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Iterated proportional fitting procedure and infinite products of stochastic matrices

Jean Brossard and Christophe Leuridan

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Abstract

The iterative proportional fitting procedure (IPFP), introduced in 1937 by Kruihof, aims to adjust the elements of an array to satisfy specified row and column sums. Thus, given a rectangular non-negative matrix X_0 and two positive marginals a and b , the algorithm generates a sequence of matrices $(X_n)_{n \geq 0}$ starting at X_0 , supposed to converge to a biproportional fitting, that is, to a matrix Y whose marginals are a and b and of the form $Y = D_1 X_0 D_2$, for some diagonal matrices D_1 and D_2 with positive diagonal entries.

When a biproportional fitting does exist, it is unique and the sequence $(X_n)_{n \geq 0}$ converges to it at an at least geometric rate. More generally, when there exists some matrix with marginal a and b and with support included in the support of X_0 , the sequence $(X_n)_{n \geq 0}$ converges to the unique matrix whose marginals are a and b and which can be written as a limit of matrices of the form $D_1 X_0 D_2$.

In the opposite case (when there exists no matrix with marginals a and b whose support is included in the support of X_0), the sequence $(X_n)_{n \geq 0}$ diverges but both subsequences $(X_{2n})_{n \geq 0}$ and $(X_{2n+1})_{n \geq 0}$ converge.

In the present paper, we use a new method to prove again these results and determine the two limit-points in the case of divergence. Our proof relies on a new convergence theorem for backward infinite products $\cdots M_2 M_1$ of stochastic matrices M_n , with diagonal entries $M_n(i, i)$ bounded away from 0 and with bounded ratios $M_n(j, i)/M_n(i, j)$. This theorem generalizes Lorenz' stabilization theorem.

We also provide an alternative proof of Touric and Nedić's theorem on backward infinite products of doubly-stochastic matrices, with diagonal entries bounded away from 0. In both situations, we improve slightly the conclusion, since we establish not only the convergence of the sequence $(M_n \cdots M_1)_{n \geq 0}$, but also its finite variation.

Keywords: infinite products of stochastic matrices - contingency matrices - distributions with given marginals - iterative proportional fitting - relative entropy - I-divergence.

MSC Classification: 15B51 - 62H17 - 62B10 - 68W40.

1 Introduction

1.1 The iterative proportional fitting procedure

The aim of the iterative proportional fitting procedure is to find a non-negative matrix with given row and columns sums and having zero entries at some given places. Fix two integers $p \geq 2$, $q \geq 2$ (namely the sizes of the matrices to be considered) and two vectors $a = (a_1, \dots, a_p)$, $b = (b_1, \dots, b_q)$ with positive components such that $a_1 + \cdots + a_p = b_1 + \cdots + b_q = 1$ (namely the target marginals). We assume that the common value of

the sums $a_1 + \dots + a_p$ and $b_1 + \dots + b_q$ is 1 for convenience only, to enable probabilistic interpretations, but this is not a true restriction.

We introduce the following notations for any $p \times q$ real matrix X :

$$X(i, +) = \sum_{j=1}^q X(i, j), \quad X(+, j) = \sum_{i=1}^p X(i, j), \quad X(+, +) = \sum_{i=1}^p \sum_{j=1}^q X(i, j),$$

and we set $R_i(X) = X(i, +)/a_i$, $C_j(X) = X(+, j)/b_j$.

The IPFP has been introduced in 1937 by Kruithof [9] to estimate telephone traffic between central stations. This procedure starts from a $p \times q$ non-negative matrix X_0 such that the sum of the entries on each row or column is positive (so X_0 has at least one positive entry on each row or column) and works as follows.

- For each $i \in [1, p]$, divide the row i of X_0 by the positive number $R_i(X_0)$. This yields a matrix X_1 satisfying the same assumptions as X_0 and having the desired row-marginals.
- For each $j \in [1, q]$, divide the column j of X_1 by the positive number $C_j(X_1)$. This yields a matrix X_2 satisfying the same assumptions as X_0 and having the desired column-marginals.
- Repeat the operations above starting from X_2 to get X_3, X_4 , and so on.

Denote by $\mathcal{M}_{p,q}(\mathbf{R}_+)$ the set of all $p \times q$ matrices with non-negative entries, and consider the following subsets:

$$\begin{aligned} \Gamma_0 &:= \{X \in \mathcal{M}_{p,q}(\mathbf{R}_+) : \forall i \in [1, p], X(i, +) > 0, \forall j \in [1, q], X(+, j) > 0\}, \\ \Gamma_1 &:= \{X \in \Gamma_0 : X(+, +) = 1\} \\ \Gamma_R &:= \Gamma(a, *) = \{X \in \Gamma_0 : \forall i \in [1, p], X(i, +) = a_i\}, \\ \Gamma_C &:= \Gamma(*, b) = \{X \in \Gamma_0 : \forall j \in [1, q], X(+, j) = b_j\}, \\ \Gamma &:= \Gamma(a, b) = \Gamma_R \cap \Gamma_C. \end{aligned}$$

For every integer $m \geq 1$, denote by Δ_m the set of all $m \times m$ diagonal matrices with positive diagonal entries.

The IPFP consists in applying alternatively the transformations $T_R : \Gamma_0 \rightarrow \Gamma_R$ and $T_C : \Gamma_0 \rightarrow \Gamma_C$ defined by

$$T_R(X)(i, j) = R_i(X)^{-1}X(i, j) \text{ and } T_C(X)(i, j) = C_j(X)^{-1}X(i, j).$$

The homogeneity of the map T_R shows that replacing X_0 with $X_0(+, +)^{-1}X_0$ does not change the matrices X_n for $n \geq 1$, so there is no restriction to assume that $X_0 \in \Gamma_1$.

Note that Γ_R and Γ_C are subsets of Γ_1 and are closed subsets of $\mathcal{M}_{p,q}(\mathbf{R}_+)$. Therefore, if $(X_n)_{n \geq 0}$ converges, its limit belongs to the set Γ . Furthermore, by construction, the matrices X_n belong to the set

$$\begin{aligned} \Delta_p X_0 \Delta_q &= \{D_1 X_0 D_2 : D_1 \in \Delta_p, D_2 \in \Delta_q\} \\ &= \{(\alpha_i X_0(i, j) \beta_j) : (\alpha_1, \dots, \alpha_p) \in (\mathbf{R}_+^*)^p, (\beta_1, \dots, \beta_q) \in (\mathbf{R}_+^*)^q\}. \end{aligned}$$

According to the terminology used by Pretzel, we will say that X_0 and X_n are diagonally equivalent. In particular, the matrices X_n have by construction the same support, where the support of a matrix $X \in \mathcal{M}_{p,q}(\mathbf{R}_+)$ is defined by

$$\text{Supp}(X) = \{(i, j) \in [1, p] \times [1, q] : X(i, j) > 0\}.$$

Therefore, for every limit point L of $(X_n)_{n \geq 0}$, we have $\text{Supp}(L) \subset \text{Supp}(X_0)$ and this inclusion may be strict. In particular, if $(X_n)_{n \geq 0}$ converges, its limit belongs to the set

$$\Gamma(X_0) := \Gamma(a, b, X_0) = \{S \in \Gamma : \text{Supp}(S) \subset \text{Supp}(X_0)\}.$$

When the set $\Gamma(X_0)$ is empty, the sequence $(X_n)_{n \geq 0}$ cannot converge, and no precise behavior was established until 2013, when Gietl and Reffel showed that both subsequences $(X_{2n})_{n \geq 0}$ and $(X_{2n+1})_{n \geq 0}$ converge [7].

In the opposite case, namely when $\Gamma(a, b)$ contains some matrix with support included in X_0 , various proofs of the convergence of $(X_n)_{n \geq 0}$ are known (Bacharach [2] in 1965, Bregman [3] in 1967, Sinkhorn [13] in 1967, Csiszár [5] in 1975, Pretzel [11] in 1980 and others (see [4] and [12] to get an exhaustive review). Moreover, the limit can be described using some probabilistic tools that we introduce now.

1.2 Probabilistic interpretations and tools

At many places, we shall identify a, b and matrices X in Γ_1 with the probability measures on $[1, p]$, $[1, q]$ and $[1, p] \times [1, q]$ given by $a(\{i\}) = a_i$, $b(\{j\}) = b_j$ and $X(\{(i, j)\}) = X(i, j)$. Through this identification, the set Γ_1 can be seen as the set of all probability measures on $[1, p] \times [1, q]$ whose marginals charge every point; the set Γ can be seen as the set of all probability measures on $[1, p] \times [1, q]$ having marginals a and b . This set is non-empty since it contains the probability $a \otimes b$.

The I -divergence, or Kullback-Leibler divergence, also called relative entropy, plays a key role in the study of the iterative proportional fitting algorithm. For every X and Y in Γ_1 , the relative entropy of Y with respect to X is

$$D(Y||X) = \sum_{(i,j) \in \text{Supp}(X)} Y(i, j) \ln \frac{Y(i, j)}{X(i, j)} \text{ if } \text{Supp}(Y) \subset \text{Supp}(X),$$

and $D(Y||X) = +\infty$ otherwise, with the convention $0 \ln 0 = 0$. Although D is not a distance, the quantity $D(Y||X)$ measures in some sense how much Y is far from X since $D(Y||X) \geq 0$, with equality if and only if $Y = X$.

In 1968, Ireland and Kullback [8] gave an incomplete proof of the convergence of $(X_n)_{n \geq 0}$ when X_0 is positive, relying on the properties of the I -divergence. Yet, the I -divergence can be used to prove the convergence when the set $\Gamma(X_0)$ is non-empty, and to determine the limit. The maps T_R and T_C can be viewed as I -projections on Γ_R and Γ_C in the sense that for every $X \in \Gamma_0$, $T_R(X)$ (respectively $T_C(X)$) is the only matrix achieving the least upper bound of $D(Y||X)$ over all Y in Γ_R (respectively Γ_C).

In 1975, Csiszár established (theorem 3.2 in [5]) that, given a finite collection of linear sets $\mathcal{E}_1, \dots, \mathcal{E}_k$ of probability distributions on a finite set, and a distribution R such that $\mathcal{E}_1 \cap \dots \cap \mathcal{E}_k$ contains some probability distribution which is absolutely continuous with regard to R , the sequence obtained from R by applying cyclically the I -projections on $\mathcal{E}_1, \dots, \mathcal{E}_k$ converges to the I -projection of R on $\mathcal{E}_1 \cap \dots \cap \mathcal{E}_k$. This result applies to our context (the finite set is $[1, p] \times [1, q]$, the linear sets \mathcal{E}_1 and \mathcal{E}_2 are Γ_R and Γ_C) and shows that if the set $\Gamma = \Gamma_R \cap \Gamma_C$ contains some matrix with support included in X_0 , then $(X_n)_{n \geq 0}$ converges to the I -projection of X_0 on Γ .

2 Old and new results

Since the behavior of the sequence $(X_n)_{n \geq 0}$ depends only on the existence or the non-existence of elements of Γ with support equal to or included in $\text{Supp}(X_0)$, we state a criterion which determines in which case we are.

Consider two subsets A of $[1, p]$ and B of $[1, q]$ such that X_0 is null on $A \times B$. Note $A^c = [1, p] \setminus A$ and $B^c = [1, q] \setminus B$. Then for every $S \in \Gamma(X_0)$,

$$a(A) = \sum_{i \in A} a_i = \sum_{(i,j) \in A \times B^c} S(i,j) \leq \sum_{(i,j) \in [1,p] \times B^c} S(i,j) = \sum_{j \in B^c} b_j = b(B^c).$$

If $a(A) = b(B^c)$, S must be null on $A^c \times B^c$. If $a(A) > b(B^c)$, we get a contradiction, so $\Gamma(X_0)$ is empty.

Actually, these causes of the non-existence of elements of Γ with support equal to or included in $\text{Supp}(X_0)$ provide necessary and sufficient conditions. We give two criterions, the first one was already stated by Bacharach [2]. Pukelsheim gave a different formulation of these conditions in theorems 2 and 3 of [12]. We use a different method, relying on the theory of linear system of inequalities, and give a more precise statement, namely item 3 of critical case below.

Theorem 1. (*Criteria to distinguish the cases*)

(*Case of incompatibility*) The set $\Gamma(X_0)$ is empty if and only if there exist two subsets $A \subset [1, p]$ and $B \subset [1, q]$ such that X_0 is null on $A \times B$ and $a(A) > b(B^c)$.

(*Critical case*) Assume now that $\Gamma(X_0)$ is not empty. Then

1. There exists a matrix $S_0 \in \Gamma(X_0)$ whose support contains the support of every matrix in $\Gamma(X_0)$.
2. The support of S_0 is strictly contained in $\text{Supp}(X_0)$ if and only if there exist two non-empty subsets A of $[1, p]$ and B of $[1, q]$ such that X_0 is null on $A \times B$ and $a(A) = b(B^c)$.
3. More precisely, the difference $\text{Supp}(X_0) \setminus \text{Supp}(S_0)$ is the union of all products $A^c \times B^c$ over all non-empty subsets $A \times B$ of $[1, p] \times [1, q]$ such that X_0 is null on $A \times B$ and $a(A) = b(B^c)$.

Note that the assumption that X_0 has at least a positive entry on each row or column prevents A and B from being full when X_0 is null on $A \times B$. The additional condition $a(A) > b(B^c)$ (respectively $a(A) = b(B^c)$) is equivalent to the condition $a(A) + b(B) > 1$ (respectively $a(A) + b(B) = 1$), so rows and column play a symmetric role.

The condition $a(A) > b(B^c)$ and the positivity of all components of a and b also prevent A and B from being empty. We will call *cause of incompatibility* any (non-empty) block $A \times B \subset [1, p] \times [1, q]$ such that X_0 is null on $A \times B$ and $a(A) > b(B^c)$. If the set $\Gamma(X_0)$ is non-empty, we will call *cause of criticality* any non-empty block $A \times B \subset [1, p] \times [1, q]$ such that X_0 is null on $A \times B$ and $a(A) = b(B^c)$.

Given a convergent sequence $(x_n)_{n \geq 0}$ of vectors in some normed vector space $(E, \|\cdot\|)$, with limit $\ell \in E$, we will say that *the rate of convergence is geometric* (respectively *at least geometric*) if $0 < \lim_n \|x_n - x_\infty\|^{1/n} < 1$ (respectively $\limsup_n \|x_n - x_\infty\|^{1/n} < 1$).

We now describe the asymptotic behavior of sequence $(X_n)_{n \geq 0}$ in each case. The first case is already well-known.

Theorem 2. (Case of fast convergence) Assume that Γ contains some matrix with same support as X_0 . Then

1. The sequences $(R_i(X_{2n})_{n \geq 0})$ and $(C_j(X_{2n+1})_{n \geq 0})$ converge to 1 at an at least geometric rate.
2. The sequence $(X_n)_{n \geq 0}$ converges to some matrix X_∞ which has the same support as X_0 . The rate of convergence is at least geometric.
3. The limit X_∞ is the only matrix in $\Gamma \cap \Delta_p X_0 \Delta_q$ (in particular X_0 and X_∞ are diagonally equivalent). It is also the unique matrix achieving the minimum of $D(Y||X_0)$ over all $Y \in \Gamma(X_0)$.

