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1 Infinitary proof theory : 2 the multiplicative additive case

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7 — Abstract —

8 Infinitary and regular proofs are commonly used in fixed point logics. Being natural intermediate
9 devices between semantics and traditional finitary proof systems, they are commonly found in
10 completeness arguments, automated deduction, verification, etc. However, their proof theory
11 is surprisingly underdeveloped. In particular, very little is known about the computational
12 behavior of such proofs through cut elimination. Taking such aspects into account has unlocked
13 rich developments at the intersection of proof theory and programming language theory. One
14 would hope that extending this to infinitary calculi would lead, *e.g.*, to a better understanding of
15 recursion and corecursion in programming languages. Structural proof theory is notably based
16 on two fundamental properties of a proof system: cut elimination and focalization. The first
17 one is only known to hold for restricted (purely additive) infinitary calculi, thanks to the work
18 of Santocanale and Fortier; the second one has never been studied in infinitary systems. In
19 this paper, we consider the infinitary proof system μMALL^∞ for multiplicative and additive
20 linear logic extended with least and greatest fixed points, and prove these two key results. We
21 thus establish μMALL^∞ as a satisfying computational proof system in itself, rather than just an
22 intermediate device in the study of finitary proof systems.

23 1 Introduction

24 Proof systems based on non-well-founded derivation trees arise naturally in logic, even more
25 so in logics featuring fixed points. A prominent example is the long line of work on tableaux
26 systems for modal μ -calculi, *e.g.*, [16, 24, 14, 11], which have served as the basis for analysing
27 the complexity of the satisfiability problem, as well as devising practical algorithms for solving
28 it. One key observation in such a setting, and many others, is that one needs not consider
29 arbitrary infinite derivations but can restrict to *regular* derivation trees (also known as *circular*
30 proofs) which are finitely representable and amenable to algorithmic manipulation. Because
31 infinitary systems are easier to work with than the finitary proof systems (or axiomatizations)
32 based on Kozen-Park (co)induction schemes, they are often found in completeness arguments
33 for such finitary systems [16, 27, 28, 29, 15, 12]. We should note, however, that those
34 arguments are far from being limited to translations from (regular) infinitary to finitary
35 proofs, since such translations are very complex and only known to work in limited cases.
36 There are many other uses of infinite (or regular) derivations, *e.g.*, to study the relationship
37 between induction and infinite descent in first-order arithmetic [8], to generate invariants for
38 program verification in separation logic [7], or as an intermediate between ludics' designs
39 and proofs in linear logic with fixed points [5]. Last but not least, Santocanale introduced
40 circular proofs [22] as a system for representing morphisms in μ -bicomplete categories [21, 23],
41 corresponding to simple computations on (co)inductive data.

42 Surprisingly, despite the elegance and usefulness of infinitary proof systems, few proof
43 theoretical studies are directly targeting these objects. More precisely, we are concerned
44 with an analysis of proofs that takes into account their computational behaviour in terms



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45 of cut elimination. In other words, we would hope that the Curry-Howard correspondence
 46 extends nicely to infinitary proofs. In this line of proof-theoretical study, two main properties
 47 stand out: cut elimination and focalization; we shall see that they have been barely addressed
 48 in infinitary proof systems. The idea of cut elimination is as old as sequent calculus, and at
 49 the heart of the proof-as-program viewpoint, where the process of eliminating cuts (indirect
 50 reasoning) in proofs is seen as computation. Considering logics with least and greatest
 51 fixed points, the computational behavior of induction and coinduction is recursion and
 52 corecursion, two important and complex programming principles that would deserve a logical
 53 understanding. Note that the many completeness results for infinitary proof systems (*e.g.*,
 54 for modal μ -calculi) only imply cut admissibility, but say nothing about the computational
 55 process of cut elimination. To our knowledge, leaving aside an early and very restrictive result
 56 of Santocanale [22], cut elimination has only been studied by Fortier and Santocanale [13]
 57 who considered an infinitary sequent calculus for lattice logic (purely additive linear logic with
 58 least and greatest fixed points) and showed that certain cut reductions converge to a limit
 59 cut-free derivation. Their proof involves a mix of combinatorial and topological arguments.
 60 So far, it has resisted attempts to extend it beyond the purely additive case. The second key
 61 property, much more recently identified than cut elimination, is focalization. It has appeared
 62 in the work of [3] on proof search and logic programming in linear logic, and is now recognized
 63 as one of the deep outcomes of linear logic, putting to the foreground the role of *polarity*
 64 in logic. In a way, focalization generalizes the invertibility results that are notably behind
 65 most deductive systems for classical μ -calculi, by bringing some key observations about
 66 non-invertible connectives. Besides its deep impact on proof search and logical frameworks,
 67 focalization resulted in important advances in all aspects of computational proof theory:
 68 in the game-semantical analysis of logic [17, 19], the understanding of evaluation order of
 69 programming languages, CPS translations, or semantics of pattern matching [10, 30], the
 70 space compression in computational complexity [26, 6], etc. Briefly, one can say that while
 71 proof nets have led to a better understanding of phenomena related to parallelism with
 72 proof-theoretical methods, polarities and focalization have led to a fine-grained understanding
 73 of sequentiality in proofs and programs. To the best of our knowledge, while reversibility
 74 has since long been a key-ingredient in completeness arguments based on infinitary proof
 75 systems, focalization has simply never been studied in such settings.

76 *Organization and contributions of the paper.* In this paper, we consider the logic μ MALL, that
 77 is multiplicative additive linear logic extended with least and greatest fixed point operators.
 78 It has been studied in finitary sequent calculus [4]: it notably enjoys cut elimination, and
 79 focalization has been shown to extend nicely (though not obviously) to it. We give in
 80 Section 2 a natural infinitary proof system for μ MALL, called μ MALL $^\infty$, which notably
 81 extends that of Santocanale and Fortier [13]. The system μ MALL $^\infty$ is also related to μ MALL
 82 in the sense that any μ MALL derivation can be turned into a μ MALL $^\infty$ proof, with cuts.
 83 We study the focalization of μ MALL $^\infty$ in Section 3. We find out that, even though fixed
 84 point polarities are not forced in the finitary sequent calculus for μ MALL, they are uniquely
 85 determined in μ MALL $^\infty$. Despite some novel aspects due to the infinitary nature of our
 86 calculus, we are able to re-use the generic *focalization graph* argument [20] to prove that
 87 focalized proofs are complete. We then turn to cut elimination in Section 4 and show that
 88 (fair) cut reductions converge to an infinitary cut free derivation. We could not apply any
 89 standard cut elimination technique (*e.g.*, induction on formulas and proofs, reducibility
 90 arguments, topological arguments as in [13]) and propose instead an unusual argument in
 91 which a coarse truth semantics is used to show that the cut elimination process cannot go
 92 wrong. We also note here that, even for the regular fragment of μ MALL $^\infty$, it would be

highly non-trivial to obtain cut elimination from the result for μMALL , since it is not known whether regular μMALL^∞ derivations can be translated to μMALL derivations (even without requiring that this translation preserves the computational behaviour of proofs). We conclude in Section 5 with directions for future work. Appendices provide technical details, proofs, and additional background material.

2 μMALL and its infinitary proof system μMALL^∞

In this section we introduce multiplicative additive linear logic extended with least and greatest fixed point operators, and an infinitary proof system for it.

► **Definition 1.** Given an infinite set of propositional variables $\mathcal{V} = \{X, Y, \dots\}$, μMALL^∞ *pre-formulas* are built over the following syntax:

$\varphi, \psi ::= \mathbf{0} \mid \top \mid \varphi \oplus \psi \mid \varphi \& \psi \mid \perp \mid \mathbf{1} \mid \varphi \wp \psi \mid \varphi \otimes \psi \mid \mu X. \varphi \mid \nu X. \varphi \mid X$ with $X \in \mathcal{V}$.

The connectives μ and ν bind the variable X in φ . From there, bound variables, free variables and capture-avoiding substitution are defined in a standard way. The subformula ordering is denoted \leq and $\text{fv}(\bullet)$ denotes free variables. Closed pre-formulas are simply called *formulas*. Note that negation is not part of the syntax, so that we do not need any positivity condition on fixed point expressions.

► **Definition 2.** *Negation* is the involution on pre-formulas written φ^\perp and satisfying $(\varphi \wp \psi)^\perp = \psi^\perp \otimes \varphi^\perp$, $(\varphi \oplus \psi)^\perp = \psi^\perp \& \varphi^\perp$, $\perp^\perp = \mathbf{1}$, $\mathbf{0}^\perp = \top$, $(\nu X. \varphi)^\perp = \mu X. \varphi^\perp$, $X^\perp = X$.

Having $X^\perp = X$ might be surprising, but it is harmless since our proof system will only deal with closed pre-formulas. Our definition yields, *e.g.*, $(\mu X. X)^\perp = (\nu X. X)$ and $(\mu X. \mathbf{1} \oplus X)^\perp = (\nu X. X \& \perp)$, as expected [4]. Note that we also have $(\varphi[\psi/X])^\perp = \varphi^\perp[\psi^\perp/X]$.

Sequent calculi are sometimes presented with sequents as sets or multisets of formulas, but most proof theoretical observations actually hold in a stronger setting where one distinguishes between several *occurrences* of a formula in a sequent, which gives the ability to precisely *trace* the provenance of each occurrence. This more precise viewpoint is necessary, in particular, when one views proofs as programs. In this work, due to the nature of our proof system and because of the operations that we perform on proofs and formulas, it is also crucial to work with occurrences. There are several ways to formally treat occurrences; for the sake of clarity, we provide below a concrete presentation of that notion which is well suited for our needs.

► **Definition 3.** An *address* is a word over $\Sigma = \{l, r, i\}$, which stands for left, right and inside. We define a *duality* over Σ^* as the morphism satisfying $l^\perp = r$, $r^\perp = l$ and $i^\perp = i$. We say that α' is a *sub-address* of α when α is a prefix of α' , written $\alpha \sqsubseteq \alpha'$. We say that α and β are *disjoint* when α and β have no upper bound wrt. \sqsubseteq .

► **Definition 4.** A *(pre)formula occurrence* (denoted by F, G, H) is given by a (pre)formula φ and an address α , and written φ_α . We say that occurrences are *disjoint* when their addresses are. The occurrences φ_α and ψ_β are *structurally equivalent*, written $\varphi_\alpha \equiv \psi_\beta$, if $\varphi = \psi$. Operations on formulas are extended to occurrences as follows: $(\varphi_\alpha)^\perp = (\varphi^\perp)_{\alpha^\perp}$; for any $\star \in \{\wp, \otimes, \oplus, \&\}$, $F \star G = (\varphi \star \psi)_\alpha$ if $F = \varphi_{\alpha l}$ and $G = \psi_{\alpha r}$; for any $\sigma \in \{\mu, \nu\}$, $\sigma X. F = (\sigma X. \varphi)_\alpha$ if $F = \varphi_{\alpha i}$; we also allow ourselves to write units as formula occurrences without specifying their address, which can be chosen arbitrarily. Finally, *substitution of occurrences* forgets addresses: $(\varphi_\alpha)[\psi_\beta/X] = (\varphi[\psi/X])_\alpha$.

► **Example.** Let $F = \varphi_{\alpha l}$ and $G = \psi_{\alpha r}$. We have, on the one hand, $(F \otimes G)^\perp = ((\varphi \otimes \psi)_\alpha)^\perp = (\psi^\perp \wp \varphi^\perp)_{\alpha^\perp}$ and, on the other hand, $G^\perp \wp F^\perp = (\psi^\perp)_{\alpha^\perp l} \wp (\varphi^\perp)_{\alpha^\perp r} = (\psi^\perp \wp \varphi^\perp)_{\alpha^\perp}$. Thus,

$$\begin{array}{cccc}
\frac{\vdash F, \Gamma \quad \vdash G, \Gamma}{\vdash F \& G, \Gamma} (\&) & \frac{\vdash F, G, \Gamma}{\vdash F \wp G, \Gamma} (\wp) & \frac{\vdash F_i, \Gamma}{\vdash F_1 \oplus F_2, \Gamma} (\oplus) & \frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta} (\otimes) \\
\frac{}{\vdash \top, \Gamma} (\top) & \frac{\vdash \Gamma}{\vdash \perp, \Gamma} (\perp) & \text{(no rule for } \mathbf{0}) & \frac{}{\vdash \mathbf{1}} (\mathbf{1}) \\
\frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma} (\mu) & \frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma} (\nu) & \frac{F \equiv G}{\vdash F, G^\perp} (\text{Ax}) & \frac{\vdash \Gamma, F \quad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta} (\text{Cut})
\end{array}$$

■ **Figure 1** Rules of the proof system μMALL^∞ .

136 $(F \otimes G)^\perp = G^\perp \wp F^\perp$. We could have designed our system to obtain $(F \otimes G)^\perp = F^\perp \wp G^\perp$
137 instead; this choice is inessential for the present work but makes our definitions suitable, in
138 principle, for a treatment of non-commutative logic.

139 ► **Definition 5.** The *Fischer-Ladner closure* of a formula occurrence F , denoted by $\text{FL}(F)$,
140 is the least set of formula occurrences such that $F \in \text{FL}(F)$ and, whenever $G \in \text{FL}(F)$,

- 141 ■ $G_1, G_2 \in \text{FL}(F)$ if $G = G_1 \star G_2$ for any $\star \in \{\oplus, \&, \wp, \otimes\}$;
- 142 ■ $B[G/X] \in \text{FL}(F)$ if $G = \sigma X.B$ for $\sigma \in \{\nu, \mu\}$.

143 We say that G is a *sub-occurrence* of F if $G \in \text{FL}(F)$. Note that, for any F and α , there
144 is at most one φ such that φ_α is a sub-occurrence of F .

145 We are now ready to introduce our infinitary sequent calculus. Details regarding formula
146 occurrences can be ignored at first read, and will only make full sense when one starts
147 permuting inferences and eliminating cuts.

