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# Event–Triggered Observation of Nonlinear Lipschitz Systems via Impulsive Observers

L. Etienne, S. Di Gennaro,

## Abstract

In this paper we investigate the observation problems for a class of nonlinear Lipschitz systems, subject to network constraints. In order to address this problem, an impulsive observer is designed, making use of the event–triggered technique in order to diminish the network usage. The proposed observer ensures practical state estimation. The output sampling is done on a periodic basis, but the data transmission is regulated by an event–triggering mechanism. The performance of the observer is tested in simulations of a flexible joint.

*Impulsive observer, Event triggered sampling, Nonlinear systems.*

## I. INTRODUCTION

State observation, or estimation, is one of the central problem in system theory. Since the seminal work of Luenberger (1966), many works have treated this problem, both for linear and nonlinear systems. Recently, impulsive observers for continuous–time systems have been proposed (Raff, 2007; Chen, 2011, 2013) and, using discontinuous Lyapunov functions, sufficient conditions for the existence of these observers have been derived in terms of linear matrix inequalities (LMIs). Further recent developments in the design of impulsive observers can be find in Khaled (2013) for linear systems and in Andrieu (2010), Guillen (2013), Dinh (2015) for nonlinear systems.

In this work this kind of observers is used to estimate the state of a nonlinear Lipschitz system, where the sensor data are communicated via a wireless channel. Regarding this latter aspect, the theoretic assumption of continuous output sensing is not verified in practice. In fact, with the use of digital devices, the output is sampled at discrete time instants. The output sampling can be performed on the basis of a periodic/aperiodic sampling, or on an event–triggered basis. The advantage of the classic approach with periodic sampling is to allow the closed–loop system to be analyzed on the basis of sampled–data formalism (Astrom (1997) for linear systems, and Karafyllis (2009), Postoyan (2012)). On the other hand, the main drawback is the fact that the sampling instant does not depend on the output value, and therefore the output is sampled also if its value is not varied sensibly since the last sampling. In this sense, the classic sampling is done in “open–loop”. In order to make the sampling time dependent on the output value, one possibility is the use of the event–triggered policy.

Event–triggered sampling has been already considered in other contexts, e.g. those in which some specific tasks have to be completed before sampling a variable. More recently, driven by the necessity of reducing the bandwidth used in the wireless channel, such an event–triggering mechanism has been reconsidered to reduce the amount of data transmitted. In fact, while the output can be continuously or, rather, periodically or aperiodically sampled, the data transmission is decided by an event–triggering mechanism. Moreover, event–triggered techniques allow the execution of computation tasks as rarely as possible, so minimizing the energy consumption and leaving the digital processor available for other tasks.

In this paper this mechanism is considered as key technique, along with the impulsive observer to design an observer ensuring practical state estimation. The output sampling is done on a periodic basis, but the data transmission is regulated by the aforementioned event–triggering mechanism. This mechanism is usually called *periodic event–triggered sampling*. The periodicity ensures implicitly the existence of a minimum iter–event time, i.e. a nonzero time interval between two transmission events. In this work, in particular, the impact of the event–triggered technique on the design of an impulsive observer is analyzed. Some studies are available on this particular subject (see Donkers (2010), Lehmann (2011), Tallapradaga (2012), Tallapradaga (2013) and the references therein). The vast majority of the existing literature is dedicated to linear systems. On the contrary, the field of observation of nonlinear system with event triggered communications has been subject of little investigation (Etienne (2015), Abdelrahim (2015)).

The event–triggered sampling technique has been developed in recent years (Arzen, 1999), as substitute of the time–periodic sampling, see for instance Heemels (2012) for an introduction to the topic and, for further details, Astrom (2003), Heemels (2008), Lunze (2010), Tabuada (2007), Wang (2011). In the case of the event–triggering the sampling is performed in order to ensure that some desired properties established in the continuous–time design, primarily the asymptotic convergence to zero

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of the observation error, can be maintained under sampling. Conceptually, this means to introduce a feedback in the sampling process, with a constant monitoring of the output to determine if the desired properties is ensured.

