{MC}(t)) / (N_{max} - 1) \quad (12)$$

The MC simulation method is widely used in practice to evaluate system reliability (Zio, 2013). It is based on the strong law of large numbers and the central limit theorem and provides an unbiased estimator (Zio, 2013). The error on the estimate can be controlled within a confidence interval built based on the sample variance given in eq. (12), which can guarantee the consistency of the estimate. The accuracy of the MC simulation method increases as the number of replications increases. The MC simulation method is efficient in solving high-dimensional problems, since the sample variance does not depend on the number of dimensions. There are techniques to further improve the efficiency of MC simulation method (such as importance sampling, sequential MC, linear sampling, subset sampling, etc.) (Zio, 2013), which have to be designed according to the specific problems and have not been considered in our general reliability assessment framework.

4.2. FV method

The FV method is an alternative for the approximated solution of the system reliability, based on a discretization of the state space of the continuous variables and time space (Eymard and Mercier, 2008). Here, we employ an explicit FV scheme developed by Coccozza-Thivent *et al.* (Coccozza-Thivent, *et al.*, 2006). The numerical scheme aims at constructing an approximate value $\rho_t(\mathbf{x}, \mathbf{y}_i) d\mathbf{x}$ for $p_t(d\mathbf{x}, \mathbf{y}_i)$. The estimated system reliability at time t , then, can be calculated as follows:

$$\widehat{R}_{FV}(t) = \int_{\mathbf{z} \in \mathcal{F}} \rho_t(\mathbf{z}) d\mathbf{z} \quad (13)$$

Appendix A contains a detailed description of the FV method. Due to the complexity of the Chapman-Kolmogorov equation (eq. (9)), there is no explicit expression for the variance or uncertainty associated with the estimate. However, the convergence of the method is proven in Coccozza-Thivent, *et al.* (2006) under the condition that $\Delta t \rightarrow 0$ and $|\mathcal{M}|/\Delta t \rightarrow 0$, where $|\mathcal{M}|$ is the space step and Δt is the time step. The efficiency and the accuracy of the method have been shown through the numerical example in Coccozza-Thivent, *et al.* (2006).

5. CASE STUDY

We consider a subsystem of a residual heat removal system (RHRS) in a nuclear power plant, which consists of a pneumatic valve and a centrifugal pump in series shown in Fig. 4.

For the degradation model of the pump, we consider a MSM modified from the one originally supplied by EDF (Lin, *et al.*, 2015a), while for the valve we take the PBM proposed in Daigle and Goebel (2011).

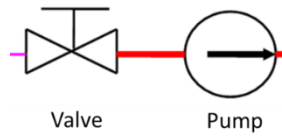


Fig. 4. Subsystem of RHRS, consisting of a centrifugal pump and a pneumatic valve.

5.1. Centrifugal pump

The degradation process of the pump is modeled by a four-state, continuous-time, homogeneous Markov chain as shown in Fig. 5.

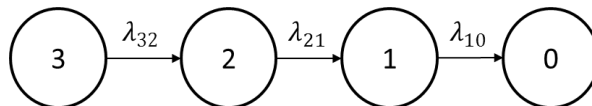


Fig. 5. Degradation process of the pump.

Among the four states of the pump, state3 is the perfect functioning state and state0 is the complete failure state. Let $Y_p(t)$ denote the degradation state of the pump at time t and $\mathcal{S}_p = \{3, 2, 1, 0\}$ denote the degradation states set. The pump is functioning until $Y_p(t) = 0$. The parameters λ_{32} , λ_{21} and λ_{10} are the transition rates between the degradation states, estimated from the available degradation and/or failure data. The pump vibrates when it reaches the degradation states 2 and 1; the intensity of the vibration of the pump on states 2 and 1 is evaluated by the experts as ‘smooth’ and ‘rough’, respectively. The set of the failure states of the pump is $\mathcal{F}_p = \{0\}$.

5.2. Pneumatic valve

The simplified scheme of the pneumatic valve is shown in Fig. 6. The degradation of the valve is the external leak at the actuator connections to the bottom pneumatic port due to corrosion, and is modeled by a PBM due to limited statistical degradation data on the valve behavior. It is much more significant than the other degradation mechanisms according to the results shown in Daigle and Goebel (2011).

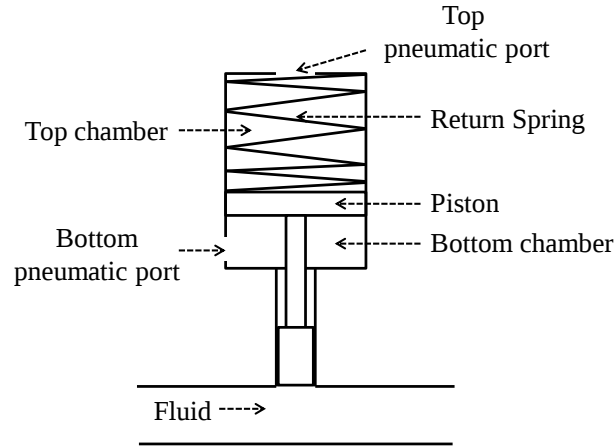


Fig.6. Simplified scheme of the pneumatic valve (Daigle and Goebel, 2011).

Let $D_b(t)$ denote the area of the leak hole at the bottom pneumatic port of the valve at time t . The development of the leak size is described by $\dot{D}_b(t) = \omega_b$, where ω_b is the original wear coefficient. The valve is considered failed when the size of the external leak exceeds a predefined threshold D_b^* . The set of the failure states of the valve is $\mathcal{F}_v = [D_b^*, +\infty)$.

5.3. Dependency between degradation processes

Dependency in the degradation processes of the two components has been indicated as a relevant problem by the experts of EDF: the pump vibrates due to degradation (Zhang, *et al.*, 2006) which, in turn, leads the valve to vibrate, aggravating its own degradation processes (Moussou, *et al.*, 2001). The development of the leak size of the valve is, then, reformulated as follows (Lin, *et al.*, 2015a):

$$\dot{D}_b(t) = \omega_b(1 + \beta(Y_p(t))) \quad (14)$$

where $\beta(Y_p(t))$ is the function indicating the relative increment of the growth rate of the external leak caused by the vibration of the pump at the degradation state $Y_p(t)$.

5.4. Random shocks

According to the experts of EDF, random shocks like water hammers and internal thermal shocks (Salonen, *et al.*, 2007) can worsen the degradation condition of both components of the subsystem considered or even immediately lead them to failures.

Random shocks can deteriorate the pump from its current state i to a degraded state j , as $p_{ij} = \frac{9 \times (0.1)^{(i-j+1)}}{1 - (0.1)^{(i+1)}}$, $i \geq j$, where p_{i0} denotes the probability of an extreme random shock leading the pump from state i directly to failure state 0. The formulation is taken from Yang, *et al.* (2011), which satisfies that $\sum_{j=i}^0 p_{ij} = 1$. By combining the degradation process of the pump with the random shock process, the resulting process takes the form shown in Fig. 7. The state of the process is represented by $Y(t) = (Y_p(t), m)$, $m \in \mathbb{N}$, where m is the number of shocks experienced by the pump. The state space of the new process is denoted by $\mathcal{S} = \{(a, b), \forall a \in \mathcal{S}_p, b \in \mathbb{N}\}$ and the set of failure states of the pump is $\mathcal{F}'_p = \{(0, b), \forall b \in \mathbb{N}\}$.

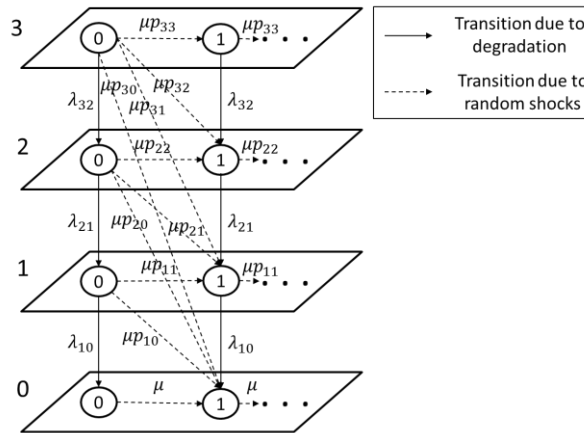


Fig. 7. Degradation and random shock processes of the pump.

For the valve, the i -th shock becomes extreme if the shock load W_i exceeds the maximal material strength D , otherwise, it can bring an instantaneous random increase H_i to the total external leak size (Song, *et al.*, 2014b). Since the shock load and the damage cannot be negative, W_i and H_i are assumed to be i.i.d. random variables following folded normal distributions (Leone, *et al.*, 1961), $W_i = |a|$ and $H_i = |b|$, where $a \sim N(\mu_h, \sigma_h^2)$ and $b \sim N(\mu_w, \sigma_w^2)$.

5.5. PDMP for the system considering dependency

An illustration of the composite degradation process of the valve considering random shocks and the degradation state of the pump is shown in Fig. 8, where the system experienced a random shock at time t_i , with the shock load W_i , $i = 1, 3, 4$. The first two shocks cause instantaneous random increases on $D(t)$, the last shock leads the valve to failure. The vibration of the pump accelerates the degradation process of the valve at time t_2 and t_3 , when the pump stepped to a further degraded state. The development of the leak size of the valve does not depend on other physical variables according to the given physics equations. Therefore, no physical variables are considered in Fig. 8 and no randomness is observed between two changing points. Nevertheless, there is actually randomness, i.e. the random changing points and the random increments. Moreover, if we consider the randomness in physical variables, then there will be randomness on the curve between two changing points.

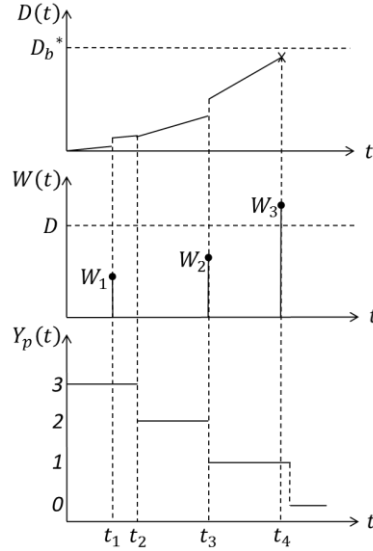


Fig. 8. An illustration of the degradation of the valve considering random shocks and the degradation state of the pump. Top Figure: degradation process of the valve; Center Figure: random shock processes; Bottom Figure: degradation process of the pump.