148 ► **Definition 6.** A *sequent*, written $\vdash \Gamma$, is a finite set of pairwise disjoint, closed formula
149 occurrences. A *pre-proof* of μMALL^∞ is a possibly infinite tree, coinductively generated
150 by the rules of Figure 1, subject to the following conditions: any two formulas occurrences
151 appearing in different branches must be disjoint except if the branches first differ right after a
152 $(\&)$ inference; if φ_α and ψ_{α^\perp} occur in a pre-proof, they must be the respective sub-occurrences
153 of the formula occurrences F and F^\perp introduced by a (Cut) rule.

154 The disjointness condition on sequents ensures that two formula occurrences from the
155 same sequent will never engender a common sub-occurrence, *i.e.*, we can define traces uniquely.
156 The disjointness condition on pre-proofs is there to ensure that the proof transformations
157 used in focusing and cut elimination preserve the disjointness condition on sequents. Note
158 that these conditions are not restrictive. Clearly, the condition on sequents never prevents
159 the (backwards) application of a propositional rule. Moreover, there is an infinite supply of
160 disjoint addresses, *e.g.*, $\{r^n l : n > 0\}$. One may thus pick addresses from that supply for
161 the conclusion sequent of the derivation, and then carry the remaining supply along proof
162 branches, splitting it on branching rules, and consuming a new address for cut rules.

163 Pre-proofs are obviously unsound: the pre-proof schema shown
164 on the right allows to derive any formula. In order to obtain proper
165 proofs from pre-proofs, we will add a validity condition. This
166 condition will reflect the nature of our two fixed point connectives.

$$\frac{\frac{\vdots}{\vdash \mu X.X} (\mu) \quad \frac{\vdots}{\vdash \nu X.X, F} (\nu)}{\vdash F} (\text{Cut})$$

167 ► **Definition 7.** Let $\gamma = (s_i)_{i \in \omega}$ be an infinite branch in a pre-proof of μMALL^∞ . A *thread*
168 t in γ is a sequence of formula occurrences $(F_i)_{i \in \omega}$ with $F_i \in s_i$ and $F_i \sqsubseteq F_{i+1}$. The set of
169 formulas that occur infinitely often in $(F_i)_{i \in \omega}$ (when forgetting addresses) admits a minimum

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197 branch, validated by the thread starting with N . If we cut that proof against an arbitrary
 198 cut-free proof of N , and perform cut elimination steps, we obtain in finite time a cut-free
 199 proof of $N_1 \otimes N_2$ which consists of two copies (up-to adresses) of the original proof of N .

200 Now let $\varphi_{\text{stream}} = \nu X. \varphi_{\text{nat}} \otimes X$

201 be the formula representing in-
 202 finite streams of natural num-
 203 bers, whose occurrences will be
 204 denoted by S, S' , etc. Let us
 205 consider the derivation shown
 206 on the right, where F is an ar-
 207 bitrary, useless formula occur-
 208 rence for illustrative purposes.

$$\begin{array}{c}
 \frac{\pi_{\text{succ}}}{N_2 \vdash N''} \quad \frac{(\star)}{N'', F \vdash S'} \\
 \hline
 \frac{N_1 \vdash N'}{N_2, F \vdash S'} \text{ (Cut)} \\
 \hline
 \frac{\pi_{\text{dup}}}{N \vdash N_1 \otimes N_2} \quad \frac{N_1, N_2, F \vdash N' \otimes S'}{N_1 \otimes N_2, F \vdash N' \otimes S'} \\
 \hline
 \frac{N, F \vdash N' \otimes S'}{N, F \vdash S} \text{ (Cut)} \\
 \hline
 \frac{(\star)}{N, F \vdash S}
 \end{array}$$

209 It is a valid proof thanks to the thread on S . By cut elimination, the computational behaviour
 210 of that proof is to take a natural number n , and some irrelevant f , and compute the stream
 211 $n :: (n + 1) :: (n + 2) :: \dots$. However, unlike in the two previous examples, the result of the
 212 computation is not obtained in finite time; instead, we are faced with a productive process
 213 which will produce any finite prefix of the stream when given enough time. The presence of
 214 the useless formula F illustrates here that weakening may be admissible in μMALL^∞ under
 215 some circumstances, and that cutting against some formulas (F in this case) will form a
 216 redex that will be delayed forever. These subtleties will show up in the next two sections,
 217 devoted to showing our two main results.

218 3 Focalization

219 *Focalization in linear logic.* MALL connectives can be split in two classes: *positive* ($\otimes, \oplus, \mathbf{0}, \mathbf{1}$)
 220 and *negative* ($\wp, \&, \top, \perp$) connectives. The distinction can be easily understood in terms
 221 of proof search: negative inferences (\wp), ($\&$), (\top) and (\perp) are *reversible* (meaning that
 222 provability of the conclusion transfers to the premisses) while positive inferences require
 223 choices (splitting the context in (\otimes) or choosing between (\oplus_1) and (\oplus_2) rules) resulting in a
 224 possible of loss of provability. Still, positive inferences satisfy the **focalization** property [3]:
 225 in any provable sequent containing no negative formula, some formula can be chosen as a
 226 **focus**, hereditarily selecting its positive subformulas as principal formulas until a negative
 227 subformula is reached. It induces the following complete proof search strategy:

Sequent Γ <i>contains a negative</i> formula	Sequent Γ <i>contains no negative</i> formula
Choose any negative formula (e.g. the leftmost one) and decompose it using the only possible negative rule.	Choose some positive formula and decompose it (and its subformulas) hereditarily until we get to atoms or negative subformulas.

229 *Focalization graphs.* Focused proofs are complete for proofs, not only provability: any linear
 230 proof is equivalent to a focused proof, up to cut-elimination. Indeed, focalization can be
 231 proved by means of proof transformations [18, 20, 6] preserving the denotation of the proof.
 232 A flexible, modular method for proving focalization that we shall apply in the next sections
 233 has been introduced by Miller and the third author [20] and relies on **focalization graphs**.
 234 The heart of the focalization graph proof technique relies on the fact the positive inference,
 235 while not reversible, all permute with each other. As a consequence, if the positive layer of
 236 some positive formula is completely decomposed within the lowest part of the proof, below
 237 any negative inference, then it can be taken as a focus. Focalization graphs ensure that it is
 238 always possible: their acyclicity provides a source which can be taken as a focus.

239 *Focusing infinitary proofs.* The infinitary nature of our proofs interferes with focalization
 240 in several ways. First, while in μMALL μ and ν can be set to have an arbitrary polarity,
 241 we will see that in μMALL^∞ , ν must be negative. Second, permutation properties of the
 242 negative inferences, which can be treated locally in μMALL , now require a global treatment
 243 due to infinite branches. Last, focalization graphs strongly rely on the finiteness of maximal
 244 positive subtrees of a proof: this invariant must be preserved in μMALL^∞ .

245 For simplicity reasons, we restrict our attention to cut-free proofs in the rest of this
 246 section. The result holds for proofs with cuts thanks to the usual trick of viewing cuts as \otimes .

247 3.1 Polarity of connectives

248 Let us first consider the question of polarizing μMALL^∞ connectives. Unlike in μMALL , we
 249 are not free to set the polarity of fixed points formulas: consider the proof π of sequent
 250 $\vdash \mu X.X, \nu Y.Y$ which alternates inferences (ν) and (μ). Assigning opposite polarities to
 251 dual formulas (an invariant necessary to define properly cut-elimination in focused proof
 252 systems), this sequent contains a negative formula; each polarization of fixed points induces
 253 one focused pre-proof, either π_μ which always unrolls μ or π_ν which repeatedly unrolls ν .
 254 Only π_ν happens to be valid, leaving but one possible choice, $\nu X.F$ negative and $\mu X.F$
 255 positive, resulting in the following polarization:

256 ► **Definition 10.** *Negative formulas* are formulas of the form $\nu X.F, F \wp G, F \& G, \perp$ and
 257 \top , *positive formulas* are formulas of the form $\mu X.F, F \otimes G, F \oplus G, \mathbf{1}$ and $\mathbf{0}$. A μMALL^∞
 258 sequent containing only positive formulas is said to be *positive*. Otherwise, it is *negative*.

259 The following proposition will be useful in the following:

260 ► **Proposition 11.** *An infinite branch of a pre-proof containing only negative (resp. positive)*
 261 *rules is always valid (resp. invalid).*

262 3.2 Reversibility of negative inferences

263 The following example with $F = \nu X.(X \& X) \oplus \mathbf{0}$ shows that, unlike
 264 in (MA)LL, negative inferences cannot be permuted down locally: no
 265 occurrence of a negative inference (\wp) on $P \wp Q$ can be permuted below
 266 a ($\&$) since it is never available in the left premise. We thus introduce
 267 a global proof transformation (which could be realized by means of cut, as is usual).

Negative rules have a uniform structure: $\frac{(\vdash \Gamma, \mathcal{N}_i^N)_{1 \leq i \leq n}}{\vdash \Gamma, N}$ (r_N). **Sub-occurrence families**
 of N are thus defined as $\mathcal{N}(N) = (\mathcal{N}_i^N)_{1 \leq i \leq n}$, its **slicing index** being $\text{sl}(N) = \#\mathcal{N}(N)$.

N	$F_1 \wp F_2$	\perp	$F_1 \& F_2$	\top	$\nu X.F$
$\mathcal{N}(N)$	$\{1 \mapsto \{F_1, F_2\}\}$	$\{1 \mapsto \emptyset\}$	$\{1 \mapsto \{F_1\}, 2 \mapsto \{F_2\}\}$	\emptyset	$\{1 \mapsto \{F[\nu X.F/X]\}\}$

268 The following two definitions define what the reversibility of a proof π , $\text{rev}(\pi)$, is:

269 ► **Definition 12** ($\pi(i, N)$). Let π be a proof of $\vdash \Gamma$ of last rule (r) and premises π_1, \dots, π_n .
 270 If $1 \leq i \leq \text{sl}(N)$, we define $\pi(i, N)$ coinductively:

- 271 ■ if N does not occur in $\vdash \Gamma$, $\pi(i, N) = \pi$;
- 272 ■ if r is the inference on N , then $\pi(i, N) = \pi_i$; (which is legal since in this case $n = \text{sl}(N)$);
- 273 ■ if r is not the inference on N , then $\pi(i, N) = \frac{\pi_1(i, N) \quad \dots \quad \pi_n(i, N)}{\vdash \Gamma, \mathcal{N}_i^N}$ (r).

274 ▶ **Definition 13** ($\text{rev}(\pi)$). Let π be a μMALL^∞ proof of $\vdash \Gamma$. $\text{rev}(\pi)$ is a pre-proof non-
 275 deterministically defined as π if $\vdash \Gamma$ is positive and, otherwise, when $N \in \Gamma$ and $n = \text{sl}(N)$,
 276 as $\text{rev}(\pi) = \frac{\text{rev}(\pi(1, N)) \quad \dots \quad \text{rev}(\pi(n, N))}{\vdash \Gamma} \text{ (r}_N\text{)}$.

277 Reversed proofs formalize the requirement for the whole
 278 negative layer to be reversed:

279 ▶ **Definition 14. Reversed pre-proofs** are defined to be
 280 the largest set of pre-proofs such that: (i) every pre-proof of
 281 a positive sequent is reversed; (ii) a pre-proof of a negative
 282 sequent is reversed if it ends with a negative inference and
 283 if each of its premises is reversed.

284 ▶ **Example 15.** rev is illustrated on the proof starting this
 285 subsection ($N = P \wp Q$, $\text{sl}(N) = 1$) in Figure 2

286 ▶ **Theorem 16.** Let π be a μMALL^∞ proof. $\text{rev}(\pi)$ is a
 287 reversed proof of the same sequent.

$$\begin{aligned} \text{rev}(\pi) &= \frac{\pi(1, N)}{\vdash F, P \wp Q} \text{ (}\wp\text{)} \\ &= \frac{\frac{(\star)}{\vdash F, P, Q} \quad \frac{\pi}{\vdash F, P, Q}}{\vdash F \& F, P, Q} \text{ (}\&\text{)} \\ &= \frac{\vdash (F \& F) \oplus \mathbf{0}, P, Q}{(\star) \vdash F, P, Q} \text{ (}\oplus_1\text{)} \\ &= \frac{\vdash F, P, Q}{\vdash F, P \wp Q} \text{ (}\wp\text{)} \end{aligned}$$

■ **Figure 2** $\text{rev}(\pi)$

288 3.3 Focalization Graph

289 In this section, we adapt the focalization graphs introduced
 290 in [20] to our setting. Considering the permutability prop-
 291 erties of positive inferences in μMALL^∞ , finiteness of positive trunks and acyclicity of
 292 focalization graphs will be sufficient to make the proof technique of [20] applicable. In order
 293 to illustrate this subsection, an example is fully explained in appendix B.5

294 ▶ **Definition 17** (Positive trunk, positive border, active formulas). Let π be a μMALL^∞ proof
 295 of \mathcal{S} . The **positive trunk** π^+ of π is the tree obtained by cutting (finite or infinite) branches
 296 of π at the first occurrence of a negative rule. The **positive border** of π is the collection
 297 of lowest sequents in π which are conclusions of negative rules. **P-active** formulas of π are
 298 those formulas of \mathcal{S} which are principal formulas of an inference in π^+ .

299 ▶ **Proposition 18.** The positive trunk of a μMALL^∞ proof is always finite.

300 ▶ **Definition 19** (Focalization graph). Given a μMALL^∞ proof π , we define its **focalization**
 301 **graph** $\mathcal{G}(\pi)$ to be the graph whose vertices are the P-active formulas of π and such that
 302 there is an edge from F to G iff there is a sequent \mathcal{S}' in the positive border containing a
 303 negative sub-occurrence F' of F and a positive sub-occurrence G' of G .

304 μMALL^∞ positive inferences are those of MALL extended with (μ) which is not branching;
 305 this ensures both that any two positive inferences permute and that the proof of acyclicity of
 306 MALL focalization graphs can easily be adapted, from which we conclude that:

307 ▶ **Proposition 20.** Focalization graphs are acyclic.

308 Acyclicity of the focalization graph implies in particular that it has a source, that is a
 309 formula P of the conclusion sequent such that whenever one of its subformulas F appears in
 310 a border sequent, F is negative. This remark, together with the fact that the trunk is finite
 311 ensures that the positive layer of P is completely decomposed in the positive trunk.