The structure of the article is the following. In Section 2, the observation problem is formulated and the structure of the impulsive observer is given. In Section 3, this problem is solved for a class of Lipschitz nonlinear systems, in terms of LMIs. In Section 4, this impulsive observer is applied to a one-link manipulator with flexible joint. Finally, in Section 5, some conclusions are drawn.

*Notations:*  $\|x\|$ ,  $\|M\|$  denote the Euclidean norms when applied to a vector  $x$  and to a matrix  $M$ , while  $\|x\|_P = x^T P x$  is the norm induced by a symmetric positive definite matrix  $P$ . Moreover,  $\lambda_{\min}^P$ ,  $\lambda_{\max}^P$  are the smallest and the biggest eigenvalues of a square matrix  $P$ . Furthermore,  $\mathbb{R}^+$ ,  $\mathbb{R}_0^+$  will denote the set of positive real numbers and the set of positive real numbers including zero, and  $\mathbb{N} = \{0, 1, 2, \dots\}$  the set of natural numbers including zero. With  $I$  we denote the identity matrix. Finally,  $t^+ = \lim_{h \rightarrow 0, h > 0} (t + h)$ , and  $t_k^+ = \lim_{h \rightarrow 0, h > 0} (t_k + h)$ ,  $k \in \mathbb{N}$ . When not necessary, the dependence on time  $t$  will be omitted.

## II. PROBLEM STATEMENT

The class of systems under study is characterized by the following equations

$$\begin{aligned} \dot{x} &= Ax + Bu + D\phi(Hx) \\ y &= Cx \end{aligned} \quad (1)$$

with  $x \in \mathbb{R}^n$  the state,  $u \in \mathbb{R}^m$  the input,  $y \in \mathbb{R}^q$  the output,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{m \times n}$ ,  $C \in \mathbb{R}^{q \times n}$ ,  $D \in \mathbb{R}^{n \times \nu}$ ,  $H \in \mathbb{R}^{p \times n}$  are constant matrices.

In the following one assumes that the pair  $(A, C)$  is observable. Note that milder results can be easily expressed in terms of detectability.

In (1),  $D\phi(Hx)$  gives the structure of the nonlinearity acting on the system, with  $\phi : \mathbb{R}^p \rightarrow \mathbb{R}^\nu$  a nonlinear function satisfying the following condition

$$\begin{aligned} \|\phi(H\chi_1) - \phi(H\chi_2)\| &\leq \gamma \|H(\chi_1 - \chi_2)\| \\ \forall (\chi_1, \chi_2) &\in \mathbb{R}^n \times \mathbb{R}^n. \end{aligned} \quad (2)$$

It is clear that (2) implies that (1) is Lipschitz with respect to  $x$ .

*Remark 2.1:* For the sake of simplicity we considered a global Lipschitz constant and  $\mathbb{R}^n$  as the domain of definition of the system. However, under appropriate technical assumptions, the results exposed here remain valid for compact sets.

Output sensors sample the outputs periodically, but the data are transmitted to the observer for elaboration at time instants that are a subset of the sampling instants. More precisely, while  $\mathcal{S}_s = \{t_k = k\delta\}_{k \in \mathbb{N}}$  is the periodic sampling sequence for the sensor, with  $\delta = t_{k+1} - t_k > 0$  the sensor sampling period, the data transmission sequence is  $\{t_\ell^s\}_{\ell \in \mathbb{N}} \subseteq \mathcal{S}_s$ . The situation is depicted in Fig. 1. The data communications between the plant and the observer, assumed instantaneous, take place only at the discrete time instants  $t_\ell^s$ .