For example, if $p = q = 2$, $a_1 = b_1 = 2/3$, $a_2 = b_2 = 1/3$, and

$$X_0 = \frac{1}{4} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then for every $n \geq 1$, X_n or X_n^\top is equal to

$$\frac{1}{3(2^n - 1)} \begin{pmatrix} 2^n & 2^n - 2 \\ 2^n - 1 & 0 \end{pmatrix},$$

depending on whether n is odd or even. The limit is

$$X_\infty = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The second case is also well-known, except the fact that the quantities $R_i(X_{2n}) - 1$ and $C_j(X_{2n+1}) - 1$ are $o(n^{-1/2})$.

Theorem 3. (Case of slow convergence) Assume that Γ contains some matrix with support included in $\text{Supp}(X_0)$ but contains no matrix with support equal to $\text{Supp}(X_0)$. Then

1. The series

$$\sum_{n \geq 0} (R_i(X_{2n}) - 1)^2 \text{ and } \sum_{n \geq 0} (C_j(X_{2n+1}) - 1)^2$$

converge.

2. The sequences $(\sqrt{n}(R_i(X_n) - 1))_{n \geq 0}$ and $(\sqrt{n}(C_j(X_n) - 1))_{n \geq 0}$ converge to 0. In particular, the sequences $(R_i(X_n))_{n \geq 0}$ and $(C_j(X_n))_{n \geq 0}$ converge to 1.
3. The sequence $(X_n)_{n \geq 0}$ converges to some matrix X_∞ whose support contains the support of every matrix in $\Gamma(X_0)$.
4. The limit X_∞ is the unique matrix achieving the minimum of $D(Y||X_0)$ over all $Y \in \Gamma(X_0)$.
5. If $(i, j) \in \text{Supp}(X_0) \setminus \text{Supp}(X_\infty)$, the infinite product $R_i(X_0)C_j(X_1)R_i(X_2)C_j(X_3) \cdots$ is infinite.

Actually the assumption that Γ contains no matrix with support equal to $\text{Supp}(X_0)$ can be removed; but when this assumption fails, theorem 2 applies, so theorem 3 brings nothing new. When this assumption holds, the last conclusion of theorem 3 shows that the rate of convergence cannot be in $o(n^{-1-\varepsilon})$ for any $\varepsilon > 0$.

However, a rate of convergence in $\Theta(n^{-1})$ is possible, and we do not know whether other rates of slow convergence may occur. For example, consider $p = q = 2$, $a_1 = a_2 = b_1 = b_2 = 1/2$, and

$$X_0 = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then for every $n \geq 1$, X_n or X_n^\top is equal to

$$\frac{1}{2n+2} \begin{pmatrix} 1 & n \\ n+1 & 0 \end{pmatrix},$$

depending on whether n is odd or even. The limit is

$$X_\infty = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

When Γ contains no matrix with support included in $\text{Supp}(X_0)$, we already know by Gietl and Reffel's theorem [7] that both sequences $(X_{2n})_{n \geq 0}$ and $(X_{2n+1})_{n \geq 0}$ converge. The convergence may be slow, so Aas gives in [1] an algorithm to fasten the convergence. Aas' algorithm finds and exploits the block structure associated to the inconsistent problem of finding a non-negative matrix whose marginals are a and b and whose support is contained in $\text{Supp}(X_0)$. The next two theorems give a complete description of the two limit points and how to find them.

Theorem 4. (*Case of divergence*) Assume that Γ contains no matrix with support included in $\text{Supp}(X_0)$.

Then there exist some positive integer $r \leq \min(p, q)$, some partitions $\{I_1, \dots, I_r\}$ of $[1, p]$ and $\{J_1, \dots, J_r\}$ of $[1, q]$ such that:

1. $(R_i(X_{2n}))_{n \geq 0}$ converges to $\lambda_k = b(J_k)/a(I_k)$ whenever $i \in I_k$;
2. $(C_j(X_{2n+1}))_{n \geq 0}$ converges λ_k^{-1} whenever $j \in J_k$;
3. $X_n(i, j) = 0$ for every $n \geq 0$ whenever $i \in I_k$ and $j \in J_{k'}$ with $k < k'$;
4. $X_n(i, j) \rightarrow 0$ as $n \rightarrow +\infty$ at a geometric rate whenever $i \in I_k$ and $j \in J_{k'}$ with $k > k'$;
5. The sequence $(X_{2n})_{n \geq 0}$ converges to the unique matrix achieving the minimum of $D(Y||X_0)$ over all $Y \in \Gamma(a', b, X_0)$, where $a'_i/a_i = \lambda_k$ whenever $i \in I_k$;
6. The sequence $(X_{2n+1})_{n \geq 0}$ converges to the unique matrix achieving the minimum of $D(Y||X_0)$ over all $Y \in \Gamma(a, b', X_0)$, where $b'_j/b_j = \lambda_k^{-1}$ whenever $j \in J_k$;
7. For every $k \in [1, r]$, $a'(I_k) = b(J_k)$ and $a(I_k) = b'(J_k)$. Moreover, the support of any matrix in $\Gamma(a', b, X_0) \cup \Gamma(a, b', X_0)$ is contained in $I_1 \times J_1 \cup \dots \cup I_r \times J_r$.
8. Let $D_1 = \text{Diag}(a'_1/a_1, \dots, a'_p/a_p)$ and $D_2 = \text{Diag}(b_1/b'_1, \dots, b_q/b'_q)$. Then for every $S \in \Gamma(a, b', X_0)$, $D_1 S = S D_2 \in \Gamma(a', b, X_0)$, and all matrices in $\Gamma(a', b, X_0)$ can be written in this way.

For example, if $p = q = 2$, $a_1 = b_1 = 1/3$, $a_2 = b_2 = 2/3$, and

$$X_0 = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then for every $n \geq 1$, X_n or X_n^\top is equal to

$$\frac{1}{3(3 \times 2^{n-1} - 1)} \begin{pmatrix} 1 & 3 \times 2^{n-1} - 2 \\ 2(3 \times 2^{n-1} - 1) & 0 \end{pmatrix},$$

depending on whether n is odd or even. We get $a'_1 = b'_1 = 2/3$ and $a'_2 = b'_2 = 1/3$ since the two limit points are

$$\frac{1}{3} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \text{ and } \frac{1}{3} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

The symmetry in theorem 4 shows that the limit points of $(X_n)_{n \geq 0}$ would be the same if we would applying T_C first instead of T_R .

Actually, the assumption that $\Gamma(X_0)$ is empty can be removed and it is not used in the proof of theorem 4. Indeed, when $\Gamma(X_0)$ is non-empty, the conclusions still hold with $r = 1$ and $\lambda_1 = 1$, but theorem 4 brings nothing new in this case.

Theorem 4 does not indicate what the partitions $\{I_1, \dots, I_r\}$ and $\{J_1, \dots, J_r\}$ are. Actually the integer r , the partitions $\{I_1, \dots, I_r\}$ and $\{J_1, \dots, J_r\}$ depend only on a , b and on the support of X_0 . and can be determined recursively as follows. This gives a complete description of the two limit points mentioned in theorem 4.

Theorem 5. *(Determining the partitions in case of divergence) Keep the assumption and the notations of theorem 4. Fix $k \in [1, r]$, set $P = [1, p] \setminus (I_1 \cup \dots \cup I_{k-1})$, $Q = [1, q] \setminus (J_1 \cup \dots \cup J_{k-1})$, and consider the restricted problem associated to the marginals $a(\cdot|P) = (a_i/a(P))_{i \in P}$, $b(\cdot|Q) = (b_j/b(Q))_{j \in Q}$ and to the initial condition $(X_0(i, j))_{(i, j) \in P \times Q}$.*

If $k = r$, this restricted problem admits some solution.

If $k < r$, the set $A_k \times B_k := I_k \times (Q \setminus J_k)$ is a cause of incompatibility of this restricted problem. More precisely, among all causes of incompatibility $A \times B$ maximizing the ratio $a(A)/b(Q \setminus B)$, it is the one which maximizes the set A and minimizes the set B .

Note that if a cause of incompatibility $A \times B$ maximizes the ratio $a(A)/b(B^c)$, then it is maximal for the inclusion order. We now give an example to illustrate how theorem 5 enables us to determine the partitions $\{I_1, \dots, I_r\}$ and $\{J_1, \dots, J_r\}$. In the following array, the * and 0 indicate the positive and the null entries in X_0 ; the last column and row indicate the target sums on each row and column.

$$\begin{array}{cccccc|c} * & * & 0 & 0 & 0 & 0 & .25 \\ 0 & * & * & 0 & 0 & 0 & .25 \\ 0 & * & * & * & 0 & 0 & .25 \\ * & * & * & * & 0 & * & .15 \\ * & 0 & * & * & * & * & .10 \\ \hline .05 & .05 & .1 & .2 & .2 & .4 & 1 \end{array}$$

We indicate below in underlined boldface characters some blocks $A \times B$ of zeroes which

are causes of incompatibility, and the corresponding ratios $a(A)/b(B^c)$.

| | | | | | | | |
|---|-----|-----|----|----|----------|----|-----|
| $A = \{1, 2, 3, 4\}$ | * | * | 0 | 0 | <u>0</u> | 0 | .25 |
| $B = \{5\}$ | 0 | * | * | * | <u>0</u> | 0 | .25 |
| $\frac{a(A)}{b(B^c)} = \frac{0.9}{0.8}$ | * | * | * | * | <u>0</u> | * | .15 |
| | * | 0 | * | * | * | * | .10 |
| | .05 | .05 | .1 | .2 | .2 | .4 | 1 |

| | | | | | | | |
|--|----------|-----|----|----|----------|----------|-----|
| $A = \{2, 3\}$ | * | * | 0 | 0 | 0 | 0 | .25 |
| $B = \{1, 5, 6\}$ | <u>0</u> | * | * | 0 | <u>0</u> | <u>0</u> | .25 |
| $\frac{a(A)}{b(B^c)} = \frac{0.5}{0.35}$ | <u>0</u> | * | * | * | <u>0</u> | <u>0</u> | .25 |
| | * | * | * | * | 0 | * | .15 |
| | * | 0 | * | * | * | * | .10 |
| | .05 | .05 | .1 | .2 | .2 | .4 | 1 |

| | | | | | | | |
|--|-----|-----|----------|----------|----------|----------|-----|
| $A = \{1\}$ | * | * | <u>0</u> | <u>0</u> | <u>0</u> | <u>0</u> | .25 |
| $B = \{3, 4, 5, 6\}$ | 0 | * | * | 0 | 0 | 0 | .25 |
| $\frac{a(A)}{b(B^c)} = \frac{0.25}{0.1}$ | 0 | * | * | * | 0 | * | .15 |
| | * | 0 | * | * | * | * | .10 |
| | .05 | .05 | .1 | .2 | .2 | .4 | 1 |

| | | | | | | | |
|---|-----|-----|----|----------|----------|----------|-----|
| $A = \{1, 2\}$ | * | * | 0 | <u>0</u> | <u>0</u> | <u>0</u> | .25 |
| $B = \{4, 5, 6\}$ | 0 | * | * | <u>0</u> | <u>0</u> | <u>0</u> | .25 |
| $\frac{a(A)}{b(B^c)} = \frac{0.5}{0.2}$ | 0 | * | * | * | 0 | * | .15 |
| | * | 0 | * | * | * | * | .10 |
| | .05 | .05 | .1 | .2 | .2 | .4 | 1 |

One checks that the last two blocks are those which maximize the ratio $a(A)/b(B^c)$. Among these two blocks, the latter has a bigger A and a smaller B , so it is $A_1 \times B_1$. Therefore, $I_1 = \{1, 2\}$ and $J_1 = \{1, 2, 3\}$, and we look at the restricted problem associated to the marginals $a(\cdot|I_1^c)$, $b(\cdot|J_1^c)$ and to the initial condition $(X_0(i, j))_{(i, j) \in I_1^c \times J_1^c}$. The dots below indicate the removed rows and columns.

| | | | | | | |
|---|---|---|-----|-----|----|----|
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| . | . | . | * | 0 | 0 | .5 |
| . | . | . | * | 0 | * | .3 |
| . | . | . | * | * | * | .2 |
| . | . | . | .25 | .25 | .5 | 1 |

Two causes of impossibility have to be considered, namely $\{3, 4\} \times \{5\}$ and $\{3\} \times \{5, 6\}$. The latter maximizes the ratio $a(A)/b(J_1^c \setminus B)$, so it is $A_2 \times B_2$. Therefore, $I_2 = \{3\}$ and $J_2 = \{4\}$, and we look at the restricted problem below.

| | | | | | | |
|---|---|---|---|-----|-----|----|
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| . | . | . | . | 0 | * | .6 |
| . | . | . | . | * | * | .4 |
| . | . | . | . | .33 | .67 | 1 |

This time, there is no cause of impossibility, so $r = 3$, and the sets $I_3 = \{4, 5\}$, $J_3 = \{5, 6\}$ contain all the remaining indices. We shall indicate with dashlines the block structure defined by the partitions $\{I_1, I_2, I_3\}$ and $\{J_1, J_2, J_3\}$ (for readability, our example was chosen in such a way that each block is made of consecutive indices). By theorem 4, the limit of the sequence $(X_{2n})_{n \geq 0}$ admits marginals a' and b , its support is included in $\text{Supp}(X_0)$ and also in $(I_1 \times J_1) \cup (I_2 \times J_2) \cup (I_3 \times J_3)$, namely it solves the problem below.

$$\begin{array}{cccccc|c}
* & * & 0 & 0 & 0 & 0 & .1 \\
0 & * & * & 0 & 0 & 0 & .1 \\
\hline
0 & 0 & 0 & * & 0 & 0 & .2 \\
\hline
0 & 0 & 0 & 0 & 0 & * & .36 \\
\hline
0 & 0 & 0 & 0 & * & * & .24 \\
\hline
.05 & .05 & .1 & .2 & .2 & .4 & 1
\end{array}$$

We observe that each cause of incompatibility $A \times B$ related to the block structure, namely $I_1 \times (J_2 \cup J_3)$ or $(I_1 \cup I_2) \times J_3$, becomes a cause of criticality with regard to the margins a' and b , namely $a'(A) = b(B^c)$, so $\lim_n X_{2n}$ has zeroes on $A^c \times B^c$. This statement fails for the “weaker” cause of incompatibility $\{1, 2, 3, 4\} \times \{5\}$. Yet, it still holds for the cause of incompatibility $\{1\} \times \{3, 4, 5, 6\}$; this rare situation occurs because there were two causes of incompatibility maximizing the ratio $a(A)/b(B^c)$, namely $\{1\} \times \{3, 4, 5, 6\}$ and $\{1, 2\} \times \{4, 5, 6\}$. Hence $\lim_n X_{2n}$ has another additional zero at position $(2, 2)$. We add dashlines below to make visible this refinement of the block structure. Here, no minimization of I -divergence is required to get the limit of $(X_{2n})_{n \geq 0}$ since the set $\Gamma(a', b, X_0)$ contains only one matrix, namely

$$\begin{array}{cccccc|c}
.05 & .05 & 0 & 0 & 0 & 0 & .1 \\
\hline
0 & 0 & 0.1 & 0 & 0 & 0 & .1 \\
\hline
0 & 0 & 0 & .2 & 0 & 0 & .2 \\
\hline
0 & 0 & 0 & 0 & 0 & .36 & .36 \\
\hline
0 & 0 & 0 & 0 & .2 & .04 & .24 \\
\hline
.05 & .05 & .1 & .2 & .2 & .4 & 1
\end{array}$$

We note that the convergence of $X_{2n}(2, 2)$ to 0 is slow since

$$\lim_{n \rightarrow +\infty} \frac{X_{2n+2}(2, 2)}{X_{2n}(2, 2)} = \lim_{n \rightarrow +\infty} \frac{1}{R_2(X_{2n})C_2(X_{2n+1})} = \frac{1}{\lambda_1 \lambda_1^{-1}} = 1.$$

This is a typical situation in which the following observation is useful.