312 ▶ **Definition 21** ($\text{foc}(\pi, P)$). Let π be a μMALL^∞ proof of $\vdash \Gamma, P$ with P a source of π 's
 313 focalization graph. One defines $\text{foc}(\pi, P)$ as the μMALL^∞ proof obtained by permuting down
 314 all the positive inferences on P and its positive subformulas (all occurring in π^+).

315 ► **Proposition 22.** *Let \mathcal{S} be a lowest sequent of $\text{foc}(\pi, P)$ which is not conclusion of a rule on*
 316 *a positive subformula of P . Then \mathcal{S} contains exactly one subformula of P , which is negative.*

317 3.4 Productivity and validity of the focalization process

318 Reversibility of the negative inferences and focalization of the positive inferences allow to
 319 consider the following (non-deterministic) proof transformation process:

320 **Focalization Process:** Let π be a μMALL^∞ proof of \mathcal{S} . Define $\text{Foc}(\pi)$ as follows:

321 ■ **Asynchronous phase:** If \mathcal{S} is negative, transform π into $\text{rev}(\pi)$ which is reversed. At
 322 least one negative inference has been brought to the root of the proof. Apply (corecursively)
 323 the synchronous phase to the proofs rooted in the lowest positive sequents of $\text{rev}(\pi)$.

324 ■ **Synchronous phase:** If \mathcal{S} is positive, let $P \in \mathcal{S}$ be a source of the associated focalization
 325 graph. Transform π into a proof $\text{foc}(\pi, P)$. At least one positive inference on P has been
 326 brought to the root of the proof. Apply (corecursively) the asynchronous phase to the
 327 proofs rooted in the lowest negative sequents of $\text{foc}(\pi, P)$.

328 Each of the above phases produces one non-empty phase, the above process is thus
 329 productive. It is actually a pre-proof thanks to theorem 16 and by definition of $\text{foc}(\pi, P)$. It
 330 remains to show that the resulting pre-proof is actually a proof. The following property is
 331 easily seen to be preserved by both transformations foc and rev and thus holds for $\text{Foc}(\pi)$:

332 ► **Proposition 23.** *Let π be a μMALL^∞ proof, r a positive rule occurring in π and r' be a*
 333 *negative rule occurring below r in π . If r occurs in $\text{Foc}(\pi)$, then r' occurs in $\text{Foc}(\pi)$, below r .*

334 ► **Lemma 24.** *For any infinite branch γ of $\text{Foc}(\pi)$ containing an infinite number of positive*
 335 *rules, there exists an infinite branch in π containing infinitely many positive rules of γ .*

336 ► **Theorem 25.** *If π is a μMALL^∞ proof then $\text{Foc}(\pi)$ is also a μMALL^∞ proof.*

337 **Proof sketch, see appendix.** An infinite branch γ of $\text{Foc}(\pi)$ may either be obtained by
 338 reversibility only after a certain point, or by alternating infinitely often synchronous and
 339 asynchronous phases. In the first case it is valid by proposition 11 while in the latter case,
 340 lemma 24 ensures the existence of a branch δ of π containing infinitely many positive rules
 341 of γ , with a valid thread t of minimal formula F_m : every rule r of δ in which F_m is principal
 342 is below a positive rule occurring in γ . Thus r occurs in γ , which is therefore valid. ◀

343 4 Cut elimination

344 In this section, we show that any μMALL^∞ proof can be transformed into an equivalent
 345 cut-free derivation. This is done by applying the cut reduction rules described in Section 2,
 346 possibly in infinite reductions converging to cut-free proofs. As usual with infinitary reductions
 347 it is not the case that any reduction sequence converges: for instance, one could reduce
 348 only deep cuts in a proof, leaving a cut untouched at the root. We avoid this problem by
 349 considering a form of head reduction where we only reduce cuts at the root.

350 Cut reduction rules are of two kinds, *principal* reductions and *auxiliary* ones. In the
 351 infinitary setting, principal cut reductions do not immediately contribute to producing a
 352 cut-free pre-proof. On the contrary, auxiliary cut reductions are productive in that sense. In
 353 other words, principal rules are seen as internal computations of the cut elimination process,
 354 while auxiliary rules are seen as a partial output of that process. Accordingly, the former
 355 will be called *internal rules* and the latter *external rules*.

$$\begin{array}{c}
\frac{\frac{\frac{\vdash \Gamma, F \quad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta} \text{ (Cut)}}{\vdash \Sigma} \dots \text{ (mcut)}}{\vdash \Sigma} \longrightarrow \frac{\frac{\vdash \Gamma, F \quad \vdash F^\perp, \Delta \quad \dots}{\vdash \Sigma} \text{ (mcut)}}{\vdash \Sigma} \\
\\
\frac{\frac{\frac{\frac{\vdash \Gamma, F}{\vdash \Gamma, F \oplus G} \quad \frac{\frac{\vdash G^\perp, \Delta \quad \vdash F^\perp, \Delta}{\vdash G^\perp \& F^\perp, \Delta}}{\vdash \Sigma} \dots \text{ (mcut)}}{\vdash \Sigma}}{\vdash \Sigma} \longrightarrow \frac{\frac{\vdash \Gamma, F \quad \vdash F^\perp, \Delta \quad \dots}{\vdash \Sigma} \text{ (mcut)}}{\vdash \Sigma} \\
\\
\frac{s_1 \dots s_n \quad \frac{\frac{\frac{\vdash \Gamma, F \quad \vdash \Gamma, G}{\vdash \Gamma, F \& G} \text{ (\&)}}{\vdash \Sigma, F \& G} \text{ (mcut)}}{\vdash \Sigma, F \& G} \longrightarrow \frac{s_1 \dots s_n \quad \frac{\vdash \Gamma, F}{\vdash \Sigma, F} \text{ (mcut)} \quad \frac{s_1 \dots s_n \quad \vdash \Gamma, G}{\vdash \Sigma, G} \text{ (mcut)}}{\vdash \Sigma, F \& G} \text{ (\&)}
\end{array}$$

■ **Figure 3** (Cut)/(mcut) and $(\oplus_1)/(\&)$ internal reductions and $(\&)/(mcut)$ external reduction.

356 When analyzing cut reductions, cut commutations can be troublesome. A $\frac{s_1 \dots s_n}{s}$ (mcut)
357 common way to avoid this technicality [13], which we shall follow, is to introduce
358 a **multicut** rule which merges multiple cuts, avoiding cut commutations.

359 ► **Definition 26.** Given two sequents s and s' , we say that they are cut-connected on a
360 formula occurrence F when $F \in s$ and $F^\perp \in s'$. We say that they are cut-connected when
361 they are connected for some F . We define the **multicut** rule as shown above with conclusion
362 s and premisses $\{s_i\}_i$, where the set $\{s_i\}_i$ is connected and acyclic with respect to the
363 cut-connection relation, and s is the set of all formula occurrences F that appear in some s_i
364 but such that no s_j is cut-connected to s_j on F .

365 From now on we shall work with μMALL_m^∞ derivations, which are μMALL^∞ derivations
366 in which the multicut rule may occur, though only at most once per branch. The notions
367 of thread and validity are unchanged. In μMALL_m^∞ we only reduce multicuts, in a way that
368 is naturally obtained from the cut reductions of μMALL^∞ . A complete description of the
369 rules is given in Definition 49, appendix C.1; only the (Cut)/(mcut) and $(\oplus_1)/(\&)$ internal
370 reduction cases and the $(\&)/(mcut)$ external reduction case are shown in figure 3. As is
371 visible in the last reduction, applying an external rule on a multicut may yield multiple
372 multicuts, though always on disjoint subtrees.

373 We will be interested in a particular kind of multicut reduction sequences, the **fair**
374 ones, which are such that any redex which is available at some point of the sequence will
375 eventually have disappeared from the sequence (being reduced or erased), details are provided
376 in appendix C.1. We will establish that these reductions eliminate multicuts:

377 ► **Theorem 27.** *Fair multicut reductions on μMALL_m^∞ proofs produce μMALL^∞ proofs.*

378 Additionnally, if all cuts in the initial derivation are above multicuts, the resulting
379 μMALL^∞ derivation must actually be cut-free: indeed, multicut reductions never produce
380 a cut. Thus Theorem 27 gives a way to eliminate cuts from any μMALL^∞ proof π of $\vdash \Gamma$
381 by forming a multicut with conclusion $\vdash \Gamma$ and π as unique subderivation, and eliminating
382 multicuts (and cuts) from that μMALL_m^∞ proof. The proof of Theorem 27 is in two parts. We
383 first prove that fair internal multicut reductions cannot diverge (Proposition 37), hence fair
384 multicut reductions are productive, *i.e.*, reductions of μMALL_m^∞ proofs converge to μMALL^∞
385 pre-proofs. We then establish that the obtained pre-proof is a valid proof (Proposition 38).

386 Regarding productivity, assuming that there exists an infinite sequence σ of internal
387 cut-reductions from a given proof π of Γ , we obtain a contradiction by extracting from π a

388 proof of the empty sequent in a suitably defined proof-system. More specifically, we observe
 389 that no formula of Γ is principal in the subtree π_σ of π visited by σ . Hence, by erasing every
 390 formula of Γ from π_σ , local correctness of the proof is preserved, resulting in a tree deriving
 391 the empty sequent. This tree can be viewed as a proof in a new proof-system $\mu\text{MALL}_\tau^\infty$ which
 392 is shown to be sound (Proposition 34) with respect to the traditional boolean semantics of
 393 the μ -calculus, thus the contradiction. The proof of validity of the produced pre-proof is
 394 similar: instead of extracting a proof of the empty sequent from π we will extract, for each
 395 invalid branch of π , a $\mu\text{MALL}_\tau^\infty$ proof of a formula containing neither $\mathbf{1}$, \top , nor ν formulas,
 396 contradicting soundness again.

397 4.1 Extracting proofs from reduction paths

398 We define now a key notion to analyze the behaviour of multicut-elimination: given a
 399 multicut reduction starting from π , we extract a (slightly modified) subderivation of π which
 400 corresponds to the part of the derivation that has been explored by the reduction. More
 401 precisely, we are interested in *reduction paths* which are sequences of proofs that end with
 402 a multicut rule, obtained by tracing one multicut through its evolution, selecting only one
 403 sibling in the case of $(\&)$ and (\otimes) external reductions. Given such a reduction path starting
 404 with π , we consider the subtree of π whose sequents occur in the reduction path as premises
 405 of some multicut. This subtree is obviously not always a μMALL^∞ derivation since some of
 406 its nodes may have missing premises. We will provide an extension of μMALL^∞ where these
 407 trees can be viewed as proper derivations by first characterizing when this situation arises.

408 ► **Definition 28** (Useless sequents, distinguished formula). Let \mathcal{R} be a reduction path starting
 409 with π . A sequent $s = (\vdash \Gamma, F)$ of π is said to be *useless* with *distinguished formula* F
 410 when in one of the following cases:

- 411 1. The sequent eventually occurs as a premise of all multicuts of \mathcal{R} and F is the principal
 412 formula of s in π . (Note that the distinguished formula F of a useless sequent s of sort
 413 (1) must be a sub-occurrence of a cut formula in π . Otherwise, the fair reduction path
 414 \mathcal{R} would eventually have applied an external rule on s . Moreover, F^\perp never becomes
 415 principal in the reduction path, otherwise by fairness the internal rule reducing F and
 416 F^\perp would have been applied.)
- 417 2. At some point in the reduction, the sequent is a premise of $(\&)$ on $F\&F'$ or $F'\&F$ which
 418 is erased in an internal $(\&)/(\oplus)$ multicut reduction. (In the $(\oplus_1)/(\&)$ internal reduction
 419 of figure 3, the sequent $\vdash G^\perp, \Delta$ is useless of sort (2).)
- 420 3. The sequent is ignored at some point in the reduction path because it is not present in the
 421 selected multicut after a branching external reduction on $F\star F'$ or $F'\star F$, for $\star \in \{\otimes, \&\}$.
 422 (In the $(\&)/(\text{mcut})$ external reduction of figure 3, if one is considering a reduction path
 423 that follows the multicut having $\vdash \Gamma, F$ as a premise, then the sequent $\vdash \Gamma, G$ is useless
 424 of sort (3), and vice versa.)
- 425 4. The sequent is ignored at some point in the reduction path because a $(\otimes)/(\text{mcut})$ external
 426 reduction distributes s to the multicut that is not selected in the path. This case will be
 427 illustrated next, and is described in full details in appendix C.1.

428 Note that, although the external reduction for \top erases sequents, we do not need to
 429 consider such sequents as useless: indeed, we will only need to work with useless sequents in
 430 infinite reduction paths, and the external reduction associated to \top terminates a path.

431 ► **Example.** Consider a multicut composed of the last example of Section 2 and an arbitrary
 432 proof of $\vdash F, \Delta$ where F is principal. In the reduction paths which always select the right

433 premise of an external $(\otimes)/(\text{mcut})$ corresponding to the $N' \otimes S'$ formulas, the sequent $\vdash F, \Delta$
 434 will always be present and thus useless by case (1). In the reduction paths which eventually
 435 select a left premise, the sequent $N_2, F \vdash S'$ is useless of sort (3) with S' distinguished, and
 436 $\vdash F, \Delta$ is useless of sort (4) with F distinguished.

437 In order to obtain a proper pre-proof from the sequents occurring in a reduction path,
 438 we need to close the derivation on useless sequents. This is done by replacing distinguished
 439 formulas by \top formulas. However, a usual substitution is not appropriate here as we are
 440 really replacing formula occurrence, which may be distributed in arbitrarily complex ways
 441 among sub-occurrences.

442 ► **Definition 29.** A *truncation* τ is a partial function from Σ^* to $\{\top, \mathbf{0}\}$ such that:

- 443 ■ For any $\alpha \in \Sigma^*$, if $\alpha \in \text{Dom}(\tau)$, then $\alpha^\perp \in \text{Dom}(\tau)$ and $\tau(\alpha) = \tau(\alpha^\perp)^\perp$.
- 444 ■ If $\alpha \in \text{Dom}(\tau)$ then for any $\beta \in \Sigma^+$, $\alpha.\beta \notin \text{Dom}(\tau)$.