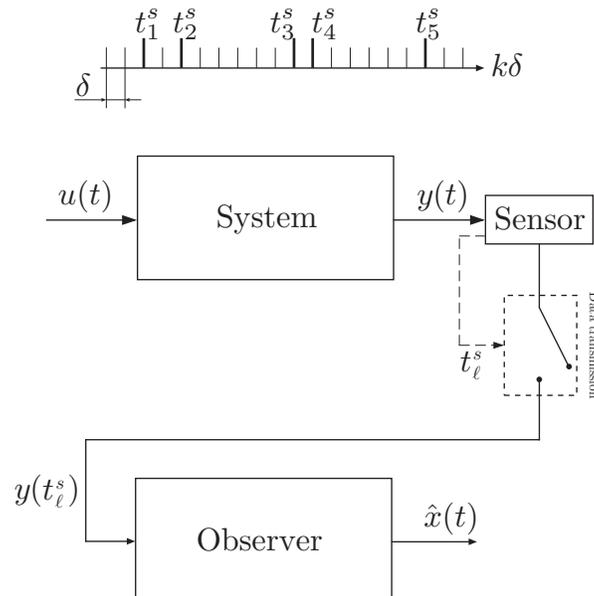


Fig. 1. Sensor sampling and transmission time instants.

The *observation problem* consists of determining an *event-triggering condition*, fixing the time instants in which the sampled output data are sent to the observer, and an *impulsive state observer* having the structure

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + D\phi(H\hat{x}) \\ \hat{x}(k\delta^+) &= \hat{x}(k\delta) + \delta G(y(t_\ell^s) - \hat{y}(k\delta)) \\ &= (I - \delta GC)\hat{x}(k\delta) + \delta GCx(t_\ell^s)\end{aligned}\quad (3)$$

such that  $\hat{x}(t)$  tends asymptotically to  $x(t)$  in a practical sense. In (3),  $\hat{x}(k\delta^+)$  is the left limit of  $\hat{x}(t)$ , and  $G \in \mathbb{R}^{n \times q}$  is the observer gain matrix. Note that the right limit is  $\hat{x}(k\delta^-) = \hat{x}(k\delta)$ . The observer dynamics correspond to a copy of the system dynamics between sampling instants  $k\delta$ ,  $(k+1)\delta$ , while it undergoes a jump in the state at the sampling instants.

### III. AN EVENT-TRIGGERED IMPULSIVE OBSERVER

The impulsive observer (3) will be implemented making use of a triggering mechanism, determining when the sensor transmits the sampled data to the plant. More precisely, these data are sent to the plant at the time instants  $t = t_\ell^s$ ,  $\ell \in \mathbb{N}$ , such that the following event-triggering condition is satisfied

$$t_{\ell+1}^s = \min_k \left\{ k\delta > t_\ell^s \mid \|y(k\delta) - y(t_\ell^s)\| \geq \varepsilon_s \right\} \quad (4)$$

where  $\varepsilon_s > 0$  is a threshold value on the output error  $y(k\delta) - y(t_\ell^s)$ .

Given the system and the observer dynamics (1), (3), one has to consider both the continuous dynamics of the observation error  $e(t) = x(t) - \hat{x}(t)$ , given by

$$\dot{e} = Ae + D(\phi(Hx) - \phi(H\hat{x})) \quad (5)$$

and the error discrete dynamics, due to the impulses on the observer state. These latter have the expression

$$\begin{aligned}e(k\delta^+) &= x(k\delta^+) - \hat{x}(k\delta^+) \\ &= (I - \delta GC)e(k\delta) + \delta GC(x(k\delta) - x(t_\ell^s))\end{aligned}\quad (6)$$

since  $x(k\delta^+) = x(k\delta^-) = x(k\delta)$ . It is worth noting that, at the triggering instants  $t_\ell^s$  in which (4) is satisfied and the system output sensor sends the sampled data to the observer, this expression reduces to

$$e(t_\ell^{s+}) = e(k\delta^+) = (I - \delta GC)e(k\delta) = (I - \delta GC)e(t_\ell^s)$$

since  $x(k\delta) = x(t_\ell^s)$ . In all the other discrete time instants, the term  $\delta GC(x(k\delta) - x(t_\ell^s)) = \delta G(y(k\delta) - y(t_\ell^s))$  appears, which can be seen as a perturbation induced by the absence of communications from the sensor.

As stated by the following result, the event-triggering condition (4) along with an appropriate choice of the observer gain ensure the exponential practical stability of the observation error (Lakshmikantham, 1990). It is worth noting that this is true also when the system dynamics (1) are not stable.

*Theorem 3.1:* Let us consider the system (1), with  $(A, C)$  observable and  $\phi$  satisfying (2). If, for a fixed sampling time  $\delta > 0$ , the following LMIs

$$N_1 \leq -\varepsilon I, \quad N_2 \leq -\varepsilon I, \quad N_3 \leq 0 \quad (7)$$

where

$$\begin{aligned}N_1 &= \begin{pmatrix} P_1 A + A^T P_1 + \gamma^2 H^T H + \frac{P_2 - P_1}{\delta} & P_1 D \\ P_1 D & -I \end{pmatrix} \\ N_2 &= \begin{pmatrix} P_2 A + A^T P_2 + \gamma^2 H^T H + \frac{P_2 - P_1}{\delta} & P_2 D \\ P_2 D & -I \end{pmatrix} \\ N_3 &= \begin{pmatrix} -P_2 & P_1 - \delta P_3 C \\ (P_1 - \delta P_3 C)^T & -P_1 \end{pmatrix}\end{aligned}$$

have solutions  $P_1, P_2, P_3$ , with  $P_i = P_i^T > 0$ ,  $i = 1, 2$ , for an  $\varepsilon > 0$ , then the observer (3), with the event-triggering condition (4), and the gain  $G = P_1^{-1} P_3$ , ensures that the origin of error dynamics (5), (6) is globally practically exponentially stable, with attractive set

$$\mathcal{I}_{\varepsilon_s} = \left\{ \|e\| \leq \rho_s \right\}, \quad \rho_s = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \frac{\delta \|G\|}{1 - e^{-\delta \bar{\varepsilon}/2}} \varepsilon_s \quad (8)$$

where  $\bar{\varepsilon} = \varepsilon / \lambda_{\max}$ , and  $\lambda_{\min} = \min \{ \lambda_{\min}^{P_1}, \lambda_{\min}^{P_2} \}$ ,  $\lambda_{\max} = \max \{ \lambda_{\max}^{P_1}, \lambda_{\max}^{P_2} \}$ .

*Remark 3.2:* It is worth noting that in (7) the sampling period  $\delta$  is fixed *a priori*. Therefore, (7) constitute a set of classic LMIs.

*Proof.* The error dynamics (5), (6) can be rewritten as follows

$$\begin{aligned}\dot{e} &= Ae + Dd_2 \\ e(k\delta^+) &= (I - \delta GC)e(k\delta) + \delta GCd_3\end{aligned}\quad (9)$$

with  $d_2 = \phi(Hx) - \phi(H\hat{x})$  and  $d_3 = x(k\delta) - x(t_\ell^s)$ . Following Suykens (1998), Raff (2007), let us consider the Lyapunov candidate  $V_o(e, t) = \|e\|_{P(t)}^2$ , where  $P(t)$  is a time-varying matrix given by the following convex combination of the matrices  $P_1, P_2$

$$\begin{aligned}P(t) &= P_1 + \frac{t - k\delta}{\delta}(P_2 - P_1) = \lambda P_1 + (1 - \lambda)P_2 \\ \lambda &= \frac{(k+1)\delta - t}{\delta} \in [0, 1]\end{aligned}\quad (10)$$

defined for  $t \in (k\delta, (k+1)\delta]$ . This combination allows constructing an appropriate Lyapunov function which, under the hypotheses (7), can show the practical exponential stability. For, first note that  $P(k\delta^+) = P_1$  and  $P((k+1)\delta) = P_2$ . Considering that  $P(t)$  is periodic with period  $\delta$ , its definition can be extended for all  $t \geq 0$ . Note also that  $\lambda_{\min}\|e\|^2 \leq V_o \leq \lambda_{\max}\|e\|^2$ . Using (9), and

$$\dot{P}(t) = \frac{P_2 - P_1}{\delta}$$

obtained from (10), one works out

$$\begin{aligned}\dot{V}_o &= e^T \left( PA + A^T P + \frac{P_2 - P_1}{\delta} \right) e \\ &\quad + e^T P D d_2 + d_2^T D^T P e + d_2^T d_2 - d_2^T d_2 \\ &\quad \forall t \in (k\delta, (k+1)\delta].\end{aligned}$$

Using (2)

$$\|d_2\|^2 \leq \gamma^2 e^T H^T H e$$

and considering the definition of  $P(t)$  one gets

$$\begin{aligned}\dot{V}_o &\leq e^T \left( PA + A^T P + \frac{P_2 - P_1}{\delta} + \gamma^2 H^T H \right) e \\ &\quad + e^T P D d_2 + d_2^T D^T P e - d_2^T d_2 \\ &= \xi^T N_1 \xi + (t - k\delta) \xi^T \bar{N} \xi \\ &= \frac{(k+1)\delta - t}{\delta} \xi^T N_1 \xi + \frac{t - k\delta}{\delta} \xi^T N_2 \xi\end{aligned}$$

for  $t \in (k\delta, (k+1)\delta]$ , where  $\xi = (e^T \quad d_2^T)^T$  and  $\bar{N} = (N_2 - N_1)/\delta$ . Hence, using (7), one finally obtains

$$\begin{aligned}\dot{V}_o &\leq -\varepsilon \frac{(k+1)\delta - t}{\delta} \|\xi\|^2 - \varepsilon \frac{t - k\delta}{\delta} \|\xi\|^2 = -\varepsilon \|\xi\|^2 \\ &\leq -\varepsilon \|e\|^2 \leq -\bar{\varepsilon} V_o\end{aligned}$$

i.e.  $\dot{V}_o$  is bounded by a negative definite function for all  $t \in (k\delta, (k+1)\delta]$ . Therefore,

$$\begin{aligned}V_o(t) &= e^{-\bar{\varepsilon}(t-t_0)} V_o(t_0) \\ \forall t_0, t &\in (k\delta, (k+1)\delta], \quad t_0 \leq t\end{aligned}\quad (11)$$

and, in particular,

$$V_o((k+1)\delta) \leq e^{-\delta\bar{\varepsilon}} V_o(k\delta^+).\quad (12)$$

Note also that in the inter-sampling

$$\|e(t)\| = e^{-\bar{\varepsilon}(t-t_0)/2} \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \|e(t_0)\| \quad (13)$$

$\forall t_0, t \in (k\delta, (k+1)\delta], \quad t_0 \leq t$ .

Let us now analyze the stability of the discrete error dynamics, i.e. the error dynamics in the discontinuity, considering the same Lyapunov candidate  $V_o(t) = \|e(t)\|_{P(t)}^2$  and recalling that

$$P(k\delta^+) = P_1, \quad P(k\delta) = P((k+1)\delta) = P_2$$

as already noted. Using (9), one gets

$$\begin{aligned}\Delta V_o &:= V_o(k\delta^+) - V_o(k\delta) = -e^T(k\delta)S_2e(k\delta) \\ &\quad + 2\delta\zeta^T(P_1 - \delta P_3C)e(k\delta) + \delta^2\zeta^T P_1\zeta\end{aligned}$$

with

$$G = P_1^{-1}P_3, \quad \zeta = GCd_3 = G(y(k\delta) - y(t_\ell^s))$$

and

$$S_2 = P_2 - (P_1 - \delta P_3C)^T P_1^{-1}(P_1 - \delta P_3C)$$

the Schur complement of the element (2,2) of the matrix  $-N_3$ .

It is well-known that for Hermitian matrices,  $-N_3 \geq 0$  is equivalent to  $P_2 > 0$  and  $S_2 \geq 0$  (see Zhang (2005), Theorem 1.12, pag. 34). Hence, since  $-N_3 \geq 0$  and  $P_2 > 0$ , then  $S_2 \geq 0$ . Therefore,  $-e^T(k\delta)S_2e(k\delta) \leq 0$ . Furthermore, note that

$$\begin{aligned}e^T(k\delta)(I - \delta GC)^T P_1(I - \delta GC)e(k\delta) \\ &= e^T(k\delta)(P_1 - \delta P_3C)^T P_1^{-1}(P_1 - \delta P_3C)e(k\delta) \\ &= e^T(k\delta)(P_2 - S_2)e(k\delta) \leq e^T(k\delta)P_2e(k\delta) = V_o(k\delta).\end{aligned}$$

This last observation allows writing

$$\begin{aligned}\zeta^T(P_1 - \delta P_3C)e(k\delta) &= \zeta^T P_1(I - \delta GC)e(k\delta) \\ &\leq \sqrt{\zeta^T P_1 \zeta} \sqrt{e^T(k\delta)(I - \delta GC)^T P_1(I - \delta GC)e(k\delta)} \\ &\leq \sqrt{\lambda_{\max}^{P_1} \|\zeta\|} \sqrt{V_o(k\delta)} \leq \sqrt{\lambda_{\max}} \|G\| \varepsilon_s \sqrt{V_o(k\delta)}\end{aligned}$$

where (4) has been used. Finally,

$$\begin{aligned}\zeta^T P_1 \zeta &= \|G(y(k\delta) - y(t_\ell^s))\|_{P_1}^2 \\ &\leq \lambda_{\max}^{P_1} \|G\|^2 \|y(k\delta) - y(t_\ell^s)\|^2 \\ &\leq \lambda_{\max}^{P_1} \|G\|^2 \varepsilon_s^2 \leq \lambda_{\max} \|G\|^2 \varepsilon_s^2.\end{aligned}$$

Therefore,

$$\begin{aligned}V_o(k\delta^+) &\leq V_o(k\delta) + 2\delta\sqrt{\lambda_{\max}} \|G\| \varepsilon_s \sqrt{V_o(k\delta)} \\ &\quad + \delta^2 \lambda_{\max} \|G\|^2 \varepsilon_s^2 = \left(\sqrt{V_o(k\delta)} + c\right)^2 \\ &\leq \left(a\sqrt{V_o((k-1)\delta^+)} + c\right)^2\end{aligned}$$

$a = e^{-\delta\varepsilon/2}$ ,  $c = \sqrt{\lambda_{\max}} \delta \|G\| \varepsilon_s$ , where (12) has been used, so that

$$\sqrt{V_o(k\delta^+)} \leq a\sqrt{V_o((k-1)\delta^+)} + c.$$

This linear discrete-time dynamics is exponentially stable to the origin since the dynamic matrix is Schur. Moreover, its solution is given by

$$\sqrt{V_o(k\delta^+)} \leq a\sqrt{V_o(0^+)} + c \sum_{j=1}^{k-1} a^j \quad (14)$$

and for  $k \rightarrow \infty$

$$\begin{aligned}\limsup_{k \rightarrow \infty} \sqrt{\lambda_{\min}} \|e(k\delta^+)\| &\leq \limsup_{k \rightarrow \infty} \sqrt{V_o(k\delta^+)} \leq \frac{c}{1-a} \\ &= \frac{c}{1 - e^{-\delta\varepsilon/2}} = \sqrt{\lambda_{\max}} \frac{\delta \|G\|}{1 - e^{-\delta\varepsilon/2}} \varepsilon_s\end{aligned}$$

where the series sum exists since  $a < 1$ . Hence,  $\limsup_{k \rightarrow \infty} \|e(k\delta^+)\| \leq \rho_s$ . Considering that (13) ensures that  $\|e(t)\|$  decreases exponentially between  $k\delta^+$  and  $(k+1)\delta$ , one can conclude that the error dynamics (5), (6) converges exponentially to the attractive set  $\mathcal{I}_{\varepsilon_s}$ . The convergence is global since all the passages do not depend on the initial state. This concludes the proof.  $\diamond$

It is worth noting that the convergence ensured by Theorem 3.1 to  $\mathcal{I}_{\varepsilon_s}$  is asymptotic. If one requires a finite-time convergence one needs to enlarge  $\mathcal{I}_{\varepsilon_s}$ . This is stated in the following result.

*Corollary 3.3:* Let us consider the system (1), with  $(A, C)$  observable and  $\phi$  satisfying (2). Under the same hypotheses and notations of Theorem 3.1, the observer (3), with the event-triggering condition (4), and the gain  $G = P_1^{-1}P_3$ , ensures that the origin of error dynamics (5), (6) globally practically converge to the set

$$\mathcal{I}_{\varepsilon_b} = \left\{ \|e\| \leq (1 + \varepsilon_b)\rho_s \right\}$$

$\rho_s$  given by (8), in a time

$$T \leq T_{\varrho, \varepsilon_b} = \frac{2}{\varepsilon} \ln \frac{d - (1+d)e^{-\delta\varepsilon/2}}{\varepsilon_b}, \quad d = \frac{1}{\delta\|G\|\varepsilon_s}\varrho$$

for any fixed  $\varepsilon_b, \varrho > 0$ , and for any observer initial condition such that  $\|e(0)\| \leq \varrho$ .

*Proof.* Since

$$V_o(0) = \|e(0)\|_{P_2}^2 \leq \lambda_{\max}^{P_2} \varrho^2 \leq \lambda_{\max} \varrho^2$$

one has

$$\sqrt{V_o(0^+)} \leq \sqrt{V_o(0)} + c \leq \sqrt{\lambda_{\max}} \varrho + c$$

and, from (14),

$$\begin{aligned} \sqrt{\lambda_{\min}} \|e(k\delta^+)\| &\leq \sqrt{V_o(k\delta^+)} \leq a\sqrt{V_o(0^+)} + c \sum_{i=0}^{k-1} a^i \\ &\leq a^k \left( \sqrt{\lambda_{\max}} \varrho + c \right) + c \frac{1 - a^k}{1 - a} \end{aligned}$$

$a = e^{-\delta\varepsilon/2}$ ,  $c = \sqrt{\lambda_{\max}} \delta\|G\|\varepsilon_s$ . Dividing by  $\sqrt{\lambda_{\min}}$  and imposing

$$\begin{aligned} \|e(k\delta^+)\| &\leq a^k \left( \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \varrho + \frac{c}{\sqrt{\lambda_{\min}}} \right) + (1 - a^k)\rho_s \\ &= (1 + \varepsilon_b)\rho_s, \quad \frac{c}{\sqrt{\lambda_{\min}}} \frac{1}{1 - a} = \rho_s \end{aligned}$$

one gets the bound of  $k\delta$  for the time  $T$  in which the error trajectory enters  $\mathcal{I}_{\varepsilon_b}$ .  $\diamond$

*Remark 3.4:* It is worth noting that in the proposed scheme it is not necessary that the system is stable in order to ensure the observer convergence. However, in practice numerical error may influence the observer performance.

#### IV. SIMULATION RESULTS FOR A ROBOT WITH A FLEXIBLE JOINT

In this section a fourth-order model of the form (1) is considered, taken from Spong (1987), Rajamani (1998), Howell (2002), Raff (2007) and representing the dynamics of a one-link manipulator, with a DC motor as actuator

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{pmatrix}, & B &= \begin{pmatrix} 0 & 21.6 & 0 & 0 \end{pmatrix}^T \\ C &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ H &= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, & D &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T \end{aligned}$$

with  $\phi = 3.3 \sin x_3$ , which is a nonlinear term due to the gravity, acting on the dynamics of  $x_4$ . As commented in Spong (1987), the joint elasticity is described by a linear torsional spring, and  $x_1$  represents the rotation of the motor,  $x_2 = \dot{x}_1$  is the corresponding angular velocity,  $x_3$  is the angular position of the link,  $x_4 = \dot{x}_3$  is its angular velocity. Physically, one measures the motor position and velocity, while the measurement of the other variables is non-trivial. The input  $u = \sin t$  is applied to the system. The performance of the impulsive observer (3) is analyzed making use of this benchmark.

Solving (7), with a sensor sampling period of  $\delta = 0.15$  s, one works out

$$P_1 = \begin{pmatrix} 28.6269 & -0.3675 & -11.6820 & 1.0816 \\ -0.3675 & 0.5296 & -1.5176 & 0.0455 \\ -11.6820 & -1.5176 & 18.8721 & -1.7201 \\ 1.0816 & 0.0455 & -1.7201 & 0.7153 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 22.9349 & -0.5313 & -11.6047 & 1.0195 \\ -0.5313 & 0.4136 & -1.9257 & 0.0810 \\ -11.6047 & -1.9257 & 20.9671 & -1.8715 \\ 1.0195 & 0.0810 & -1.8715 & 0.7393 \end{pmatrix}$$

$$G = \begin{pmatrix} 9.3334 & 1.0001 \\ -48.7804 & 22.3665 \\ -0.0524 & 3.3194 \\ 19.4066 & -0.3167 \end{pmatrix}.$$

where  $\lambda_{\min} = 0.06$ ,  $\lambda_{\max} = 37$ ,  $\varepsilon = 1$ . Then, in order to impose a ball of convergence for the observation error of  $\rho_s = 0.012$ , from (8) one obtains  $\varepsilon_s \simeq 3.64 \times 10^{-7}$ . This theoretical value is highly conservative and, as a matter of fact, it is possible to fix less restrictive values. In fact, as shown by the simulation in Figs. 2, 3 obtained with  $\delta = 0.05$  s and  $\varepsilon_s = 10^{-3}$ , one can fix well bigger event-triggering thresholds still obtaining the desired balls of convergence of dimensions  $\rho_s$ . As expected, at the beginning the communications are more frequent, and become less frequent when the steady-state is reached after about 1.75 s, with an average sampling time of  $\delta_{\text{av}} = 0.31$  s.

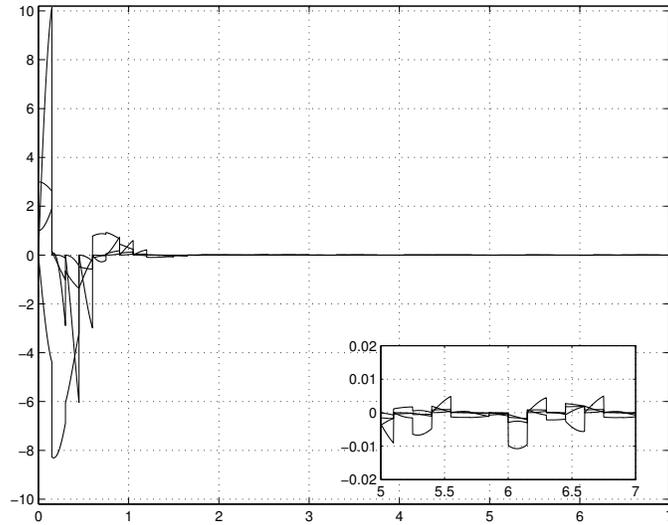


Fig. 2. Observation error  $e(t)$ , with a zoom showing  $\|e\| \leq \rho_s$ .

## V. CONCLUSIONS

In this work an impulsive observer was presented for a Lipschitz nonlinear system. The event-triggering mechanism is periodic. In the proposed scheme, for the observer convergence it is not necessary that the system is stable. The triggering parameter for the sensor can be computed in order to ensure an upper bound on the size of the attractive set. The simulation results show that this upper bound is not tight, and further work is necessary to obviate this over-approximation. The observer-based stabilization problem will be the subject of a forthcoming paper.

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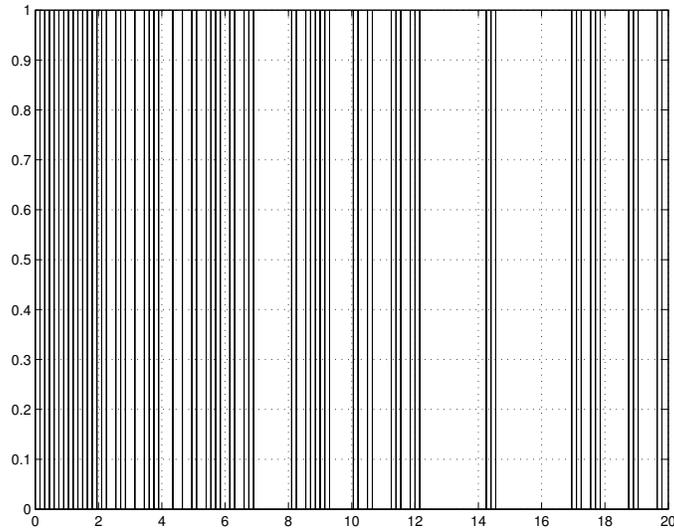


Fig. 3. Sensor transmission instants  $t_l^s$ .

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