Theorem 6. *The matrices $\lim_n X_{2n}$ and $\lim_n X_{2n+1}$ have the same support Σ and Σ is the union of the supports of all matrices in $\Gamma(a', b, X_0) \cup \Gamma(a, b', X_0)$.*

Moreover, if X'_0 denotes the matrix obtained from X_0 by setting to 0 all entries outside Σ , then the limit points provided by the IPFP starting from X_0 and from X'_0 coincide.

Aas mentions this fact (proposition 1 in [1]) as a result of Pretzel (last part of theorem 1 in [11]), although Pretzel considers only the case where the set $\Gamma(a, b, X_0)$ is not empty. We prove theorem 6 by adapting Pretzel’s proof.

The set Σ can be determined by applying theorem 1 (critical case, item 3) to the the marginals a' and b . The interest of theorem 6 is that starting from X'_0 ensures an at least geometric rate of convergence, since theorem 2 applies when one performs the IPFP on the marginals a' and b (or a and b') and the initial matrix X'_0 .

That is why Aas investigates the inherent block structure. Actually, the splitting considered by Aas is finer than the splitting provided by our theorems 4 and 5; in the example above, Aas would split I_1 into $\{1, 2\}$ and $\{3\}$, and J_1 into $\{1\}$ and $\{2\}$. Up to this distinction, most of the statements provided by our theorems 4 and 5 are explicitly or implicitly present in Aas' paper, which focuses on an algorithmic point of view.

The convergence of the sequences $(X_{2n})_{n \geq 0}$ and $(X_{2n+1})_{n \geq 0}$ was already established by Gietl and Reffel [7] with the help of I -divergence. Our proof is completely different (although I -divergence helps us to determine the limit points). Our first step is to prove the convergence of the sequences $(R_i(X_{2n}))_{n \geq 0}$ and $(C_j(X_{2n+1}))_{n \geq 0}$ by exploiting recursion relations involving stochastic matrices. The proof relies on the next general result on infinite products of stochastic matrices. Theorem 7 below will only be used for the proof of theorems 4 and 5, so apart from lemma 9, section 3 can be skipped if the reader is only interested by the new proof of theorems 1, 2 and 3.

A sequence $(x_n)_{n \geq 0}$ of vectors in some normed vector space $(E, \|\cdot\|)$ will be said to have a *finite variation* if the series $\sum_n \|x_{n+1} - x_n\|$ converges. The finite variation implies the convergence when $(E, \|\cdot\|)$ is a Banach space, in particular when E has a finite dimension.

Theorem 7. *Let $(M_n)_{n \geq 1}$ be some sequence of $d \times d$ stochastic matrices. Assume that there exists some constants $\gamma > 0$, and $\rho \geq 1$ such that for every $n \geq 1$ and i, j in $[1, d]$, $M_n(i, i) \geq \gamma$ and $M_n(i, j) \leq \rho M_n(j, i)$. Then the sequence $(M_n \cdots M_1)_{n \geq 1}$ has a finite variation, so it converges to some stochastic matrix L . Moreover, the series $\sum_n M_n(i, j)$ and $\sum_n M_n(j, i)$ converge whenever the rows of L with indexes i and j are different.*

An important literature deals with infinite products of stochastic matrices, with various motivations: study of inhomogeneous Markov chains, of opinion dynamics... See for example [14]. Backward infinite products converge much more often than forward infinite products. Many theorems involve the ergodic coefficients of stochastic matrices. For a $d \times d$ stochastic matrix M , the ergodic coefficient is the quantity

$$\tau(M) = \min_{1 \leq i, i' \leq d} \sum_{j=1}^d \min(M(i, j), M(i', j)) \in [0, 1].$$

The difference $1 - \tau(M)$ is the maximal total variation distance between the lines of M seen as probabilities on $[1, d]$. These theorems do not apply in our context.

To our knowledge, theorem 7 is new. The closest statements we found in the literature are Lorenz' stabilization theorem (theorem 2 of [10]) and a theorem of Touri and Nedić on infinite product of bistochastic matrices (theorem 7 of [15], relying on theorem 6 of [17]). The method we use to prove theorem 7 is different of theirs.

On the one hand, theorem 7 provides a stronger conclusion (namely finite variation and not only convergence) and has weaker assumptions than Lorenz' stabilization theorem. Indeed, Lorenz assumes that each M_n has a positive diagonal and a symmetric support, and that the entries of all matrices M_n are bounded below by some $\delta > 0$; this entails the assumptions of our theorem 7, with $\gamma = \delta$ and $\rho = \delta^{-1}$.

On the other hand, Lorenz' stabilization theorem gives more precisions on the limit $L = \lim_{n \rightarrow +\infty} M_n \cdots M_1$. In particular, if the support of M_n does not depends on n , then Lorenz shows that by applying a same permutation on the rows and on the columns of L , one gets a block-diagonal matrix in which each diagonal block is a consensus matrix, namely a stochastic matrix whose rows are all the same. This additional conclusion does

not hold anymore under our weaker assumptions. For example, for every $r \in [-1, 1]$, consider the stochastic matrix

$$M(r) = \frac{1}{2} \begin{pmatrix} 1+r & 1-r \\ 1-r & 1+r \end{pmatrix}.$$

One checks that for every r_1 and r_2 in $[-1, 1]$, $M(r_2)M(r_1) = M(r_2r_1)$. Let $(r_n)_{n \geq 1}$ be any sequence of numbers in $]0, 1]$ whose infinite product converges to some $\ell > 0$. Then our assumptions hold with $\gamma = 1/2$ and $\rho = 1$ and the matrices $M(r_n)$ have the same support. Yet, the limit of the products $M(r_n) \cdots M(r_1)$, namely $M(\ell)$, has only positive coefficients and is not a consensus matrix.

Note also that, given an arbitrary sequence $(M_n)_{n \geq 1}$ of stochastic matrices, assuming only that the diagonal entries are bounded away from 0 does not ensure the convergence of the infinite product $\cdots M_2 M_1$. Indeed, consider the triangular stochastic matrices

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

A recursion shows that for every $n \geq 1$ and $(\varepsilon_1, \dots, \varepsilon_n) \in \{0, 1\}^n$,

$$T_{\varepsilon_n} \cdots T_{\varepsilon_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & 1 - 2^{-n} - r & 2^{-n} \end{pmatrix}, \quad \text{where } r = \sum_{k=1}^n \frac{\varepsilon_k}{2^{n+1-k}}.$$

Hence, one sees that the infinite product $\cdots T_0 T_1 T_0 T_1 T_0 T_1$ diverges.

Yet, for a sequence of *doubly-stochastic* matrices, it is sufficient to assume that the diagonal entries are bounded away from 0. This result was proved by Touri and Nedić (theorem 5 of [16] or theorem 7 of [15], relying on theorem 6 of [17]). We provide a simpler proof and a slight improvement, showing that the sequence $(M_n \cdots M_1)_{n \geq 1}$ not only converges but also has a finite variation.

Theorem 8. *Let $(M_n)_{n \geq 1}$ be some sequence of $d \times d$ doubly-stochastic matrices. Assume that there exists some constant $\gamma > 0$ such that for every $n \geq 1$ and i in $[1, d]$, $M_n(i, i) \geq \gamma$. Then the sequence $(M_n \cdots M_1)_{n \geq 1}$ has a finite variation, so it converges to some stochastic matrix L . Moreover, the series $\sum_n M_n(i, j)$ and $\sum_n M_n(j, i)$ converge whenever the rows of L with indexes i and j are different.*

Our proof relies on the following fact: for every column vector $V \in \mathbf{R}^d$, set

$$D(V) = \sum_{1 \leq i, j \leq d} |V(i) - V(j)| \quad \text{and} \quad \|V\|_1 = \sum_{1 \leq i \leq d} |V(i)|.$$

We will call *dispersion of V* the quantity $D(V)$. Then, under the assumptions of theorem 8, the inequality

$$\gamma \|M_{n+1} \cdots M_1 V - M_n \cdots M_1 V\|_1 \leq D(M_n \cdots M_1 V) - D(M_{n+1} \cdots M_1 V).$$

holds for every $n \geq 0$.

3 Infinite products of stochastic matrices

3.1 Proof of theorem 7

We begin with an elementary lemma.

Lemma 9. *Let M be any $m \times n$ stochastic matrix and $V \in \mathbf{R}^n$ be a column vector. Denote by \underline{M} the smallest entry of M , by \underline{V} , \bar{V} and $\text{diam}(V) = \bar{V} - \underline{V}$ the smallest entry, the largest entry and the diameter of V . Then*

$$\underline{MV} \geq (1 - \underline{M}) \underline{V} + \underline{M} \bar{V} \geq \underline{V},$$

$$\overline{MV} \leq \underline{M} \underline{V} + (1 - \underline{M}) \bar{V} \leq \bar{V},$$

so

$$\text{diam}(MV) \leq (1 - 2\underline{M})\text{diam}(V).$$

Proof. Call $M(i, j)$ the entries of M and $V(1), \dots, V(n)$ the entries of V . Let j_1 and j_2 be indexes such that $V(j_1) = \underline{V}$ and $V(j_2) = \bar{V}$. Then for every $i \in [1, m]$,

$$\begin{aligned} (MV)_i &= \sum_{j \neq j_2} M(i, j)V(j) + M(i, j_2)V(j_2) \\ &\geq \sum_{j \neq j_2} M(i, j)\underline{V} + M(i, j_2)\bar{V} \\ &= \underline{V} + M(i, j_2)(\bar{V} - \underline{V}) \\ &\geq \underline{V} + \underline{M}(\bar{V} - \underline{V}) \\ &\geq \underline{V}. \end{aligned}$$

The first inequality follows. Applying it to $-V$ yields the second inequality. \square

The interesting case is when $n \geq 2$, so $\underline{M} \leq 1/2$ and $1 - 2\underline{M} \geq 0$. Yet, the lemma and the proof above still apply when $n = 1$, since $\overline{MV} = \underline{MV} = \bar{V} = \underline{V}$ in this case.

We now restrict ourselves to square stochastic matrices. To every column vector $V \in \mathbf{R}^d$, we associate the column vector $V^\uparrow \in \mathbf{R}^d$ obtained by ordering the components in non-decreasing order. In particular $V^\uparrow(1) = \underline{V}$ and $V^\uparrow(d) = \bar{V}$.

In the next lemmas and corollary, we establish inequalities that will play a key role in the proof of theorem 7.

Lemma 10. *Let M be some $d \times d$ stochastic matrix with diagonal entries bounded below by some constant $\gamma > 0$, and $V \in \mathbf{R}^d$ be a column vector with components in increasing order $V(1) \leq \dots \leq V(d)$. Let σ be a permutation of $[1, d]$ such that $(MV)(\sigma(1)) \leq \dots \leq (MV)(\sigma(d))$. For every $i \in [1, d]$, set*

$$A_i = \sum_{j=1}^{i-1} M(\sigma(i), j) [V(i) - V(j)], \quad B_i = \sum_{j=i+1}^d M(\sigma(i), j) [V(j) - V(i)],$$

with the natural conventions $A_1 = B_d = 0$. The following statements hold.

1. For every $i \in [1, d]$, $(MV)^\uparrow(i) - V^\uparrow(i) = B_i - A_i$.
2. All the terms in the sums defining A_i and B_i are non-negative.

3. $B_i \geq M(\sigma(i), \sigma(i)) [V(\sigma(i)) - V(i)] \geq \gamma [V(\sigma(i)) - V(i)]$ whenever $i < \sigma(i)$.
4. Let $a < b$ in $[1, d]$. If the orbit $O(a)$ of a associated to the permutation σ contains some integer at least equal to b , then

$$V(b) - V(a) \leq \gamma^{-1} \sum_{i \in O(a) \cap [1, b-1]} B_i \mathbf{1}_{[i < \sigma(i)]} \leq \gamma^{-1} \sum_{i \in O(a) \cap [1, b-1]} B_i.$$

5. One has

$$\sum_{i=1}^d |V(\sigma(i)) - V(i)| \leq 2\gamma^{-1} \sum_{i=1}^d B_i \mathbf{1}_{[i < \sigma(i)]} \leq 2\gamma^{-1} \sum_{i=1}^{d-1} B_i.$$

Proof. By assumption,

$$(MV)^\uparrow(i) - V^\uparrow(i) = (MV)(\sigma(i)) - V(i) = \sum_{j=1}^d M(\sigma(i), j) [V(j) - V(i)] = -A_i + B_i,$$

which yields the first item. The next two items follow directly from the assumptions $V(1) \leq \dots \leq V(d)$ and $M(j, j) \geq \gamma$ for every $j \in [1, d]$.

Under the assumptions of item 4, the integer $n(a, b) = \min\{n \geq 1 : \sigma^n(a) \geq b\}$ is well-defined and

$$\begin{aligned} V(b) - V(a) &\leq V(\sigma^{n(a,b)}(a)) - V(a) \\ &\leq \sum_{k=0}^{n(a,b)-1} [V(\sigma^{k+1}(a)) - V(\sigma^k(a))] \mathbf{1}_{\sigma^k(a) < \sigma^{k+1}(a)} \\ &\leq \gamma^{-1} \sum_{k=0}^{n(a,b)-1} B_{\sigma^k(a)} \mathbf{1}_{\sigma^k(a) < \sigma^{k+1}(a)} \end{aligned}$$

by item 3. Item 4 follows.