445 ► **Definition 30** (Truncation of a reduction path). Let \mathcal{R} be a reduction path. The truncation
 446 τ associated to \mathcal{R} is defined by setting $\tau(\alpha) = \top$ and $\tau(\alpha^\perp) = \mathbf{0}$ for every formula occurrence
 447 φ_α that is distinguished in some useless sequent of \mathcal{R} .

448 The above definition is justified because F and F^\perp cannot both be distinguished, by
 449 fairness of \mathcal{R} . We can finally obtain the pre-proof associated to a reduction path, in a proof
 450 system slightly modified to take truncations into account.

451 ► **Definition 31** (Truncated proof system). Given a truncation τ , the
 452 infinitary proof system $\mu\text{MALL}_\tau^\infty$ is obtained by taking all the rules of μMALL^∞ , with the proviso that they only apply when the address of their
 453 principal formula is not in the domain of τ , with the following extra rule: $\frac{\vdash \tau(\alpha)_{\alpha i}, \Delta}{\vdash F, \Delta} (\tau)$
 454 if $\alpha \in \text{Dom}(\tau)$

453 The adress $\alpha.i$ associated with $\tau(\alpha)$ in the rule (τ) forbids loops on a (τ) rule. Indeed if
 454 $\alpha \in \text{Dom}(\tau)$ then $\alpha.i \notin \text{Dom}(\tau)$.

455 ► **Definition 32** (Truncated proof associated to a reduction path). Let \mathcal{R} be a fair infinite
 456 reduction path starting with π and τ be the truncation associated to it. We define $TR(\mathcal{R})$
 457 to be the $\mu\text{MALL}_\tau^\infty$ proof obtained from π by keeping only sequents that occur as premise of
 458 some multicut in \mathcal{R} , using the same rules as in π whenever possible, and deriving useless
 459 sequents by rules (τ) and (\top) .

460 This definition is justified by definition of τ and because only useless sequents may be
 461 selected without their premises (in π) being also selected. Notice that the dual F^\perp of a
 462 distinguished formula F may only occur in \mathcal{R} for distinguished formulas of type (1) and (4); in
 463 these cases F^\perp is never principal in \mathcal{R} by fairness. Thus, there is no difficulty in constructing
 464 $TR(\mathcal{R})$ with a truncature defined on the address of F^\perp . Finally, note that $TR(\mathcal{R})$ is indeed
 465 a valid $\mu\text{MALL}_\tau^\infty$ pre-proof, because its infinite branches are infinite branches of π .

466 ► **Example.** Continuing the previous example, we consider the path where the left premise of the tensor is selected immediately. The associated truncation is such that $\tau(S') = \top$ and $\tau(F) = \top$ by (3) and (4) respectively. The derivation $TR(\mathcal{R})$ is shown below, where Π_{ax} denotes the expansion of the axiom given by Prop 9.

$$\frac{\frac{\frac{\frac{\Pi_{\text{ax}}}{N_1 \vdash N'} \quad \frac{\Pi_{\text{ax}}}{N_2, F \vdash S'}}{N_1, N_2, F \vdash N' \otimes S'} (\tau), (\top)}{\frac{\Pi_{\text{dup}}}{N \vdash N_1 \otimes N_2} \quad \frac{\Pi_{\text{ax}}}{N_1 \otimes N_2, F \vdash N' \otimes S'}} (\text{Cut})}{\frac{\frac{\Pi_{\text{ax}}}{N, F \vdash N' \otimes S'}}{N, F \vdash S}} (\tau), (\top)}{N \vdash S, \Delta} (\text{mcut})$$

4.2 Truncated truth semantics

We fix a truncation τ and define a truth semantics with respect to which $\mu\text{MALL}_\tau^\infty$ will be sound. The semantics is classical, assigning a boolean value to formula occurrences. For convenience, we take $\mathcal{B} = \{\mathbf{0}, \top\}$ as our boolean lattice, with \wedge and \vee being the usual meet and join operations on it. The following definition provides an interpretation of μMALL formulas which consists in the composition of the standard interpretation of μ -calculus formulas with the obvious linearity-forgetting translation from μMALL to classical μ -calculus.

► **Definition 33.** Let φ_α be a pre-formula occurrence. We call *environment* any function \mathcal{E} mapping free variables of φ to (total) functions of $E := \Sigma^* \rightarrow \mathcal{B}$. We define $[\varphi_\alpha]^\mathcal{E} \in \mathcal{B}$, the *interpretation* of φ_α in the environment \mathcal{E} , by $[\varphi_\alpha]^\mathcal{E} = \tau(\alpha)$ if $\alpha \in \text{Dom}(\tau)$, and otherwise:

- $[X_\alpha]^\mathcal{E} = \mathcal{E}(X)(\alpha)$, $[\top_\alpha]^\mathcal{E} = [\mathbf{1}_\alpha]^\mathcal{E} = \top$ and $[\mathbf{0}_\alpha]^\mathcal{E} = [\perp_\alpha]^\mathcal{E} = \mathbf{0}$.
- $[(\varphi \otimes \psi)_\alpha]^\mathcal{E} = [\varphi_{\alpha.l}]^\mathcal{E} \wedge [\psi_{\alpha.r}]^\mathcal{E}$, for $\otimes \in \{\&, \otimes\}$.
- $[(\varphi \oplus \psi)_\alpha]^\mathcal{E} = [\varphi_{\alpha.l}]^\mathcal{E} \vee [\psi_{\alpha.r}]^\mathcal{E}$, for $\oplus \in \{\oplus, \wp\}$.
- $[(\mu X.\varphi)_\alpha]^\mathcal{E} = \text{lfp}(f)(\alpha)$ and $[(\nu X.\varphi)_\alpha]^\mathcal{E} = \text{gfp}(f)(\alpha)$ where $f : E \rightarrow E$ is given by $f : h \mapsto \beta \mapsto (\tau(\beta)$ if $\beta \in \text{Dom}(\tau)$ and $[\varphi_{\beta.i}]^\mathcal{E} :: X \mapsto h$ otherwise).

When F is closed, we simply write $[F]$ for $[F]^\emptyset$.

We refer the reader to the appendix for details on the construction of the interpretation. We simply state here the main result about it.

► **Proposition 34.** *If $\vdash \Gamma$ is provable in $\mu\text{MALL}_\tau^\infty$, then $[F] = \top$ for some $F \in \Gamma$.*

We only sketch the soundness proof (see appendix C for details) which proceeds by contradiction. Assuming we are given a proof π of a formula F such that $[F] = \mathbf{0}$, we exhibit a branch β of π containing only formulas interpreted by $\mathbf{0}$. A validating thread of β unfolds infinitely often some formula $\nu X.\varphi$. Since the interpretation of $\nu X.\varphi$ is defined as the gfp of a monotonic operator f we have, for each occurrence $(\nu X.\varphi)_\alpha$ in β , an ordinal λ such that $[(\nu X.\varphi)_\alpha] = f^\lambda(\bigvee E)(\alpha)$, where $\bigvee E$ is the supremum of the complete lattice E . We show that this ordinal can be forced to decrease along β at each fixed point unfolding, contradicting the well-foundedness of the class of ordinals.

► **Definition 35.** A truncation τ is *compatible* with a formula φ_α if $\alpha \notin \text{dom}(\tau)$ and, for any $\alpha \sqsubseteq \beta.d \in \text{Dom}(\tau)$ where $d \in \{l, r, i\}$, we have that φ_α admits a sub-occurrence ψ_β with \otimes or $\&$ as the toplevel connective of ψ , $d \in \{l, r\}$, and $\alpha.d' \notin \text{Dom}(\tau)$ for any $d' \neq d$.

In other words, a truncation τ is compatible with a formula F if it truncates only sons of \otimes or $\&$ nodes in the tree of the formula F and at most one son of each such node.

► **Proposition 36.** *If F is a formula compatible with τ and containing no ν binders, no \top and no $\mathbf{1}$, then $[F] = \mathbf{0}$.*

4.3 Proof of cut elimination

Multicut reduction is shown productive and then to result in a valid cut-free proof.

► **Proposition 37.** *Any fair reduction sequence produces a $\mu\text{MALL}_\tau^\infty$ pre-proof.*

Proof. By contradiction, consider a fair infinite sequence of internal multicut reductions. This sequence is a fair reduction path \mathcal{R} . Let τ and $TR(\mathcal{R})$ be the associated truncations and truncated proof. Since no external reduction occurs, it means that conclusion formulas of $TR(\mathcal{R})$ are never principal in the proof, thus we can transform it into a proof of the empty sequent, which contradicts soundness of $\mu\text{MALL}_\tau^\infty$. ◀

510 ► **Proposition 38.** *Any fair mcut-reduction produces a μMALL^∞ proof.*

511 **Proof.** Let π be a μMALL_m^∞ proof of conclusion $\vdash \Gamma$, and π' the cut-free pre-proof obtained
 512 by Prop. 37, *i.e.*, the limit of the multicut reduction process. Any branch of π' corresponds
 513 to a multicut reduction path. For the sake of contradiction, assume that π' is invalid. It
 514 must thus have an invalid infinite branch, corresponding to an infinite reduction path \mathcal{R} . Let
 515 τ and $\theta := TR(\mathcal{R})$ be the associated truncation and truncated proof in $\mu\text{MALL}_\tau^\infty$.

516 We first observe that formulas of Γ cannot have suboccurrences of the form $\mathbf{1}_\alpha$ or \top_α
 517 that are principal in π' . Indeed, this could only be produced by an external rule (\top)/(mcut)
 518 in the reduction path \mathcal{R} , but that would terminate the path, contradicting its infiniteness.

519 Next, we claim that all threads starting from formulas in Γ are invalid. Indeed, all rules
 520 applied to those formulas are transferred (by means of external rules) to the branch produced
 521 by the reduction path. The existence of a valid thread starting from the conclusion sequent
 522 in θ would thus imply the existence of a valid thread in our branch of π' .

523 By the first observation, we can replace all $\mathbf{1}$ and \top subformulas of Γ by $\mathbf{0}$ without changing
 524 the derivation, and obviously without breaking its validity. By the second observation, we
 525 can further modify Γ by changing all ν combinators into μ combinators. The derivation
 526 is easily adapted (using rule (μ) instead of (ν)) and it remains valid, since the validity of θ
 527 could not have been caused by a valid thread starting from the root. We thus obtain a valid
 528 pre-proof θ' of $\vdash \Gamma'$ in $\mu\text{MALL}_\tau^\infty$, where Γ' contains no ν , $\mathbf{1}$ and \top .

529 We finally show that τ is compatible with any formula occurrence from Γ . Indeed, if $\tau(\beta)$
 530 is defined for some suboccurrence ψ_β of a formula $\varphi_\alpha \in \Gamma$, then it can only be because of
 531 a useless sequent of sort (3), *i.e.*, a truncation due to the fact that the reduction path has
 532 selected only one sibling after a branching external rule. We thus conclude, by Proposition 36,
 533 that all formulas of Γ are interpreted as $\mathbf{0}$ in the truncated semantics associated to τ , which
 534 contradicts the validity of θ' and Proposition 34. ◀

535 **5 Conclusion**

536 We have established focalization and cut elimination for μMALL^∞ , the infinitary sequent
 537 calculus for μMALL . Our cut elimination result extends that of Santocanale and Fortier [13],
 538 but this extension has required the elaboration of a radically different proof technique.

539 An obvious direction for future work is now to go beyond linear logic, and notably
 540 handle structural rules in infinitary cut elimination. But many interesting questions are
 541 also left in the linear case. First, it will be natural to relax the hypothesis on fairness in
 542 the cut-elimination result. Other than cut elimination, the other long standing problem
 543 regarding μMALL^∞ and similar proof systems is whether regular proofs can be translated, in
 544 general, to finitary proofs. Further, one can ask the same question, requiring in addition
 545 that the computational content of proofs is preserved in the translation. It may well be that
 546 regular μMALL^∞ contains more computations than μMALL ; even more so if one considers
 547 other classes of finitely representable infinitary proofs. It would be interesting to study how
 548 this could impact the study of programming languages for (co)recursion, and understanding
 549 links with other approaches to this question [1, 2]. In this direction, we will be interested
 550 in studying the computational interpretation of focused cut-elimination, providing a logical
 551 basis for inductive and coinductive matching in regular and infinitary proof systems.

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A Appendix relative to Section 2

In this appendix we provide a proof of Proposition 9, but also supplementary material that may be useful to better understand μMALL^∞ , its validity condition and its relationship to μMALL . Most of this material is adapted directly from classical observations about μ -calculi, with the exception of the translation from μMALL to μMALL^∞ : it is unpublished, but we view it more as folklore than as a contribution of this paper.

A.1 Details on the validity condition

We first provide more details and intuitions about the notion of valid thread. If a thread $(F_i)_{i \in \omega}$ is eventually constant in terms of formula occurrences, it simply means that it traces a formula that is never principal in the branch: this formula plays no role in the proof, and there is no reason to declare the thread valid. Otherwise, addresses keep growing along the thread: at any point in the thread there is a later point where the address increases. Forgetting addresses and considering the set S of formulas that appear infinitely often in the thread, we immediately see that any two formulas $\varphi, \psi \in S$ are *co-accessible*, *i.e.*, $\psi \in \text{FL}(\varphi)$. Indeed, if $F_i = \varphi_\alpha$, there must be some $j > i$ such that $F_j = \psi_\beta$. In that case, the thread is valid iff the minimum of S wrt. the subformula ordering is a ν -formula. As we shall see, this definition makes sense because that minimum is always defined. Moreover, it is always a fixed point formula, so what the definition really says is that this minimum fixed point must be a greatest fixed point for the thread to be valid. All this is justified by the following classical observation about μ -calculi, which we restate next in our setting.

► **Proposition 39.** *Let $t = (F_i)_{i \in \omega}$ be a thread that is not eventually constant. The set S of formulas that occur infinitely often in t admits a minimum with respect to the subformula ordering, and that minimum is a fixed point formula.*

Proof. We assume that all formulas of t occur infinitely often in t , and that $F_i = \psi_\alpha$ implies $F_{i+1} = \psi'_{\alpha a}$ for some $a \in \Sigma$, *i.e.*, F_{i+1} is an immediate descendant of F_i . This is without loss of generality, by extracting from t the infinite sub-thread of occurrences F_i whose formulas are in S and which are principal, *i.e.*, for which $F_{i+1} \neq F_i$.