The degradation processes of the whole system can be represented by:

$$\mathbf{Z}(t) = (D(t), Y(t)) \in \mathbb{R}^+ \times \mathbf{S} = \mathbf{E} \quad (15)$$

Let $T_k, k \in \mathbb{N}$ denote the k -th jump time in $Y(t)$ and $\mathbf{Z}_k = (D_k, Y_k) = \mathbf{Z}(T_k)$. The evolution of $\mathbf{Z}(t)$ between two consecutive jumps of $Y(t)$, between which no shock occurs to the system and the degradation state of the pump does not change, can be written as follows:

$$\begin{aligned}\dot{\mathbf{Z}}(t) &= (\dot{D}(t), \dot{Y}(t)) \\ &= (\omega_b(1 + \beta(Y_p(t))), (0, 0)) \\ &= (v(Y(t)), (0, 0)), \text{ for } t \in [T_k, T_{k+1}[(16)\end{aligned}$$

where $v(\cdot)$ is used to denote the corresponding equation.

By integrating eq. (25), we can obtain that:

$$\begin{aligned}\mathbf{Z}(t) &= (D_k + (t - T_k)\omega_b(1 + \beta(Y_p(T_k))), Y_k) \\ &= (\varphi_1(\mathbf{Z}_k, t - T_k), Y_k), \text{ for } t \in [T_k, T_{k+1}[\\ &= \varphi(\mathbf{Z}_k, t - T_k), \text{ for } t \in [T_k, T_{k+1}[(17)\end{aligned}$$

where $\varphi_1(\cdot)$ and $\varphi(\cdot)$ are used to denote the corresponding equations.

Let $p_t(dx, \mathbf{y}_i)$ denote the probability distribution of $\mathbf{Z}(t)$. Given the series logic configuration of the system considered, the system fails when one of the two components fails; the reliability of the system at time t is, then, defined as follows:

$$R(t) = P[\mathbf{Z}(s) \notin \mathcal{F}, \forall s \leq t] = \int_{x \notin \mathcal{F}_v} \sum_{\mathbf{y}_i \notin \mathcal{F}_p'} p_t(dx, \mathbf{y}_i) (18)$$

where $\mathcal{F} = \mathbb{R}^+ \times \mathcal{F}_p' \cup \mathcal{F}_v \times \mathbf{S}$ is the set of the failure states of the system.

The parameter values related to the system degradation processes and random shocks under accelerated aging conditions are presented in Table I. The first eight parameter values related to the degradation processes are taken from Lin, *et al.* (2014), the values of μ_w , σ_w and D are taken from Peng, *et al.* (2010) and those of μ , μ_h and σ_h are assumed by expert judgment. Indeed, all parameter values are set upon the discussion with the experts from EDF. Indeed, all parameter values are set upon the discussion with the experts from EDF.

Table I Parameter values

Parameter	Value
λ_{32}	3e-3 /s
λ_{21}	3e-3 /s
λ_{10}	3e-3 /s
ω_b	1e-8 m ² /s
$\beta(3)$	0
$\beta(2)$	10%
$\beta(1)$	20%
$\beta(0)$	0

D_b^*	$1.06\text{e-}5 \text{ m}^2$
μ	$5\text{e-}3 / \text{s}$
μ_w	1.2 Gpa
σ_w	0.2 Gpa
D	1.5 Gpa.
μ_h	$1\text{e-}7 \text{ m}^2$
σ_h	$2\text{e-}8 \text{ m}^2$

6. NUMERICAL RESULTS AND ANALYSIS

The MC simulation and the FV methods are employed to estimate the system reliability. All the experiments are carried out in MATLAB on a PC with an Intel Core 2 Duo CPU at 3.06 GHz and a RAM of 3.07 GB. MC simulations with 10^3 , 10^4 and 10^5 replications (named MC1, MC2 and MC3, respectively) are applied over a time horizon of $T_{miss} = 1000 \text{ s}$ for the system reliability estimation. System holding time, arrival state for stochastic process $Y(t)$ and arrival state for process $D(t)$ can be sampled by using the probability distribution eq. (28), the probability mass function eq. (30) and the probability distribution eq. (31), respectively. See Appendix B for detailed descriptions of these equations.

The results are shown in Fig. 9. It is seen that the MC simulation method requires a number of replications to achieve the desired level of accuracy. The average computation times of MC1, MC2 and MC3 are 0.21 s, 2.17 s and 21.77 s, respectively.

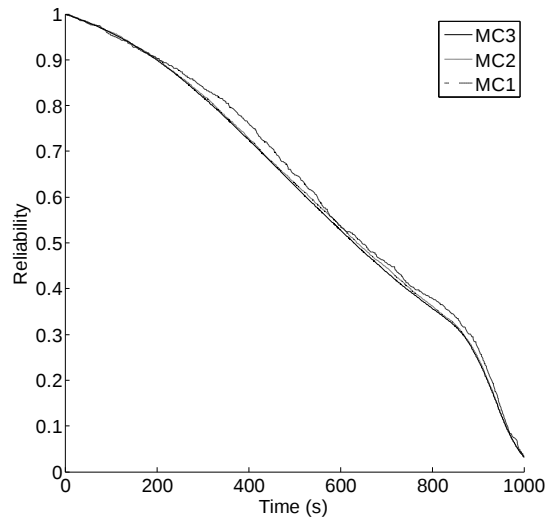


Fig. 9. System reliability estimated by MC1, MC2 and MC3.

For the FV method, the state space $\mathbb{R}^+$ of $D(t)$ has been divided into an admissible mesh

$\mathcal{M} = \cup_{m=0,1,2,\dots} [m\Delta x, (m+1)\Delta x[$ and the time space $\mathbb{R}^+$ has been divided into small intervals $\mathbb{R}^+ = \cup_{n=0,1,2,\dots} [n\Delta t, (n+1)\Delta t[$. See Appendix C for the application of FV method.

The system reliability estimated by the FV method, is shown in Fig. 10 with the following different parameter settings: (1) FV1: $\Delta x = 5e-9, \Delta t = 0.5$; (2) FV2: $\Delta x = 1.5e-8, \Delta t = 1.5$ and (3) FV3: $\Delta x = 4.5e-8, \Delta t = 4.5$. The accuracy of the FV scheme increases as the space step Δx and the time step Δt are reduced. The average computation times of FV1, FV2 and FV3 are 0.19 s, 1.93 s and 26.39 s, respectively.

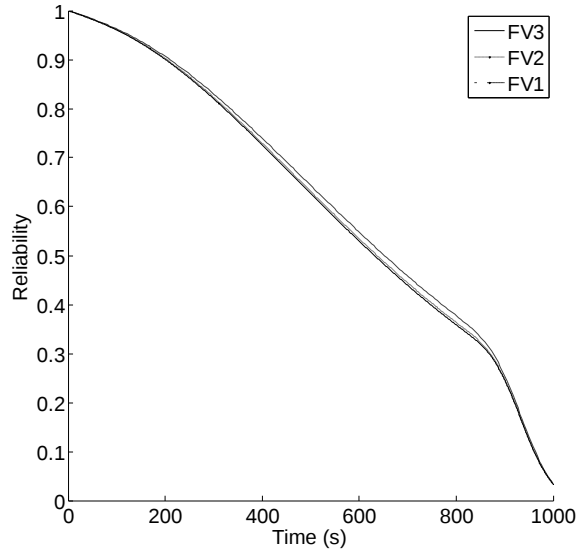


Fig. 10. System reliability estimated by FV1, FV2 and FV3.

The quantitative comparison of the most accurate results obtained by MC3 with those obtained by FV3 is shown in Table II. The sample variances associated with system reliability values estimated by MC3 are less than $2.5e-6$ according to eq. (12), which means the results are sufficiently consistent and accurate. The quantitative comparison of results obtained by MC3 and FV3 shown in Table II is only used to show that FV scheme can achieve comparable results to the MC simulation method (relative error less than 0.9%) in the illustrative case. Note that FV3 gives deterministic results since the values of Δx and Δt do not change, which guarantees the accuracy and consistency of the quantitative comparison. To provide more information, we have added Fig. 11 to compare the results obtained by MC3 with those obtained by FV3 over the time horizon. For this case study, the computational expense of the two methods is similar.

Table II Quantitative comparison of the results obtained by MC3 and FV3

Method Time	MC3	FV3	Relative error
100s	0.9611	0.9607	0.0438%
200s	0.9021	0.9011	0.1162%
300s	0.8230	0.8205	0.3027%
400s	0.7285	0.7263	0.2974%
500s	0.6284	0.6271	0.2109%
600s	0.5312	0.5300	0.2394%
700s	0.4395	0.4397	0.0365%
800s	0.3576	0.3591	0.4157%
900s	0.2467	0.2459	0.3204%
1000s	0.0335	0.0332	0.8955%

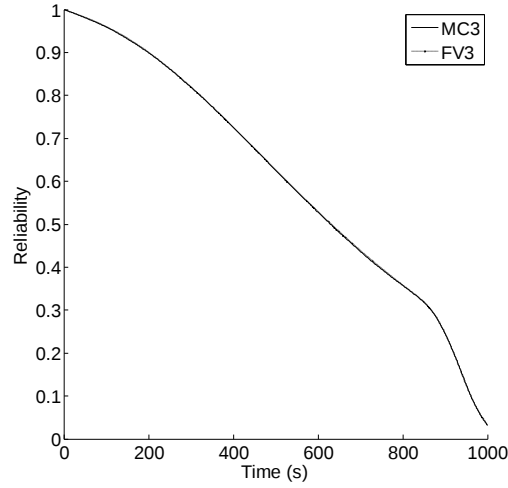


Fig. 11. Comparison of the results obtained by MC3 and FV3.

The reliability values of the valve, the pump and the system with/without random shocks, obtained by MC3, are shown in Fig. 12. The numerical comparisons on the reliability of the system, the valve and the pump with/without random shocks at the final time of 1000 s are presented in Table III.

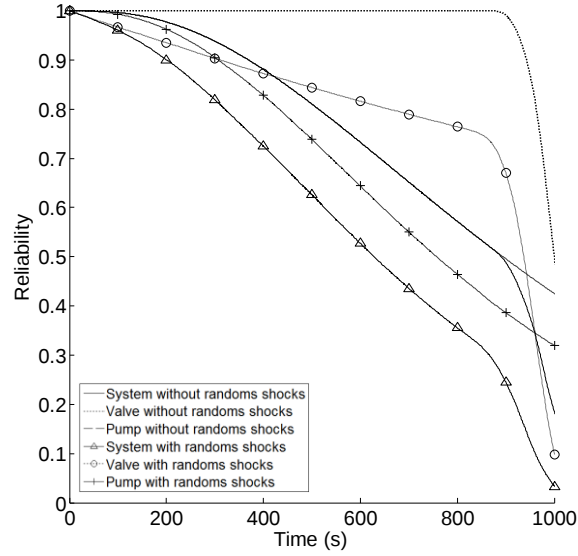


Fig. 12. The reliability of the system, the valve and the pump with/without random shocks.

When random shocks are ignored, the system reliability is basically determined by the pump before around 870 s, since the valve is highly reliable. After that, the sharp decrease of the valve reliability due to degradation leads to the same behavior in the system reliability. When random shocks are considered, the system reliability is determined by both the pump reliability and the valve reliability from the beginning until around 850 s, since the valve is no longer as highly reliable as before. Then, the valve reliability decreases sharply due to the joint effects of random shocks and degradation, and this drives also the sharp decrease of the system reliability. We can see from the results that neglecting random shocks can result in an underestimation of the reliability of the system and of the components.

Table III Comparison of reliability with/without random shocks at 1000 s

	Reliability without random shocks	Reliability with random shocks	Relative change
System	0.18	0.033	81.67%
Valve	0.50	0.099	80.20%
Pump	0.43	0.32	25.58%

The uncertainties associated with the parameter estimates can influence the estimation of system reliability. The actual effect depends on the types and degrees of uncertainties, which are problem specific. Following one assumption of our work (i.e. limited historical data), epistemic uncertainty can arise due to the incomplete or imprecise knowledge about the degradation processes and the governing parameters of the pump and the valve, which has been considered in Lin, *et al.* (2015a) by

describing the degradation model parameters as intervals (or fuzzy numbers). In the revised manuscript, we follow the settings in Lin, *et al.* (2015a) where a relative deviation of $\pm 10\%$ to the original parameters values has been considered for λ_{32} , λ_{21} , λ_{10} , ω_b , $\beta(3)$, $\beta(2)$, $\beta(1)$ and $\beta(0)$ upon the discussions with the domain experts from EDF. The lower and upper bounds of system reliability under uncertainty, and the original values without uncertainty obtained by MC3 are shown in Fig. 13. The lower bound of system reliability with uncertainty decreases more sharply after around 790 s, earlier than that without uncertainty. It is seen that the system fails after around 964 s, because at that time the valve is completely failed. The upper bound of system reliability with uncertainty does not experience a rapid decrease because the valve is mostly functioning over the time horizon.

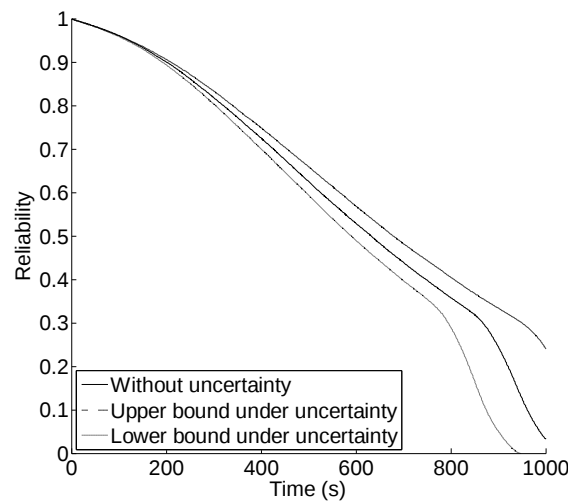


Fig. 13. The lower and upper bounds of system reliability with uncertainty, and the original values without uncertainty obtained by MC3.

7. CONCLUSIONS

In this paper, we presented reliability models for systems experiencing both degradation processes and random shocks. The degradation processes involve both continuous and multi-state processes, which are modeled by MSMs and PBMs, respectively. The dependencies between degradation processes and random shocks and among degradation processes are addressed by PDMP modeling. The procedures of the MC simulation and FV methods to solve the model are developed. A subsystem of a RHRS in a nuclear power plant, which consists of a pneumatic valve and a centrifugal pump, is considered as the illustrative example to demonstrate the effectiveness and modeling capabilities of the proposed framework. As original contribution and differently from our previous work (Lin, *et al.*, 2015a), this work is first in considering system reliability under both continuous and multi-state degradation processes, random shocks and their dependencies.

As future work, we will include maintenance in the model and derive optimal maintenance policies under the conditions considered.

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Appendix A: FV method

Assumptions

The FV method for determining the approximated solution of the system reliability can be developed under the following assumptions (Cocozza-Thivent, *et al.*, 2006):

- The transition rates $\lambda_{y_i, y_j}(\cdot | \theta_K), \forall y_i, y_j \in \mathcal{S}'$ are continuous and bounded functions from $\mathbb{R}^{d_L}$ to $\mathbb{R}^+$.
- The physics equations $f_L^{y_i}(\cdot | \theta_L), \forall y_i \in \mathcal{S}'$ are continuous functions from $\mathbb{R}^{d_L}$ to $\mathbb{R}^{d_L}$ and locally Lipschitz continuous.
- The physics equations $f_L^{y_i}(\cdot | \theta_L), \forall y_i \in \mathcal{S}'$ are sub-linear, i.e. there are some $V_1 > 0$ and $V_2 > 0$ such that

$$\forall x \in \mathbb{R}^{d_L}, t \in \mathbb{R}^+ |f_L^{y_i}(x | \theta_L)| \leq V_1 \|x\| + V_2$$

- The functions $\text{div}(f_L^{y_i}(\cdot | \theta_L)), \forall y_i \in \mathcal{S}'$ are almost everywhere bounded in absolute value by some real value $D > 0$ (independent of y_i).
- If $\phi(\cdot)$ is a continuous and bounded function from $\mathbb{R}^{d_L}$ to $\mathbb{R}$, then, $x \rightarrow \int \phi(\vec{y}) \mu(y_i, y_j, x)(dy)$ is continuous from $\mathbb{R}^{d_L}$ to $\mathbb{R}$.

Solution approach

For ease of notation, we let $g^{y_i}(\cdot, \cdot | \theta_L): \mathbb{R}^{d_L} \times \mathbb{R} \rightarrow \mathbb{R}^{d_L}$ denote the solution of

$$\frac{\partial}{\partial t} g^{y_i}(x, t | \theta_L) = f_L^{y_i}(g^{y_i}(x, t | \theta_L) | \theta_L), \forall y_i \in \mathcal{S}', x \in \mathbb{R}^{d_L}, t \in \mathbb{R} \quad (19)$$

with

$$g^{y_i}(x, 0 | \theta_L) = x, \forall y_i \in \mathcal{S}', x \in \mathbb{R}^{d_L} \quad (20)$$

and $g^{y_i}(x, t | \theta_L)$ being the result of the deterministic behavior of $X(t)$ after time t , starting from the point x while the processes $Y'(t)$ hold on state y_i .

The state space $\mathbb{R}^{d_L}$ of continuous variables $X'(t)$ is divided into an admissible mesh $\mathcal{M}$, which is a family of measurable subsets of $\mathbb{R}^{d_L}$, i.e., $\mathcal{M}$ is a partition of $\mathbb{R}^{d_L}$ such that:

- (1) $\cup_{A \in \mathcal{M}} A = \mathbb{R}^{d_L}$.
- (2) $\forall A, B \in \mathcal{M}, A \neq B \Rightarrow A \cap B = \emptyset$.
- (3) $m_A = \int_A dx > 0, \forall A \in \mathcal{M}$, where m_A is the volume of grid A .
- (4) $\sup_{A \in \mathcal{M}} \text{diam}(A) < +\infty$ where $\text{diam}(A) = \sup_{x, y \in A} |x - y|$.

Additionally, the time space $\mathbb{R}^+$ is divided into small intervals $\mathbb{R}^+ = \cup_{n=0,1,2,\dots} [n\Delta t, (n+1)\Delta t[$ by setting the time step $\Delta t > 0$ (the length of each interval).

The numerical scheme aims at constructing an approximate value $\rho_t(x, \cdot) dx$ for $p_t(dx, \cdot)$, such that $\rho_t(x, \cdot)$ is constant on each $[n\Delta t, (n+1)\Delta t[\times A \times \{y_i\}, \forall A \in \mathcal{M}, y_i \in \mathcal{S}'$:

$$\rho_t(x, y_i) = P_n(A, y_i), \forall y_i \in \mathcal{S}', x \in A, t \in [n\Delta t, (n+1)\Delta t[\quad (21)$$

$P_0(A, y_i), \forall y_i \in \mathcal{S}', A \in \mathcal{M}$ is defined as follows:

$$P_0(A, y_i) = \int_A p_0(dx, y_i) / m_A \quad (22)$$

Then, $P_{n+1}(A, y_i)$ can be calculated considering the deterministic evaluation of $X(t)$ and the stochastic evolution of $Y'(t)$ based on $P_n(\mathcal{M}, y_i)$ by the Chapman-Kolmogorov forward equation, as

follows:

$$P_{n+1}(A, \mathbf{y}_i) = \frac{1}{1+\Delta t b_A^{\mathbf{y}_i}} \widehat{P_{n+1}}(A, \mathbf{y}_i) + \Delta t \sum_{B \in \mathcal{M}} \sum_{\mathbf{y}_j \in \mathcal{S}'} \frac{a_{B,A}^{\mathbf{y}_j, \mathbf{y}_i}}{1+\Delta t b_A^{\mathbf{y}_j}} \widehat{P_{n+1}}(B, \mathbf{y}_j) \quad (23)$$

where

$$a_{B,A}^{\mathbf{y}_j, \mathbf{y}_i} = \int_A \lambda_{\mathbf{y}_j, \mathbf{y}_i}(\mathbf{x} \mid \boldsymbol{\theta}_K) \int_B \mu(\mathbf{y}_j, \mathbf{y}_i, \mathbf{x})(d\mathbf{y}) \mathbf{x} / m_A \quad (24)$$

is the average transition rate from state $\mathbf{y}_j$ and grid B to state $\mathbf{y}_i$ and grid A ,

$$b_A^{\mathbf{y}_i} = \int_A \sum_{\mathbf{y}_j \in \mathcal{S}'} \lambda_{\mathbf{y}_i, \mathbf{y}_j}(\mathbf{x} \mid \boldsymbol{\theta}_K) d\mathbf{x} / m_A \quad (25)$$

is the average transition rate out of state $\mathbf{y}_i$ for grid A ,

$$\widehat{P_{n+1}}(A, \mathbf{y}_i) = \sum_{B \in \mathcal{M}} m_{BA}^{\mathbf{y}_i} P_n(B, \mathbf{y}_i) / m_A \quad (26)$$

is the approximate value of probability density function on $[(n+1)\Delta t, (n+2)\Delta t] \times A \times \{\mathbf{y}_i\}$ according to the deterministic evolution of $\mathbf{X}(t)$,

$$m_{BA}^{\mathbf{y}_i} = \int_{\{\mathbf{y} \in B \mid g^{\mathbf{y}_i}(\mathbf{y}, \Delta t \mid \boldsymbol{\theta}_L) \in A\}} d\mathbf{y} \quad (27)$$

is the volume of the part of grid B which will enter grid A after time Δt , according to the deterministic evolution of $\mathbf{X}(t)$.

The approximated solution $\rho_t(\mathbf{x}, \cdot) d\mathbf{x}$ weakly converges towards $p_t(d\mathbf{x}, \cdot)$ when $\Delta t \rightarrow 0$ and $|\mathcal{M}| / \Delta t \rightarrow 0$ where $|\mathcal{M}| = \sup_{A \in \mathcal{M}} \text{diam}(A)$.

Appendix B: Equations for MC simulation method in case study

The semi-Markov kernel of $\{\mathbf{Z}_n, T_n\}_{n \geq 0}$ is $(\mathbf{i} = (x, \mathbf{y}_i), (dx, \mathbf{y}_j), dt) = Q(\varphi(\mathbf{i}, t), (dx, \mathbf{y}_j)) dF_i(t), \forall k \in \mathbb{N}, \mathbf{y}_i, \mathbf{y}_j \in \mathcal{S}, x \in \mathbb{R}^+, dx \rightarrow 0, dt \rightarrow 0$. According to the degradation models of the system, we can obtain that:

$$dF_{\mathbf{i}=(x, \mathbf{y}_i)}(t) = \lambda_{\mathbf{y}_i} e^{-\lambda_{\mathbf{y}_i} t} dt \quad (28)$$

where $\lambda_{\mathbf{y}_i}$ is the sum of the outgoing transition rates of $Y(t)$ from state $\mathbf{y}_i$, and

$$\begin{aligned} & Q(\varphi(\mathbf{i}, t), (dx, \mathbf{y}_i)) \\ &= P[D_{k+1} \in [x, x+dx] \mid Y_{k+1} = \mathbf{y}_j, T_{k+1} - T_k \in [t, t+dt], \mathbf{Z}_k = \mathbf{i}] \\ & \quad \cdot P[Y_{k+1} = \mathbf{y}_j \mid T_{k+1} - T_k \in [t, t+dt], \mathbf{Z}_k = \mathbf{i}] \end{aligned} \quad (29)$$

where

$$\begin{aligned} & P[Y_{k+1} = \mathbf{y}_j \mid T_{k+1} - T_k \in [t, t+dt], \mathbf{Z}_k = \mathbf{i}] \\ &= P[Y_{k+1} = \mathbf{y}_j \mid Y_k = \mathbf{y}_i] \\ &= \frac{\lambda_{\mathbf{y}_i, \mathbf{y}_j}}{\lambda_{\mathbf{y}_i}} \end{aligned} \quad (30)$$

where $\lambda_{\mathbf{y}_i, \mathbf{y}_j}$ is the transition rate of $Y(t)$ from state $\mathbf{y}_i$ to state $\mathbf{y}_j$, and

$$\begin{aligned}
& P[D_{k+1} \in [x, x + dx] \mid Y_{k+1} = \mathbf{y}_j, T_{k+1} - T_k \in [t, t + dt], \mathbf{Z}_k = \mathbf{i}] \\
&= \begin{cases} P[\varphi_1(\mathbf{i}, t) + H(T_k + t) \in [x, x + dx]], \\ \text{if transition from } \mathbf{y}_i \text{ to } \mathbf{y}_j \text{ is due to random shock} \\ \delta_{\varphi_1(\mathbf{i}, t)}(dx), \\ \text{if transition from } \mathbf{y}_i \text{ to } \mathbf{y}_j \text{ is due to degradation} \end{cases} \quad (31)
\end{aligned}$$

where $H(T_k + t)$ is the instantaneous random increase caused by shock at time $T_k + t$, δ is the Dirac delta function and

$$\begin{aligned}
& P[\varphi_1(\mathbf{i}, t) + H(T_k + t) \in [x, x + dx]] \\
&= \begin{cases} \Phi\left(\frac{D - \mu_w}{\sigma_w}\right) \cdot \frac{1}{\sigma_h} \phi\left(\frac{x - \varphi_1(\mathbf{i}, t) - \mu_h}{\sigma_h}\right) dx, \\ \text{if } x < D_b^* \\ (1 - \Phi\left(\frac{D - \mu_w}{\sigma_w}\right)) \cdot \delta_{\varphi_1(\mathbf{i}, t) + D_b^*}(dx) + \Phi\left(\frac{D - \mu_w}{\sigma_w}\right) \cdot \frac{1}{\sigma_h} \phi\left(\frac{x - \varphi_1(\mathbf{i}, t) - \mu_h}{\sigma_h}\right) dx, \\ \text{if } x \geq D_b^* \end{cases} \quad (32)
\end{aligned}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cumulative distribution function and the probability density function of a folded normal distribution related to the standard normal distribution, respectively. Here, since an extreme shock can directly lead the valve to failure, we assume each extreme shock increase the total external leak size by D_b^* to formulate the problem within the settings of PDMP. Note that this assumption will not change the reliability of the valve.

Appendix C: Application of FV method in case study

The probability distribution of $\mathbf{Z}(t), p_t(dx, \mathbf{y}_i)$, obeys the Chapman-Kolmogorov equation (Davis, 1993) as follows:

$$\begin{aligned}
& \int_0^t \sum_{\mathbf{y}_i \in \mathcal{S}} \int_{\mathbb{R}^+} \sum_{\mathbf{y}_j \in \mathcal{S}} \lambda_{\mathbf{y}_i, \mathbf{y}_j} \left(\int_{\mathbb{R}^+} \psi(\mathbf{y}_j, y) \mu(\mathbf{y}_i, \mathbf{y}_j, x)(dy) - \psi(\mathbf{y}_i, x) \right) p_s(dx, \mathbf{y}_i) ds + \\
& \int_0^t \sum_{\mathbf{y}_i \in \mathcal{S}} \int_{\mathbb{R}^+} v(\mathbf{y}_i) \text{div}(\psi(\mathbf{y}_i, x)) p_s(dx, \mathbf{y}_i) ds - \sum_{\mathbf{y}_i \in \mathcal{S}} \int_{\mathbb{R}^+} \psi(\mathbf{y}_i, x) p_t(dx, \mathbf{y}_i) + \\
& \sum_{\mathbf{y}_i \in \mathcal{S}} \int_{\mathbb{R}^+} \psi(\mathbf{y}_i, x) p_0(dx, \mathbf{y}_i) = 0 \quad (33)
\end{aligned}$$

where $\psi(\cdot, \cdot)$ is any continuously differentiable function from $\mathcal{S} \times \mathbb{R}^+$ to $\mathbb{R}$ with a compact support and $\mu(\mathbf{y}_i, \mathbf{y}_j, x)(dy)$ is the probability of $D(t) \in [y, y + dy]$ after jumping from x when $Y(t)$ steps to state $\mathbf{y}_j$ from state $\mathbf{y}_i$ as follows:

$$\mu(\mathbf{y}_i, \mathbf{y}_j, x)(dy) = \begin{cases} \Phi\left(\frac{D - \mu_w}{\sigma_w}\right) \cdot \frac{1}{\sigma_h} \phi\left(\frac{y - x - \mu_h}{\sigma_h}\right) dy, & \text{if transition from } \mathbf{y}_i \text{ to } \mathbf{y}_j \text{ is due to random shock and } y < D_b^* \\ (1 - \Phi\left(\frac{D - \mu_w}{\sigma_w}\right)) \cdot \delta_{x+D_b^*}(dy) + \Phi\left(\frac{D - \mu_w}{\sigma_w}\right) \cdot \frac{1}{\sigma_h} \phi\left(\frac{y - x - \mu_h}{\sigma_h}\right) dy & \text{if transition from } \mathbf{y}_i \text{ to } \mathbf{y}_j \text{ is due to random shock and } y \geq D_b^* \\ \delta_x(dy), & \text{if transition from } \mathbf{y}_i \text{ to } \mathbf{y}_j \text{ is due to degradation} \end{cases} \quad (34)$$

$P_0(m, \mathbf{y}_i)$ is defined as follows:

$$P_0(m, \mathbf{y}_i) = \int_{m\Delta x}^{(m+1)\Delta x} p_0(dx, \mathbf{y}_i) / \Delta x \quad (35)$$

where $p_0(dx, \mathbf{y}_i) = \delta_0(dx) \cdot \mathbf{1}_{\{\mathbf{y}_i=(3,0)\}}$. Then, $P_{n+1}(m, \mathbf{y}_i)$, $n \in \mathbb{N}$ can be calculated considering the deterministic evolution of $D(t)$ and the stochastic evolution of $Y(t)$ based on $P_n(\cdot, \cdot)$ by the Chapman-Kolmogorov forward equation, as follows:

$$\begin{aligned} & P_{n+1}(m, \mathbf{y}_i) \\ &= \frac{1}{1+\Delta t \lambda_{\mathbf{y}_i}} \widehat{P}_{n+1}(m, \mathbf{y}_i) + \Delta t \sum_{m' \in \mathbb{N}} \sum_{\mathbf{y}_j \in S} \frac{a_{m',m}^{\mathbf{y}_j, \mathbf{y}_i}}{1+\Delta t \lambda_{\mathbf{y}_j}} \widehat{P}_{n+1}(m', \mathbf{y}_j) \end{aligned} \quad (36)$$

where

$$a_{m',m}^{\mathbf{y}_j, \mathbf{y}_i} = \lambda_{\mathbf{y}_j, \mathbf{y}_i} \int_{m'\Delta x}^{(m'+1)\Delta x} \int_{m\Delta x}^{(m+1)\Delta x} \mu(\mathbf{y}_j, \mathbf{y}_i, x)(dy) dx / \Delta x \quad (37)$$

is the average transition rate from state $\mathbf{y}_j$ and grid $[m'\Delta x, (m'+1)\Delta x[$ to state $\mathbf{y}_i$ and grid $[m\Delta x, (m+1)\Delta x[$,

$$\widehat{P}_{n+1}(m, \mathbf{y}_i) = \sum_{m' \in \mathbb{N}} v_{m',m}^{\mathbf{y}_i} P_n(m', \mathbf{y}_i) / \Delta x \quad (38)$$

is the approximate value of probability density function on $[m\Delta x, (m+1)\Delta x[\times \{\mathbf{y}_i\}$ according to the deterministic evolution of $D(t)$ between jumps of $Y(t)$ and

$$v_{m',m}^{\mathbf{y}_i} = \int_{\{x \in [m'\Delta x, (m'+1)\Delta x[\mid (\varphi_1((x, \mathbf{y}_i), \Delta t) \in [m\Delta x, (m+1)\Delta x[\}} dx \quad (39)$$

is the volume of the part of grid $[m'\Delta x, (m'+1)\Delta x[$ which will enter grid $[m\Delta x, (m+1)\Delta x[$ after time Δt according to the deterministic evaluation of $D(t)$.