Since the sum of $V(\sigma(i)) - V(i)$ over all $i \in [1, d]$ is null and since $V(1) \leq \dots \leq V(d)$, one has

$$\sum_{i=1}^d |V(\sigma(i)) - V(i)| = 2 \sum_{i=1}^d (V(\sigma(i)) - V(i)) \mathbf{1}_{[i < \sigma(i)]} \leq 2\gamma^{-1} \sum_{i=1}^{d-1} B_i \mathbf{1}_{[i < \sigma(i)]},$$

by item 3. The proof is complete. \square

We denote by $\|\cdot\|_1$ the norm on \mathbf{R}^d defined as the sum of the absolute values of the components.

Lemma 11. *Keep the assumptions and the notations of lemma 10. Assume that there exists some constant $\rho \geq 1$ such that for every $n \geq 1$ and i, j in $[1, d]$, $M(i, j) \leq \rho M(j, i)$. Set $C = d(d-1) \max(\gamma^{-1}, \rho)$. Then the following statements hold*

1. For every $i \in [1, d]$, $A_i \leq (i-1) \max(\gamma^{-1}, \rho)(B_1 + \dots + B_{i-1})$.
2. For every $m \in [1, d]$, $B_1 + \dots + B_m \leq (1 + C + \dots + C^{m-1}) \|(MV)^\uparrow - V^\uparrow\|_1$.

Proof. Given $i \in [2, d]$ and $j \in [1, i-1]$, let us check that

$$A_{i,j} := M(\sigma(i), j) [V(i) - V(j)] \leq \max(\gamma^{-1}, \rho)(B_1 + \dots + B_{i-1}).$$

We distinguish two cases.

- If the orbit of some $k \in [1, j]$ contains some integer at least equal to i , then inequality 4 of lemma 10 applied with $(a, b) = (k, i)$ yields

$$A_{i,j} \leq V(i) - V(j) \leq V(i) - V(k) \leq \gamma^{-1} \sum_{z \in O(k) \cap [1, i-1]} B_z.$$

- Otherwise, the orbit of every element of $[1, j]$ is contained in $[1, i-1]$, so the orbit of every element of $[i, d]$ is contained in $[j+1, d]$. In particular, the orbits $O(\sigma(i)) = O(i)$ and $O(j)$ are disjoint. Applying inequality 3 and inequality 4 of lemma 10, once with $(a, b) = (\sigma(i), i)$, once with $(a, b) = (j, \sigma^{-1}(j))$ yields

$$\begin{aligned} A_{i,j} &= M(\sigma(i), j) [V(i) - V(\sigma(i))] + M(\sigma(i), j) [V(\sigma(i)) - V(\sigma^{-1}(j))] \\ &\quad + M(\sigma(i), j) [V(\sigma^{-1}(j)) - V(j)] \\ &\leq \mathbf{1}_{\sigma(i) < i} [V(i) - V(\sigma(i))] + \mathbf{1}_{\sigma^{-1}(j) < \sigma(i)} \rho M(j, \sigma(i)) [V(\sigma(i)) - V(\sigma^{-1}(j))] \\ &\quad + \mathbf{1}_{j < \sigma^{-1}(j)} [V(\sigma^{-1}(j)) - V(j)] \\ &\leq \gamma^{-1} \sum_{z \in O(i) \cap [1, i-1]} B_z + \rho B_{\sigma^{-1}(j)} + \gamma^{-1} \sum_{z \in O(j) \cap [1, \sigma^{-1}(j)-1]} B_z \\ &\leq \max(\gamma^{-1}, \rho) \sum_{z \in [1, i-1]} B_z. \end{aligned}$$

In both cases, we have got the inequality $A_{i,j} \leq \max(\gamma^{-1}, \rho)(B_1 + \dots + B_{i-1})$. Summing over all $j \in [1, i-1]$ yields item 1.

Let $m \in [1, d]$. Equality $(MV)^\uparrow(i) - V^\uparrow(i) = B_i - A_i$ and item 1 yield

$$\begin{aligned} \sum_{i=1}^m B_i &\leq \sum_{i=1}^m |(MV)^\uparrow(i) - V^\uparrow(i)| + \sum_{i=1}^m A_i \\ &\leq \|(MV)^\uparrow - V^\uparrow\|_1 + \sum_{i=1}^m (i-1) \max(\gamma^{-1}, \rho)(B_1 + \dots + B_{i-1}) \\ &\leq \|(MV)^\uparrow - V^\uparrow\|_1 + \sum_{i=1}^m (d-1) \max(\gamma^{-1}, \rho)(B_1 + \dots + B_{m-1}) \\ &\leq \|(MV)^\uparrow - V^\uparrow\|_1 + C \sum_{i=1}^{m-1} B_i \end{aligned}$$

The particular case where $m = 1$ yields $B_1 \leq \|(MV)^\uparrow - V^\uparrow\|_1$. Item 2 follows by induction. \square

Corollary 12. *Let M be some $d \times d$ stochastic matrix with diagonal entries bounded below by some $\gamma > 0$. Assume that there exists some constant $\rho \geq 1$ such that for every $n \geq 1$ and i, j in $[1, d]$, $M(i, j) \leq \rho M(j, i)$. Set $C = d(d-1) \max(\gamma^{-1}, \rho)$. Then for any column $V \in \mathbf{R}^d$ the following statements hold.*

1. Fix $m \in [1, d]$ and $s \geq 1 + (m - 1) \max(\gamma^{-1}, \rho)$.

$$\sum_{i=1}^m s^{-i} (MV)^\uparrow(i) \geq \sum_{i=1}^m s^{-i} V^\uparrow(i).$$

2. $\|MV - V\|_1 \leq (2 + C + \dots + C^{d-2}) \|(MV)^\uparrow - V^\uparrow\|_1$.

Proof. By applying a same permutation to the components of V , to the rows and to the columns of M , one may assume that $V(1) \leq \dots \leq V(d)$. Let σ be a permutation of $[1, d]$ such that $(MV)(\sigma(1)) \leq \dots \leq (MV)(\sigma(d))$. Then lemmas 10 and 11 apply.

For every $i \in [1, m]$,

$$(MV)^\uparrow(i) - V^\uparrow(i) = B_i - A_i \geq B_i - (m - 1) \max(\gamma^{-1}, \rho) (B_1 + \dots + B_{i-1}).$$

Summing over i yields

$$\begin{aligned} \sum_{i=1}^m s^{-i} ((MV)^\uparrow(i) - V^\uparrow(i)) &\geq \sum_{j=1}^m s^{-j} B_j - \sum_{i=2}^m \sum_{j=1}^{i-1} s^{-i} (m - 1) \max(\gamma^{-1}, \rho) B_j \\ &= \sum_{j=1}^m \left(s^{-j} - (m - 1) \max(\gamma^{-1}, \rho) \sum_{i=j+1}^m s^{-i} \right) B_j \\ &\geq \sum_{j=1}^m \left(s^{-j} - (m - 1) \max(\gamma^{-1}, \rho) \frac{s^{-(j+1)}}{1 - s^{-1}} \right) B_j \\ &= \sum_{j=1}^m \frac{s^{-j}}{s - 1} [s - 1 - (m - 1) \max(\gamma^{-1}, \rho)] B_j \\ &\geq 0. \end{aligned}$$

Furthermore, for every $i \in [1, d]$,

$$|(MV)(\sigma(i)) - V(\sigma(i))| - |(MV)(\sigma(i)) - V(i)| \leq |V(\sigma(i)) - V(i)|.$$

Summing over i and using the last statements of lemma 10 and corollary 11 yield

$$\begin{aligned} \|MV - V\|_1 - \|(MV)^\uparrow - V^\uparrow\|_1 &\leq \sum_{i=1}^d |V(\sigma(i)) - V(i)| \\ &\leq 2\gamma^{-1} \sum_{i=1}^{d-1} B_i \\ &\leq (1 + C + \dots + C^{d-2}) \|(MV)^\uparrow - V^\uparrow\|_1 \end{aligned}$$

The proof is complete. \square

We now derive the last step of the proof of theorem 7. Indeed, applying the next corollary to each vector of the canonical basis on \mathbf{R}^d yields theorem 7.

Corollary 13. *Let $(M_n)_{n \geq 1}$ be some sequence of $d \times d$ stochastic matrices. Assume that there exists some constants $\gamma > 0$, and $\rho \geq 1$ such that for every $n \geq 1$ and i, j in $[1, d]$, $M_n(i, i) \geq \gamma$ and $M_n(i, j) \leq \rho M_n(j, i)$. For every column vector $V \in \mathbf{R}^d$, the sequence of vectors $(V_n)_{n \geq 0} := (M_n \dots M_1 V)_{n \geq 0}$ has a finite variation, so it converges. Moreover, the series $\sum_n M_n(i, j)$ and $\sum_n M_n(j, i)$ converge whenever the two sequences $(V_n(i))_{n \geq 0}$ and $(V_n(j))_{n \geq 0}$ have a different limit.*

Proof. Fix $s \geq 1 + (d-1) \max(\gamma^{-1}, \rho)$. For each n , one can apply corollary 12 to the matrix M_{n+1} and to the vector V_n .

For every $m \in [1, d]$, the sequence $(s^{-1}V_n^\uparrow(1) + \dots + s^{-m}V_n^\uparrow(m))_{n \geq 0}$ is non-decreasing by corollary 12 (first part) and bounded above by $s^{-1}V^\uparrow(d) + \dots + s^{-m}V^\uparrow(d)$, thanks to lemma 9, so it has a finite variation and converges. By difference, each sequence $(V_n^\uparrow(i))_{n \geq 0}$ has a finite variation. The convergence of the series $\sum_n \|V_{n+1}^\uparrow - V_n^\uparrow\|_1$ follows, and also $\sum_n \|V_{n+1} - V_n\|_1$ by corollary 12 (second part).

Call $\lambda_1 < \dots < \lambda_r$ the distinct values of $\lim_{n \rightarrow \infty} V_n(i)$ for $i \in [1, d]$. For every $k \in [1, r]$, set $I_k = \{i \in [1, d] : \lim_{n \rightarrow \infty} V_n(i) = \lambda_k\}$, $J_k = I_1 \cup \dots \cup I_k$ and call m_k the size of J_k . Fix $\varepsilon > 0$ such that $2\varepsilon < \gamma \min(\lambda_2 - \lambda_1, \dots, \lambda_r - \lambda_{r-1})$, so that the intervals $[\lambda_k - \varepsilon, \lambda_k + \varepsilon]$ are pairwise disjoint. Then one can find some non-negative integer N , such that $V_n(i) \in [\lambda_k - \varepsilon, \lambda_k + \varepsilon]$ for every $n \geq N$, $k \in [1, r]$, and $i \in I_k$.

Given $k \in [1, r-1]$ and $n \geq N$, we show below that

$$\sum_{i=1}^{m_k} s^{-i} [V_{n+1}^\uparrow(i) - V_n^\uparrow(i)] \geq (\lambda_{k+1} - \lambda_k - 2\varepsilon) s^{-m_k} \sum_{i \in J_k} \sum_{j \in J_k^c} M_{n+1}(i, j).$$

This inequality together with the convergence of the sequence $(V_n^\uparrow)_{n \geq 0}$ and the inequalities $M_n(j, i) \leq \rho M_n(i, j)$ will yield the convergence of the series $\sum_n M_n(i, j)$ and $\sum_n M_n(j, i)$ for every $(i, j) \in J_k \times J_k^c$.

Fix $n \geq N$ and a permutation σ of $[1, d]$ such that $V_{n+1}(\sigma(1)) \leq \dots \leq V_{n+1}(\sigma(d))$. Note that $\sigma([1, m_k]) = J_k$.

The column vector U_n defined by $U_n(j) = \min(V_n(j), \lambda_k + \varepsilon)$ has the same m_k least components as V_n (corresponding to the indexes $j \in J_k$), so U_n^\uparrow have the same m_k first components as V_n^\uparrow . Furthermore, $V_n(j) - U_n(j) \geq \lambda_{k+1} - \lambda_k - 2\varepsilon$ for every $j \in J_k^c$. Hence, for every $i \in [1, d]$,

$$\begin{aligned} V_{n+1}^\uparrow(i) - (M_{n+1}U_n)(\sigma(i)) &= V_{n+1}(\sigma(i)) - (M_{n+1}U_n)(\sigma(i)) \\ &= (M_{n+1}V_n - M_{n+1}U_n)(\sigma(i)) \\ &= \sum_{j=1}^d M_{n+1}(\sigma(i), j) (V_n(j) - U_n(j)) \\ &\geq (\lambda_{k+1} - \lambda_k - 2\varepsilon) \sum_{j \in J_k^c} M_{n+1}(\sigma(i), j). \end{aligned}$$

Fix $s \geq 1 + (m_k - 1) \max(\gamma^{-1}, \rho)$. Then

$$\begin{aligned} \sum_{i=1}^{m_k} s^{-i} (V_{n+1}^\uparrow(i) - (M_{n+1}U_n)(\sigma(i))) &\geq (\lambda_{k+1} - \lambda_k - 2\varepsilon) \sum_{i=1}^{m_k} s^{-i} \sum_{j \in J_k^c} M_{n+1}(\sigma(i), j) \\ &\geq (\lambda_{k+1} - \lambda_k - 2\varepsilon) s^{-m_k} \sum_{i=1}^{m_k} \sum_{j \in J_k^c} M_{n+1}(\sigma(i), j) \\ &= (\lambda_{k+1} - \lambda_k - 2\varepsilon) s^{-m_k} \sum_{i \in J_k} \sum_{j \in J_k^c} M_{n+1}(i, j). \end{aligned}$$

But the rearrangement inequality ¹ and the first part of corollary 12 yield

$$\begin{aligned} \sum_{i=1}^{m_k} s^{-i}(M_{n+1}U_n)(\sigma(i)) &\geq \sum_{i=1}^{m_k} s^{-i}(M_{n+1}U_n)^\uparrow(i) \\ &\geq \sum_{i=1}^{m_k} s^{-i}U_n^\uparrow(i) \\ &= \sum_{i=1}^{m_k} s^{-i}V_n^\uparrow(i). \end{aligned}$$

We get the desired inequality by additioning the last two inequalities.

The proof is complete. □

3.2 Proof of theorem 8

The proof we give is simpler than the proof of theorem 7, although some arguments are very similar. We begin with the key lemma.

Lemma 14. *Let M be some $d \times d$ doubly-stochastic matrix with diagonal entries bounded below by some constant $\gamma > 0$, and $V \in \mathbf{R}^d$ be any column vector. Call dispersion of V the quantity*

$$D(V) = \sum_{1 \leq i, j \leq d} |V(i) - V(j)|.$$

Then $D(V) - D(MV) \geq \gamma \|MV - V\|_1$.