Let $|\varphi|$ be the size of a formula, *i.e.*, the number of connectives used to construct the formula. Take any $\varphi \in S$ that has minimum size, *i.e.*, $|\varphi| \leq |\psi|$ for all $\psi \in S$. We shall establish that φ must in fact be a minimum for the subformula ordering, *i.e.*, $\varphi \leq \psi$ for all $\psi \in S$. It suffices to prove that if $F_i = \psi_\alpha$ and $F_j = \varphi_{\alpha\beta}$, then $\varphi \leq \psi$. We proceed by induction on β . The result is obvious if β is empty, since one then has $\varphi = \psi$. Otherwise, we distinguish two cases:

- If $\psi = \psi^l \star \psi^r$ and $F_{i+1} = (\psi^a)_{\alpha a}$ for some $a \in \{l, r\}$, we have $\beta = a\beta'$. By induction hypothesis (with $\alpha := \alpha a$ and $\beta := \beta'$) we obtain that $\varphi \leq \psi^a$, and thus $\varphi \leq \psi$.
- Otherwise, $\psi = \sigma X.\psi'$, $F_{i+1} = (\psi'[\psi/X])_{\alpha i}$ and $\beta = i\beta'$. By induction hypothesis, $\varphi \leq \psi'[\psi/X]$. Since $|\varphi| \leq |\psi|$, φ is a subformula of $\psi'[\psi/X]$ which cannot strictly contain ψ . Thus we either have $\varphi = \psi$ or $\varphi \leq \psi'$. In both cases, we conclude immediately.

We finally show that φ must be a fixed point formula. Take any i such that $F_i = \varphi_\alpha$. We have $F_{i+1} = \psi_{\alpha a}$. Assuming that φ is not a fixed point expression, it would be of the form $\varphi_1 \star \varphi_2$ with $\psi = \varphi_i$ for some $1 \leq i \leq 2$, contradicting $|\varphi| \leq |\psi|$. ◀

A.2 Admissibility of the axiom

We now prove the admissibility of (Ax), by showing that infinite η -expansions are valid.

671 ► **Proposition (9).** *Rule (Ax) is admissible in μMALL^∞ .*

672 **Proof.** As is standard, any instance of (Ax) can be expanded by introducing two dual connectives and concluding by (Ax) on the sub-occurrences. For instance, (Ax) on $\vdash (\varphi \otimes \psi)_\alpha, (\psi^\perp \wp \varphi^\perp)_\beta$
 673 is expanded by using rules (\wp), (\otimes), and then axioms on $\vdash \varphi_{\alpha l}, \varphi_{\beta r}^\perp$ and $\vdash \psi_{\alpha r}, \psi_{\beta l}^\perp$. In μMALL^∞
 674 we can co-iterate this expansion to obtain an axiom-free pre-proof from any instance of (Ax)
 675 on $\vdash F, G^\perp$. On any infinite branch of that pre-proof, there are exactly two threads and
 676 they are not eventually constant. Let $t = (F_i)_{i \in \omega}$ and $t' = (G_i)_{i \in \omega}$ be the corresponding
 677 sequences of distinct sub-occurrences, *i.e.*, keeping an occurrence only when it is principal.
 678 We actually have that, for all i , $F_i \equiv G_i^\perp$. The minimum of a thread that is not eventually
 679 constant is necessarily a fixed point formula, thus $\min(t)$ is a ν formula iff $\min(t')$ is a μ , and
 680 one of the two threads validates the branch. ◀

682 A.3 Translating from μMALL to μMALL^∞

683 Generalizing the previous construction, we now introduce the functoriality construction,
 684 which shall be useful to present the translation from the finitary sequent calculus μMALL to
 685 its infinitary counterpart μMALL^∞ .

686 ► **Definition 40.** Let F be a pre-formula such that $\text{fv}(F) \subseteq \{X_i\}_{1 \leq i \leq n}$, and let $\vec{\Pi} = (\Pi_i)_{1 \leq i \leq n}$
 687 be a collection of pre-proofs of respective conclusions $\vdash P_i, Q_i$. We define coinductively the
 688 pre-proof $F(\vec{\Pi})$ of conclusion $\vdash F^\perp[P_i/X_i]_{1 \leq i \leq n}, F[Q_i/X_i]_{1 \leq i \leq n}$ as follows:

- 689 ■ If $F = X_i$ then $F(\vec{\Pi}) = \Pi_i$ up to relocalization, *i.e.*, changing the addresses of occurrences
 690 in Π_i to match the required ones.
 691 ■ If $F = F_1 \otimes F_2$, then $F(\vec{\Pi})$ is:

$$\frac{\frac{\frac{F_1(\vec{\Pi})}{\vdash F_1^\perp[P_i/X_i]_i, F_1[Q_i/X_i]_i} \quad \frac{F_2(\vec{\Pi})}{\vdash F_2^\perp[P_i/X_i]_i, F_2[Q_i/X_i]_i}}{\vdash F_2^\perp[P_i/X_i]_i, F_1^\perp[P_i/X_i]_i, (F_1 \otimes F_2)[Q_i/X_i]_i} (\otimes)}{\vdash (F_2^\perp \wp F_1^\perp)[P_i/X_i]_i, (F_1 \otimes F_2)[Q_i/X_i]_i} (\wp)$$

- 692 ■ If $F = F_1 \oplus F_2$, then $F(\vec{\Pi})$ is:

$$\frac{\frac{\frac{F_1(\vec{\Pi})}{\vdash F_1^\perp[P_i/X_i]_i, F_1[Q_i/X_i]_i} \quad \frac{F_2(\vec{\Pi})}{\vdash F_2^\perp[P_i/X_i]_i, F_2[Q_i/X_i]_i}}{\vdash F_1^\perp[P_i/X_i]_i, (F_1 \oplus F_2)[Q_i/X_i]_i} (\oplus_1) \quad \frac{\frac{F_1(\vec{\Pi})}{\vdash F_1^\perp[P_i/X_i]_i, F_1[Q_i/X_i]_i} \quad \frac{F_2(\vec{\Pi})}{\vdash F_2^\perp[P_i/X_i]_i, F_2[Q_i/X_i]_i}}{\vdash F_2^\perp[P_i/X_i]_i, (F_1 \oplus F_2)[Q_i/X_i]_i} (\oplus_2)}{\vdash (F_2^\perp \& F_1^\perp)[P_i/X_i]_i, (F_1 \oplus F_2)[Q_i/X_i]_i} (\&)$$

- 693 ■ If $F = \mu X.G$ then $F(\vec{\Pi})$ is obtained from applying functoriality on G with $F(\vec{\Pi})$ as the
 694 derivation for the new free variable $X_{n+1} := X$:

$$\frac{\frac{\frac{G(\vec{\Pi}, F(\vec{\Pi}))}{\vdash G^\perp[(\nu X.G^\perp)/X][P_i/X_i]_i, G[(\mu X.G)/X][Q_i/X_i]_i}}{\vdash G^\perp[(\nu X.G^\perp)/X][P_i/X_i]_i, (\mu X.G)[Q_i/X_i]_i} (\mu)}{\vdash (\nu X.G^\perp)[P_i/X_i]_i, (\mu X.G)[Q_i/X_i]_i} (\nu)$$

- 695 ■ If $F = \mathbf{0}$ then $F(\vec{\Pi})$ is directly obtained by applying (\top) on $F^\perp[P_i/X_i]_i$.

- 696 ■ If $F = \mathbf{1}$ then $F(\vec{\Pi})$ is obtained by applying rule (\perp) followed by ($\mathbf{1}$).

693 ■ Other cases are treated symmetrically.

694 As said above, the construction $F(\vec{\Pi})$ is a generalization of the infinitary η -expansion,
 695 where the derivations Π_i are plugged where free variables are encountered. In fact, if F is a
 696 closed pre-formula, then $F()$ is the derivation constructed in the proof of Proposition 9.

697 Also note that, since only finitely many sequents may arise in the process of constructing
 698 $F(\vec{\Pi})$, and since the construction is entirely guided by its end sequent, the derivation $F(\vec{\Pi})$
 699 is actually regular as long as the derivations Π_i are regular as well.

700 An infinite branch of $F(\vec{\Pi})$ either has an infinite branch of some Π_i as a suffix, or is only
 701 visiting sequents of $F(\vec{\Pi})$ that are not sequents of the input derivations $\vec{\Pi}$. In the former
 702 case, the branch is valid provided that the input derivations are valid. In the latter case, the
 703 branch contains exactly two dual threads (as in the proof of Proposition 9), one of which must
 704 be valid. Thus, $F(\vec{\Pi})$ is a proof provided that the input derivations are proofs. This result is
 705 however not usable directly to prove the validity of a pre-proof in which we make repeated
 706 use of functoriality, *i.e.*, one where branches may go through infinitely many successive uses
 707 of functoriality.

708 We now make use of functoriality to translate finitary μMALL proofs (corresponding to
 709 the propositional fragment of [4]) to infinitary derivations.

► **Definition 41** (μMALL sequent calculus). The sequent calculus for the propositional
 fragment of μMALL is a finitary sequent calculus whose rules are the same as those of
 μMALL^∞ , except that the ν rule is as follows:

$$\frac{\vdash S^\perp, F[S/X]}{\vdash S^\perp, \nu X.F}$$

The ν rule corresponds to reasoning by coinduction. In [4] it is found in a slightly different
 form, which can be obtained from the above version by means of cut:

$$\frac{\vdash \Gamma, S \quad \vdash S^\perp, F[S/X]}{\vdash \Gamma, \nu X.F}$$

710 ► **Definition 42** (Translation from μMALL to μMALL^∞). Given a μMALL proof Π of $\vdash \Gamma$, we
 711 define coinductively the μMALL^∞ pre-proof Π^i of $\vdash \Gamma$, as follows:

■ If Π starts with an inference that is present in μMALL^∞ , we use the same inference and
 proceed co-recursively. For instance,

$$\Pi = \frac{\frac{\Pi_1}{\vdash \Gamma', F} \quad \frac{\Pi_2}{\vdash G, \Gamma''}}{\vdash \Gamma', F \otimes G, \Gamma''} \quad \text{yields} \quad \Pi^i = \frac{\frac{\Pi_1^i}{\vdash \Gamma', F} \quad \frac{\Pi_2^i}{\vdash G, \Gamma''}}{\vdash \Gamma', F \otimes G, \Gamma''} .$$

■ Otherwise, Π starts with an instance of the ν rule of μMALL :

$$\Pi = \frac{\frac{\Pi_1}{\vdash S^\perp, F[S/X]}}{\vdash S^\perp, \nu X.F}$$

We transform it as follows, where (F) denotes a use of the functoriality construction:

$$\Pi^i = \frac{\frac{\frac{\Pi_1^i}{\vdash S^\perp, F[S/X]} \quad \frac{\frac{\Pi^i}{\vdash S^\perp, \nu X.F}}{\vdash F^\perp[S^\perp/X], F[(\nu X.F)/X]}}{\vdash S^\perp, F[(\nu X.F)/X]} \quad (\text{F})}{\vdash S^\perp, \nu X.F} \quad (\text{Cut})$$

712 This construction induces infinite branches, some of which being contained in the functori-
 713 ality construct, and some of which that encounter infinitely often the sequent $\vdash S^\perp, \nu X.F$
 714 (up-to structural equivalence). Note that a branch that eventually goes to the left of
 715 the above (cut) cannot cycle back to $\vdash S^\perp, \nu X.F$ anymore. It may still be infinite, going
 716 through other cycles obtained from the translation of other coinduction rules in Π_1 .

717 As a side remark, note that if Π is cut-free, then so is Π^i . Of course, if Π is cut-free but
 718 uses the version of the ν rule that embeds a cut, this is not true anymore.

719 ► **Proposition 43.** *For any μMALL derivation Π , its translation Π^i is a μMALL^∞ proof.*

720 **Proof sketch.** We have to check that all infinite branches of Π^i are valid. Consider one such
 721 infinite branch. After a finite prefix, the branch must be contained in the pre-proof obtained
 722 from the translation of a coinduction rule (second case in the above definition). If the branch
 723 is eventually contained in a functoriality construct, then it contains two dual threads, and is
 724 thus valid. Otherwise, the branch visits infinitely often (up-to structural equivalence) the
 725 sequent $\vdash S^\perp, \nu X.F$ corresponding the our translated coinduction rule. The branch in Π^i
 726 contains a thread that contains the successive sub-occurrences of $\nu X.F$ in those sequents.
 727 More specifically, that formula is principal infinitely often in the thread. It only remains to
 728 show that it is minimal among formulas that appear infinitely often: this simply follows from
 729 the fact that formulas encountered along the thread inside the functoriality construct (F) all
 730 contain $\nu X.F$ as a subformula. ◀

731 **B Appendix relative to Section 3**

732 In this appendix, we first prove results corresponding to Section 3 and then develop a
 733 complete example of focusing process, in order to exemplify the different concepts and objects
 734 defined in Section 3:

- 735 ■ reversibility of negative inference;
- 736 ■ focalization graph;
- 737 ■ focusing on positive inference;
- 738 ■ stepwise construction, by alternation of the two above – asynchronous and synchronous –
- 739 phases, of a focusing proof from any given proof.

740 **B.1 Polarity of connectives**

741 ► **Proposition (11).** *An infinite branch of a pre-proof containing only negative (resp. positive)
 742 rules is always valid (resp. invalid).*

743 **Proof.** An infinite negative branch contains only greatest fixed points. Among the threads,
 744 some are not eventually constant and their minimal formulas are ν -formulas: they are valid
 745 threads.

746 An infinite positive branch cannot be valid since for any non-constant thread t , $\min(t)$,
 747 its minimal formula, is a μ -formula. ◀

748 **B.2 Reversibility**

749 Before proving that **rev** actually builds a reversed proof, we first consider a simplified proof
 750 transformation for a proof π of a sequent $\vdash \Gamma, N$, $\text{rev}_0(\pi, N)$, the effect of which being to
 751 reverse only the topmost connective of N . It is defined similarly to **rev** except that the
 752 procedure is not called on the subproofs contrarily to definition 13.

► **Definition 44** ($\text{rev}_0(\pi, N)$). We define $\text{rev}_0(\pi, N)$ to be the pre-proof

$$\frac{\pi(1, N) \quad \dots \quad \pi(\text{sl}(N), N)}{\vdash \Gamma, N} \text{ (r}_N\text{)}.$$

753 ► **Proposition 45.** *Let π be a μMALL^∞ proof of $\vdash \Gamma, N$. $\text{rev}_0(\pi, N)$ is a μMALL^∞ proof.*

754 **Proof.** The reader will easily check that any infinite branch β of $\text{rev}_0(\pi, N)$ is obtained from
 755 a branch α of π , either of the form $(r_N) \cdot \alpha$ when α does not contain an inference on N or
 756 $(r_N) \cdot \alpha_1 \dots \alpha_{n-1} \cdot \alpha_{n+1} \dots$ where α_n has N a principal formula (occurrence). Validating
 757 threads are therefore preserved. ◀

758 We can now consider the general case of rev :

759 ► **Theorem (16).** *Let π be a μMALL^∞ proof. $\text{rev}(\pi)$ is a reversed proof of the same sequent.*

760 **Proof.** rev is obviously productive: each recursive call is guarded. Inferences of $\text{rev}(\pi)$ are
 761 locally valid: if π is a preproof, so is $\text{rev}(\pi)$.

762 If moreover π is a proof, infinite branches of $\text{rev}(\pi)$ are valid: indeed, infinite branches of
 763 $\text{rev}(\pi)$ are either fully negative (and therefore valid) or after a certain point they coincide
 764 with inferences of an infinite branch of π and their validity follows that of π .

765 The resulting proof is obviously shown to be reversed: we do not find any positive
 766 inference on any branch of $\text{rev}(\pi)$, until the first positive sequent is reached. ◀

767 B.3 Focalization graphs

768 ► **Proposition (18).** *The positive trunk of a μMALL^∞ proof is always finite.*

769 **Proof.** The positive trunk of a proof cannot have infinite branches, because they would be
 770 infinite positive branches of the original proof, thus necessarily invalid by proposition 11. ◀

771 ► **Proposition (20).** *Focalization graphs are acyclic.*

772 Even though the proof directly adapts the argument from [20], we provide it for com-
 773 pleteness:

774 **Proof.** We prove the result by *reductio ad absurdum*. Let \mathcal{S} be a positive sequent with a
 775 proof π . Let π^+ be the corresponding positive trunk and \mathcal{G} the associated Focalization Graph.
 776 Suppose that \mathcal{G} has a cycle and consider such a cycle of minimal length $(F_1 \rightarrow F_2 \rightarrow \dots \rightarrow$
 777 $F_n \rightarrow F_1)$ in \mathcal{G} and let us consider $\mathcal{S}_1, \dots, \mathcal{S}_n$ sequents of the border justifying the arrows of
 778 the cycle.

779 These sequents are actually uniquely defined or the exact same reason as in MALL [20].
 780 With the same idea we can immediately notice that the cycle is necessarily of length $n \geq 2$
 781 since two \prec -subformulas of the same formula can never be in the same sequent in the border
 782 of the positive trunk.

783 Let \mathcal{S}_0 be $\bigwedge_{i=1}^n \mathcal{S}_i$ be the highest sequent in π such that all the \mathcal{S}_i are leaves of the tree
 784 rooted in \mathcal{S}_0 . We will obtain the contradiction by studying \mathcal{S}_0 and we will reason by case on
 785 the rule applied to this sequent \mathcal{S}_0 :

786 ■ the rule cannot be (1) rule since this rule produces no premiss and thus we would have
 787 an empty cycle which is non-sens. Any rule with no premiss would lead to the same
 788 contradiction.

789 ■ If the rule is one of (\oplus_i) or (μ) , then the premiss \mathcal{S}'_0 of the rule would also satisfy
 790 the condition required for \mathcal{S}_0 (all the \mathcal{S}_i would be part of the proof tree rooted in \mathcal{S}'_0)
 791 contradicting the maximality of \mathcal{S}_0 . If the rule is any other non-branching rule, maximality
 792 of \mathcal{S}_0 would also be contradicted.

793 ■ Thus the rule shall be branching: it shall be a (\otimes) . Write \mathcal{S}_L and \mathcal{S}_R for the left and
 794 right premisses of \mathcal{S}_0 . Let $G = G_L \otimes G_R$ be the principal formula in \mathcal{S}_0 and let F be the
 795 active formula of the Trunk such that $F \prec G$.

796 There are two possibilities:

797

798 (i) either $F \in \{F_1, \dots, F_n\}$ and F is the only formula of the cycle having at the same
 799 time \prec -subformulas in the left premiss and in the right premiss,

800

801 (ii) or $F \notin \{F_1, \dots, F_n\}$ and no formula of the cycle has \prec -subformulas in both premisses.

802 Let thus I_L (resp. I_R) be the sets of indices of the active formulas of the root \mathcal{S} having
 803 (\prec -related) subformulas only in the left (resp. right) premiss. Clearly neither I_L nor I_R
 804 is empty since it would contradict the maximality of \mathcal{S}_0 . Indeed if $I_L = \emptyset$, then \mathcal{S}_R
 805 satisfies the condition of being dominated by all the $\mathcal{S}_i, 1 \leq i \leq n$ and \mathcal{S}_0 is not maximal
 806 anymore. By definition of the two sets of indices we have of course $I_L \cap I_R = \emptyset$ and the
 807 only formula of the cycle possibly not in $I_L \cup I_R$ is F if we are in the case (i): all other
 808 formulas in the cycle have their index either in I_L or in I_R .

809 As a consequence there must be an arrow in the cycle (and thus in the graph) from a
 810 formula in I_L to a formula in I_R (or the opposite). Let $i \in I_L$ and $j \in I_R$ be such indexes
 811 (say for instance $F_i \rightarrow F_j$ in \mathcal{G}) and let \mathcal{S}' be the sequent of the border responsible for
 812 this edge. \mathcal{S}' contains F'_i and F'_j and by definition of the sets I_L and I_R , \mathcal{S}' cannot be in
 813 the tree rooted in \mathcal{S}_0 which is in contradiction with the way we constructed \mathcal{S}_0 .

814 Then there cannot be any cycle in the focalization graph. ◀

815 ► **Proposition (22).** *Let \mathcal{S} be a lowest sequent of $\text{foc}(\pi, P)$ which is not conclusion of a*
 816 *rule on a positive subformula of P . Then \mathcal{S} contains exactly one subformula of P , which is*
 817 *negative.*

818 **Proof.** $\text{foc}(\pi, P)$ is such that the maximal prefix containing only rules applied to P and
 819 its positive subformulas decomposes P up to its negative subformulas. Uniqueness of the
 820 subformula in the case of MALL, treated in [20], can be directly adapted here. ◀

821 B.4 Productivity and validity of the focalization process

822 ► **Proposition (23).** *Let π be a μMALL^∞ proof, r a positive rule occurring in π and r' be a*
 823 *negative rule occurring below r in π . If r occurs in $\text{Foc}(\pi)$, then r' occurs in $\text{Foc}(\pi)$, below r .*

824 **Proof.** The proposition amounts to the simple remark that none of the transformation we
 825 do, for foc and rev , will ever permute a positive **below** a negative.

826 The proposition is thus satisfied by both transformations foc and rev and thus holds for
 827 $\text{Foc}(\pi)$ which results from the iteration of the reversibility and focalization processes. ◀

828 ► **Lemma (24).** *For any infinite branch γ of $\text{Foc}(\pi)$ containing an infinite number of positive*
 829 *rules, there exists an infinite branch in π containing infinitely many positive rules of γ .*

830 **Proof.** The lemma results from a simple application of Koenig's lemma. ◀

831 ► **Theorem (25).** *If π is a μMALL^∞ proof then $\text{Foc}(\pi)$ is also a μMALL^∞ proof.*

832 **Proof.** Let γ be an infinite branch of $\text{Foc}(\pi)$. If, at a certain point, γ is obtained by
 833 reversibility only, then it contains only negative rules and is therefore valid.

834 Otherwise, γ has been obtained by alternating infinitely often focalization phases **foc** and
 835 reversibility phases **rev** as described above. It therefore contains infinitely many positive
 836 inferences. By Lemma 24, there exists an infinite branch δ of π containing an infinite number
 837 of positive rules of γ . Since δ is valid, it contains a valid thread t .

838 Let F_m be the minimal formula of thread t , a ν -formula, and $(r_i)_{i \in \omega}$ the rules of δ in
 839 which F_m is the principal formula.

840 For any i , there exists a positive rule r'_i occurring in γ which is above r_i and r_i therefore
 841 also appears in γ by Proposition 23, which is therefore valid. ◀

842 B.5 An Example of Focalization

843 To conclude this section of the appendices, we present a detailed example of a focalization
 844 process in order to illustrate the material developed in the section of the paper devoted to
 845 focalization.

846

Let us consider the following proof of sequent

$$\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}), (\mu X.X) \otimes \mathbf{1}.$$

$$\frac{\frac{\frac{\vdots}{\vdash \nu X.X, \mathbf{1}} (\nu)}{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1}, \mathbf{0}} (\otimes)}{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1} \wp \mathbf{0}} (\wp)}{\frac{\frac{\frac{\vdots}{\vdash \nu X.X, \mathbf{0}} (\nu)}{\vdash \nu X.X, \mu X.X} (\nu), (\mu)}{\vdash \nu X.X, \mu X.X} (\nu)}{\vdash (\nu X.X) \otimes (\nu X.X), (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}), \mu X.X} (\otimes)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}), \mu X.X} (\oplus_2)}{\frac{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}), \mu X.X} (\mu)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}), \mu X.X} (\otimes)} \overline{\vdash \mathbf{1}} (\mathbf{1})$$

The Positive Trunk corresponding to this proof is:

$$\frac{\frac{\frac{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1} \wp \mathbf{0} \quad \vdash \nu X.X, \mu X.X} (\otimes)}{\vdash (\nu X.X) \otimes (\nu X.X), (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}), \mu X.X} (\oplus_2)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}), \mu X.X} (\mu)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}), \mu X.X} (\otimes)} \overline{\vdash \mathbf{1}} (\mathbf{1})$$

847 and the Border is made of only two sequents:

$$\{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1} \wp \mathbf{0} \quad ; \quad \vdash \nu X.X, \mu X.X\}$$

848 the Active Formulas of the positive trunk are thus:

- 849 ■ $\mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X))$
- 850 ■ $(\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0})$
- 851 ■ $(\mu X.X) \otimes \mathbf{1}$

852 the Focalization Graph, which has thus those three formulas as vertices, is the following:

$$(\mu X.X) \otimes \mathbf{1} \leftarrow (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}) \longrightarrow \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X))$$

853 which is indeed acyclic and has a single source, $(\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0})$, which we pick as focus. By rewriting the Propositive Trunk we arrive at

$$\frac{\frac{\pi_1}{\frac{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), \mathbf{1} \wp \mathbf{0}}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}), (\mu X.X) \otimes \mathbf{1}}}}{\vdash \nu X.X, (\mu X.X) \otimes \mathbf{1}} \quad \frac{\pi_2}{\vdash \nu X.X, (\mu X.X) \otimes \mathbf{1}} \quad (\otimes)$$

with

$$\pi_1 = \frac{\frac{\frac{\frac{\vdots}{\vdash \nu X.X, \mathbf{1}} (\nu) \quad \frac{\vdots}{\vdash \nu X.X, \mathbf{0}} (\nu)}{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1}, \mathbf{0}} (\otimes)}{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1} \wp \mathbf{0}} (\wp)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), \mathbf{1} \wp \mathbf{0}} (\oplus_2)} \quad \text{and} \quad \pi_2 = \frac{\frac{\frac{\frac{\vdots}{\vdash \nu X.X, \mu X.X} (\nu), (\mu)}{\vdash \nu X.X, \mu X.X} (\nu)}{\vdash \nu X.X, \mu X.X} (\mu)}{\vdash \nu X.X, (\mu X.X) \otimes \mathbf{1}} (\otimes)}{\vdash \mathbf{1}} (\mathbf{1})$$

854 and we continue by focalizing π_1 and π_2 .

As for π_1 , its conclusion is a negative sequent, so that one first considers $\text{rev}(\pi_1)$:

$$\text{rev}(\pi_1) = \frac{\frac{\frac{\frac{\frac{\vdots}{\vdash \nu X.X, \mathbf{1}} (\nu) \quad \frac{\vdots}{\vdash \nu X.X, \mathbf{0}} (\nu)}{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1}, \mathbf{0}} (\otimes)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), \mathbf{1}, \mathbf{0}} (\oplus_2)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), \mathbf{1} \wp \mathbf{0}} (\wp)}$$

$\text{rev}(\pi_1)$ is actually already focused: the conclusion of

$$\frac{\frac{\frac{\frac{\vdots}{\vdash \nu X.X, \mathbf{1}} (\nu) \quad \frac{\vdots}{\vdash \nu X.X, \mathbf{0}} (\nu)}{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1}, \mathbf{0}} (\otimes)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), \mathbf{1}, \mathbf{0}} (\oplus_2)}$$

is a positive sequent and its positive trunk is:

$$\frac{\frac{\vdash \nu X.X, \mathbf{1} \quad \vdash \nu X.X, \mathbf{0}}{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1}, \mathbf{0}} (\otimes)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), \mathbf{1}, \mathbf{0}} (\oplus_2)$$

855 This positive trunk contains only one active formula which therefore is automatically chosen
856 as a focus (and the positive trunk actually already focused on it).

Subproofs

$$\frac{\vdots}{\vdash \nu X.X, \mathbf{1}} (\nu) \quad \frac{\vdots}{\vdash \nu X.X, \mathbf{0}} (\nu)$$

857 are infinite negative branches and therefore reversed, focused proofs.

858 As for π_2 , its conclusion is also a negative sequent so that we build $\text{rev}(\pi_2)$ which turns
859 out to be focused as it is reduced to an infinite negative branch of (ν) rules:

$$\text{rev}(\pi_2) = \frac{\vdots}{\vdash \nu X.X, (\mu X.X) \otimes \mathbf{1}} (\nu)$$

860 To sum up, the focused proof associated with our starting proof object is:

$$\frac{\frac{\frac{\frac{\vdots}{\vdash \nu X.X, \mathbf{1}} (\nu) \quad \frac{\vdots}{\vdash \nu X.X, \mathbf{0}} (\nu)}{\vdash (\nu X.X) \otimes (\nu X.X), \mathbf{1}, \mathbf{0}} (\otimes)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), \mathbf{1}, \mathbf{0}} (\oplus_2)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), \mathbf{1} \wp \mathbf{0}} (\wp) \quad \frac{\vdots}{\vdash \nu X.X, (\mu X.X) \otimes \mathbf{1}} (\nu)}{\vdash \mathbf{0} \oplus ((\nu X.X) \otimes (\nu X.X)), (\nu X.X) \otimes (\mathbf{1} \wp \mathbf{0}), (\mu X.X) \otimes \mathbf{1}} (\otimes)}$$

861 **C** Appendix relative to Section 4

862 C.1 Detailed definitions

863 We first give a detailed description of the multicut reduction rules. In order to treat the
864 external reduction for the tensor, we first need to introduce a few preliminary definitions.
865 Given a sequent $\vdash \Gamma, \Delta, F \otimes G$ that is a premise of a multicut, we need to define which part
866 of the multicut is connected to Γ and which part is connected to Δ . These two sub-nets,
867 respectively called \mathcal{C}_Γ and \mathcal{C}_Δ , will be split apart in the external tensor reduction.

868 ► **Definition 46.** We call *cut net* any set of sequents $\{s_i\}_i$ that forms a valid set of premises
869 for the multicut rule, *i.e.*, a connected acyclic graph for the cut-connection relation. The
870 conclusion of a cut net is the conclusion that the multicut rule would have with the cut net as
871 premise, *i.e.*, the set of formula occurrences that appear in the net but not as cut formulas.

872 ► **Definition 47.** Let \mathcal{M} be a cut net, and F be a formula occurrence appearing in some
873 $s \in \mathcal{M}$. We define $\mathcal{C}_F \subseteq \mathcal{M}$ as follows. If $F^\perp \in s'$ for some $s' \in \mathcal{M}$, then \mathcal{C}_F is the connected
874 component of $\mathcal{M} \setminus \{s\}$ containing s' . Otherwise, $\mathcal{C}_F = \emptyset$. If Δ is a set of formula occurrences,
875 we define $\mathcal{C}_\Delta := \bigcup_{F \in \Delta} \mathcal{C}_F$.

876 ► **Proposition 48.** Let $s = \vdash F, \Delta, \Gamma$ be a sequent, and $\mathcal{M} = \{s\} \cup \mathcal{C}$ be a cut net of conclusion
877 $\vdash F, \Sigma$. One has $\mathcal{C} = \mathcal{C}_\Delta \uplus \mathcal{C}_\Gamma$. Moreover, $\{\vdash \Gamma\} \cup \mathcal{C}_\Gamma$ and $\{\vdash \Delta\} \cup \mathcal{C}_\Delta$ are cut nets and, if
878 Σ_Γ and Σ_Δ are their respective conclusions, we have $\Sigma = \Sigma_\Delta \uplus \Sigma_\Gamma$.

► **Definition 49** (Multicut reduction rules). *Principal and external reductions* are re-
spectively defined in Figure 4 and 5. *Internal reduction* is the union of merge and principal
reductions. *Merge reduction* is defined as follows, with $r = (\text{merge}, \{F, F^\perp\})$:

$$\frac{\frac{\frac{\vdash \Delta, F \quad \vdash \Gamma, F^\perp}{\vdash \Delta, \Gamma} (\text{Cut})}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} \xrightarrow{r} \frac{\mathcal{C} \quad \vdash \Delta, F \quad \vdash \Gamma, F^\perp}{\vdash \Sigma} (\text{mcut})$$

879 We can now provide more explicit notions of reduction sequences and fairness.

880 ► **Definition 50.** A *multicut reduction sequence* is a finite or infinite sequence $\sigma =$
881 $(\pi_i, r_i)_{i \in \lambda}$, with $\lambda \in \omega + 1$, where the π_i, r_i are pairs of μMALL_m^∞ proofs and r_i is label
882 identifying a multicut reduction rule and, whenever $i + 1 \in \lambda$, $\pi_i \xrightarrow{r_i} \pi_{i+1}$.

883 The following definition of fair reduction is standard from rewriting theory (see for
884 instance chapter 9 of [25], definition 4.9.10):

$$\begin{array}{c}
\frac{\frac{\mathcal{C} \quad \frac{\frac{\vdash \Delta, F \quad \vdash \Gamma, G}{\vdash \Delta, \Gamma, F \otimes G} (\otimes) \quad \frac{\vdash \Theta, G^\perp, F^\perp}{\vdash \Theta, G^\perp \wp F^\perp} (\wp)}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})} \xrightarrow{r} \frac{\mathcal{C} \quad \frac{\frac{\vdash \Delta, F \quad \vdash \Gamma, G \quad \vdash \Theta, G^\perp, F^\perp}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} (\text{mcut})} \\
\frac{\mathcal{C} \quad \frac{\frac{\frac{\vdash \Delta, F_2 \quad \vdash \Delta, F_1}{\vdash \Delta, F_2 \& F_1} (\&) \quad \frac{\vdash \Gamma, F_i^\perp}{\vdash \Gamma, F_1^\perp \oplus F_2^\perp} (\oplus_i)}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})} \xrightarrow{r} \frac{\mathcal{C} \quad \frac{\frac{\vdash \Delta, F_i \quad \vdash \Gamma, F_i^\perp}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} (\text{mcut})} \\
\frac{\mathcal{C} \quad \frac{\frac{\frac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F} (\mu) \quad \frac{\vdash \Gamma, F^\perp[\nu X.F^\perp/X]}{\vdash \Gamma, \nu X.F^\perp} (\nu)}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})} \xrightarrow{r} \frac{\mathcal{C} \quad \frac{\frac{\vdash \Delta, F[\mu X.F/X] \quad \vdash \Gamma, F^\perp[\nu X.F^\perp/X]}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} (\text{mcut})} \\
\frac{\mathcal{C} \quad \frac{\frac{\frac{\vdash \Gamma}{\vdash \Gamma, \perp} (\perp) \quad \overline{\vdash \mathbf{1}} (1)}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})}{\vdash \Sigma} \quad (\text{mcut})} \xrightarrow{r} \frac{\mathcal{C} \quad \frac{\vdash \Gamma}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma} (\text{mcut})}
\end{array}$$

■ **Figure 4** Principal reductions, where $r = (\text{principal}, \{F, F^\perp\})$ with $\{F, F^\perp\}$ the principal formulas that have been reduced.

$$\begin{array}{c}
\frac{\mathcal{C} \quad \frac{\frac{\frac{\vdash \Delta, F \quad \vdash \Gamma, G}{\vdash \Delta, \Gamma, F \otimes G} (\otimes)}{\vdash \Sigma_\Delta, \Sigma_\Gamma, F \otimes G} (\text{mcut})}{\vdash \Sigma_\Delta, \Sigma_\Gamma, F \otimes G} (\text{mcut})}{\vdash \Sigma_\Delta, \Sigma_\Gamma, F \otimes G} (\text{mcut})} \xrightarrow{r} \frac{\frac{\mathcal{C}_\Delta \quad \frac{\vdash \Delta, F}{\vdash \Sigma_\Delta, F} (\text{mcut}) \quad \frac{\mathcal{C}_\Gamma \quad \frac{\vdash \Gamma, G}{\vdash \Sigma_\Gamma, G} (\text{mcut})}{\vdash \Sigma_\Delta, \Sigma_\Gamma, F \otimes G} (\otimes)}{\vdash \Sigma_\Delta, \Sigma_\Gamma, F \otimes G} (\text{mcut})}{\vdash \Sigma_\Delta, \Sigma_\Gamma, F \otimes G} (\text{mcut})} \\
\frac{\mathcal{C} \quad \frac{\frac{\frac{\vdash \Delta, F, G}{\vdash \Delta, F \wp G} (\wp)}{\vdash \Sigma, F \wp G} (\text{mcut})}{\vdash \Sigma, F \wp G} (\text{mcut})}{\vdash \Sigma, F \wp G} (\text{mcut})} \xrightarrow{r} \frac{\frac{\mathcal{C} \quad \frac{\frac{\vdash \Delta, F, G}{\vdash \Sigma, F, G} (\text{mcut})}{\vdash \Sigma, F \wp G} (\wp)}{\vdash \Sigma, F \wp G} (\text{mcut})}{\vdash \Sigma, F \wp G} (\text{mcut})}{\vdash \Sigma, F \wp G} (\text{mcut})} \\
\frac{\mathcal{C} \quad \frac{\frac{\frac{\frac{\vdash \Delta, F \quad \vdash \Delta, G}{\vdash \Delta, F \& G} (\&)}{\vdash \Sigma, F \& G} (\text{mcut})}{\vdash \Sigma, F \& G} (\text{mcut})}{\vdash \Sigma, F \& G} (\text{mcut})} \xrightarrow{r} \frac{\frac{\mathcal{C} \quad \frac{\frac{\vdash \Delta, F}{\vdash \Sigma, F} (\text{mcut}) \quad \frac{\mathcal{C} \quad \frac{\vdash \Delta, G}{\vdash \Sigma, G} (\text{mcut})}{\vdash \Sigma, G} (\text{mcut})}{\vdash \Sigma, F \& G} (\&)}{\vdash \Sigma, F \& G} (\text{mcut})}{\vdash \Sigma, F \& G} (\text{mcut})} \\
\frac{\mathcal{C} \quad \frac{\frac{\frac{\vdash \Delta, F_i}{\vdash \Delta, F_1 \oplus F_2} (\oplus_i)}{\vdash \Sigma, F_1 \oplus F_2} (\text{mcut})}{\vdash \Sigma, F_1 \oplus F_2} (\text{mcut})}{\vdash \Sigma, F_1 \oplus F_2} (\text{mcut})} \xrightarrow{r} \frac{\frac{\mathcal{C} \quad \frac{\frac{\vdash \Delta, F_i}{\vdash \Sigma, F_i} (\text{mcut})}{\vdash \Sigma, F_1 \oplus F_2} (\oplus_i)}{\vdash \Sigma, F_1 \oplus F_2} (\text{mcut})}{\vdash \Sigma, F_1 \oplus F_2} (\text{mcut})}{\vdash \Sigma, F_1 \oplus F_2} (\text{mcut})} \\
\frac{\mathcal{C} \quad \frac{\overline{\vdash \Delta, \top_\alpha} (\top)}{\vdash \Sigma, \top_\alpha} (\text{mcut})}{\vdash \Sigma, \top_\alpha} (\text{mcut})} \xrightarrow{r} \frac{\overline{\vdash \Sigma, \top_\alpha} (\top)}{\vdash \Sigma, \top_\alpha} (\top)} \\
\frac{\mathcal{C} \quad \frac{\frac{\frac{\vdash \Delta}{\vdash \Delta, \perp_\alpha} (\perp)}{\vdash \Sigma, \perp_\alpha} (\text{mcut})}{\vdash \Sigma, \perp_\alpha} (\text{mcut})} \xrightarrow{r} \frac{\frac{\mathcal{C} \quad \frac{\vdash \Delta}{\vdash \Sigma} (\text{mcut})}{\vdash \Sigma, \perp_\alpha} (\perp)}{\vdash \Sigma, \perp_\alpha} (\perp)} \quad \frac{\overline{\vdash \mathbf{1}} (1)}{\vdash \mathbf{1}} (\text{mcut}) \xrightarrow{r} \frac{\overline{\vdash \mathbf{1}} (1)}{\vdash \mathbf{1}} (1)}
\end{array}$$

■ **Figure 5** External reductions rules, where $r = (\text{ext}, F)$ and F is the formula occurrence that is principal after the rule application.

885 ► **Definition 51** (Fair reduction sequences). A *multicut reduction sequence* $(\pi_i, r_i)_{i \in \lambda}$ is
 886 *fair* if for every $i \in \lambda$ and r such that $\pi_i \xrightarrow[r]{} \pi'$, there is some $j \geq i$, $j \in \lambda$, such that π_j
 887 contains no residual of r .

888 Fairness is defined in the same way for a reduction path rather than a reduction sequence.
 889 In that case, fairness can be rephrased in a simpler way: A *multicut reduction path*
 890 $(\pi_i, r_i)_{i \in \lambda}$ is *fair* if for every $i \in \lambda$ and r such that $\pi_i \xrightarrow[r]{} \pi'$, there is some $j \geq i$, $j \in \lambda$,
 891 such that r has disappeared from π_{j+1} (or: r_j is r or r_j erases r).

892 Note that reduction paths issued from a fair reduction sequence are always fair.

We end this section with more details on definition 28, which defines useless sequents. Useless sequents of sort (3) and (4) are useless only because we are considering a reduction path and not a reduction sequence. Writing \Rightarrow for the reduction steps associated to reduction paths, we can more explicitly say that the sequent $\vdash \Gamma, F_i$ is useless of sort (3) with distinguished formula F_i if, at some point in the reduction path, one of the following reductions is performed (with $\{i, j\} = \{1, 2\}$):

$$\begin{array}{c} \frac{\frac{\vdash \Gamma, F_1 \quad \vdash \Gamma, F_2}{\vdash \Gamma, F_1 \& F_2} (\&) \quad \mathcal{C}}{\vdash \Sigma, F_1 \& F_2} (\text{mcut}) \quad \xRightarrow{r_i} \quad \frac{\mathcal{C} \quad \vdash \Gamma, F_j}{\vdash \Sigma, F_j} (\text{mcut}) \\ \\ \frac{\frac{\vdash \Gamma, F_i \quad \vdash \Delta, F_j}{\vdash \Delta, \Gamma, F_1 \otimes F_2} (\otimes) \quad \mathcal{C}}{\vdash \Sigma_{\Delta}, \Sigma_{\Gamma}, F_1 \otimes F_2} (\text{mcut}) \quad \xRightarrow{r_i} \quad \frac{\mathcal{C}_{\Delta} \quad \vdash \Delta, F_j}{\vdash \Sigma_{\Delta}, F_j} (\text{mcut}) \end{array}$$

893 Moreover, the second reduction renders all sequents of \mathcal{C}_{Γ} useless of sort (4). Their
 894 distinguished formulas are cut formulas, chosen based on a traversal of the acyclic graph \mathcal{C}_{Γ} ,
 895 in a way which ensures that G and G^{\perp} are never both distinguished. In particular, for each
 896 $s' \in \mathcal{C}_{\Gamma}$ that is cut-connected to $\vdash \Gamma, F_i$ on G , we choose G^{\perp} as the distinguished formula of
 897 s' . More precisely, we define the distinguished formulas of \mathcal{C}_{Γ} inductively as follows:

- 898 ■ The distinguished formula of Γ, F_i is F_i .
 - 899 ■ If the distinguished formula of a sequent s has been defined, and if s' cut-connected to s
 900 on $G \in s'$, we choose G as the distinguished formula of s' .
- 901 Notice that two dual cut formulas G and G^{\perp} can never both be distinguished.

902 C.2 Truncated truth semantics

903 In order to develop the soundness argument for the interpretation of truncated formula
 904 occurrences, we need to work with a slightly enriched notion of formula. We thus introduce
 905 below a generalization of formulas and of the interpretation of Definition 33.

► **Definition 52.** *Marked pre-formulas* are built over the following syntax, where θ is an ordinal:

$$\varphi, \psi ::= \mathbf{0} \mid \top \mid \varphi \oplus \psi \mid \varphi \& \psi \mid \perp \mid \mathbf{1} \mid \varphi \wp \psi \mid \varphi \otimes \psi \mid \mu X. \varphi \mid \nu^{\theta} X. \varphi \mid X \text{ with } X \in \mathcal{V}.$$

906 A marked formula is a marked pre-formula with no free variables. A marked formula
 907 occurrence is given by a marked formula φ and an address α and is written φ_{α} .

908 ► **Definition 53.** Let $\bigvee E$ be the truncation $\alpha \mapsto \top$. Let f be an operator over E . We define
 909 the iterations of f starting from $\bigvee E$ by:

- 910 ■ $f^0(\bigvee E) = \bigvee E$;
 911 ■ $f^\delta(\bigvee E) = f(f^\lambda(\bigvee E))$ for every successor ordinal $\delta = \lambda + 1$;
 912 ■ $f^\delta(\bigvee E) = \bigcap_{\lambda < \delta} f^\lambda(\bigvee E)$ for every limit ordinal δ .

913 We define the interpretation of a marked formula occurrence as follows, generalizing
 914 Definition 33:

915 ► **Definition 54.** Let φ_α be a marked formula occurrence and \mathcal{E} be an environment, *i.e.*,
 916 a function mapping every free variable of φ to an element of E . We define $[\varphi_\alpha]^\mathcal{E} \in \mathcal{B}$, the
 917 interpretation of φ_α in the environment \mathcal{E} as follows: if $\alpha \in \text{Dom}(\tau)$ then $[\varphi_\alpha]^\mathcal{E} = \tau(\alpha)$;
 918 otherwise:

- 919 ■ $[X_\alpha]^\mathcal{E} = \mathcal{E}(X)(\alpha)$, $[\top_\alpha]^\mathcal{E} = \top$, $[\mathbf{0}_\alpha]^\mathcal{E} = \mathbf{0}$, $[\mathbf{1}_\alpha]^\mathcal{E} = \top$ and $[\perp_\alpha]^\mathcal{E} = \mathbf{0}$.
 920 ■ $[(\varphi \otimes \psi)_\alpha]^\mathcal{E} = [\varphi_{\alpha.l}]^\mathcal{E} \wedge [\psi_{\alpha.r}]^\mathcal{E}$, for $\otimes \in \{\&, \otimes\}$.
 921 ■ $[(\varphi \oplus \psi)_\alpha]^\mathcal{E} = [\varphi_{\alpha.l}]^\mathcal{E} \vee [\psi_{\alpha.r}]^\mathcal{E}$, for $\oplus \in \{\oplus, \wp\}$.
 922 ■ $[(\mu X.\varphi)_\alpha]^\mathcal{E} = \text{lfp}(f)(\alpha)$ and $[(\nu X^\theta.\varphi)_\alpha]^\mathcal{E} = f^\theta(\bigvee E)(\alpha)$ where $f : E \rightarrow E$ is defined by:

$$f : h \mapsto \beta \mapsto \begin{cases} \tau(\beta) & \text{if } \beta \in \text{Dom}(\tau) \\ [\varphi_{\beta.i}]^{\mathcal{E}, X \mapsto h} & \text{otherwise.} \end{cases}$$

922 We denote by $\mathcal{O}(\varphi, X, \mathcal{E})$ the operator f and we set $[\varphi]^\mathcal{E} := (\alpha \mapsto [\varphi_\alpha]^\mathcal{E})$.

923 As is standard, the least fixed point of f is guaranteed to exist in the above definition
 924 because $[\varphi]^\mathcal{E}$ is a monotonic operator in the complete lattice E , obtained by lifting the lattice
 925 \mathcal{B} where $\mathbf{0} \leq \top$ with a pointwise ordering.

926 ► **Proposition 55 (Cousot & Cousot).** *Let λ the least ordinal such that the class $\{\delta : \delta \in \lambda\}$
 927 has a cardinality greater than the cardinality $\text{Card}(E)$. Let f be a monotonic operator over
 928 E . The sequence $(f^\delta(\bigvee E))_{\delta \in \lambda}$ is a stationary decreasing chain, its limit $f^\lambda(\bigvee E)$ is the
 929 greatest fixed point of f .*

930 Let \overline{F} be the marked formula occurrence obtained from F by marking every ν binder by
 931 λ . As a consequence of Proposition 55, one has that $[F] = [\overline{F}]$.

► **Lemma 56.** *Let φ, ψ be marked pre-formulas such that $X \notin \text{fv}(\psi)$. One has:*

$$[\varphi_\alpha]^\mathcal{E}, X \mapsto [\psi]^\mathcal{E} = [(\varphi[\psi/X])_\alpha]^\mathcal{E}.$$

932 **Proof.** The proof is by induction on φ . We treat only the cases where φ is a fixed point
 933 formula; the other cases are immediate.

934 Suppose that $\varphi = \nu Y^\theta.\xi$ and let $f = \mathcal{O}(\xi, Y, \mathcal{E}, X \mapsto [\psi]^\mathcal{E})$ and $g = \mathcal{O}(\xi[\psi/X], Y, \mathcal{E})$. By
 935 induction hypothesis one has $f^\theta(\bigvee E) = g^\theta(\bigvee E)$, which concludes this case.

Suppose now that $\varphi = \mu Y.\xi$, then we have:

$$\begin{aligned} [(\mu Y.\xi)_\alpha]^\mathcal{E}, X \mapsto [\psi]^\mathcal{E} &= \text{lfp}(\mathcal{O}(\xi, Y, \mathcal{E}, X \mapsto [\psi]^\mathcal{E}))(\alpha) \\ &\stackrel{*}{=} \text{lfp}(\mathcal{O}(\xi, Y, \mathcal{E}, X \mapsto [\psi]^\mathcal{E}, Y \mapsto h))(\alpha) \\ &\stackrel{IH}{=} \text{lfp}(\mathcal{O}(\xi[\psi/X], Y, \mathcal{E}))(\alpha) \\ &= [(\mu Y.\xi[\psi/X])_\alpha]^\mathcal{E} \end{aligned}$$

936 (*) We are considering capture-free substitutions, hence $Y \notin \text{fv}(\psi)$ and $[\psi]^\mathcal{E}, Y \mapsto f = [\psi]^\mathcal{E}$. ◀

937 An immediate consequence of this proposition is that the interpretation of a least fixed
 938 point formula is equal to the interpretation of its unfolding:

939 ► **Lemma 57.** *If $\alpha \notin \text{Dom}(\tau)$, $[(\mu X.\varphi)_\alpha]^\mathcal{E} = [(\varphi[\mu X.\varphi/X])_{\alpha.i}]^\mathcal{E}$*

Proof. We set $f = \mathcal{O}(\varphi, X, \mathcal{E})$. Let us notice first that for all $\alpha \in \Sigma^*$, one has $[(\mu X.\varphi)_\alpha]^\mathcal{E} = \text{lfp}(f)(\alpha)$. Indeed, one has the equality by definition when $\alpha \notin \text{Dom}(\tau)$ and it is easy to prove it when $\alpha \in \text{Dom}(\tau)$ since both sides are equal to $\tau(\alpha)$.

$$\begin{aligned} [(\mu X.\varphi)_\alpha]^\mathcal{E} &= \text{lfp}(f)(\alpha) \\ &= [\varphi_{\alpha.i}]^{\mathcal{E}, X \mapsto \text{lfp}(f)} \\ &= [\varphi_{\alpha.i}]^{\mathcal{E}, X \mapsto [\mu X.\varphi]^\mathcal{E}} \\ &= [(\varphi[\mu X.\varphi/X])_{\alpha.i}]^\mathcal{E} \end{aligned}$$

940

941 ► **Lemma 58.** *If $[(\nu X^\theta.\varphi)_\alpha]^\mathcal{E} = \mathbf{0}$ and $\alpha \notin \text{Dom}(\tau)$ then there is an ordinal $\gamma < \theta$ s.t.*
942 *$[(\varphi[\nu X^\gamma.\varphi/X])_{\alpha.i}]^\mathcal{E} = \mathbf{0}$.*

Proof. We set $f = \mathcal{O}(\varphi, X, \mathcal{E})$. If θ is a successor ordinal $\delta + 1$, then:

$$\begin{aligned} [(\nu X^\theta.\varphi)_\alpha]^\mathcal{E} &= f^{\delta+1}(\bigvee E)(\alpha) \\ &= [\varphi_{\alpha.i}]^{\mathcal{E}, X \mapsto f^\delta(\bigvee E)} \\ &= [\varphi_{\alpha.i}]^{\mathcal{E}, X \mapsto [\nu X^\delta.\varphi]^\mathcal{E}} \\ &= [(\varphi[\nu X^\delta.\varphi/X])_{\alpha.i}]^\mathcal{E} \end{aligned}$$

943 We take γ to be the ordinal δ and we have obviously that $[(\varphi[\nu X^\gamma.\varphi/X])_{\alpha.i}]^\mathcal{E} = \mathbf{0}$.

If θ is a limit ordinal, then:

$$\begin{aligned} [(\nu X^\theta.\varphi)_\alpha]^\mathcal{E} &= f^\theta(\bigvee E)(\alpha) \\ &= \bigcap_{\beta < \theta} f^\beta(\bigvee E) \\ &= \bigcap_{\delta+1 < \theta} f^{\delta+1}(\bigvee E) \end{aligned}$$

944 Hence there is a successor ordinal $\delta + 1$ such that $[(\nu X^\theta.\varphi)_\alpha]^\mathcal{E} = f^{\delta+1}(\bigvee E)(\alpha)$ and we
945 continue as before. ◀

946 We prove easily the following lemma by induction on F :

947 ► **Lemma 59.** *Let F be an (unmarked) formula occurrence. One has $[F^\perp] = [F]^\perp$.*

948 We can finally establish our soundness result:

949 ► **Proposition (34).** *If $\vdash \Gamma$ is provable in $\mu\text{MALL}_\tau^\infty$, then $[F] = \top$ for some $F \in \Gamma$.*

950 **Proof.** If F is a marked formula occurrence, we denote by F^* the formula occurrence obtained
951 by forgetting the marking information.

952 Suppose that $\vdash \Gamma$ has a $\mu\text{MALL}_\tau^\infty$ proof π and that $[F] = \mathbf{0}$ for all $F \in \Gamma$. We will
953 construct a branch $\gamma = s_0 s_1 \dots$ of π and a sequence of functions f_0, f_1, \dots where f_i maps
954 every formula occurrence G of s_i to a marked formula occurrence $f_i(G)$ such that $[f_i(G)] = \mathbf{0}$
955 and $f_i(G)^* = G$ unless $G = \varphi_{\alpha.i}$ with $\alpha \in \text{Dom}(\tau)$. We set $s_0 = \Gamma$ and $f_0(F) = \bar{F}$. One has
956 $[\bar{F}] = [F] = \mathbf{0}$. Suppose that we have constructed s_i and f_i . We construct s_{i+1} depending
957 on the rule applied to s_i :

958 ■ If the rule is a logical rule, G being principal in s_i , we set $G_m := f_i(G)$, we have the
959 following cases:

- 960 ■ If $G = H \wp K$, then G_m is of the form $G_m = H_m \wp K_m$. We set s_{i+1} to be the
961 unique premise of s_i , $f_{i+1}(H) = H_m$ and $f_{i+1}(K) = K_m$. Since $[G_m] = \mathbf{0}$ and
962 $[G_m] = [H_m] \vee [K_m]$, one has $[G_m] = \mathbf{0}$ and $[K_m] = \mathbf{0}$. For every other formula
963 occurrence L of s_{i+1} we set $f_{i+1}(L) = f_i(L)$.
- 964 ■ If $G = H \oplus K$, we proceed exactly in the same way as above.
- 965 ■ If $G = H \otimes K$, then G_m is of the form $G_m = H_m \otimes K_m$. Since $[G_m] = \mathbf{0}$ and $[G_m] =$
966 $[H_m] \wedge [K_m]$, one has $[H_m] = \mathbf{0}$ or $[K_m] = \mathbf{0}$. Suppose wlog that $[H_m] = \mathbf{0}$. We set
967 s_{i+1} to be the premise of s_i that contains H and $f_{i+1}(H) = H_m$. For every other
968 formula occurrence L of s_{i+1} we set $f_{i+1}(L) = f_i(L)$.
- 969 ■ If $G = H \& K$, we proceed exactly in the same way as above.
- 970 ■ If $G = \mu X.K$, then G_m is of the form $G_m = \mu X.K_m$. We set s_{i+1} to be the unique
971 premise of s_i and $f_{i+1}(K[G/X]) = K_m[G_m/X]$. By Corollary 57 and since $[G_m] = \mathbf{0}$,
972 one has $[K_m[G_m/X]] = \mathbf{0}$. For every other formula occurrence L of s_{i+1} , we set
973 $f_{i+1}(L) = f_i(L)$.
- 974 ■ If $G = \nu X.H$, then G_m is of the form $G_m = \nu X^\theta.K_m$. