Micro/macro viability analysis of individual-based models: Investigation into the viability of a stylized agricultural cooperative

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Micro/Macro Viability Analysis of Individual-based Models: Investigation into the Viability of a Stylized Agricultural Cooperative

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Abstract

The mathematical viability theory proposes methods and tools to study at a global level how controlled dynamical systems can be confined in a desirable subset of the state space. Multi-level viability problems are rarely studied since they induce combinatorial explosion (the set of N agents each evolving in a p-dimensional state space, can evolve in a Np dimensional state space). In this paper, we propose an original approach which consists in solving first local viability problems and then studying the real viability of the combination of the local strategies, by simulation where necessary. In this article we consider as multi-level viability problem a stylized agricultural cooperative which has to keep a minimum of members. Members have an economical constraint and some members have a simple model of the functioning of the cooperative and make assumptions on other members' behavior, especially pro-viable agents which are concerned about their own viability. In this framework, the model assumptions allow us to solve the local viability problem at the agent level. At the cooperative level, considering mixture of agents, simulation results indicate if and when including pro-viable agents increases the viability of the whole cooperative.

Keywords: Viability theory, individual-based models, local/global viability problems
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1. Introduction

For Hardin [1], common goods are doomed to be overexploited and devastated. Noting that there exists thousands of practical examples preventing the "tragedy of the commons" in reality, Ostrom [2] prefers to address the "issue of how to improve the skills of participants to change the rules of the game coercive to achieve results different from the relentless tragedies". Schuster [3] considers that no matter how nice the successful examples of self-governance are we should also not forget the large numbers of cases where self-organization failed terribly and his conclusion is that the evolution of interactions between governmental interference and self-governance is still not sufficiently well understood yet.

Mathematical methods and models can help to understand the general processes of interactions and regulations happening in groups made of individuals. The complex system is viewed as a network of interacting discrete entities (individuals or objects) and individual-based model group includes models ranging from cellular automata to very detailed agent-based models on various social network graphs. These individual-based models are becoming the tools of choice for investigating the behavior of groups of individuals in many fields [4, 5]: Scheffran and Hannon [6] introduce a general framework for modeling the interactions between agents in which agents can adjust their actions and resources to those of other agents and study by simulations the transition from conflict to cooperation in several cases; considering a model of groundwater use, López et al. [7] show that the success of the optimal management program depends heavily on the information that the users have about the resource.

The problem addressed in this paper is the investigation of the links between microscopic/individual viability and macroscopic/global viability in individual-based mathematical models. The issue is the persistence of a group and the entities that make it up over time.

The mathematical viability theory has been developing for thirty years methods and tools to study the compatibility between dynamics and constraints [8, 9]. This framework has been applied to renewable resources management and especially to the regulation of fisheries [10, 11, 12, 13], forest preservation [14, 15] or lake eutrophication management [16] as well as to broader(eco)-system dynamics [17, 18, 19] or to pure economic or social ones [20, 21].

In all these works, systems are described by global variables and the con-

trol regulation also operates at the global scale, for example the total amount of harvest or the total amount of phosphorus inputs in the lake. However, the total amount of harvest is the sum of harvest of individual fisheries, as the total amount of phosphorus inputs is the sum of phosphorus inputs of individual farmers. And these entities may have their own dynamics and constraints at a local scale. Viability studies including macro and micro scales are rare. Studying quotas in marine fisheries, Péreau *et al.* [22] recently determined viable global quotas strategies integrating a constraint at the global level which takes into account the social and economic constraints at the level of individual fisheries (considering the less efficient). Considering agents harvesting a renewable resource, Doyen and Péreau [23] considered local controls (the effort of each agent i) but the viability problem they solve only concerns the global level.

As far as individual-based models are concerned, viability constraints can both rely on local and global scales. Once agent behaviors and interaction rules are defined, the evolution of the system can be analysed by statistical mechanics techniques (master equation in the mean-field limit in [24] for instance) or shown by extensive computer simulations as in [25].

The inverse problem of the individual strategy design is addressed by differential game theory: Every agent considered as a player chooses a control to maximize independently from the others his own performance objective. In the zero-sum differential games framework with two players, Bettiol *et al.* [26] have included state constraints; nonzero-sum differential games with a very large number of players have been investigated in the terminology of mean-field games [27, 28]. But the advances on nonzero-sum differential games with N players have been scarcer, and mainly restricted to linear quadratic games for the main reason that there was very little understanding of the system of Hamilton-Jacobi equations naturally attached to these games [29].

In this article, we then propose to combine viability theory and simulations to study viability properties of an individual based model (N players in game theory terminology) facing both local and global constraints: We use viability theory tools to derive viable local individual strategies and simulations to study the impact of these strategies on collective viability.

Agricultural cooperatives are world-wide well-known farmers organizations (see for instance [30] for German cooperatives). An agricultural cooperative and its members may constitute a relatively simple example of two

scales systems: They are characterized by open membership societies, collective ownership, equal voting power, principles of equality and solidarity. This cooperative model is constructed for making cooperative business efficient under specific economic conditions especially when the average cost curve is continually declining with size [31]. Cooperative members, however, are rather heterogeneous. In general, members can be differentiated according to geographic dispersion, variances in age and education, farm size and type, as well as business objectives and strategies. Furthermore, when members do not see a strong connection between the success of the cooperative and their own business, Fulton and Giannakas [32] showed that member heterogeneity concerning the trust and loyalty toward the cooperative firm may lead to substantial financial pressures for the cooperatives. For example in wine cooperatives, usually the same price is paid for the same grape. Therefore, each viticulturist that belongs to a wine cooperative can produce the grapes for every wine that he wishes, regardless whether the grapes match the consumer quality criteria in taste. As a result, members select their grapes adversely so that they sell the better-quality grapes to other mostly private owned enterprises [33].

Several works apply economic theory to agricultural cooperative, they develop models using neoclassical theory solving members' utility maximization problem to find optimal pricing rules or game theory for instance to examine the issue of member commitment in the context of a mixed oligopoly where cooperatives and investor-owned firms compete with each other in supplying a consumer good [34]. In this article we build a model that includes the evolution with time of the cooperative and its members to be able to study the viability of this two scales system that is its persistence over time.

In the stylized cooperative we propose to study, we consider that members heterogeneous behavior is described by their loyalty toward the cooperative: The N members of the cooperative earn an amount proportional to the provided quantity. They have the possibility to fraud by increasing artificially the provided amount without being discovered until a given threshold. However, the proportionality coefficient used to evaluate their earnings depends on the global quality of the cooperative production which is altered by fraud.

From a viability viewpoint, if the cooperative members get no information and are not subjected to any pressure from the aggregated level, it is then only a juxtaposition of individual independent local viability problems (as it is rather the case in consumer's buying groups). When the global level makes the local controls compulsory, then it is a classical global viability problem.

In this paper we consider a partial influence of the global level on the local level: The payment of each member depends on the average level of fraud.

In this framework we study the impact of the individual local strategies on the viability at the global level. A naive approach consists in considering the viability problem of the global system of N agents. In general it is impossible to solve: It deals with a dynamical system of at least N state variables. Moreover, it is very often unrealistic to assume that the global level can enforce the behavior of each agent. In order to make the problem tractable, we propose to provide each agent with a model of the impact of the other members' behavior on its income. Making assumptions about the other members' behavior, the agent is able to solve its own local viability problem. Here we consider simple models and behaviors, and very elementary hypothesis about the possible behaviors. The objective of the paper is to show that even in this stylized framework, individual viability concerns can generate collective viability.

The paper is organized as follows: In the next section we present the particular viability problem we propose to address. In section three we describe the illustrative cooperative model from which we derive local viability problems depending on three agent types. We present the theoretical results concerning the viability of the strategies followed by the three types of agents in section four. The results concerning the global viability of the whole cooperative are described in section five. Finally, the main conclusions of this work are summarized in section six.

2. Problem statement: Interlinked local viability problems

We consider N agents, members of the same global entity.
Each agent i is characterized by its own individual state variables, $x_i \in X_i \subset R^{n_i}$, and control variables, $u_i \in R^{p_i}$. Since all agents are members of the same global entity, each of the individual dynamics depends on the individual state and control variables but it is also influenced by the state variables of the global entity $x \in X \subset R^n$:

$$\begin{array}{rcl} x_i'(t) & = & f_i(x_i(t), u_i(t), x(t)) \\ u_i(t) & \in & U_i(x_i(t), x(t)) \end{array} \tag{1}$$

where $U_i(x_i, x) \subset R^{p_i}$ is the set of admissible controls for agent i when its state is x_i and the global entity state is x. Local agents face local viability

constraints described by $L_i \subset X_i$:

$$\forall t \ , \quad x_i(t) \in L_i. \tag{2}$$

Moreover, the variations of the macroscopic variables $x \in X \subset R^n$ describing the global entity depend on their own values but also on local variables and controls :

$$x'(t) = f(x(t), \{x_i(t)\}_{1,\dots,N}, \{u_i(t)\}_{1,\dots,N}). \tag{3}$$

To solve a viability problem is to find control strategies such that all the constraints are satisfied over time. Here the dimension of the state space equals $\sum_1^N n_i + n$, so the classical approach of [8] is intractable.
We then propose to study this problem in two steps. First we solve the viability problems at the local level, allowing each agent to make assumptions on the evolution of the other members' behavior. With these hypotheses, it is possible to compute for each agent viable strategies for its own local viability problem. Second, we propose to study by simulation the real viability of the combination of these local strategies when they are implemented by the agents.

3. The case study: The cooperative and the micro/macro viability problem

3.1. The model of the cooperative

In our stylized cooperative, the cooperative is described by a single global variable which is the price at which the production is sold; each member has one state variable, its capital, and one control variable, its fraud level. However, the evolution with time of all these variables are correlated that makes the local and global viability issues intertwined.

3.1.1. The agent model

Each agent i has only one state variable: Its capital K_i, which evolves with the simple following dynamics:

(i) Each agent i produces a quantity w_i.
(ii) Each agent i can control partially the quality of its production. Here we consider that agents can alter the quality by fraud, which means

artificially increasing the production quantity brought to the cooperative. Actually, each member brings a quantity $\alpha_i w_i$ to the cooperative, $\alpha_i = 1$ if he does not fraud and $\alpha_i \in]1; \bar{\alpha}]$ otherwise. The value $\bar{\alpha}$ corresponds to the maximal fraud level which avoids detection.

(iii) Each member i pays fix charges with amount C_i associated with its production activity including the minimal salary for its subsistence; Each member benefit equals $p\alpha_i w_i - C_i$ where p is the price per quantity unit paid by the cooperative to all members.

The capital, $K_i(t)$, of each member i evolves with time according to its benefits:

$$K_i(t+1) = K_i(t) + p(t+1)\alpha_i(t+1)w_i(t+1) - C_i(t+1). \tag{4}$$

3.1.2. The global price dynamics

We consider a single global variable: The price per quantity unit $p(t)$. This price depends on:

(i) The cooperative production quality, Q, which depends at each time step on the global fraud amount that can be measured by comparing $\sum_i \alpha_i w_i$ and $\sum_i w_i$.
If no member frauds, $\sum_i \alpha_i w_i = \sum_i w_i$ and the global cooperative production quality is high equal to Q_0. However, when some members fraud, the quality is lower. If all members fraud with highest value $\bar{\alpha}$, $\sum_i \alpha_i w_i = \bar{\alpha} \sum_i w_i$ and the quality of the production of the cooperative is the worst quality the cooperative can produce, $\underline{Q}$. We consider here that a highly degraded global quality results in a decrease of individual incomes. Indeed, if the price is proportional to the quality, the individual income is proportional to the product of the global quality by the individual production. This income equals $Q_0.w_i$ if no member frauds. When all members fraud with highest value $\bar{\alpha}$, the global production is $\bar{\alpha} \sum_i w_i$, but the quality is degraded in a higher proportion we set to $\underline{Q} = Q_0/\bar{\alpha}^2$. Hence the individual income is lower if all the members fraud than if no member frauds, since with $\bar{\alpha} > 1$,

$$\frac{Q_0}{\bar{\alpha}^2}.\bar{\alpha}.w_i = \frac{1}{\bar{\alpha}} Q_0.w_i < Q_0.w_i.$$

Once set the extreme values, we assume that the quality, Q, varies linearly with the global fraud amount $\frac{\sum_i \alpha_i w_i}{\sum_i w_i}$. And consequently,

$$Q = -Q_0 A \frac{\sum_i \alpha_i w_i}{\sum_i w_i} + Q_0(1+A) \tag{5}$$

with

$$A = \frac{\bar{\alpha}+1}{\bar{\alpha}^2}. \tag{6}$$

Actually, $Q(\sum_i \alpha_i w_i = \bar{\alpha} \sum_i w_i) = \frac{Q_0}{\bar{\alpha}^2}$ (i.e. maximal fraud) and $Q(\sum_i \alpha_i w_i = \sum_i w_i) = Q_0$ (no fraud).

(ii) The cooperative reputation: When the quality decreases, the effect is instantaneous, the price per quantity unit paid to each member corresponds to the global production quality of the cooperative; However, when the quality increases, the price does not increase as fast as it has decreased, it is a matter of reputation. The effect of a quality decrease is instantaneous, so we set that $\Delta p(t) := p(t+1) - p(t)$ is proportional to $\Delta Q(t) := Q(t+1) - Q(t)$. When the quality increases, that is $\Delta Q(t) \geq 0$, the price will increase less quickly since getting a good reputation takes time. This inertia is represented by the weight $\delta \in [0;1[$ allocated to the previous price, $p(t)$, when evaluating $p(t+1)$:

$$\begin{array}{rcll} p(t+1) & = & p(t) + \frac{p_0}{Q_0}\Delta Q(t) & \text{if } \Delta Q(t) \leq 0 \\ p(t+1) & = & \delta p(t) + (1-\delta)(p(t) + \frac{p_0}{Q_0}\Delta Q(t)) & \text{if } \Delta Q(t) \geq 0 \end{array} \tag{7}$$

with $\frac{p_0}{Q_0}$ the proportionality coefficient: p_0 is the price corresponding to the highest quality Q_0.

From (5),

$$\Delta Q = -Q_0 A \Delta \left(\frac{\sum_i \alpha_i w_i}{\sum_i w_i} \right). \tag{8}$$

Consequently, from (7),

$$\begin{array}{rcl} p(t+1) & = & p(t) - p_0 A \Delta \left(\frac{\sum_i \alpha_i(t) w_i(t)}{\sum_i w_i(t)} \right), \text{ if } \Delta \left(\frac{\sum_i \alpha_i(t) w_i(t)}{\sum_i w_i(t)} \right) \geq 0 \\ p(t+1) & = & p(t) - (1-\delta) p_0 A \Delta \left(\frac{\sum_i \alpha_i(t) w_i(t)}{\sum_i w_i(t)} \right), \text{ if } \Delta \left(\frac{\sum_i \alpha_i(t) w_i(t)}{\sum_i w_i(t)} \right) \leq 0 \end{array} \tag{9}$$

3.2. The model of the other members' behavior

We provide the agents with a simple assumption on the other members' future behavior based on the idea that other agents may probably do the same reasoning as they does and then choose the same fraud level and others

will behave in average as in the previous period: Agent i anticipates that if it chooses for its next fraud level the value $\alpha(t+1)$, the next average fraud level of the whole cooperative measured by $\frac{\sum_i \alpha_i(t+1)w_i(t+1)}{\sum_i w_i(t+1)}$ will be equal to

$$\frac{1/B_i(\sum_{j=1}^N w_j(t+1))\alpha_i(t+1) + (1-1/B_i)(\sum_{j=1}^N w_j(t+1))\alpha_m(t)}{\sum_{j=1}^N w_j(t+1)}$$

with $B_i > 1$.
Member i anticipates that in the mean, at time $t+1$, other members will behave as if its new fraud level, $\alpha_i(t+1)$, would apply to a proportion $1/B_i$ of the global production $(\sum_{j=1}^N w_j(t+1))$; Whereas the remaining proportion, $1-1/B_i$, of the global production would experience a fraud level equal to the previous fraud average, $\alpha_m(t)$.

The agents estimate the previous fraud average, $\alpha_m(t)$, from the price, $p(t)$, they have sold their last production: if $p(t) = p_0$, then they estimate $\alpha_m(t) = 1$, if $p(t) = p_0/\bar{\alpha}^2$, then they estimate $\alpha_m(t) = \bar{\alpha}$, and in general using a linear extrapolation:

$$\alpha_m(t) = \frac{p_0(1+A) - p(t)}{p_0 A}. \tag{10}$$

Consequently, member i anticipates that:

$$\begin{aligned} \Delta\left(\frac{\sum_{j=1}^N \alpha_j(t)w_j(t)}{\sum_{j=1}^N w_j(t)}\right) &= \frac{1/B_i \sum_{j=1}^N w_j(t+1)\alpha_i(t+1) + (1-1/B_i)\sum_{j=1}^N w_j(t+1)\alpha_m(t)}{\sum_{j=1}^N w_j(t+1)} - \alpha_m(t) \\ &= \frac{1}{B_i}(\alpha_i(t+1) - \alpha_m(t)) \\ &= \frac{1}{B_i}(\alpha_i(t+1) - \frac{p_0(1+A)-p(t)}{p_0 A}). \end{aligned} \tag{11}$$

3.3. The local viability problems

With these models of cooperative functioning and other members' behavior, i local viability problems can be defined.

3.3.1. The dynamics

From (9) and (11), the global price value forecast by member i equals:

$$\begin{array}{rcl} p(t+1) & = & p(t) - \frac{p_0 A}{B_i}(\alpha_i(t+1) - \frac{p_0(1+A)-p(t)}{p_0 A}), \\ & & \text{if } (\alpha_i(t+1) - \frac{p_0(1+A)-p(t)}{p_0 A}) \geq 0 \\ p(t+1) & = & p(t) - (1-\delta)\frac{p_0 A}{B_i}(\alpha_i(t+1) - \frac{p_0(1+A)-p(t)}{p_0 A}), \\ & & \text{if } (\alpha_i(t+1) - \frac{p_0(1+A)-p(t)}{p_0 A}) \leq 0. \end{array} \tag{12}$$

Equations (4) and (12) give the values of the global price and of the capital of member i at time $t+1$ as a function of their values at time t and of member i fraud value, $\alpha_i(t+1)$. Member i chooses is next fraud level between 1 and $\bar{\alpha}$. Thus, let $u(t) \in [1; \bar{\alpha}]$,

$$\alpha(t+1) = u(t) \in [1; \bar{\alpha}]. \tag{13}$$

Variable u is called control variable in the mathematical viability theory framework and the model of agent i, member of the cooperative, is a control dynamical system discrete with time.

The dynamics of each local viability problem is the following:

$$\left\{ \begin{array}{rcl} K_i(t+1) & = & K_i(t) + p(t+1)u_i(t)w_i(t+1) - C_i(t+1) \\ p(t+1) & = & p(t) - \left(1 - \delta * 1_{u_i(t) - \frac{(p_0(1+A)-p(t))}{p_0 A} \leq 0}\right) \frac{p_0 A}{B_i}(u_i(t) - \frac{(p_0(1+A)-p(t))}{p_0 A}) \\ u_i(t) & \in & [1; \bar{\alpha}] \end{array} \right. \tag{14}$$

with $1_{u_i(t) - \frac{(p_0(1+A)-p(t))}{p_0 A} \leq 0} = 1$ when $u_i(t) - \frac{(p_0(1+A)-p(t))}{p_0 A} \leq 0$ and 0 otherwise.

For sake of simplicity, we consider in the following sections that $w_i(t) = w_i$ and $C_i(t) = C_i$, to reduce the dimension of the dynamical system to 2 state variables K_i and p. Otherwise it would be necessary to add the time dimension.

3.3.2. The constraints

The constraint on the global variable, $p(t) \in [\frac{p_0}{\bar{\alpha}^2}; p_0]$ is necessarily satisfied for all $t \in \mathbb{N}$ according to global price evolution equation (14).

The simplest constraint is imposed on the capital of each individual member, K_i, which is to remain above a given threshold, $K_{i_{min}}$, with time, that is:

$$\forall t \in \mathbb{N}, \quad K_i(t) \geq K_{i_{min}}. \tag{15}$$

This second constraint is not necessarily satisfied, it depends on the price the cooperative can sell its production which depends itself on the quality of the production and on the successive fraud amounts.

The constraint set is then defined by:

$$L_i \subset \mathbb{R}^2 := \{(K_i, p) \in [K_{i_{min}}; +\infty[\times[\frac{p_0}{\bar{\alpha}^2}; p_0]\}. \tag{16}$$

3.4. Different individual strategies

In this paper, we aim at showing that individual viability concerns can generate collective viability. We will then consider three agent types:

1. Myopic agent replicates the previous average fraud level;
2. Maximizing agent maximizes its income;
3. Pro-viable agent maximizes its income among the viable fraud levels.

It is worth noticing that the first type of agent does not need any assumption on the other members' behavior, whereas the second and third ones do. Obviously many other types of agents and strategies could be studied.

4. Theoretical results

The myopic agents situation is easy to describe: The fraud level remains constant. Theoretical results concerns the maximal and the maximal viable individual strategies. In the following, we omit index i in $K_i(t)$, $u_i(t)$, B_i, w_i, C_i, $K_{i_{min}}$ and L_i.

4.1. The strategy of maximizing agents does not always correspond to maximal fraud

From (14), $K(t+1) - K(t)$ depends on $p(t)$ and $u(t)$ but not on $K(t)$, so the strategy of the maximizing agent is described by the function, u_M, which only depends on p:

$$\begin{array}{rccl} u_M : [\frac{p_0}{\bar{\alpha}^2}; p_0] & \rightarrow & [1, \bar{\alpha}] \\ p & \rightarrow & \text{argmax}_{u \in [1;\bar{\alpha}]} K(t+1) - K(t) \end{array} \tag{17}$$

We also define the function, $\Delta p_M : [p_0/\bar{\alpha}^2; p_0] \rightarrow \mathbb{R}$, which is the variation of p when $u = u_M(p)$, which also only depends on p.

The following propositions specify the link between $u_M(p)$ and p and assess sufficient conditions for the maximal strategy not to be viable.

Proposition 1. *If $B > \bar{\alpha}(\bar{\alpha}+1)$, $u_M(p) = \bar{\alpha}$ for all $p \in [p_0/\bar{\alpha}^2; p_0]$.*

If $B \leq \bar{\alpha}(\bar{\alpha}+1)$,

(i) *when* $p < p_0\frac{1+A}{1+BD}$, $u_M(p) = \frac{p_0(1+A)-p(1-BD)}{2p_0A}$,
(ii) *when* $p \in [\frac{p_0(1+A)}{1+BD}; \frac{p_0(1+A)}{1+B}]$, $u_M(p) = \frac{p_0(1+A)-p}{p_0A}$,
(iii) *when* $p \in [p_0\frac{1+A}{1+B}; p_0\frac{\bar{\alpha}(\bar{\alpha}+1)-1}{\bar{\alpha}^2(B-1)}]$, $u_M(p) = \frac{p_0(1+A)-p(1-B)}{2p_0A}$,
(iv) *when* $p > p_0\frac{\bar{\alpha}(\bar{\alpha}+1)-1}{\bar{\alpha}^2(B-1)}$, $u_M(p) = \bar{\alpha}$.

PROOF. From (14), with $\Delta p(t) := p(t+1) - p(t)$, and omitting the (t), we get:

$$\Delta p = -(1 - \delta * 1_{\{\Delta p>0\}})\frac{p_0A}{B}u + (1 - \delta * 1_{\{\Delta p>0\}})\frac{1}{B}(p_0(1+A) - p) \quad (18)$$

and u as a function of p and Δp:

$$u(p, \Delta p) = -\frac{1}{1 - \delta * 1_{\{\Delta p>0\}}}\frac{B}{p_0A}\Delta p + \frac{1}{p_0A}(p_0(1+A) - p). \quad (19)$$

Moreover,

$$\begin{array}{rcl} K(t+1) - K(t) & = & p(t+1)u(t)w - C \\ & = & \Delta p(t)u(t)w + p(t)u(t)w - C\,. \end{array} \quad (20)$$

Let define $\Delta K(p(t), \Delta p(t)) := K(t+1) - K(t)$ and, let $D := 1/(1-\delta) \geq 1$. Omitting the (t) again,

$$\begin{array}{rcl} \Delta K(p, \Delta p) & = & -\frac{BDw}{p_0A}\Delta p^2 + \frac{w}{p_0A}(p_0(1+A) - p(1+BD))\Delta p + \frac{pw}{p_0A}(p_0(1+A) - p) - C \\ & & \text{if } \Delta p \geq 0 \\ \Delta K(p, \Delta p) & = & -\frac{Bw}{p_0A}\Delta p^2 + \frac{w}{p_0A}(p_0(1+A) - p(1+B))\Delta p + \frac{pw}{p_0A}(p_0(1+A) - p) - C \\ & & \text{if } \Delta p \leq 0. \end{array} \quad (21)$$

Moreover, when u ranges from 1 to $\bar{\alpha}$, Δp ranges from $\Delta p_{min}(p)$ to $\Delta p_{max}(p)$. From (14), the minimum is reached for $u = \bar{\alpha}$, and the maximum for $u = 1$, so we have:

$$\Delta p_{min}(p) = 1/B(p_0/\bar{\alpha}^2 - p) \quad (22)$$

and

$$\Delta p_{max}(p) = 1/(BD)(p_0 - p). \quad (23)$$

So, for a situation described by the pair $(K,\ p)$, the set of reachable situations at the next time step according to the fraud level chosen by member

i, u, and its anticipation model of other members' behavior (14) is represented in the (K, p)-plane by two sections of parabola curves which meet when $\Delta p = 0$.

We first consider $\max_{\Delta p \in [0;\Delta p_{max}(p)]} \Delta K(p, \Delta p)$.

If $p_0(1+A) - p(1+BD) > 0$, that is $p < p_0 \frac{1+A}{1+BD}$, the top of the parabola is at $\Delta p^*(p) := \frac{p_0(1+A)-p(1+BD)}{2BD} > 0$. Moreover, $\Delta p^*(p) \leq \Delta p_{max}(p)$, consequently,

$$\max_{\Delta p \in [0;\Delta p_{max}(p)]} \Delta K(p, \Delta p) = \Delta K(p, \Delta p^*(p)). \tag{24}$$

If $p_0(1+A) - p(1+BD) \leq 0$,

$$\max_{\Delta p \in [0;\Delta p_{max}(p)]} \Delta K(p, \Delta p) = \Delta K(p, 0). \tag{25}$$

We consider now $\max_{\Delta p \in [\Delta p_{min}(p);0]} \Delta K(p, \Delta p)$.

If $p_0(1+A) - p(1+B) < 0$, that is $p > p_0 \frac{1+A}{1+B}$, the top of the parabola is at $\Delta p^{**}(p) := \frac{p_0(1+A)-p(1+B)}{2B} < 0$.
Moreover, $\Delta p^{**}(p) \geq \Delta p_{min}(p)$, that is $\frac{p_0(1+A)-p(1+B)}{2B} \geq \frac{p_0/\bar{\alpha}^2-p}{B}$, when $p \leq p_0 \frac{\bar{\alpha}(\bar{\alpha}+1)-1}{\bar{\alpha}^2(B-1)}$ and then :

$$\max_{\Delta p \in [\Delta p_{min}(p);0]} \Delta K(p, \Delta p) = \Delta K(p, \Delta p^{**}(p)). \tag{26}$$

Otherwise,

$$\max_{\Delta p \in [\Delta p_{min}(p);0]} \Delta K(p, \Delta p) = \Delta K(p, \Delta p_{min}(p)). \tag{27}$$

If $p_0(1+A) - p(1+B) \geq 0$,

$$\max_{\Delta p \in [\Delta p_{min}(p);0]} \Delta K(p, \Delta p) = \Delta K(p, 0)). \tag{28}$$

Consequently,

$$\begin{array}{ll} (i) & \text{when } p < p_0 \frac{1+A}{1+BD},\ \Delta p_M(p) = \Delta p^*(p) > 0, \\ (ii) & \text{when } p \in [\frac{p_0(1+A)}{1+BD}; \frac{p_0(1+A)}{1+B}],\ \Delta p_M(p) = 0, \\ (iii) & \text{when } p \in]p_0 \frac{1+A}{1+B}; p_0 \frac{\bar{\alpha}(\bar{\alpha}+1)-1}{\bar{\alpha}^2(B-1)}],\ \Delta p_M(p) = \Delta p^{**}(p) < 0, \\ (iv) & \text{when } p > p_0 \frac{\bar{\alpha}(\bar{\alpha}+1)-1}{\bar{\alpha}^2(B-1)},\ \Delta p_M(p) = \Delta p_{min}(p) < 0. \end{array} \tag{29}$$

Finally, from (19), when $\Delta p_M(p) = \Delta p^*(p)$, the fraud value, $u_M(p)$, that corresponds to the price variation $\Delta p_M(p)$ equals

$$\begin{array}{rcl} u_M(p) & = & -\frac{BD}{p_0 A}\Delta p^*(p) + \frac{1}{p_0 A}(p_0(1+A) - p) \\ & = & \frac{1}{2p_0 A}(p_0(1+A) - p(1-BD)), \end{array} \tag{30}$$

when $\Delta p_M(p) = \Delta p^{**}(p)$, the fraud value, $u_M(p)$, that corresponds to the price variation $\Delta p_M(p)$ equals

$$\begin{array}{rcl} u_M(p) & = & -\frac{B}{p_0 A}\Delta p^{**}(p) + \frac{1}{p_0 A}(p_0(1+A) - p) \\ & = & \frac{1}{2p_0 A}(p_0(1+A) - p(1-B)), \end{array} \tag{31}$$

when $\Delta p_M(p) = 0$, $u_M(p) = \frac{p_0(1+A)-p}{p_0 A}$, and when $\Delta p_M(p) = \Delta p_{min}(p)$, $u_M(p) = \bar{\alpha}$. ■

Corollary 1. *Let $(K(n), p(n))$, $n \in \mathbb{N}$, an evolution governed by the maximal strategy.*

- *When $B > \bar{\alpha}(\bar{\alpha}+1)$, if $\frac{C}{p_0 w} > \frac{1}{\bar{\alpha}}$ or*
- *when $B \leq \bar{\alpha}(\bar{\alpha}+1)$, if $\frac{C}{p_0 w} > \frac{(1+A)^2}{A}\frac{B}{(1+B)^2}$*

then $\exists N \in \mathbb{N}$ and $\exists c < 0$ such that $\forall n \geq N$, $K(n+1) - K(n) < c$.
Consequently, the evolution $(K(n), p(n))$ is not included in any constraint set (16), the evolutions governed by the maximal strategy are never viable.

Proof. We first show that under the conditions of corollary 1, $\Delta K(p, 0)$ is strictly negative when $p \in [p_0/\bar{\alpha}^2, \max(p_0\frac{1+A}{1+B}, p_0/\bar{\alpha}^2)]$.
Let $c(p) := \Delta K(p, 0) = \frac{pw}{p_0 A}(p_0(1+A) - p) - C$. The top of the parabola is reached for $p = p_0(1+A)/2 \geq \max(p_0\frac{1+A}{1+B}, p_0/\bar{\alpha}^2)$, so c is increasing between $p_0/\bar{\alpha}^2$ and $\max(p_0\frac{1+A}{1+B}, p_0/\bar{\alpha}^2)$.
Consequently, $\forall p \in [p_0/\bar{\alpha}^2, \max(p_0\frac{1+A}{1+B}, p_0/\bar{\alpha}^2)]$, $c(p) \leq c(\max(p_0\frac{1+A}{1+B}, p_0/\bar{\alpha}^2))$.

- when $B > \bar{\alpha}(\bar{\alpha}+1)$, we have $p_0/\bar{\alpha}^2 > p_0\frac{1+A}{1+B}$.
  Since $c(p_0/\bar{\alpha}^2) = \frac{wp_0}{\bar{\alpha}} - C$, if $\frac{wp_0}{\bar{\alpha}} - C < 0$, $c(\max(p_0\frac{1+A}{1+B}, p_0/\bar{\alpha}^2)) < 0$ and

$$\forall p \in [p_0/\bar{\alpha}^2; \max(p_0\frac{1+A}{1+B}, p_0/\bar{\alpha}^2)],\ \Delta K(p, 0) = c(p) < 0. \tag{32}$$

- when $B \leq \bar{\alpha}(\bar{\alpha}+1)$, we have $p_0/\bar{\alpha}^2 \leq p_0\frac{1+A}{1+B}$.
  Since $c(p_0\frac{1+A}{1+B}) = \frac{(1+A)^2 w p_0}{A}\frac{B}{(1+B)^2} - C$, if $\frac{(1+A)^2 w p_0}{A}\frac{B}{(1+B)^2} - C < 0$, $c(\max(p_0\frac{1+A}{1+B}, p_0/\bar{\alpha}^2)) < 0$ and

$$\forall p \in [p_0/\bar{\alpha}^2; \max(p_0\frac{1+A}{1+B}, p_0/\bar{\alpha}^2)],\ \Delta K(p,0) = c(p) < 0. \qquad (33)$$

Now, let $(K(n), p(n))$ an evolution governed by the maximal strategy. Necessarily, $p(0) \in [p_0/\bar{\alpha}^2; p_0]$. Moreover, from (29),

- if $p(0) > p_0\frac{1+A}{1+B}$, then $p(n)$ tends toward $\max(p_0\frac{1+A}{1+B}, p_0/\bar{\alpha}^2)$,
- if $p_0\frac{1+A}{1+BD} \leq p(0) \leq p_0\frac{1+A}{1+B}$, then for all n, $p(n) = p(0)$,
- if $p(0) < p_0\frac{1+A}{1+BD}$, then $p(n)$ tends toward $p_0\frac{1+A}{1+BD}$.

Thus, for all $p(0) \in [p_0/\bar{\alpha}^2; p_0]$, $\lim_n p(n) = \hat{p} \in [p_0/\bar{\alpha}^2; \max(p_0/\bar{\alpha}^2, p_0(1+A)/(1+B))]$ and $\lim_n p(n+1) - p(n) = 0$.

Consequently, the capital variation $K(n+1) - K(n)$ tends toward $\Delta K(\hat{p}, 0) = c(\hat{p})$ which is strictly negative from (32) and (33) since $\hat{p} \in [p_0/\bar{\alpha}^2; \max(p_0/\bar{\alpha}^2, p_0(1+A)/(1+B))]$. Consequently, the evolution $(K(n), p(n))$ is not viable. ■

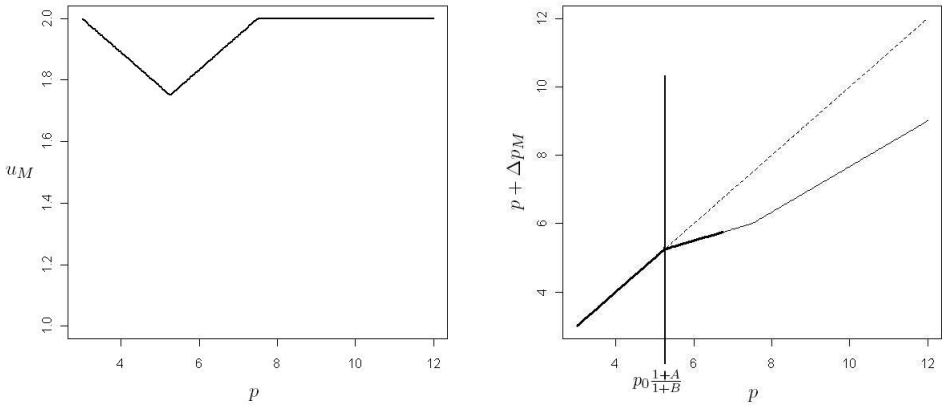
Figures 1(a) et 1(b) display u_M and Δp_M according to p with parameter values such that according to Corollary 1, the maximal strategy is never viable. Figure 1(a) shows that such a strategy implies high fraud values for all p. Moreover, figure 1(b) shows that necessarily such a strategy converges toward a fraud level associated with a negative value of ΔK and so leads to ruin.

4.2. The maximal viable strategy

The maximal viable individual benefit strategy consists in maximizing at each time step the agent benefit while ensuring the constraint satisfaction over time.

From [8], given a control dynamical system

$$\left\{ \begin{array}{lcl} x(t+1) & = & f(x(t), u(t)) \\ u(t) & \in & U(x(t)) \end{array} \right. \qquad (34)$$



(a) (b)

Figure 1: The parameter values are $p_0 = 12$, $\bar{\alpha} = 2$, $w = 1$, $C = 11$, $B = 3$ and $D = 2$. Figure 1(a) displays u_M according to p. With these values $p_0 \frac{1+A}{1+B} = 5.25$. Figure 1(b) displays $p + \Delta p_M(p)$ according to p with bold line when $\Delta K(p, \Delta p_M(p)) < 0$.

and a constraint set, L, the first step to determine viable strategies is to compute the viability kernel that is the subset of L gathering all states of the system such that there exists at least one control function that allows the system to remain in the constraint set indefinitely[1]:

$$\text{Viab}_{(34)}(L) = \{x \in L \mid \exists u(.) \mid x(t) \in L \; \forall t \in \mathbb{N}\}).$$

The states belonging to the viability kernel are called viable states and the control functions that allow to remain in the viability kernel are called viable control functions. Such functions can be derived from viability kernel determination: These are the control functions such that system state remains in the viability kernel at next time step.

Consequently, the computation of the viability kernel of the local viability problem of one cooperative member, $\text{Viab}_{(14)}(L) \subset R^2$, associated with the control dynamical system (14) and the constraint set (15) is a preliminary to maximal viable strategy determination. Actually, viable strategies can only be defined for pairs (K, p) belonging to the viability kernel. Outside,

[1]In the discrete time case, the assumptions are that the control dynamical system has to be Lipschitz with closed images and the constraint set has to be closed.

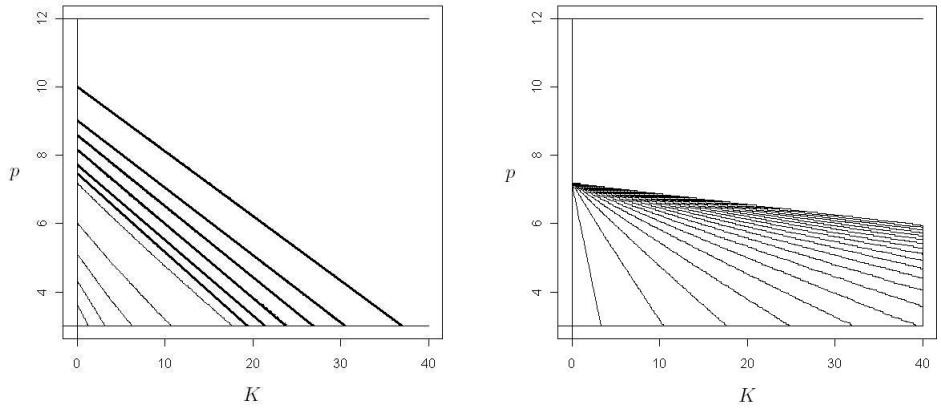
constraints will be violated whatever the fraud strategy. In particular, if the viability kernel is empty, from any point, there is no viable strategy. The following theorem specifies necessary and sufficient conditions for the viability kernel to be non empty.

Theorem 1. *Let* $f :]0; 2] \to \mathbb{R}$ *such that* $f(A) = \frac{(1+A)^2}{4A}$ *if* $A \leq 1$ *and* $f(A) = 1$ *otherwise.*
$Viab_{(14)}(L) \neq \emptyset$ *if and only if* $\frac{C}{p_0 w} \leq f(A)$.

Proof. We give here a sketch of the proof, the complete proof is in Appendix A.
When the condition on the parameter values of Theorem 1 is satisfied, we can exhibit a viable evolution so the viability kernel is not empty.
When the condition on the parameter values of Theorem 1 is not satisfied, the third term of the quadratic function describing $\Delta K(p, .)$ according to Δp is strictly negative whatever the value of p (21). The first consequence is that if $\Delta p = 0$, $\Delta K(p, 0) < 0$. The first part of the proof consists in comparing the marginal capital growth associated with an increase of the price with the marginal capital growth associated with a decrease of the price. This comparison allows to show in the second part of the proof that any capital evolution is lower than a sequence which tends toward $-\infty$. ■

We compute the viability kernels following the algorithm of [35]. Since the variations of the state variables (K, p) do not depend on K, the results for $K_{min} > 0$ can be deduced from those with $K_{min} = 0$ by a translation of $+K_{min}$ on the K-axis. We then set $K_{min} = 0$ in the following, the constraint to be satisfied is then $K \geq 0$.
Given a maximal exploration value for K, K_{max}, given discretization steps for p and K, all pairs $(K, p) \in [0; K_{max}] \times [p_0/\bar{\alpha}^2; p_0]$ belonging to the discretization grid constitute the set to be tested at the initialization of the algorithm. At each iteration of the algorithm, pairs are removed if they have no successor by (14) in the current set to be tested. Algorithm stops when no pair is removed during an iteration.

Figure 2 displays the viability kernel for dynamics (14) and constraint set (15) for different values of C and for different values of B. The viability kernel volume decreases with C and becomes empty when $\frac{C}{p_0 w} > \frac{(1+A)^2}{4A}$ as stated in Theorem 1 (Fig. 2(a)). It decreases with B (Fig. 2(b)) since the effect of agent i's fraud on the global quality is inversely proportional to B.



(a) (b)

Figure 2: The parameter values are $p_0 = 12$, $\bar{\alpha} = 2$, $w = 1$, $D = 2$, so $\frac{p_0 w(1+A)^2}{4A} = 12.25$. Figure 2(a) displays the lower boundary of the viability kernel for $B = 3$ and $C = 7, 8, 9, 10, 11$ for the plain lines and $C = 11.2, 11.4, 11.6, 11.8, 12, 12.2$ for the bold lines. The viability kernel volume decreases with C. Figure 2(b) displays the lower boundary of the viability kernel for $C = 11$ and $B = 2, 3, \ldots, 40$. The viability kernel volume decreases with B.

Let $\Delta p_{MV}(K,p)$ and $u_{MV}(K,p)$ the values of the global price variation and the values of the fraud levels of member i associated with the maximal viable benefit strategy, these functions are only defines on $\text{Viab}_{(14)}(L)$ and contrary to the maximal strategy, these values depend on K:

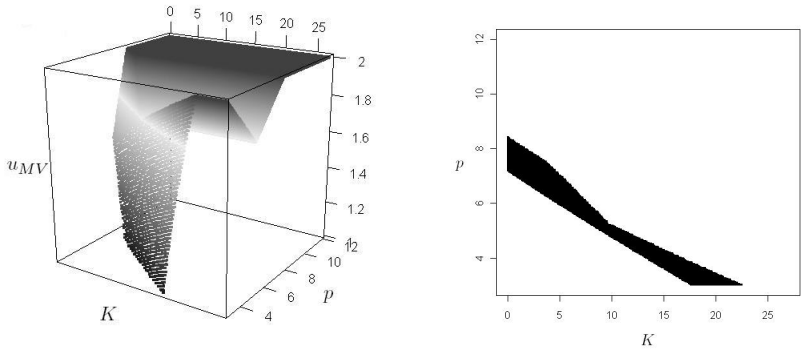
$$\Delta p_{MV}(K,p) = \underset{\substack{\Delta p \in [\Delta p_{min}(p); \Delta p_{max}(p)] \\ (K+\Delta K(p,\Delta p), p+\Delta p) \in \text{Viab}_{(14)}(L)}}{\text{argmax}} \Delta K(p, \Delta p). \tag{35}$$

From (19),

$$\begin{array}{rcl} u_{MV}(K,p) & = & -\frac{BD}{p_0 A}\Delta p_{MV}(K,p) + \frac{1}{p_0 A}(p_0(1+A) - p) \\ & & \text{if } \Delta p_{MV}(K,p) \geq 0 \\ u_{MV}(K,p) & = & -\frac{B}{p_0 A}\Delta p_{MV}(K,p) + \frac{1}{p_0 A}(p_0(1+A) - p) \\ & & \text{if } \Delta p_{MV}(K,p) < 0. \end{array} \tag{36}$$

The values of u_{MV} according to (K,p) are displayed in figure 3(a). Figures 3(a) and 3(b) show that when K is high, maximal and maximal viable

strategies merge: Constraint set boundary $K = 0$ is too far to have an influence. Figure 3(a) shows that for smaller K values, halting fraud is necessary to avoid ruin.



(a) (b)

Figure 3: The parameter values are $p_0 = 12$, $\bar{\alpha} = 2$, $w = 1$, $D = 2$, $B = 3$ and $C = 11$. Figure 3(a) displays u_{MV} according to (K, p). In figure 3(b), the area where $u_{MV}(K, p)$ differs from $u_M(p)$ is colored black.

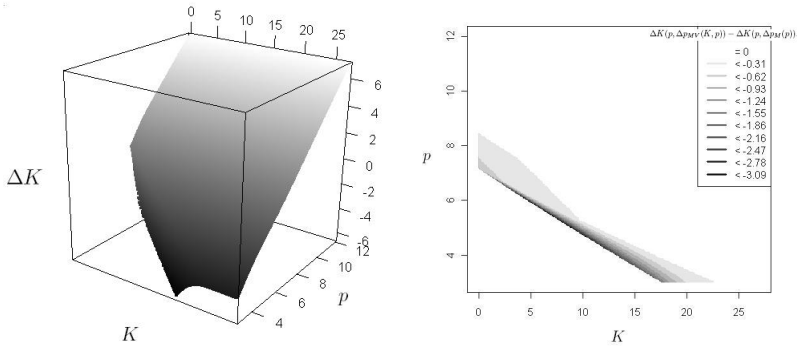
Figure 4(a) gives the capital variation values, $\Delta K(p, \Delta p_{MV}(K, p))$, associated with the maximal viable strategy according to (K, p). In the area where the maximal and the maximal viable strategy differ, the capital variation obviously also differs. Figure 4(b) shows the capital lost during one time step by pro-viable agent compared to a maximal one according to K and p, $\Delta K(p, \Delta p_{MV}(K, p)) - \Delta K(p, \Delta p_M(p))$.

5. Practical issue: Can local strategies of pro-viable agents ensure collective viability?

In the previous sections we have studied the local behavior of the agents, when they work out their strategy with their own model of the cooperative. We now study the real viability of the combination of the local strategies of the members of the cooperative.

5.1. Definition of the viability of the cooperative

The viability of the cooperative depends on the number of agents that remains in the cooperative. $N(t)$ is the state variable of the cooperative on which the viability constraint is defined.



(a) (b)

Figure 4: The parameter values are $p_0 = 12$, $\bar{\alpha} = 2$, $w = 1$, $D = 2$, $B = 3$ and $C = 11$. Figure 4(a) displays capital variations according to (K, p) when $u = u_{MV}$. In figure 4(b), the area where $u_{MV}(K, p)$ differs from $u_M(p)$ is colored according to the difference between the capital variations associated with $u = u_M$ and the capital variations associated with $u = u_{MV}$, $\Delta K(p, \Delta p_{MV}(K, p)) - \Delta K(p, \Delta p_M(p))$.

Definition 1. A cooperative is strongly viable when $N(t) = N(0)$, $\forall t \in \mathbb{N}$.

Definition 2. A cooperative is weakly viable when $N(t) \geq N_{min}$, $\forall t \in \mathbb{N}$.

We consider here that agents leave the cooperative when their capital K_i becomes too low ($K_i < K_{i_{min}}$), and we suppose that no new agent enters the cooperative.

5.2. Cooperative of Myopic agents

When all members of the cooperative are myopic agents, the price remains constant, as all agents' capital variation and fraud level. From (19), an agent i is viable (according to the local definition) if:

$$K_i(1) - K_i(0) = \frac{p(0)w_i(p_0(1 + A) - p(0))}{p_0 A} - C_i \geq 0\,, \tag{37}$$

since $\forall t$, $K_i(t+1) - K_i(t) = K_i(1) - K_i(0)$. Consequently the cooperative is strongly viable if all agents remain in the cooperative, that is when:

$$\forall i,\ \frac{p(0)w_i(p_0(1 + A) - p(0))}{p_0 A} - C_i \geq 0. \tag{38}$$

Let $M := \text{card}\{i/\Delta K_i(0) \geq 0\}$, then the cooperative is weakly viable for $N_{min} \leq M$. In all other cases, the cooperative is not viable.

Proposition 2. *Let $f :]0;2] \to \mathbb{R}$ such that $f(A) = \frac{(1+A)^2}{4A}$ if $A \leq 1$ and $f(A) = 1$ otherwise (as in theorem 1).*
When all members of the cooperative are myopic agents, a myopic agent is never viable whatever the initial price value when $\frac{C_i}{w_i p_0} > f(A)$.
When $\frac{C_i}{w_i p_0} \leq f(A)$, this myopic agent is viable if and only if

$$p(0) \in [p_{Z_i min}; p_{Z_i max}] := [\max(p_0/\bar{\alpha}^2, \frac{p_0(1+A) - \sqrt{\hat{\Delta}}}{2}); \min(\frac{p_0(1+A) + \sqrt{\hat{\Delta}}}{2}, p_0)] \tag{39}$$

where

$$\hat{\Delta} := p_0^2(1+A)^2 - 4\frac{p_0 A C_i}{w_i}. \tag{40}$$

Proof. Proposition 2 is a consequence of the study of the quadratic equation (37). ■

We note Z_i the area of the state space (K, p) delimited by the values of $(K_i(0), p(0))$ for which myopic agent i is viable in a cooperative of myopic agents (see figure 5),

$$Z_i = [0; +\infty[\times [p_{Z_i min}; p_{Z_i max}]$$

with $p_{Z_i min}$ and $p_{Z_i max}$ defined in (39).

5.3. Mixture of agents

For sake of simplicity, we assume from here on that all agents have the same production quantity and charge:

$$\forall i, \quad w_i = w \text{ and } C_i = C. \tag{41}$$

In that case, B_i can be interpreted as the inverse of the proportion of cooperative members that act like agent i. Let k_i such that $B_i = N/k_i$, this means that agent i assumes that:

- k_i members among N will make the same fraud level choice as itself;
- the $N - k_i$ other members are myopic agents (they choose on average a next fraud level equal to the previous one).

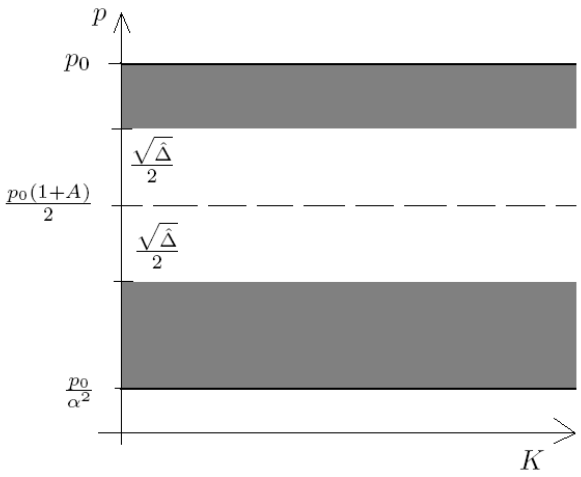


Figure 5: The area Z_i is colored white: If $(K_i(0), p(0)) \in Z_i$, then the myopic agent i is viable in a cooperative of myopic agents.

5.3.1. Mixture of myopic and maximizing agents

We consider now a mixture of k maximizing agents and $N-k$ myopic agents. We assume that the maximizing agents have an accurate perception of the other members' behavior, that is that their parameter $B_i = B = N/k$. The following proposition is a consequence of corollary 1:

Proposition 3. *Let a mixture of myopic and maximizing agents satisfying condition (41).*

- *When* $B > \bar{\alpha}(\bar{\alpha}+1)$, *if* $\frac{C}{p_0 w} > \frac{1}{\bar{\alpha}}$ *or*
- *when* $B \leq \bar{\alpha}(\bar{\alpha}+1)$, *if* $\frac{C}{p_0 w} > \frac{(1+A)^2}{A}\frac{B}{(1+B)^2}$

then neither maximizing nor myopic agents of the mixture are viable.

From the previous subsection we know that there are values of $(K_i(0), p(0))$ for which myopic agents alone are viable when $\frac{C}{wp_0} \leq f(A)$.
Incorporating maximizing agents provoke then a complete loss of viability (myopic agents alone are viable for $(K_i(0), p(0)) \in Z$, whereas all the agents of the mixture are never viable whatever the value of $(K_i(0), p(0))$) when:

$$\frac{C}{wp_0} \in [\tfrac{1}{\bar{\alpha}}; f(A)] \text{ with } B > \bar{\alpha}(\bar{\alpha}+1), \tag{42}$$

or

$$\frac{C}{wp_0} \in [\frac{(1+A)^2}{A}\frac{B}{(1+B)^2}, f(A)] \text{ with } B \leq \bar{\alpha}(\bar{\alpha}+1). \quad (43)$$

Figure 6 illustrates this loss of global viability: the area colored in grey represents the values of the initial price for which a cooperative of myopic agents is viable according to the value of $\frac{C}{p_0 w}$. When maximizing agents are included in the mixture, the area of initial price for which the mixture is viable is reduced to the left side of the vertical lines; their positions depend on the proportion of maximazing agents described by parameter B.

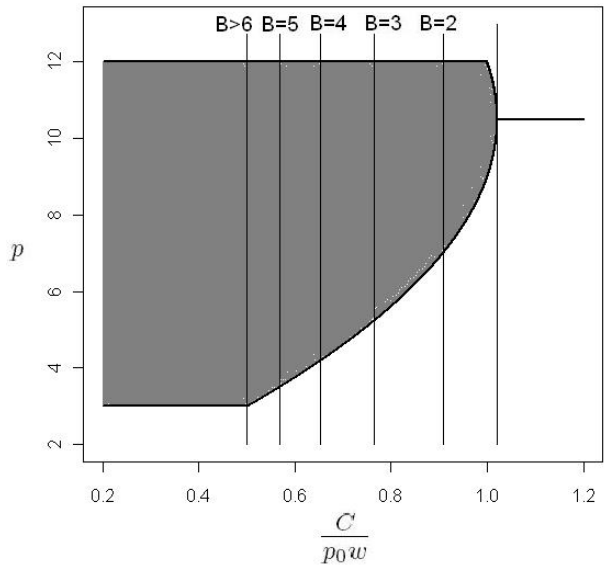


Figure 6: The upper curve represents $p_{Z\,max}$ according to $\frac{C}{p_0 w}$. The lower curve represents $p_{Z\,min}$ according to $\frac{C}{p_0 w}$. The area which is colored grey represents the values of the initial price for which a cooperative of myopic agents is viable. When maximizing agents are included in the mixture, the area of initial price for which the mixture is viable is reduced to the left side of the vertical lines. $p_0 = 12$, $\bar{\alpha} = 2$ and from (43), the position of the vertical lines is defined by $\frac{C}{p_0 w} = \frac{(1+A)^2}{A}\frac{B}{(1+B)^2}$ when $B \leq \bar{\alpha}(\bar{\alpha}+1) = 6$ and by $\frac{C}{p_0 w} = \frac{1}{\bar{\alpha}}$ when $B > \bar{\alpha}(\bar{\alpha}+1) = 6$.

5.3.2. Mixture of myopic and pro-viable agents

Theorems on pro-viable agents individual viability cannot help to derive any conclusion on the viability of a mixture of myopic and pro-viable agents. It is necessary to study the viability of the mixture through simulation. However,

Lemma 1. *With condition (41), the area Z for which the myopic agents are viable is included in the viability kernel of each pro-viable agent.*

PROOF. Let $(K, p) \in Z$, $K \geq K_{min}$ and $c(p) \geq 0$. So, $\Delta K(p, 0) = c(p) \geq 0$ and the evolution such that $(K(0) = K, p(0) = p)$ and for all $t \in \mathbb{N}$, $\Delta p(t) = 0$ is viable. ■

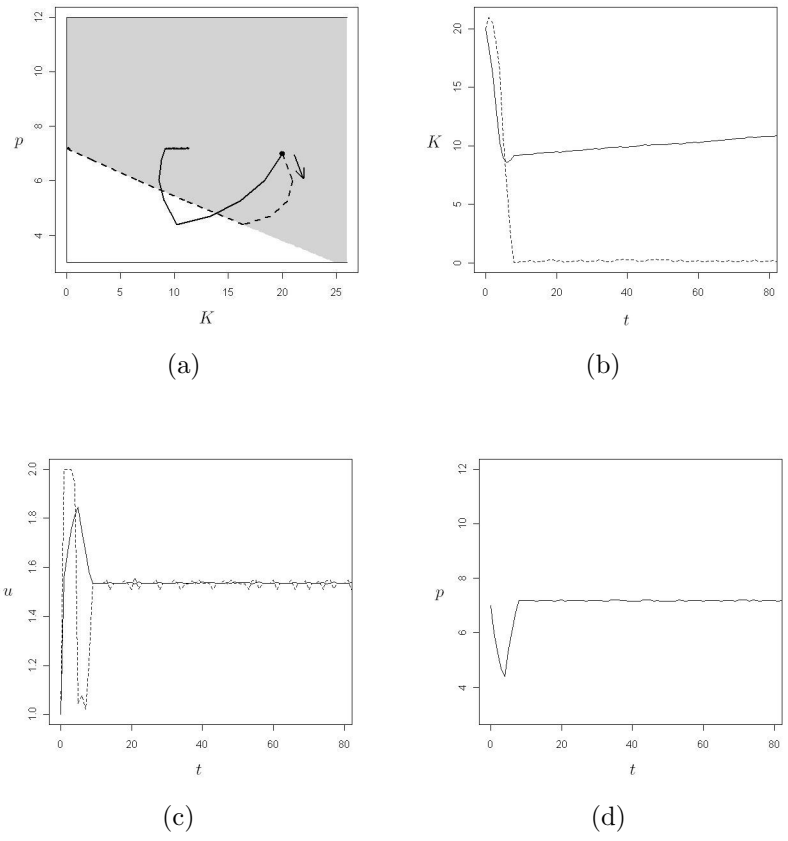
To highlight the influence of the pro-viable agents on the collective viability, we now study the behavior of a mixture of myopic and pro-viable agents in a parameter range such that myopic agents alone can be viable (depending on the initial value of p) and such that no mixture with maximizing agents is ever viable. From the preceding subsection, we choose then simulation parameters which satisfy conditions (42) or (43).

In particular, we have chosen:

- $p_0 = 12$
- $\bar{\alpha} = 2$, and therefore $A = \frac{3}{4}$
- $w = 1$
- $C = 11$
- $\delta = 0.5$
- $K_{min} = 0$ for the pro-viable agents
- $K(0) = 20$ is the same for all pro-viable agents
- $N = 24$ (similar results are obtained with other values for N)
- $k = 4$ $(B = 6)$, $k = 6$ $(B = 4)$, $k = 8$ $(B = 3)$, $k = 10$ $(B = 2.4)$.

With these parameters, the conditions expressed in (43) are verified and we have $p_{Z_{min}} = \frac{21-3\sqrt{5}}{2}$ (39).

As in the mixture with myopic and maximizing agents, we assume that pro-viable agents have an accurate perception of the other members' behavior $(B_i = B = N/k)$.
All simulation results indicate that:

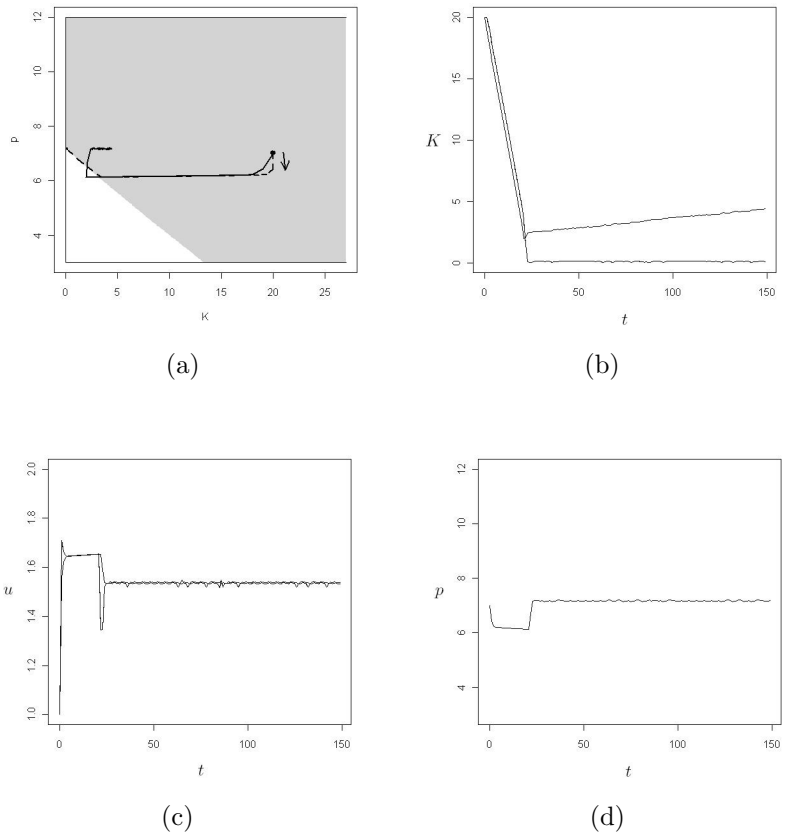


(a) (b)

(c) (d)

Figure 7: The parameter values are $N = 24$, $k = 6$ ($B = 4$), $K(0) = 20$ and $p(0) = 7$. Figure 7(a) displays the trajectory of the evolution of the global price and the capital of the pro-viable agents (dotted line) and of the global price and the capital of the myopic agents (plain line) in the (K, p)-plane. The viability kernel of the pro-viable agents is colored gray. Figure 7(b) show the evolution of the capital of the pro-viable agents (dotted line) and of the capital of the myopic agents (plain line) according to time. Figure 7(c) show the evolution of the fraud level of the pro-viable agents (dotted line) and of the fraud level of the myopic agents (plain line) according to time. Figure 7(d) show the evolution of the global price according to time.

- Pro-viable agents begin with increasing their capital when it is possible (at least when $p(t) \geq p_{Z_{min}}$ but not only, see figure 9). This is achieved by increasing their fraud level, therefore $p(t)$ decreases as does the myopics' capital (since their fraud level is always lower than the pro-



(a) (b)

(c) (d)

Figure 8: The parameter values are $N = 24$, $k = 10$ ($B = 2.4$), $K(0) = 20$ and $p(0) = 7$. Figure 8(a) displays the trajectory of the evolution of the global price and the capital of the pro-viable agents (dotted line) and of the global price and the capital of the myopic agents (plain line) in the (K, p)-plane. The viability kernel of the pro-viable agents is colored gray. Figure 8(b) show the evolution of the capital of the pro-viable agents (dotted line) and of the capital of the myopic agents (plain line) according to time. Figure 8(c) show the evolution of the fraud level of the pro-viable agents (dotted line) and of the fraud level of the myopic agents (plain line) according to time. Figure 8(d) show the evolution of the global price according to time.

viable agents fraud level at this stage) (Fig. 7(a), 8(a), 9(a) and 10(a)).

- When $p(t)$ becomes too low, pro-viable agents behave as maximizing agents as long as they are not near the boundary of the viability kernel: They keep an almost constant fraud level, therefore $p(t)$ remains

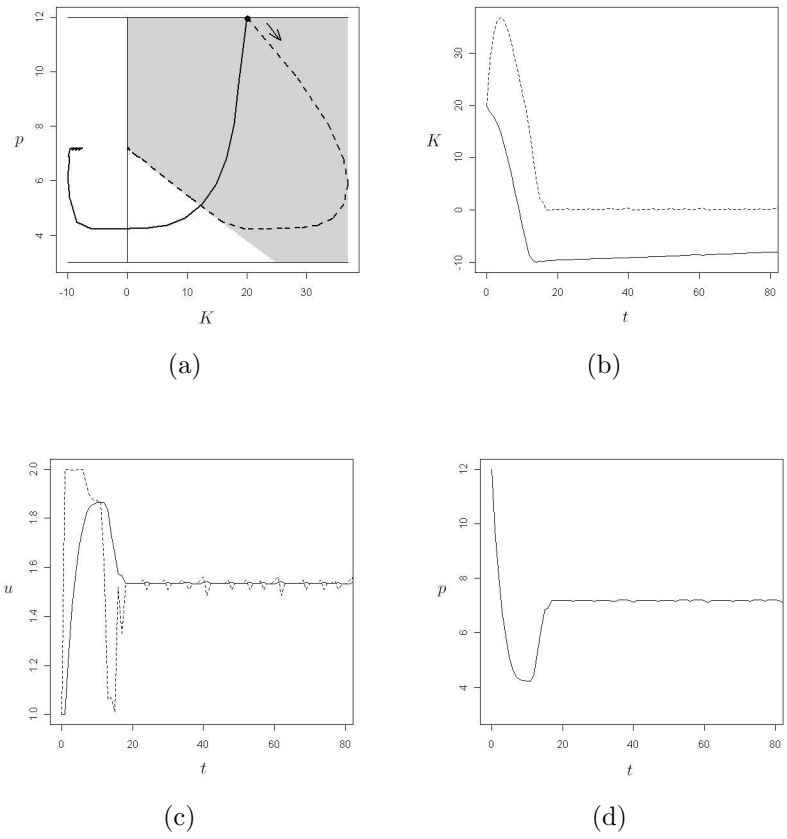


Figure 9: The parameter values are $N = 24$, $k = 6$ ($B = 4$), $K(0) = 20$ and $p(0) = 12$. Figure 9(a) displays the trajectory of the evolution of the global price and the capital of the pro-viable agents (dotted line) and of the global price and the capital of the myopic agents (plain line) in the (K, p)-plane. The viability kernel of the pro-viable agents is colored gray. Figure 9(b) show the evolution of the capital of the pro-viable agents (dotted line) and of the capital of the myopic agents (plain line) according to time. Figure 9(c) show the evolution of the fraud level of the pro-viable agents (dotted line) and of the fraud level of the myopic agents (plain line) according to time. Figure 9(d) show the evolution of the global price according to time.

constant (Fig. 7(d), 8(d), 9(d) and 10(d)) but all agents loose money (Fig. 7(b), 8(b), 9(b) and 10(b)).

- When the pro-viable agents reach the boundary of their viability kernel, they change radically their behavior (Fig. 7(a), 8(a), 9(a) and

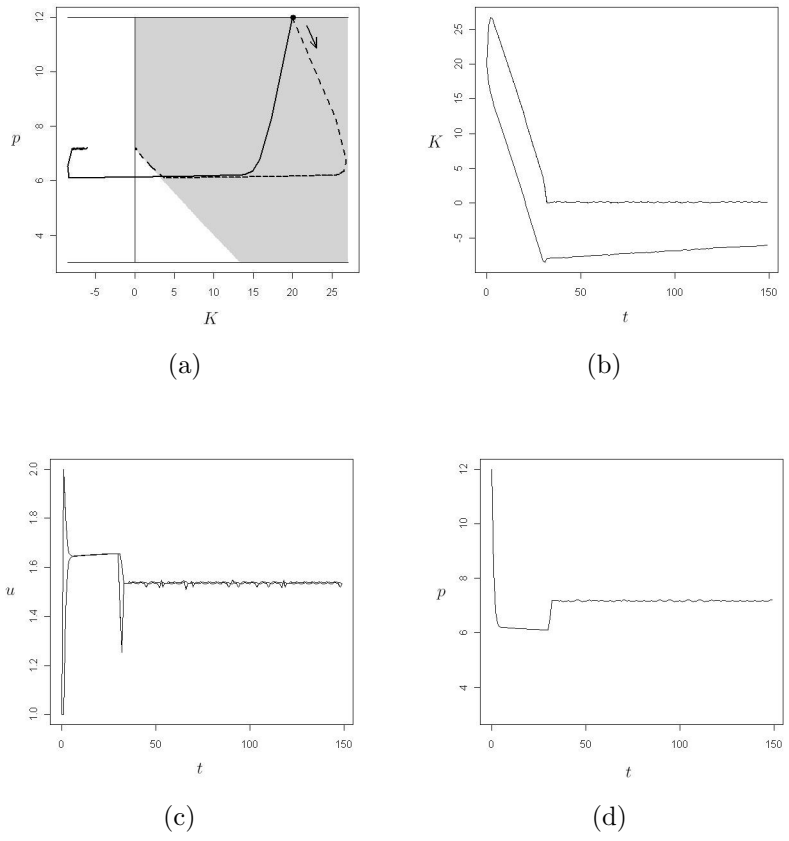


Figure 10: The parameter values are $N = 24$, $k = 10$ ($B = 2.4$), $K(0) = 20$ and $p(0) = 12$. Figure 10(a) displays the trajectory of the evolution of the global price and the capital of the pro-viable agents (dotted line) and of the global price and the capital of the myopic agents (plain line) in the (K, p)-plane. The viability kernel of the pro-viable agents is colored gray. Figure 10(b) show the evolution of the capital of the pro-viable agents (dotted line) and of the capital of the myopic agents (plain line) according to time. Figure 10(c) show the evolution of the fraud level of the pro-viable agents (dotted line) and of the fraud level of the myopic agents (plain line) according to time. Figure 10(d) show the evolution of the global price according to time.

10(a)). They reduce their fraud level until $p(t)$ becomes close to $p_{Z_{min}}$ (Fig. 7(d), 8(d), 9(d) and 10(d)). In this phase, myopic agents loose less money than the pro-viable ones. Pro-viable agents evolve on the boundary of the viability kernel until they reach the constraint bound-

ary $K = 0$. After that their fraud level stays very close to $u(p_{Z_{min}})$ (19) (Fig. 7(c), 8(c), 9(c) and 10(c)). Myopic agents no longer loose money.

- Myopic agents' capital has a lower bound, which can be negative, depending on $p(0)$, $K(0)$, and $\frac{k}{N}$ (Fig. 7(b), 8(b), 9(b) and 10(b)).

The experiment suggests that when $p(0)$ and $K(0)$ are chosen inside the viability kernel of pro-viable agents, then the myopic agents' capital has always a lower bound. This means that **pro-viable agents increase the viability area of myopic agents**. The viable area gained by myopic agents depends on the proportion of pro-viable agents, as shown in Figure 2(b). With one single pro-viable agent here $B = 24$ the area gained is rather small, but with $B = 2.4$ and $K(0) = 20$ it is possible to choose as initial value of price the lowest price $p(0) = \frac{p_0}{\bar{\alpha}^2}$.

We then conjecture that the cooperative is strongly viable as soon as the myopic agents K_{min} is lower than this lower bound.

Figures 7 and 8 show how the cooperative agents evolve for different proportions of pro-viable agents (respectively 1/4 and 5/12, other parameters give similar results and figures). These are cases where myopic agents alone are not viable ($p(0) = 7$ does not belong to Z). In all cases the cooperative becomes viable (since $(p(0) = 7, K(0) = 20)$ is inside the local viability kernel of pro-viable agents). However the minimum capital of myopic agents is lower as the proportion of pro-viable agents increases as shown in Table 1. When there are only pro-viable agents left, their minimum capital is 0.

N	k	$\min_t\{K(t)\}$ for all myopic agents when $p(0) = 7$	$\min_t\{K(t)\}$ for all myopic agents when $p(0) = 12$
24	4	10.83	-17.17
24	6	8.56	-10.05
24	8	4.46	-9.06
24	10	2.00	-8.54

Table 1: Minimal capital value over time, $\min_t\{K(t)\}$, for all myopic agents depending on the proportion of pro-viable agents.

When $p(0) \in Z$, for example $p(0) = 12$, myopic agents are viable by themselves (their income is positive at each time step). In a mixture with pro-viable agents their capital can be lower than when they are by themselves. It can even be negative. Figures 9 and 10 show for different proportions

of pro-viable agents (respectively 1/4 and 5/12) how myopic agents capital first decreases to a minimum and then increases slightly (other values for k/N satisfying condition (43) give similar shapes). The minimum capital of myopic agents is negative but increases with the proportion of pro-viable agents as shown in Table 1.

6. Conclusion

We have studied a non classical viability problem which encompasses N linked local viability problems and which is then in general non tractable by the existing viability algorithms due to the high dimensionality. We consider the case of a stylized agricultural cooperative. The cooperative members income is a function of their economic ratio $\lambda = \frac{C}{wp_0}$, their fraud level (between 1 and $\bar{\alpha}$) and the price which depends on the average fraud level. Three kinds of strategies are taken into account: Myopic agents (which imitate the average behavior), maximizing agents and pro-viable agents. These last two types of agents use a model of the other's behavior characterized by the proportion $(1/B)$ of agents acting like themselves.

We have proved that the strategy of maximizing agents does not always correspond to maximal fraud, and we have identified the range of values of parameters for which this behavior prevails. In particular, this happens when the proportion of maximizing agents is high and the price is not too high. We have also proved that for some range of values of the proportion $1/B$ and of the micro-economic ratio $\lambda = \frac{C}{wp_0}$, the maximizing strategy is never viable.

Regarding pro-viable agents, which maximize their income while ensuring their own viability, we have proved that the local viability kernel is not empty if and only if the micro-economic ratio λ is below a threshold, $\hat{\lambda}$. This threshold does not depend on the proportion $1/B$, it depends only on the maximum fraud level $\bar{\alpha}$.

Regarding a cooperative of myopic agents with same micro-economic ratio, we have proved that the cooperative is never viable when the micro-economic ratio is above the same threshold value $\hat{\lambda}$. When it is lower than this threshold, we have proved that the cooperative is viable if and only if the initial price belongs to a specific interval.

Regarding a cooperative composed of myopic and maximizing agents with identical economic ratio and with an accurate perception of the other members' behavior, we have proved that when the maximizing strategy is not

viable, then the myopic agents are no more viable either.

Regarding a cooperative composed of myopic agents and identical (same $K(0)$, w, C, B) pro-viable agents, with same micro-economic ratio $\frac{C}{wp_0}$ for both pro-viable and myopic type of agents, we have proved that the local viability kernel of pro-viable agents contains the viability area of the cooperative with only myopic agents.

Besides, we have illustrated by simulation the behavior of such a cooperative of myopic and pro-viable agents with an accurate perception of the other members' behavior. The simulation results show how in some cases pro-viable agents increase the viability area of myopic agents.

Acknowledgment

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Appendix A. Proof of theorem 1

We recall the statement of the theorem :

Let $f :]0;2] \to \mathbb{R}$ such that $f(A) = \frac{(1+A)^2}{4A}$ if $A \leq 1$ and $f(A) = 1$ otherwise.
$\text{Viab}_{(14)}(L) \neq \emptyset$ if and only if $\frac{C}{p_0 w} \leq f(A)$.

We recall that the equation which governs the evolution of K according to p and the variation of p, Δp, is :

$$\begin{array}{rcl} \Delta K(p,\Delta p) & = & -\frac{BDw}{p_0 A}\Delta p^2 + \frac{w}{p_0 A}(p_0(1+A) - p(1+BD))\Delta p + \frac{pw}{p_0 A}(p_0(1+A)-p) - C \\ & & \text{if } \Delta p \geq 0 \\ \Delta K(p,\Delta p) & = & -\frac{Bw}{p_0 A}\Delta p^2 + \frac{w}{p_0 A}(p_0(1+A) - p(1+B))\Delta p + \frac{pw}{p_0 A}(p_0(1+A)-p) - C \\ & & \text{if } \Delta p \leq 0. \end{array} \tag{A.1}$$

where $A = \frac{\bar{\alpha}+1}{\bar{\alpha}^2} > 0$, $D := 1/(1-\delta) \geq 1$ since δ which measures the influence of reputation belongs to $[0;1[$, and $B > 1$ is the inverse of the production proportion $1/B$ that experiences the new fraud level, whereas the remaining proportion, $1 - 1/B$ experiences a fraud level equal to the previous fraud average.

We also recall that $c(p) := \Delta K(p,0)$ and :

$$c(p) := \frac{pw}{p_0 A}(p_0(1+A) - p) - C \tag{A.2}$$

Moreover, when u ranges from 1 to $\bar{\alpha}$, Δp ranges from $\Delta p_{min}(p)$ to $\Delta p_{max}(p)$ with, from (14),

$$\Delta p_{min}(p) = 1/B(p_0/\bar{\alpha}^2 - p) \tag{A.3}$$

and

$$\Delta p_{max}(p) = 1/(BD)(p_0 - p). \tag{A.4}$$

Proof. Let $\tilde{C} = \max_{p \in [\frac{p_0}{\alpha^2}; p_0]} c(p)$ and $\tilde{p} = \text{argmax}_{p \in [\frac{p_0}{\alpha^2}; p_0]} c(p)$.

If the condition of the theorem is satisfied, that is $\frac{C}{p_0 w} \leq f(A)$, then $\tilde{C}$ is positive.
Let define the evolution $(K(n), p(n))$, $n \in \mathbb{N}$ governed by (14) with $K(0) = K_{min}$ and $p(0) = \tilde{p}$ and $\forall n$, $u(n) = \frac{1}{p_0 A}(p_0(1+A) - p(0))$ such that $p(n+1) = p(n) = p(0) = \tilde{p}$ from (19).

Then, $\forall n,\ K(n+1) - K(n) = \Delta K(\tilde{p}, 0) = c(\tilde{p}) = \tilde{C} \geq 0$.
Consequently, $\forall n,\ K(n) \geq K(0) = K_{min}$. The evolution $(K(n), p(n))$ remains in the constraint set L, $(K_{min}, \tilde{p}) \in \text{Viab}_{(14)}(L)$ and $\text{Viab}_{(14)}(L) \neq \emptyset$.

If the condition of the theorem is not satisfied, $\frac{C}{p_0 w} > f(A)$, then $\tilde{C} < 0$ and $\forall p \in [\frac{p_0}{\alpha^2}; p_0]$, $\Delta K(p, 0) = c(p) \leq \tilde{C} < 0$. Moreover, since the parabola which represents $\Delta K(p, .)$ as a function of Δp is located below its tangents, from (A.1), $\Delta K(p, \Delta p)$ remains lower than $\tilde{C}/2$ when

$$|\Delta p| \leq \frac{|\tilde{C}|}{2|\frac{w}{p_0 A}(p_0(1+A) - p(1+BD))|}. \tag{A.5}$$

Since function $p \to |\frac{w}{p_0 A}(p_0(1+A) - p(1+BD))|$ is bounded on $[\frac{p_0}{\alpha^2}; p_0]$, there exists $\tilde{\Delta p} > 0$ such that, for all $p \in [\frac{p_0}{\alpha^2}; p_0]$, $|\Delta p| \leq \tilde{\Delta p}$ implies $\Delta K(p, \Delta p) < \tilde{C}/2$:

$$\tilde{\Delta p} = \frac{|\tilde{C}|}{2 \max_{p \in [p_0/\alpha^2; p_0]} |\frac{w}{p_0 A}(p_0(1+A) - p(1+BD))|}. \tag{A.6}$$

Let define S by:

$$S := \{(\Delta, p) \in \mathbb{R}^+ \times [\frac{p_0}{\alpha^2}; p_0] \mid \Delta > 0 \text{ and } p + \Delta \leq p_0 \text{ and } p - \Delta \geq \frac{p_0}{\alpha^2}\}, \tag{A.7}$$

For all pairs $(\Delta, p) \in S$, we consider all pairs $(p_1, \Delta p_1) \in S_1(\Delta, p)$ where

$$\begin{aligned} S_1(\Delta, p) := \{(p_1, \Delta p_1) \in\ & [\tfrac{p_0}{\alpha^2}; p_0] \times [0; \Delta p_{max}(p_1)] \mid \\ & p_1 \leq p - \Delta \text{ and } p_1 + \Delta p_1 \geq p + \Delta\}. \end{aligned} \tag{A.8}$$

More precisely, from (A.4), $p_1 \in [\max(\frac{p0}{\alpha^2}, \frac{p + \Delta - p_0/(BD)}{1 - 1/(BD)}), p - \Delta]$ and $\Delta p_1 \in [p + \Delta - p_1; 1/(BD)(p_0 - p_1)]$ (Figure A.11).

We now aim at maximizing $\frac{\Delta K(p_1, \Delta p_1)}{\Delta p_1}$, the marginal capital growth, for $(p_1, \Delta p_1) \in S_1(\Delta, p)$.

$\frac{\Delta K(p_1, \Delta p_1)}{\Delta p_1} = -\frac{BDw}{p_0 A}\Delta p_1 + \frac{w}{p_0 A}(p_0(1+A) - p_1(1+BD)) + \frac{\frac{p_1 w}{p_0 A}(p_0(1+A) - p_1) - C}{\Delta p_1}$

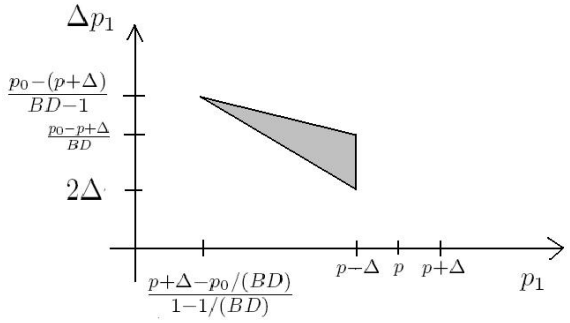

Figure A.11: The area $S_1(\Delta, p)$ is colored grey.

We differentiate $\frac{\Delta K(p_1,\Delta p_1)}{\Delta p_1}$ according to p_1 and Δp_1:

$$\frac{\partial}{\partial \Delta p_1}\frac{\Delta K(p_1,\Delta p_1)}{\Delta p_1} \quad = \quad -\frac{BDw}{p_0 A}+\frac{C-\frac{p_1 w}{p_0 A}(p_0(1+A)-p_1)}{\Delta p_1^2}. \tag{A.9}$$

Then,

$$\frac{\partial}{\partial \Delta p_1}\frac{\Delta K(p_1,\Delta p_1)}{\Delta p_1}=0 \Leftrightarrow \Delta p_1^2 = 1/(BD)(p_1^2-p_0(1+A)p_1+\frac{p_0 AC}{w}). \tag{A.10}$$

Moreover,

$$\frac{\partial}{\partial p_1}\frac{\Delta K(p_1,\Delta p_1)}{\Delta p_1} \quad = \quad -\frac{w}{p_0 A}(1+BD)+\frac{w}{A\Delta p_1}(1+A)-\frac{2w}{p_0 A\Delta p_1}p_1. \tag{A.11}$$

Then,

$$\frac{\partial}{\partial p_1}\frac{\Delta K(p_1,\Delta p_1)}{\Delta p_1}=0 \Leftrightarrow \Delta p_1 = 1/(1+BD)(p_0(1+A)-2p_1). \tag{A.12}$$

Thus,

$$\frac{\partial}{\partial \Delta p_1}\frac{\Delta K(p_1,\Delta p_1)}{\Delta p_1}=\frac{\partial}{\partial p_1}\frac{\Delta K(p_1,\Delta p_1)}{\Delta p_1}=0 \tag{A.13}$$

amounts to

$$\left(1/(1+BD)(p_0(1+A)-2p_1)\right)^2 = 1/(BD)(p_1^2-p_0(1+A)p_1+\frac{p_0 AC}{w}) \tag{A.14}$$

and

$$-(BD-1)^2 p_1^2+p_0(1+A)(BD-1)^2 p_1+p_0(p_0(1+A)^2 BD-(1+BD)^2 AC/w)=0. \tag{A.15}$$

Let $g(p_1) = -(BD-1)^2 p_1^2 + p_0(1+A)(BD-1)^2 p_1 + p_0(p_0(1+A)^2 BD - (1+BD)^2 AC/w)$. The top of the parabola g is reached for $p_0 \frac{1+A}{2}$.
Since $\frac{C}{p_0 w} > f(A)$,

- case 1 : $A \leq 1$, then the discriminant of the quadratic trinomial, g, equals

  $$p_0^2(1+A)^2(BD-1)^4 + 4(BD-1)^2 p_0(p_0(1+A)^2 BD - (1+BD)^2 AC/w)$$

  and is strictly negative since $\frac{C}{p_0 w} > \frac{(1+A)^2}{4A}$, which implies

  $$p_0(1+A)^2 BD - (1+BD)^2 AC/w < -p_0 \frac{(1+A)^2}{4}(BD-1)^2.$$

  Consequently, $\frac{\Delta K(p_1, \Delta p_1)}{\Delta p_1}$ does not reach any local maximum in the interior of $S_1(\Delta, p)$,

- case 2 : $A \geq 1$, then g reaches its maximum for $p_1 = p_0 \frac{1+A}{2} > p_0$. The discriminant of g can be positive, but then the p_1-value for which g is null corresponds to a local minimum of $\frac{\Delta K(p_1, \Delta p_1)}{\Delta p_1}$ and again, $\frac{\Delta K(p_1, \Delta p_1)}{\Delta p_1}$ does not reach any local maximum in the interior of $S_1(\Delta, p)$.

Consequently, the maximum of $\frac{\Delta K(p_1, \Delta p_1)}{\Delta p_1}$ for $(p_1, \Delta p_1) \in S_1(\Delta, p)$ belongs to the boundary of the domain $S_1(\Delta, p)$ (Figure A.11).
On the upper boundary, $\Delta p_1 = 1/(BD)(p_0 - p_1)$, then,

$$\frac{\Delta K(p_1, 1/(BD)(p_0-p_1))}{1/(BD)(p_0-p_1)} = w \frac{p_1(BD-1) + p_0 - CBD/w}{p_0 - p_1} \tag{A.16}$$

and

$$\begin{aligned} \frac{d}{dp_1} \frac{\Delta K(p_1, 1/(BD)(p_0-p_1))}{1/(BD)(p_0-p_1)} &= wBD \frac{p_0 - C/w}{(p_0-p_1)^2} \\ &= \frac{BD}{(p_0-p_1)^2} c(p_0) < 0. \end{aligned} \tag{A.17}$$

So $\frac{\Delta K(p_1, 1/(BD)(p_0-p_1))}{1/(BD)(p_0-p_1)}$ decreases with p_1 on the upper boundary of $S_1(\Delta, p)$.
On the lower boundary, $\Delta p_1 = p + \Delta - p_1$,

$$\frac{\Delta K(p_1, p+\Delta-p_1)}{p+\Delta-p_1} = \frac{w}{p_0 A} \frac{(p_1(BD-1) + p_0(1+A))(p+\Delta) - BD(p+\Delta)^2 - C}{p+\Delta-p_1} \tag{A.18}$$

and

$$\begin{aligned} \frac{d}{dp_1} \frac{\Delta K(p_1, p+\Delta-p_1)}{p+\Delta-p_1} &= \frac{w}{Ap_0} \frac{-(p+\Delta)^2 + p_0(1+a)(p+\Delta) - C}{(p+\Delta-p_1)^2} \\ &= \frac{1}{(p+\Delta-p_1)^2} c(p+\Delta) < 0. \end{aligned} \tag{A.19}$$

So $\frac{\Delta K(p_1,p+\Delta-p_1)}{p+\Delta-p_1}$ decreases with p_1 on the lower boundary of $S_1(\Delta,p)$.

Moreover, for all $\hat{p} \in [p+\Delta; p+\Delta+1/(BD)(p_0-p+\Delta)-2\Delta]$,

$$\begin{aligned} \frac{d}{dp_1}\frac{\Delta K(p_1,\hat{p}-p_1)}{\hat{p}-p_1} &= \frac{w}{Ap_0}\frac{-\hat{p}^2+p_0(1+a)\hat{p}-C}{(\hat{p}-p_1)^2} \\ &= \frac{1}{(\hat{p}-p_1)^2}c(\hat{p}) < 0. \end{aligned} \tag{A.20}$$

Thus,

$$\begin{aligned} \max_{(p_1,\Delta p_1)\in S_1(\Delta,p)} \frac{\Delta K(p_1,\Delta p_1)}{\Delta p_1} &\leq \frac{\Delta K(\frac{p+\Delta-p_0/(BD)}{1-1/(BD)}, \frac{p_0-(p+\Delta)}{BD-1})}{\frac{p_0-(p+\Delta)}{BD-1}} \\ &\leq w\frac{\frac{p+\Delta-p_0/(BD)}{1-1/(BD)}(BD-1)+p_0-CBD/w}{p_0-\frac{p+\Delta-p_0/(BD)}{1-1/(BD)}} \\ &\leq w(BD-1)\frac{p+\Delta-\frac{C}{w}}{p_0-(p+\Delta)} \end{aligned} \tag{A.21}$$

Moreover,

$$\frac{\partial\left(\frac{p+\Delta-\frac{C}{w}}{p_0-(p+\Delta)}\right)}{\partial\Delta}(\Delta,p) = \frac{p_0-\frac{C}{w}}{(p_0-(p+\Delta))^2} < 0, \tag{A.22}$$

when $\frac{C}{p_0 w} > f(A)$.

So, let define $S(p) := \{\Delta \text{ such that } (p,\Delta) \in S\}$,

$$\begin{aligned} \max_{\substack{\Delta\in S(p)\\(p_1,\Delta p_1)\in S_1(\Delta,p)}} \frac{\Delta K(p_1,\Delta p_1)}{\Delta p_1} &\leq w(BD-1)\lim_{\Delta\to 0^+}\left(\frac{p+\Delta-\frac{C}{w}}{p_0-(p+\Delta)}\right) \\ &\leq w(BD-1)\frac{p-\frac{C}{w}}{p_0-p} \\ &\leq w(B-1)\frac{p-\frac{C}{w}}{p_0-p} := f_1(p). \end{aligned} \tag{A.23}$$

$f_1(p)$ is defined for $p \in [p_0/\alpha^2; p_0[$ and admits an upper bound, $\max_{p\in[p_0/\alpha^2;p_0[} f_1(p) \leq F_1 < +\infty$.

Given $(\Delta,p) \in S$, let us now consider all pairs $(p_2,\Delta p_2) \in S_2$, where

$$\begin{aligned} S_2(\Delta,p) := \{(p_2,\Delta p_2) \in\ & [\tfrac{p_0}{\alpha^2}; p_0] \times [\Delta p_{min}(p_2); 0] \mid \\ & p_2 \geq p+\Delta \text{ and } p_2+\Delta p_2 \leq p-\Delta\}. \end{aligned} \tag{A.24}$$

More precisely, $p_2 \in [p+\Delta; \min(p_0, \frac{p-\Delta-p_0/(B\alpha^2)}{1-1/B})]$ and $\Delta p_2 \in [1/B(p_0/\alpha^2 - p_2); p-\Delta-p_2]$.

We now aim at minimizing $\frac{\Delta K(p_2,\Delta p_2)}{\Delta p_2}$, the marginal capital growth, for $(p_2, \Delta p_2) \in S_2(\Delta, p)$.

Following the same reasoning as above, we show that

$$\min_{(p_2,\Delta p_2)\in S_2(\Delta,p)} \frac{\Delta K(p_2,\Delta p_2)}{\Delta p_2} \geq \frac{\Delta K(\frac{p-\Delta-p_0/(B\alpha^2)}{1-1/B}, \frac{p_0/\alpha^2-(p-\Delta)}{B-1})}{\frac{p_0/\alpha^2-(p-\Delta)}{B-1}} \tag{A.25}$$

for all $(\Delta, p) \in S$.

On the upper boundary of S_2, $\Delta p_2 = 1/B(\frac{p_0}{\alpha^2} - p_2)$, then,

$$\frac{\Delta K(p_2, 1/B(\frac{p_0}{\alpha^2} - p_2))}{1/B(\frac{p_0}{\alpha^2} - p_2)} = \alpha w \frac{p_2(B-1) + \frac{p_0}{\alpha^2} - CB/(\alpha w)}{\frac{p_0}{\alpha^2} - p_2} \tag{A.26}$$

Then,

$$\min_{(p_2,\Delta p_2)\in S_2(\Delta,p)} \frac{\Delta K(p_2,\Delta p_2)}{\Delta p_2} \geq w(B-1)\frac{\alpha(p-\Delta) - \frac{C}{w}}{p_0/\alpha^2 - (p-\Delta)} \tag{A.27}$$

Since $\frac{\alpha(p-\Delta)-\frac{C}{w}}{p_0/\alpha^2-(p-\Delta)}$ increases with Δ,

$$\begin{array}{rcl} \min\limits_{\substack{\Delta\in S(p)\\(p_2,\Delta p_2)\in S_2(\Delta,p)}} \frac{\Delta K(p_2,\Delta p_2)}{\Delta p_2} & \geq & w(B-1)\lim_{\Delta\to 0^+}\left(\frac{\alpha(p-\Delta)-\frac{C}{w}}{p_0/\alpha^2-(p-\Delta)}\right) \\ & \geq & w(B-1)\frac{\alpha p-\frac{C}{w}}{p_0/\alpha^2-p} := f_2(p) \end{array} \tag{A.28}$$

$f_2(p)$ is defined for $p \in]p_0/\alpha^2; p_0]$ and admits a lower bound, $\max_{p\in]p_0/\alpha^2;p_0]} f_2(p) \geq F_2 > -\infty$.

Moreover,

$$\begin{array}{rcl} f_1(p) - f_2(p) & \leq & w(B-1)\left[\frac{p-C/w}{p_0-p} - \frac{\alpha p-C/w}{p_0/\alpha^2-p}\right] \\ & \leq & w(B-1)\frac{(1-\alpha)p^2+pp_0(\alpha-1/\alpha^2)+Cp_0/w(1/\alpha^2-1)}{(p_0-p)(p-p_0/\alpha^2)} \\ & \leq & w(B-1)(\alpha-1)\frac{-p^2+pp_0(1+A)-Cp_0A/w}{(p_0-p)(p-p_0/\alpha^2)} \\ & \leq & w(B-1)(\alpha-1)\frac{p_0A/w\ c(p)}{(p_0-p)(p-p_0/\alpha^2)} \\ & \leq & p_0A(B-1)(\alpha-1)\frac{\tilde{C}}{(p_0-p)(p-p_0/\alpha^2)} \\ & \leq & 4\frac{B-1}{p_0A(\alpha-1)}\tilde{C} \\ & < & 0. \end{array} \tag{A.29}$$

Let $(K(n), p(n))$, $n \in \mathbb{N}$ an evolution governed by (14), we aim at showing that $K(n)$ is lower than a sequence which tends toward $-\infty$ when n goes to infinity.

If $p(i+1) - p(i) = 0$, $K(i+1) - K(i) = \Delta K(p(i), 0) = c(p(i)) < 0$ since $\frac{C}{p_0 w} < f(A)$, so we consider the more favourable case where $\forall i\ p(i+1) - p(i) \neq 0$.

$$\begin{aligned} K(n) &= K(0) + \sum_{i=0}^{n-1} K(i+1) - K(i) \\ &= K(0) + \sum_{i=0}^{n-1} \int_{p(i)}^{p(i+1)} \frac{K(i+1)-K(i)}{p(i+1)-p(i)} dp \end{aligned} \tag{A.30}$$

If $p(i+1) - p(i) > 0$, $\frac{K(i+1)-K(i)}{p(i+1)-p(i)} \leq \max\limits_{\substack{\Delta \in S(p) \\ (p_1, \Delta p_1) \in S_1(\Delta, p)}} \frac{\Delta K(p_1, \Delta p_1)}{\Delta p_1} \leq f_1(p)$, for all $p \in [p(i); p(i+1)]$.

If $p(i+1) - p(i) < 0$, $\frac{K(i+1)-K(i)}{p(i+1)-p(i)} \geq \min\limits_{\substack{\Delta \in S(p) \\ (p_2, \Delta p_2) \in S_2(\Delta, p)}} \frac{\Delta K(p_2, \Delta p_2)}{\Delta p_2} \geq f_2(p)$, for all $p \in [p(i); p(i+1)]$.

Consequently,

$$\begin{aligned} K(n) &= K(0) + \sum_{i=0}^{n-1} \int_{p(i)}^{p(i+1)} \frac{K(i+1)-K(i)}{p(i+1)-p(i)} dp \\ &= K(0) + \sum_{\substack{i=0 \\ p(i+1)-p(i)>0}}^{n-1} \int_{p(i)}^{p(i+1)} \frac{K(i+1)-K(i)}{p(i+1)-p(i)} dp + \sum_{\substack{i=0 \\ p(i+1)-p(i)<0}}^{n-1} \int_{p(i)}^{p(i+1)} \frac{K(i+1)-K(i)}{p(i+1)-p(i)} dp \\ &= K(0) + \sum_{\substack{i=0 \\ p(i+1)-p(i)>0}}^{n-1} \int_{p(i)}^{p(i+1)} \frac{K(i+1)-K(i)}{p(i+1)-p(i)} dp + \sum_{\substack{i=0 \\ p(i+1)-p(i)<0}}^{n-1} \int_{p(i+1)}^{p(i)} -\frac{K(i+1)-K(i)}{p(i+1)-p(i)} dp \\ &\leq K(0) + \sum_{\substack{i=0 \\ p(i+1)-p(i)>0}}^{n-1} \int_{p(i)}^{p(i+1)} f_1(p) dp + \sum_{\substack{i=0 \\ p(i+1)-p(i)<0}}^{n-1} \int_{p(i+1)}^{p(i)} -f_2(p) dp \end{aligned} \tag{A.31}$$

Assume for instance that $p(n) \geq p(0)$,

$$\begin{aligned} K(n) &\leq K(0) + \sum_{\substack{i=0 \\ p(i+1)-p(i)<0}}^{n-1} \int_{p(i+1)}^{p(i)} (f_1(p) - f_2(p)) dp + \int_{p(0)}^{p(n)} f_1(p) dp \\ &\leq \sum_{\substack{i=0 \\ p(i+1)-p(i)<0}}^{n-1} \int_{p(i)}^{p(i+1)} \frac{4(B-1)}{p_0 A(\alpha-1)} \tilde{C} dp + (p(n) - p(0)) \max_{p \in [p_0/\alpha^2; p_0[} f_1(p) \\ &\leq \frac{4(B-1)}{p_0 A(\alpha-1)} \tilde{C} \frac{1}{2} [\sum_{i=0}^{n-1} |p(i+1) - p(i)| - (p(n) - p(0))] + (p_0 - p_0/\alpha^2) F_1 \end{aligned} \tag{A.32}$$

If $p(n) \leq p(0)$,

$$\begin{aligned} K(n) &\leq K(0) + \sum_{\substack{i=0 \\ p(i+1)-p(i)>0}}^{n-1} \int_{p(i)}^{p(i+1)} (f_1(p) - f_2(p))dp + \int_{p(0)}^{p(n)} -f_2(p)dp \\ &\leq \frac{4(B-1)}{p_0 A(\alpha-1)} \tilde{C} \frac{1}{2} [\sum_{i=0}^{n-1} |p(i+1) - p(i)| - (p(n) - p(0))] + (p_0 - p_0/\alpha^2) F_2. \end{aligned} \tag{A.33}$$

Consequently, there exits $M \in \mathbb{R}$ such that :

$$K(n) \leq K(0) + 2 \sum_{j=1}^{n-1} |\Delta p_j| \frac{B-1}{p_0 A(\alpha-1)} \tilde{C} + M. \tag{A.34}$$

If $\sum_{j=1}^{n-1} |\Delta p_j|$ tends toward $+\infty$ when n goes to infinity, then $K(n)$ tends toward $-\infty$ since $\frac{B-1}{p_0 A(\alpha-1)} \tilde{C} < 0$ when $\frac{C}{p_0 w} > f(A)$.

Otherwise, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $|\Delta p(n)| \leq \tilde{\Delta p}$, and then from (A.6), $K(n+1) - K(n) = \Delta K(p(n), \Delta p(n)) < \tilde{C}/2$ and $K(n)$ tends toward $-\infty$.

In both cases $(K(n), p(n))$ does not remain in the constraint set L and the viability kernel of L for dynamics (14) when $\frac{C}{p_0 w} > f(A)$ is then empty. ■