Proof. On the one hand, for every i and j in $[1, d]$,

$$\begin{aligned} (MV)(i) - (MV)(j) &= \sum_{1 \leq k \leq d} M(i, k)V(k) - \sum_{1 \leq l \leq d} M(j, l)V(l) \\ &= \sum_{1 \leq k, l \leq d} M(i, k)M(j, l)(V(k) - V(l)), \end{aligned}$$

so

$$D(MV) = \sum_{1 \leq i, j \leq d} \left| \sum_{1 \leq k, l \leq d} M(i, k)M(j, l)(V(k) - V(l)) \right|.$$

On the other hand

$$\begin{aligned} D(V) &= \sum_{1 \leq k, l \leq d} |V(k) - V(l)| \\ &= \sum_{1 \leq i, j \leq d} \sum_{1 \leq k, l \leq d} M(i, k)M(j, l)|V(k) - V(l)|. \end{aligned}$$

By difference, $D(V) - D(MV)$ is the sum over all i and j in $[1, d]$ of the non negative quantities

$$\Delta(i, j) = \sum_{1 \leq k, l \leq d} M(i, k)M(j, l)|V(k) - V(l)| - \left| \sum_{1 \leq k, l \leq d} M(i, k)M(j, l)(V(k) - V(l)) \right|.$$

¹namely $\sum_{i=1}^d U^\uparrow(d+1-i)V^\uparrow(i) \leq \sum_{i=1}^d U(i)V(i) \leq \sum_{i=1}^d U^\uparrow(i)V^\uparrow(i)$ for every U and V in \mathbf{R}^d .

Thus

$$D(V) - D(MV) \geq \sum_{1 \leq i \leq d} \Delta(i, i).$$

But for every $i \in [1, d]$,

$$\begin{aligned} \Delta(i, i) &= \sum_{1 \leq k, l \leq d} M(i, k)M(i, l)|V(k) - V(l)| - 0 \\ &\geq \sum_{1 \leq k \leq d} M(i, k)M(i, i)|V(k) - V(i)| \\ &\geq \gamma \sum_{1 \leq k \leq d} M(i, k)|V(k) - V(i)| \\ &\geq \gamma \left| \sum_{1 \leq k \leq d} M(i, k)(V(k) - V(i)) \right| \\ &= \gamma |(MV)(i) - V(i)|. \end{aligned}$$

The result follows. \square

We now derive the last step of the proof of theorem 8. Indeed, applying the next corollary to each vector of the canonical basis on \mathbf{R}^d yields theorem 8.

Corollary 15. *Let $(M_n)_{n \geq 1}$ be any sequence of $d \times d$ bistochastic matrices with diagonal entries bounded below by some $\gamma > 0$. For every column vector $V \in \mathbf{R}^d$, the sequence $(V_n)_{n \geq 0} := (M_n \dots M_1 V)_{n \geq 0}$ has a finite variation, so it converges. Moreover, the series $\sum_n M_n(i, j)$ and $\sum_n M_n(j, i)$ converge whenever the two sequences $(V_n(i))_{n \geq 0}$ and $(V_n(j))_{n \geq 0}$ have a different limit.*

Proof. Lemma 14 yields $\gamma \|V_{n+1} - V_n\|_1 \leq D(V_n) - D(V_{n+1})$ for every $n \geq 0$. In particular, the sequence $((D(V_n))_{n \geq 0})$ is non-increasing and bounded below by 0, so it converges. The convergence of the series $\sum_n \|V_{n+1} - V_n\|_1$ and the convergence of the sequence $(V_n)_{n \geq 0}$ follow.

Call $\lambda_1 < \dots < \lambda_r$ the distinct values of $\lim_{n \rightarrow \infty} V_n(i)$ for $i \in [1, d]$. For every $k \in [1, r]$, set $I_k = \{i \in [1, d] : \lim_{n \rightarrow \infty} V_n(i) = \lambda_k\}$, and $J_k = I_1 \cup \dots \cup I_k$.

The proof of the convergence of the series $\sum_n M_{n+1}(i, j)$ for every $(i, j) \in J_k \times J_k^c$ works like the proof of corollary 13, with r replaced by 1, thanks to lemma 16 stated below, so the rearrangement inequality becomes an equality.

Using the equality

$$\sum_{(i, j) \in J_k^c \times J_k} M_{n+1}(i, j) = |J_k| - \sum_{(i, j) \in J_k \times J_k} M_{n+1}(i, j) = \sum_{(i, j) \in J_k \times J_k^c} M_{n+1}(i, j),$$

we derive the convergence of the series $\sum_n M_{n+1}(i, j)$ for every $(i, j) \in J_k^c \times J_k$. The proof is complete. \square

Lemma 16. *Let M be some $d \times d$ doubly-stochastic matrix. Then for every column vector $V \in \mathbf{R}^d$ and $m \in [1, d]$*

$$\sum_{i=1}^m (MV)^\uparrow(i) \geq \sum_{i=1}^m V^\uparrow(i).$$

Proof. By applying a same permutation to the columns of M and to the components of V , one may assume that $V(1) \leq \dots \leq V(d)$. By applying a permutation to the rows of M , one may assume also that $(MV)(1) \leq \dots \leq (MV)(d)$. Since M is doubly-stochastic, the real numbers

$$S(j) = \sum_{i=1}^m M(i, j) \text{ for } j \in [1, d]$$

are in $[0, 1]$ and add up to m . Moreover

$$\sum_{i=1}^m (MV)(i) = \sum_{j=1}^d S(j)V(j).$$

Hence

$$\begin{aligned} \sum_{i=1}^m (MV)(i) - \sum_{j=1}^m V(j) &= \sum_{j=m+1}^d S(j)V(j) + \sum_{j=1}^m (S(j) - 1)V(j) \\ &\geq \sum_{j=m+1}^d S(j)V(m) + \sum_{j=1}^m (S(j) - 1)V(m) \\ &= 0. \end{aligned}$$

We are done. □

4 Proof of theorem 1

4.1 Condition for the non-existence of a solution with support included in $\text{Supp}(X_0)$

We assume that Γ contains no matrix with support included in $\text{Supp}(X_0)$, namely that the system

$$\begin{cases} \forall i \in [1, p], X(i, +) = a_i \\ \forall j \in [1, q], X(+, j) = b_j \\ \forall (i, j) \in [1, p] \times [1, q], X(i, j) \geq 0 \\ \forall (i, j) \in \text{Supp}(X_0)^c, X(i, j) = 0 \end{cases}$$

is inconsistent.

This system can be seen as a system of linear inequalities of the form $\ell(X) \leq c$ (where ℓ is some linear form and c some constant) by splitting each equality $\ell(X) = c$ into the two inequalities $\ell(X) \leq c$ and $\ell(X) \geq c$, and by transforming each inequality $\ell(X) \geq c$ into the equivalent inequality $-\ell(X) \leq -c$. But theorem 4.2.3 in [18] (a consequence of Farkas' or Fourier's lemma) states that a system of linear inequalities of the form $\ell(X) \leq c$ is inconsistent if and only if some linear combination with non-negative weights of the linear inequalities yields the inequality $0 \leq -1$.

Consider such a linear combination and call $\alpha_{i,+}$, $\alpha_{i,-}$, $\beta_{j,+}$, $\beta_{j,-}$, $\gamma_{i,j,+}$, $\gamma_{i,j,-}$ the weights associated to the inequalities $X(i, +) \leq a_i$, $-X(i, +) \leq -a_i$, $X(+, j) \leq b_j$, $-X(+, j) \leq -b_j$, $X(i, j) \leq 0$, $-X(i, j) \leq 0$. When $(i, j) \in \text{Supp}(X_0)$, the inequality $X(i, j) \leq 0$ does not appear in the system, so we set $\gamma_{i,j,+} = 0$. Then the real numbers $\alpha_i := \alpha_{i,+} - \alpha_{i,-}$, $\beta_j := \beta_{j,+} - \beta_{j,-}$, $\gamma_{i,j} := \gamma_{i,j,+} - \gamma_{i,j,-}$ satisfy the following conditions:

- for every $(i, j) \in [1, p] \times [1, q]$, $\alpha_i + \beta_j + \gamma_{i,j} = 0$,

- $\sum_{i=1}^p \alpha_i a_i + \sum_{j=1}^q \beta_j b_j = -1,$
- $\gamma_{i,j} \leq 0$ whenever $(i,j) \in \text{Supp}(X_0).$

Let U and V be two random variables with respective laws

$$\sum_{i=1}^p a_i \delta_{\alpha_i} \text{ and } \sum_{j=1}^q b_j \delta_{-\beta_j}.$$

Then

$$\int_{\mathbf{R}} (P[U > t] - P[V > t]) dt = \mathbf{E}[U] - \mathbf{E}[V] = \sum_{i=1}^p \alpha_i a_i + \sum_{j=1}^q \beta_j b_j = -1 < 0,$$

so there exists some real number t such that $P[U > t] - P[V > t] < 0$. Consider the sets $A = \{i \in [1, p] : \alpha_i \leq t\}$ and $B = \{j \in [1, q] : \beta_j < -t\}$. Then for every $(i, j) \in A \times B$, $-\gamma_{i,j} = \alpha_i + \beta_j < 0$, so $(i, j) \notin \text{Supp}(X_0)$. In other words, X_0 is null on $A \times B$. Moreover,

$$a(A) - b(B^c) = \sum_{i \in A} a_i - \sum_{j \in B^c} b_j = P[U \leq t] - P[-V \geq -t] = P[V > t] - P[U > t] > 0.$$

Hence the block $A \times B$ is a cause of incompatibility. The proof is complete.

4.2 Condition for the existence of additional zeroes shared by every solution in $\Gamma(X_0)$

We now assume that Γ contains some matrix with support included in $\text{Supp}(X_0)$.

Using the convexity of $\Gamma(X_0)$, one can construct a matrix $S_0 \in \Gamma(X_0)$ whose support contains the support of every matrix in $\Gamma(X_0)$. This yields item 1.

We now prove items 2 and 3. The observations made in the introduction show that $\text{Supp}(X_0) \setminus \text{Supp}(S_0)$ contains $A^c \times B^c$ whenever $A \times B$ is a non-empty subset of $[1, p] \times [1, q]$ such that X_0 is null on $A \times B$ and $a(A) = b(B^c)$. This yields the ‘if’ part of item 2 and one inclusion in item 3.

To prove the ‘only if’ part of item 2 and the reverse inclusion in item 3, fix $(i_0, j_0) \in \text{Supp}(X_0) \setminus \text{Supp}(S_0)$. Then for every $p \times q$ matrix X with real entries,

$$\left. \begin{array}{l} \forall i \in [1, p], X(i, +) = a_i \\ \forall j \in [1, q], X(+, j) = b_j \\ \forall (i, j) \in [1, p] \times [1, q], X(i, j) \geq 0 \\ \forall (i, j) \in \text{Supp}(X_0)^c, X(i, j) = 0 \end{array} \right\} \implies X(i_0, j_0) \leq 0.$$

The system in the left-hand side of the implication is consistent since $\Gamma(X_0)$ is non-empty by assumption. As before, the system at the left-hand side of the implication can be seen as a system of linear inequalities of the form $\ell(X) \leq c$.

We now use theorem 4.2.7 in [18] (a consequence of Farkas’ or Fourier’s lemma) which states that any linear inequation which is a consequence of some consistent system of linear inequalities of the form $\ell(X) \leq c$ can be deduced from the system and from the inequality $0 \leq 1$ by linear combinations with non-negative weights.

Consider such a linear combination and call $\alpha_{i,+}$, $\alpha_{i,-}$, $\beta_{j,+}$, $\beta_{j,-}$, $\gamma_{i,j,+}$, $\gamma_{i,j,-}$ and η the weights associated to the inequalities $X(i,+) \leq a_i$, $-X(i,+) \leq -a_i$, $X(+,j) \leq b_j$, $-X(+,j) \leq -b_j$, $X(i,j) \leq 0$, $-X(i,j) \leq 0$, and $0 \leq 1$. When $(i,j) \in \text{Supp}(X_0)$, the inequality $X(i,j) \leq 0$ does not appear in the system, so we set $\gamma_{i,j,+} = 0$. Then the real numbers $\alpha_i := \alpha_{i,+} - \alpha_{i,-}$, $\beta_j := \beta_{j,+} - \beta_{j,-}$, $\gamma_{i,j} := \gamma_{i,j,+} - \gamma_{i,j,-}$ satisfy the following conditions.

- for every $(i,j) \in [1,p] \times [1,q]$, $\alpha_i + \beta_j + \gamma_{i,j} = \delta_{i,i_0} \delta_{j,j_0}$,
- $\sum_{i=1}^p \alpha_i a_i + \sum_{j=1}^q \beta_j b_j + \eta = 0$.
- $\gamma_{i,j} \leq 0$ whenever $(i,j) \in \text{Supp}(X_0)$,
- $\eta \geq 0$

Let U and V be two random variables with respective laws

$$\sum_{i=1}^p a_i \delta_{\alpha_i} \text{ and } \sum_{j=1}^q b_j \delta_{-\beta_j}.$$

Then

$$\int_{\mathbf{R}} (P[U > t] - P[V > t]) dt = \mathbf{E}[U] - \mathbf{E}[V] = \sum_{i=1}^p \alpha_i a_i + \sum_{j=1}^q \beta_j b_j = -\eta \leq 0.$$

Set $u_0 = \min(\alpha_1, \dots, \alpha_p) = \text{ess inf } U$ and $v_0 = \max(-\beta_1, \dots, -\beta_q) = \text{ess sup } V$. Then $P[U > t] - P[V > t] \geq 0$ when $t < u_0$ or $t \geq v_0$. Hence one can find $t \in [u_0, v_0]$ such that $P[U > t] - P[V > t] \leq 0$. Consider the sets $A = \{i \in [1,p] : \alpha_i \leq t\}$ and $B = \{j \in [1,q] : \beta_j < -t\}$. For every $(i,j) \in A \times B$, $\gamma_{i,j} = \delta_{i,i_0} \delta_{j,j_0} - \alpha_i - \beta_j > 0$, so $(i,j) \notin \text{Supp}(X_0)$. In other words, X_0 is null on $A \times B$. Moreover,

$$a(A) = \sum_{i \in A} a_i = P[U \leq t] \geq P[U = u_0] > 0,$$

$$b(B) = \sum_{j \in B} b_j = P[-V < -t] \geq P[V = v_0] > 0,$$

so A and B are non-empty and

$$a(A) - b(B^c) = P[U \leq t] - P[V \leq t] = P[V > t] - P[U > t] \geq 0.$$

The last inequality is necessarily an equality, since otherwise $A \times B$ would be a cause of incompatibility. Hence $a(A) = b(B^c)$, so $A \times B$ is cause of criticality. This proves the ‘only if’ part of item 2.

We also know that $(i_0, j_0) \notin A \times B$ since $X_0(i_0, j_0) > 0$. If we had $(i_0, j_0) \in A^c \times B^c$, we would get the reverse inclusion in item 3. Unfortunately, this statement may fail with the choice of t made above.

Assume, that $(i_0, j_0) \notin A^c \times B^c$. Since $a(A) = b(B^c)$, the linear system defining $\Gamma(a, b, X_0)$ can be split into three independent consistent subsystems, namely

$$\forall (i,j) \in (A \times B) \cup (A^c \times B^c), X(i,j) = 0$$

and the two systems

$$\left\{ \begin{array}{l} \forall i \in I, X(i, J) = a_i \\ \forall j \in J, X(I, j) = b_j \\ \forall (i, j) \in I \times J, X(i, j) \geq 0 \\ \forall (i, j) \in (I \times J) \setminus \text{Supp}(X_0)^c, X(i, j) = 0 \end{array} \right.$$

where the block $I \times J$ is either $A \times B^c$ or $A^c \times B$.

If $(i_0, j_0) \notin A^c \times B^c$, then (i_0, j_0) belongs to one of these two blocks, say $I_1 \times J_1$. Since $a(I_1) = b(J_1)$ and since the equality $X(i_0, j_0) = 0$ is a consequence of the consistent subsystem above with $I \times J = I_1 \times J_1$, one can apply the proof of item 2 to the marginals $a(\cdot|I_1)$ and $b(\cdot|J_1)$ and the restriction of X_0 on $I_1 \times J_1$. This yields a subset $A_1 \times B_1$ of $I_1 \times J_1$ such that X_0 is null on $A_1 \times B_1$, $a(A_1) = b(B_1^c)$, and $(i_0, j_0) \notin A_1 \times B_1$.

If $(i_0, j_0) \in A_1^c \times B_1^c$, we are done. Otherwise, (i_0, j_0) belongs to one of the two blocks $A_1 \times B_1^c$ or $A_1^c \times B_1$, say $I_2 \times J_2$, and the recursive construction goes on. This construction necessarily stops after a finite number of steps and produces a set $A' \times B'$ such that X_0 is null on $A' \times B'$, $a(A') = b(B'^c)$, and $(i_0, j_0) \in A'^c \times B'^c$. Item 3 follows.

5 Tools and preliminary results

5.1 Results on the quantities $R_i(X_{2n})$ and $C_j(X_{2n+1})$

Lemma 17. *Let $X \in \Gamma_1$. Then*

$$\sum_{i=1}^q a_i R_i(X) = \sum_{i,j} X(i, j) = \sum_j b_j = 1$$

and for every $j \in [1, q]$,

$$C_j(T_R(X)) = \sum_{i=1}^p \frac{X(i, j)}{b_j} R_i(X)^{-1}.$$

When $X \in \Gamma_C$, this equality expresses $C_j(T_R(X))$ as a weighted (arithmetic) mean of the quantities $R_i(X)^{-1}$, with weights $X(i, j)/b_j$.

For every $X \in \Gamma_1$, call $R(X)$ the column vecteur with components $R_1(X), \dots, R_p(X)$ and $C(X)$ the column vecteur with components $C_1(X), \dots, C_q(X)$. Set

$$\underline{R}(X) = \min_i R_i(X), \quad \overline{R}(X) = \max_i R_i(X), \quad \underline{C}(X) = \min_j C_j(X), \quad \overline{C}(X) = \max_j C_j(X).$$

Corollary 18. *The intervals*

$$[\overline{C}(X_1)^{-1}, \underline{C}(X_1)^{-1}], [\underline{R}(X_2), \overline{R}(X_2)], [\overline{C}(X_3)^{-1}, \underline{C}(X_3)^{-1}], [\underline{R}(X_4), \overline{R}(X_4)], \dots$$

contain 1 and form a non-increasing sequence.

In lemma 17, one can invert the roles of the lines and the columns. Given $X \in \Gamma_C$, the matrix $T_R(X)$ is in Γ_R so the quantities $R_i(T_C(T_R(X)))$ can be written as weighted (arithmetic) means of the $C_j(T_R(X))^{-1}$. But the $C_j(T_R(X))$ can be written as a weighted (arithmetic) means of the quantities $R_k(X)^{-1}$. Putting things together, one gets weighted arithmetic means of weighted harmonic means. Next lemma shows how to transform these into weighted arithmetic means by modifying the weights.

Lemma 19. Let $X \in \Gamma_C$. Then $R(T_C(T_R(X))) = P(X)R(X)$, where $P(X)$ is the $p \times p$ matrix given by

$$P(X)(i, k) = \sum_{j=1}^q \frac{T_R(X)(i, j)T_R(X)(k, j)}{a_i b_j C_j(T_R(X))}.$$

The matrix $P(X)$ is stochastic. Moreover it satisfies for every i and k in $[1, p]$,

$$P(X)(i, i) \geq \frac{\underline{a}}{\bar{b} \bar{C}(T_R(X)) q}$$

and

$$P(X)(k, i) \leq \frac{\bar{a}}{\underline{a}} P(X)(i, k).$$

Proof. For every $i \in [1, p]$,

$$R_i(T_C(T_R(X))) = \sum_{j=1}^q \frac{T_R(X)(i, j)}{a_i} \frac{1}{C_j(T_R(X))}.$$

But the assumption $X \in \Gamma_C$ yields

$$1 = \frac{1}{b_j} \sum_{k=1}^p X(k, j) = \frac{1}{b_j} \sum_{k=1}^p T_R(X)(k, j) R_k(X).$$

Hence,

$$R_i(T_C(T_R(X))) = \sum_{j=1}^q \frac{T_R(X)(i, j)}{a_i} \sum_{k=1}^p \frac{T_R(X)(k, j)}{b_j C_j(T_R(X))} R_k(X).$$

These equalities can be written as $R(T_C(T_R(X))) = P(X)R(X)$, where $P(X)$ is the $p \times p$ matrix whose entries are given in the statement of lemma 19. By construction, the entries of $P(X)$ are non-negative and for every $i \in [1, p]$,

$$\sum_{k=1}^p P(X)(i, k) = \sum_{j=1}^q \frac{T_R(X)(i, j)}{a_i b_j C_j(T_R(X))} \sum_{k=1}^p T_R(X)(k, j) = \sum_{j=1}^q \frac{T_R(X)(i, j)}{a_i} = 1.$$

Moreover, since

$$\sum_{j=1}^q T_R(X)(i, j)^2 \geq \frac{1}{q} \left(\sum_{j=1}^q T_R(X)(i, j) \right)^2 = \frac{a_i^2}{q}$$

we have

$$\begin{aligned} P(X)(i, i) &\geq \frac{1}{a_i \bar{b} \bar{C}(T_R(X))} \sum_{j=1}^q T_R(X)(i, j)^2 \\ &= \frac{a_i}{\bar{b} \bar{C}(T_R(X)) q} \\ &\geq \frac{\underline{a}}{\bar{b} \bar{C}(T_R(X)) q}. \end{aligned}$$

The last inequality to be proved follows directly from the symmetry of the matrix $(a_i P(X)(i, k))_{1 \leq i, k \leq p}$. \square

5.2 A function associated to each element of Γ_1

Definition 20. For every X and S in Γ_1 , we set

$$F_S(X) = \prod_{(i,j) \in [1,p] \times [1,q]} X(i,j)^{S(i,j)},$$

with the convention $0^0 = 1$.

We note that $0 \leq F_S(X) \leq 1$, and that $F_S(X) > 0$ if and only if $\text{Supp}(S) \subset \text{Supp}(X)$.

Lemma 21. Let $S \in \Gamma_1$. For every $X \in \Gamma_1$, $F_S(X) \leq F_S(S)$, with equality if and only if $X = S$. Moreover, if $\text{Supp}(S) \subset \text{Supp}(X)$, then $D(S||X) = \ln(F_S(S)/F_S(X))$.

Proof. Assume that $\text{Supp}(S) \subset \text{Supp}(X)$. The definition of F_S and the arithmetic-geometric inequality yield

$$\frac{F_S(S)}{F_S(X)} = \prod_{i,j} \left(\frac{X(i,j)}{S(i,j)} \right)^{S(i,j)} \leq \sum_{i,j} S(i,j) \left(\frac{X(i,j)}{S(i,j)} \right) = \sum_{i,j} X(i,j) = 1,$$

with equality if and only if $X(i,j) = S(i,j)$ for every $(i,j) \in \text{Supp}(S)$. The result follows. \square

Lemma 22. Let $X \in \Gamma_1$.

- For every $S \in \Gamma_R$ such that $\text{Supp}(S) \subset \text{Supp}(X)$, one has $F_S(X) \leq F_S(T_R(X))$, and the ratio $F_S(X)/F_S(T_R(X))$ does not depend on S .
- For every $S \in \Gamma_C$ such that $\text{Supp}(S) \subset \text{Supp}(X)$, one has $F_S(X) \leq F_S(T_C(X))$, and the ratio $F_S(X)/F_S(T_C(X))$ does not depend on S .

Proof. Let $S \in \Gamma_R$. For every $(i,j) \in \text{Supp}(X)$, $X(i,j)/(T_R(X))(i,j) = R_i(X)$ so the arithmetic-geometric mean inequality yields

$$\frac{F_S(X)}{F_S(T_R(X))} = \prod_{i,j} R_i(X)^{S_{i,j}} = \prod_i R_i(X)^{a_i} \leq \sum_i a_i R_i(X) = \sum_{i,j} X(i,j) = 1.$$

The first statement follows. The second statement is proved in the same way. \square

Corollary 23. Assume that $\Gamma(X_0)$ is not empty. Then:

1. for every $S \in \Gamma(X_0)$, the sequence $(F_S(X_n))_{n \geq 0}$ is non-decreasing and bounded above, so it converges ;
2. for every (i,j) in the union of the supports $\text{Supp}(S)$ over all $S \in \Gamma(X_0)$, the sequence $(X_n(i,j))_{n \geq 1}$ is bounded away from 0.

Proof. Lemmas 21 and 22 yield the first item. Given $S \in \Gamma(X_0)$ and $(i,j) \in \text{Supp}(S)$, we get for every $n \geq 0$, $X_n(i,j)^{S(i,j)} \geq F_S(X_n) \geq F_S(X_0) > 0$. The second item follows. \square

The first item of corollary 23 will yield the first items of theorem 3. The second item of corollary 23 is crucial to establish the geometric rate of convergence in theorem 2 and the convergence in theorem 3.

6 Proof of theorem 2

In this section, we assume that Γ contains some matrix having the same support as X_0 , and we establish the convergences with at least geometric rate stated in theorem 2. The main tools are lemma 9 and the second item of corollary 23. Corollary 23 shows that the non-zero entries of all matrices X_n are bounded below by some positive real number γ . Therefore, the non-zero entries of all matrices $X_n X_n^\top$ are bounded below by γ^2 . These matrix have the same support as $X_0 X_0^\top$.

By lemma 19, for every $n \geq 1$, $R(X_{2n+2}) = P(X_{2n})R(X_{2n})$, where $P(X_{2n})$ is a stochastic matrix given by

$$P(X_{2n})(i, i') = \sum_{j=1}^q \frac{X_{2n+1}(i, j)X_{2n+1}(i', j)}{a_i b_j C_j(X_{2n+1})}.$$

These matrices have also the same support as $X_0 X_0^\top$. Moreover, by lemma 19 and corollary 18,

$$P(X_{2n})(i, i') \geq \frac{1}{\bar{a} \bar{b} \bar{C}(X_3)} (X_{2n+1} X_{2n+1}^\top)(i, i'),$$

so the non-zero entries of $P(X_{2n})$ are bounded below by $\gamma^2 / (\bar{a} \bar{b} \bar{C}(X_3)) > 0$.

We now define a binary relation on the set $[1, p]$ by

$$i \mathcal{R} i' \Leftrightarrow X_0 X_0^\top(i, i') > 0 \Leftrightarrow \exists j \in [1, q], X_0(i, j)X_0(i', j) > 0.$$

The matrix $X_0 X_0^\top$ is symmetric with positive diagonal (since on each line, X_0 has at least a positive entry), so the relation \mathcal{R} is symmetric and reflexive. Call I_1, \dots, I_r the connected components of the graph G associated to \mathcal{R} , and d the maximum of their diameters. For each k , set $J_k = \{j \in [1, q] : \exists i \in I_k : X_0(i, j) > 0\}$.

Lemma 24. *The sets J_1, \dots, J_k form a partition of $[1, q]$ and the support of X_0 is contained in $I_1 \times J_1 \cup \dots \cup I_r \times J_r$. Therefore, the support of $X_0 X_0^\top$ is contained in $I_1 \times I_1 \cup \dots \cup I_r \times I_r$, so one can get a block-diagonal matrix by permuting suitably the lines of X_0 .*

Proof. By assumption, the sum of the entries of X_0 on any row or any column is positive.

Given $k \in [1, r]$ and $i \in I_k$, there exists $j \in [1, q]$ such that $X_0(i, j) > 0$, so J_k is not empty.

Fix now $j \in [1, q]$. There exists $i \in [1, p]$ such that $X_0(i, j) > 0$. Such an i belongs to some connected component I_k , and j belongs to the corresponding J_k . If j also belongs to $J_{k'}$, then $X_0(i', j) > 0$ for some $i' \in I_{k'}$, so $X_0 X_0^\top(i, i') \geq X_0(i, j)X_0(i', j) > 0$, hence i and i' belong to the connected component of G , so $k' = k$.

The other statements follow. \square

Lemma 25. *For every $n \geq 1$, set $P_{2n} = P(X_{2n})$ and $M_n = P_{2n+2d-2} \cdots P_{2n+2} P_{2n}$. Call c the infimum of all positive entries of all matrices P_{2n} . Then $c > 0$, and for every $n \geq 1$, $i \in I_k$ and $i' \in I_{k'}$.*

$$\begin{aligned} M_n(i, i') &\geq c^d && \text{if } k = k'. \\ M_n(i, i') &= 0 && \text{if } k \neq k'. \end{aligned}$$

Proof. The positivity of c has already be proved at the beginning of the present section. Moreover,

$$M_n(i, i') = \sum_{1 \leq i_1, \dots, i_{d-1} \leq p} P_{2n+2d-2}(i, i_1) P_{2n+2d-4}(i_1, i_2) \cdots P_{2n+2}(i_{d-2}, i_{d-1}) P_{2n}(i_{d-1}, i').$$

If $k \neq k'$, all these products are 0, since no path can connect i and i' in the graph G .

If $k = k'$, one can find a path $i = i_0, \dots, i_\ell = i'$ in the graph G with length $\ell \leq d$. Setting $i_{\ell+1} = \dots = i_d$ if $\ell < d$, we get

$$P_{2n+2d-2}(i, i_1) P_{2n+2d-4}(i_1, i_2) \cdots P_{2n+2}(i_{d-2}, i_{d-1}) P_{2n}(i_{d-1}, i') \geq c^d.$$

The result follows. \square

Keep the notations of the last lemma. Then for every $n \geq 1$, $R(X_{2n+2d}) = M_n R(X_{2n})$. For each $k \in [1, r]$, lemma 9 applied to the submatrix $(M_n(i, i'))_{i, i' \in I_k}$ and the vector $L_{I_k}(X_{2n}) = (R_i(X_{2n}))_{i \in I_k}$ yields

$$\text{diam}(L_{I_k}(X_{2n+2d})) \leq (1 - c^d) \text{diam}(L_{I_k}(X_{2n})).$$

But lemma 9 applied to the submatrix $(P(X_{2n})(i, i'))_{i, i' \in I_k}$ shows that the intervals

$$\left[\min_{i \in I_k} R_i(X_{2n}), \max_{i \in I_k} R_i(X_{2n}) \right]$$

indexed by $n \geq 1$ form a non-increasing sequence. Therefore, each sequence $(R_i(X_{2n}))$ tends to a limit which does not depend on $i \in I_k$, and the speed of convergence it at least geometric.

Call λ_k this limit. By lemma 24, we have for every $n \geq 1$,

$$\sum_{i \in I_k} a_i R_i(X_{2n}) = \sum_{(i, j) \in I_k \times J_k} X_{2n}(i, j) = \sum_{j \in J_k} X_{2n}(+, j) = \sum_{j \in J_k} b_j$$

Passing to the limit yields

$$\lambda_k \sum_{i \in I_k} a_i = \sum_{j \in J_k} b_j,$$

whereas the assumption that Γ contains some matrix S having the same support as X_0 yields

$$\sum_{i \in I_k} a_i = \sum_{(i, j) \in I_k \times J_k} S(i, j) = \sum_{j \in J_k} b_j.$$

Thus $\lambda_k = 1$.

We have proved that each sequence $(R_i(X_{2n}))_{n \geq 0}$ tends to 1 with at least geometric rate. The same arguments work for the sequences $(C_j(X_{2n+1}))_{n \geq 0}$. Therefore, each infinite product $R_i(X_0)R_i(X_2) \cdots$ or $C_j(X_1)C_j(X_3) \cdots$ converges at an at least geometric rate. The convergence of the sequence $(X_n)_{n \geq 0}$ with at a least geometric rate follows.

Moreover, call α_i and β_j the inverses of the infinite products $R_i(X_0)R_i(X_2) \cdots$ and $C_j(X_1)C_j(X_3) \cdots$ and X_∞ the limit of $(X_n)_{n \geq 0}$. Then $X_\infty(i, j) = \alpha_i \beta_j X_0(i, j)$, so X_∞ belongs to the set $\Delta_p X_0 \Delta_q$. As noted in the introduction, we have also $X_\infty \in \Gamma$. It remains to prove that X_∞ is the only matrix in $\Gamma \cap \Delta_p X_0 \Delta_q$ and the only matrix which achieves the least upper bound of $D(Y||X_0)$ over all $Y \in \Gamma(X_0)$.

Let E_{X_0} be the vector space of all matrices in $\mathcal{M}_{p,q}(\mathbf{R})$ which are null on $\text{Supp}(X_0)^c$ (which can be identified canonically with $\mathbf{R}^{\text{Supp}(X_0)}$), and $E_{X_0}^+$ be the convex subset of all non-negative matrices in E_{X_0} . The subset $E_{X_0}^{+*}$ of all matrices in E_{X_0} which are positive on $\text{Supp}(X_0)^c$, is open in E_{X_0} , dense in $E_{X_0}^+$ and contains X_∞ . Consider the map f_{X_0} from $E_{X_0}^+$ to \mathbf{R} defined by

$$f_{X_0}(Y) = \sum_{(i,j) \in \text{Supp}(X_0)} Y(i,j) \ln \frac{Y(i,j)}{X_0(i,j)},$$

with the convention $t \ln t = 0$. This map is strictly convex since the map $t \mapsto t \ln t$ from \mathbf{R}_+ to \mathbf{R} is. Its differential at any point $Y \in E_{X_0}^{+*}$ is given by

$$df_{X_0}(Y)(H) = \sum_{(i,j) \in \text{Supp}(X_0)} \left(\ln \frac{Y(i,j)}{X_0(i,j)} + 1 \right) H(i,j).$$

Now, if Y_0 is any matrix in $\Gamma \cap \Delta_p X_0 \Delta_q$ (including the matrix X_∞), the quantities $\ln(Y_0(i,j)/X_0(i,j))$ can be written $\lambda_i + \mu_j$. Thus for every matrix $H \in E(X_0)$ with null row-sums and column-sums, $df_{X_0}(Y_0)(H) = 0$, hence the restriction of f_{X_0} to $\Gamma(X_0)$ has a strict global minimum at Y_0 . The proof is complete.

The case of positive matrices. The proof of the convergence at an at least geometric rate can be notably simplified when X_0 has only positive entries. In this case, Fienberg [6] used geometric arguments to prove the convergence of the iterated proportional fitting procedure at an at least geometric rate. We sketch another proof using the observation made by Fienberg that the ratios

$$\frac{X_n(i,j)X_n(i',j')}{X_n(i,j')X_n(i',j)}$$

are independent of n , since they are preserved by the transformations T_R and T_C . Call κ the least of these positive constants. Using corollary 18, one checks that the average of the entries of X_n on each row or column remains bounded below by some constant $\gamma > 0$. Thus for every location (i,j) and $n \geq 1$, one can find two indexes $i' \in [1,p]$ and $j' \in [1,q]$ such that $X_n(i',j) \geq \gamma$ and $X_n(i,j') \geq \gamma$, so

$$X_n(i,j) \geq X_n(i,j)X_n(i',j') \geq \kappa X_n(i,j')X_n(i',j) \geq \kappa \gamma^2.$$

This shows that the entries of the matrices X_n remain bounded away from 0, so the ratios $X_n(i,j)/b_j$ and $X_n(i,j)/a_i$ are bounded below by some constant $c > 0$ independent of $n \geq 1, i$ and j . Set

$$\rho_n = \frac{\overline{R}(X_n)}{\underline{R}(X_n)} \text{ if } n \text{ is even, } \rho_n = \frac{\overline{C}(X_n)}{\underline{C}(X_n)} \text{ if } n \text{ is odd.}$$

For every $n \geq 1$, the equalities

$$C_j(X_{2n+1}) = \sum_{i=1}^p \frac{X_{2n}(i,j)}{b_j} \frac{1}{R_i(X_{2n+1})}$$

and lemma 9 yields

$$\overline{R}(X_{2n})^{-1} \leq (1-c)\overline{R}(X_{2n})^{-1} + c\underline{R}(X_{2n})^{-1} \leq C_j(X_{2n+1}) \leq (1-c)\underline{R}(X_{2n})^{-1} + c\overline{R}(X_{2n})^{-1}.$$

Thus,

$$\begin{aligned}\rho_{2n+1} - 1 &= \frac{\overline{C}(X_{2n+1}) - \underline{C}(X_{2n+1})}{\underline{C}(X_{2n+1})} \\ &\leq \frac{(1-2c)(\underline{R}(X_{2n})^{-1} - \overline{R}(X_{2n})^{-1})}{\overline{R}(X_{2n})^{-1}} = (1-2c)(\rho_{2n} - 1).\end{aligned}$$

We prove the inequality $\rho_{2n} - 1 \leq (1-2c)(\rho_{2n-1} - 1)$ in the same way. Hence $\rho_n \rightarrow 1$ at an at least geometric rate. The result follows by corollary 18.

7 Proof of theorem 3

We now assume that Γ contains some matrix with support included in $\text{Supp}(X_0)$.

7.1 Asymptotic behavior of the sequences $(R(X_n))$ and $(C(X_n))$.

The first item of corollary 23 yields the convergence of the infinite product

$$\prod_i R_i(X_0)^{a_i} \times \prod_j C_j(X_1)^{b_j} \times \prod_i R_i(X_2)^{a_i} \times \prod_j C_j(X_3)^{b_j} \times \dots$$

Set $g(t) = t - 1 - \ln t$ for every $t > 0$. Using the equalities

$$\forall n \geq 1, \sum_{i=1}^p a_i R_i(X_{2n}) = \sum_{j=1}^q b_j C_j(X_{2n-1}) = 1,$$

we derive the convergence of the series

$$\sum_i a_i g(R_i(X_0)) + \sum_j b_j g(C_j(X_1)) + \sum_i a_i g(R_i(X_2)) + \sum_j b_j g(C_j(X_3)) + \dots$$

But g is null at 1, positive everywhere else, and tends to infinity at $0+$ and at $+\infty$. By positivity of the a_i and b_j , we get the convergence of all series

$$\sum_{n \geq 0} g(R_i(X_{2n})) \text{ and } \sum_{n \geq 0} g(C_j(X_{2n+1}))$$

and the convergence of all sequences $(R_i(X_{2n}))_{n \geq 0}$ and to $(C_j(X_{2n+1}))_{n \geq 0}$ towards 1. But $g(t) \sim (t-1)^2/2$ as $t \rightarrow 1$, so the series $\sum_n (R_i(X_{2n}) - 1)^2$ and $\sum_{n \geq 0} (C_j(X_{2n+1}) - 1)^2$ converge.

We now use a quantity introduced by Bregman [3] and called L_1 -error by Pukelsheim [12]. For every $X \in \Gamma_1$, set

$$\begin{aligned}e(X) &= \sum_{i=1}^p \left| \sum_{j=1}^q X(i, j) - a_i \right| + \sum_{j=1}^q \left| \sum_{i=1}^p X(i, j) - b_j \right| \\ &= \sum_{i=1}^p a_i |R_i(X) - 1| + \sum_{j=1}^q b_j |C_j(X) - 1|.\end{aligned}$$

The convexity of the square function yields

$$\frac{e(X)^2}{2} \leq \sum_{i=1}^p a_i (R_i(X) - 1)^2 + \sum_{j=1}^q b_j (C_j(X) - 1)^2.$$

Thus the series $\sum_n e(X_n)^2$ converges. But the sequence $(e(X_n))_{n \geq 1}$ is non-increasing (the proof of this fact is recalled below). Therefore, for every $n \geq 1$,

$$0 \leq \frac{n}{2} e(X_n)^2 \leq \sum_{n/2 \leq k \leq n} e(X_k)^2.$$

Convergences $ne(X_n)^2 \rightarrow 0$, $\sqrt{n}(R_i(X_n) - 1) \rightarrow 0$ and $\sqrt{n}(C_j(X_n) - 1) \rightarrow 0$ follow.

To check the monotonicity of $(e(X_n))_{n \geq 1}$, note that $T_R(X) \in \Gamma_R$ for every $X \in \Gamma_C$, so

$$\begin{aligned} e(T_R(X)) &= \sum_{j=1}^q \left| \sum_{i=1}^p X(i, j) R_i(X)^{-1} - b_j \right| \\ &= \sum_{j=1}^q \left| \sum_{i=1}^p X(i, j) (R_i(X)^{-1} - 1) \right| \\ &\leq \sum_{i=1}^p \sum_{j=1}^q X(i, j) |R_i(X)^{-1} - 1| \\ &= \sum_{i=1}^p a_i R_i(X) |R_i(X)^{-1} - 1| \\ &= e(X). \end{aligned}$$

In the same way, $e(T_C(Y)) \leq e(Y)$ for every $Y \in \Gamma_R$.

7.2 Convergence and limit of (X_n)

Since $\Gamma(X_0)$ is not empty, we can fix a matrix $S_0 \in \Gamma(X_0)$ whose support is maximum, like in theorem 1, critical case, item 1.

Let L be a limit point of the sequence $(X_n)_{n \geq 0}$, so L is the limit of some subsequence $(X_{\varphi(n)})_{n \geq 0}$. As noted in the introduction, $\text{Supp}(L) \subset \text{Supp}(X_0)$. But for every $i \in [1, p]$ and $j \in [1, q]$, $R_i(L) = \lim R_i(X_{\varphi(n)}) = 1$ and $C_j(L) = \lim C_j(X_{\varphi(n)}) = 1$. Hence $L \in \Gamma(X_0)$. Corollary 23 yields the inclusion $\text{Supp}(S_0) \subset \text{Supp}(L)$ hence for every $S \in \Gamma(X_0)$, $\text{Supp}(S) \subset \text{Supp}(S_0) \subset \text{Supp}(L) \subset \text{Supp}(X_0)$, so the quantities $F_S(X_0)$ and $F_S(L)$ are positive.

By lemma 22, the ratios $F_S(X_{\varphi(n)})/F_S(X_0)$ do not depend on $S \in \Gamma(X_0)$, so by continuity of F_S , the ratio $F_S(L)/F_S(X_0)$ does not depend on $S \in \Gamma(X_0)$. But by lemma 21,

$$\ln \frac{F_S(L)}{F_S(X_0)} = \ln \frac{F_S(S)}{F_S(X_0)} - \ln \frac{F_S(S)}{F_S(L)} = D(S||X_0) - D(S||L),$$

and $D(S||L) \geq 0$ with equality if and only if $S = L$. Therefore, L is the only element achieving the greatest lower bound of $D(S||X_0)$ over all $S \in \Gamma(X_0)$.

$$L = \arg \min_{S \in \Gamma(X_0)} D(S||X_0).$$

We have proved the unicity of the limit point of the sequence $(X_n)_{n \geq 0}$. By compactness of $\Gamma(X_0)$, the convergence follows.

Remark 26. *Actually, one has $\text{Supp}(S_0) = \text{Supp}(L)$. Indeed, theorem 3 shows that for every $(i, j) \in \text{Supp}(X_0) \setminus \text{Supp}(S_0)$, $X_n(i, j) \rightarrow 0$ as $n \rightarrow +\infty$. This fact could be retrived by using the same arguments as in the proof of theorem 1 to show that on the set $\Gamma(X_0)$, the linear form $X \mapsto X(i_0, j_0)$ coincides with some linear combination of the affine forms $X \mapsto R_i(X) - 1$, $i \in [1, p]$, and $X \mapsto C_j(X) - 1$, $j \in [1, q]$.*

8 Proof of theorems 4, 5 and 6

We recall that neither proof below uses the assumption that $\Gamma(X_0)$ is empty.

8.1 Proof of theorem 4

Convergence of the sequences $(R(X_{2n}))$ and $(C(X_{2n+1}))$. By lemma 19 and corollary 18, we have for every $n \geq 1$, $R(X_{2n+2}) = P(X_{2n})R(X_{2n})$, where $P(X_{2n})$ is a stochastic matrix such that for every i and k in $[1, p]$,

$$P(X_{2n})(i, i) \geq \frac{a}{\bar{b} \bar{C}(X_{2n+1})q} \geq \frac{a \underline{R}(X_2)}{\bar{b}q}$$

and

$$P(X_{2n})(k, i) \leq (\bar{a}/a)P(X_{2n})(i, k).$$

The sequence $(P(X_{2n}))_{n \geq 0}$ satisfies the assumption of corollary 13 and theorem 7, so any one of these two results ensures the convergence of the sequence $(R(X_{2n}))_{n \geq 0}$. By corollary 18, the entries of these vectors stay in the interval $[\underline{R}(X_2), \bar{R}(X_2)]$, so the limit of each entry is positive. The same arguments show that the sequence $(C(X_{2n+1}))_{n \geq 0}$ also converges to some vector with positive entries.

Relations between the components of the limits, and block structure. Denote by $\lambda_1 < \dots < \lambda_r$ the different values of the limits of the sequences $(R_i(X_{2n}))_{n \geq 0}$, and by $\mu_1 > \dots > \mu_s$ the different values of the limits of the sequences $(C_j(X_{2n+1}))_{n \geq 0}$. The values of these limits will be precised later. Consider the sets

$$I_k = \{i \in [1, p] : \lim R_i(X_{2n}) = \lambda_k\} \text{ for } k \in [1, r],$$

$$J_l = \{j \in [1, q] : \lim C_j(X_{2n+1}) = \mu_l\} \text{ for } l \in [1, s].$$

When $(i, j) \in I_k \times J_l$, the sequence $(R_i(X_{2n})C_j(X_{2n+1}))_{n \geq 0}$ converges to $\lambda_k \mu_l$. If $\lambda_k \mu_l > 1$, this entails the convergence to 0 of the sequence $(X_n(i, j))_{n \geq 0}$ with a geometric rate; and if $\lambda_k \mu_l < 1$, this entails the nullity of all $X_n(i, j)$ (otherwise the sequence $(X_n(i, j))_{n \geq 0}$ would go to infinity). But for all $n \geq 1$, $R_i(X_{2n}) = 1$ and $C_j(X_{2n+1}) = 1$, so at least one entry of the matrices X_n on each line or column does not converge to 0. This forces the equalities $s = r$ and $\mu_k = \lambda_k^{-1}$ for every $k \in [1, r]$.

Convergence of the sequences (X_{2n}) and (X_{2n+1}) . Let L be any limit point of the sequence $(X_{2n})_{n \geq 0}$, so L is the limit of some subsequence $(X_{2\varphi(n)})_{n \geq 0}$. By definition of a' , $R_i(L) = \lim R_i(X_{2\varphi(n)}) = a'_i/a_i$ for every $i \in [1, p]$. Moreover, $\text{Supp}(L) \subset \text{Supp}(X_0)$, so L belongs to $\Gamma(a', b, X_0)$.

Like in subsection 7.2, we check that the quantity

$$D(S||X_0) - D(S||L) = \ln(F_S(L)/F_S(X_0))$$

does not depend on $S \in \Gamma(a', b, X_0)$, so L is the unique matrix achieving the greatest lower bound of $D(S||X_0)$ over all $S \in \Gamma(a', b, X_0)$. The convergence of $(X_{2n})_{n \geq 0}$ follows by compactness of $\Gamma(a', b, X_0)$.

By lemma 22, the ratios $F_S(X_{2\varphi(n)})/F_S(X_0)$ do not depend on $S \in \Gamma(X_0)$, so by continuity of F_S , the ratio $F_S(L)/F_S(X_0)$. The same arguments show that the sequence $(X_{2n+1})_{n \geq 0}$ converges to the unique matrix achieving the greatest lower bound of $D(S||X_0)$ over all $S \in \Gamma(a, b', X_0)$.

Formula for λ_k . We know that the sequence $(X_n(i, j))_{n \geq 0}$ converges to 0 whenever $i \in I_k$ and $j \in J_l$ with $k \neq l$. Thus the support of $L = \lim X_{2n}$ is contained in $I_1 \times J_1 \cup \dots \cup I_r \times J_r$. But L belongs to $\Gamma(a', b, X_0)$, so for every $k \in [1, r]$

$$\lambda_k a(I_k) = \sum_{i \in I_k} a'_i = \sum_{(i, j) \in I_k \times J_k} L(i, j) = \sum_{j \in J_k} b_j = b(J_k).$$

Properties of matrices in $\Gamma(a', b, X_0)$ and $\Gamma(a, b', X_0)$. Let $S \in \Gamma(a, b', X_0)$.

Let $k \in [1, r-1]$, $A_k = I_1 \cup \dots \cup I_k$ and $B_k = J_{k+1} \cup \dots \cup J_r$. We already know that X_0 is null on $A_k \times B_k$, so S is also null on this set. Moreover, for every $l \in [1, r]$,

$$a(I_l) = \lambda_l^{-1} b(J_l) = \sum_{j \in J_l} \lambda_l^{-1} b_j = \sum_{j \in J_l} b'_j = b'(J_l).$$

Summation over all $l \in [1, k]$ yields $a(A_k) = b'(B_k^c)$. Hence by theorem 1 (critical case), S is null on the set $A_k^c \times B_k^c = (I_{k+1} \cup \dots \cup I_r) \times (J_1 \cup \dots \cup J_k)$.

This shows that the support of S is included in $I_1 \times J_1 \cup \dots \cup I_r \times J_r$. This block structure and the equalities $a'_i/a_i = b_j/b'_j = \lambda_k$ whenever $(i, j) \in I_k \times J_k$ yield the equality $D_1 S = S D_2$. This matrix has the same support as S . Moreover, its i -th row is a'_i/a_i times the i -th row of S , so its i -th row-sum is $a'_i/a_i \times a_i = a'_i$; in the same way its j -th column is b_j/b'_j times the j -th column of S , so its j -th column sum is $b_j/b'_j \times b'_j = b_j$. As symmetric conclusions hold for every matrix in $\Gamma(a', b, X_0)$, the proof is complete.

8.2 Proof of theorem 5

Let $k \in [1, r]$, $P = [1, p] \setminus (I_1 \cup \dots \cup I_{k-1})$, $Q = [1, q] \setminus (J_1 \cup \dots \cup J_{k-1})$. Fix $S \in \Gamma(a', b, X_0)$ (we know by theorem 4 that this set is not empty).

If $k = r$, then $P = I_r$ and $Q = J_r$. As $a'_i = a_i b(J_r)/a(I_r)$ for every $i \in I_r$, we have $a'(I_r) = b(J_r)$ and $a'_i/a'(I_r) = a_i b(J_r)/a(I_r)$ for every $i \in I_r$. Therefore, the matrix $(S(i, j)/a'(I_r))$ is a solution of the restricted problem associated to the marginals $a(\cdot|P) = (a_i/a(I_r))_{i \in P}$, $b(\cdot|Q) = (b_j/b(Q))_{j \in Q}$ and to the initial condition $(X_0(i, j))_{(i, j) \in P \times Q}$.

If $k < r$, then $P = I_k \cup \dots \cup I_r$ and $Q = J_k \cup \dots \cup J_r$. Let $A_k = I_k$ and $B_k = J_{k+1} \cup \dots \cup J_r$. By theorem 4, the matrix X_0 is null on product $A_k \times B_k$. Moreover, the inequalities $\lambda_1 < \dots < \lambda_r$ and $a(I_l) > 0$ for every $l \in [1, r]$ yield

$$b(Q) = \sum_{l=k}^r b(J_l) = \sum_{l=k}^r \lambda_l a(I_l) > \lambda_k \sum_{l=k}^r a(I_l) = \lambda_k a(P),$$

so

$$\frac{a(A_k|P)}{b(Q \setminus B_k|Q)} = \frac{a(I_k)/a(P)}{b(J_k)/b(Q)} = \lambda_k^{-1} \frac{b(Q)}{a(P)} > 1.$$

Hence $A_k \times B_k$ is a cause of incompatibility of the restricted problem associated to the marginals $a(\cdot|P) = (a_i/a(P))_{i \in P}$, $b(\cdot|Q) = (b_j/b(Q))_{j \in Q}$ and to the initial condition $(X_0(i, j))_{(i, j) \in P \times Q}$.

Now, assume that X_0 is null on some subset $A \times B$ of $P \times Q$. Then S is also null on $A \times B$, so for every $l \in [k, r]$,

$$\lambda_k a(A \cap I_l) \leq \lambda_l a(A \cap I_l) = a'(A \cap I_l) = S((A \cap I_l) \times ((Q \setminus B) \cap J_l)) \leq b((Q \setminus B) \cap J_l).$$

Summing this inequalities over all $l \in [k, r]$ yields $\lambda_k a(A) \leq b(Q \setminus B)$, so

$$\frac{a(A)}{b(Q \setminus B)} \leq \lambda_k^{-1} = \frac{a(I_k)}{b(I_k)} = \frac{a(A_k)}{b(Q \setminus B_k)}.$$

Moreover, if equality holds in the last inequality, then for every $l \in [k, r]$,

$$\lambda_k a(A \cap I_l) = \lambda_l a(A \cap I_l) = b((Q \setminus B) \cap J_l).$$

This yields $A \cap I_l = (Q \setminus B) \cap J_l = \emptyset$ for every $l \in [k+1, r]$, thus $A \subset I_k = A_k$ and $Q \setminus B \subset I_k$, namely $B \supset B_k$. The proof is complete.

8.3 Proof of theorem 6

The proof relies the next two lemmas, from Pretzel, relying on notion of diagonal equivalence. We provide proofs to keep the paper self-contained. The first one is different from Pretzel's original proof. Recall that two matrices X and Y in $\mathcal{M}_{p, q}(\mathbf{R}_+)$ are said to be diagonally equivalent if there exists $D' \in \Delta_p$ and $D'' \in \Delta_q$ such that $Y = D' X D''$. In particular, X and Y must have the same support to be diagonally equivalent.

Lemma 27. (Property 1 of [11]) *Let X and Y be in $\mathcal{M}_{p, q}(\mathbf{R}_+)$. If X and Y are diagonally equivalent and have the same marginals then $X = Y$.*

Proof. By assumption, $Y = D' X D''$ for some $D' \in \Delta_p$ and $D'' \in \Delta_q$. Call $\alpha_1, \dots, \alpha_p$ and β_1, \dots, β_q the diagonal entries of D' and D'' . For every $(i, j) \in [1, p] \times [1, q]$, $Y(i, j) = \alpha_i \beta_j X(i, j)$, so

$$D(Y|X) = \sum_{i, j} Y(i, j) (\ln \alpha_i + \ln \beta_j) = \sum_i Y(i, +) \ln \alpha_i + \sum_j Y(+, j) \ln \beta_j.$$

In the same way,

$$D(X|Y) = \sum_{i, j} X(i, j) (\ln \alpha_i^{-1} + \ln \beta_j^{-1}) = - \sum_i X(i, +) \ln \alpha_i - \sum_j X(+, j) \ln \beta_j.$$

Since X and Y have the same marginals, the non-negative quantities $D(Y|X)$ and $D(X|Y)$ are opposite, so they are null. Hence $X = Y$. \square

Lemma 28. (Lemma 2 of [11]) *Let X and Y be in $\mathcal{M}_{p,q}(\mathbf{R}_+)$. If Y has the same support as X and is the limit of a some sequence $(Y_n)_{n \geq 0}$ of matrices which are diagonally equivalent to X , then X and Y are diagonally equivalent.*

Proof. For every $n \geq 0$, one can find some positive real numbers $\alpha_n(1), \dots, \alpha_n(p)$ and $\beta_n(1), \dots, \beta_n(q)$ such that $Y_n(i, j) = \alpha_n(i)\beta_n(j)X(i, j)$ for every $(i, j) \in [1, p] \times [1, q]$. By assumption, the sequence $(\alpha_n(i)\beta_n(j))_{n \geq 0}$ converges to a positive number whenever $(i, j) \in \text{Supp}(X)$.

In a way similar to the beginning of the proof of theorem 4, we define a non-oriented graph G on $[1, p]$ as follows: (i, i') is an edge if and only if there exists some $j \in [1, q]$ such that $X(i, j)X(i', j) > 0$. Then the sequence $(\alpha_n(i)/\alpha_n(i'))_{n \geq 0}$ converges whenever i and i' belong to a same connected component of G .

Call I_1, \dots, I_r the connected components of the graph G . For each $k \in [1, r]$ choose $i_k \in I_k$ and set $J_k = \{j \in [1, q] : \exists i \in I_k : X(i, j) > 0\}$. Then the sets J_1, \dots, J_k form a partition of $[1, q]$ and the support of X is contained in $I_1 \times J_1 \cup \dots \cup I_r \times J_r$.

For every $n \geq 0$, set $\alpha'_n(i) = \alpha_n(i)/\alpha_n(i_k)$ whenever $i \in I_k$ and $\beta'_n(j) = \beta_n(j)\alpha_n(i_k)$ whenever $j \in J_k$. Then $Y_n(i, j) = \alpha'_n(i)\beta'_n(j)X(i, j)$ for every $(i, j) \in [1, p] \times [1, q]$. Since all sequences $(\alpha'_n(i))_{n \geq 0}$ and $(\beta'_n(j))_{n \geq 0}$ converge to a positive limit, we deduce that X and Y are diagonally equivalent. \square

We now prove theorem 6.

Set $L_{\text{even}} = \lim_n X_{2n}$ and $L_{\text{odd}} = \lim_n X_{2n+1}$. Letting n go to infinity in the equality $X_{2n+1} = T_R(X_{2n})$ yields $L_{\text{odd}} = D_1^{-1}L_{\text{even}}(i, j)$, where $D_1 = \text{Diag}(a'_1/a_1, \dots, a'_p/a_p)$. We deduce that L_{even} and L_{odd} have the same support Σ .

By theorem 4, L_{even} is the only matrix achieving the minimum of $D(Y||X_0)$ over all $Y \in \Gamma(a', b, X_0)$. Thus, by theorem 3, L_{even} is also the limit of the sequence provided by the IPFP performed with the marginals a' and b and the initial matrix X_0 , so Σ is the maximum of the supports of all matrices in $\Gamma(a', b, X_0)$.

By construction, for every $n \geq 0$, one can find $D'_n \in \Delta_p$ and $D''_n \in \Delta_q$ such that $X_{2n} = D'_n X_0 D''_n$. Since D'_n and D''_n are diagonal, $(D'_n X_0 D''_n)(i, j) = X_{2n}(i, j)$ whenever $(i, j) \in \Sigma$ and $(D'_n X_0 D''_n)(i, j) = 0$ whenever $(i, j) \in \Sigma^c$. Hence $\lim_n D'_n X_0 D''_n = L_{\text{even}}$ since $\Sigma = \text{Supp}(L_{\text{even}})$. But X'_0 and L_{even} have the same support, they are diagonally equivalent by lemma 28.

Call $(X'_n)_{n \geq 0}$ (respectively $(X''_n)_{n \geq 1}$) the sequence provided by the IPFP performed on the marginals a, b (respectively a', b) and the initial matrix X'_0 . Equivalently, one could also start from $X'_0(+, +)^{-1}X'_0$ to have an initial matrix in Γ_1 .

Since $L_{\text{even}} \in \Gamma(a', b)$ and $\text{Supp}(L_{\text{even}}) = \text{Supp}(X'_0)$, theorem 2 applies, so the limit $L'' = \lim_n X''_n$ exists, L'' belongs to $\Gamma(a', b)$ and L'' is diagonally equivalent to X'_0 and therefore to L_{even} . By lemma 27, we get $L'' = L_{\text{even}}$. Since all matrices X'_n and X''_n have the same support Σ , contained in $I_1 \times J_1 \cup \dots \cup I_1 \times J_1$ by theorem 4, a recursion shows that for every $n \geq 0$, $X'_{2n} = X''_{2n}$ and $X'_{2n+1} = D_1^{-1}X''_{2n+1}$. Hence $\lim_n X'_{2n} = L'' = L_{\text{even}}$.

A similar proof works for L_{odd} and the set $\Gamma(a, b', X_0)$.

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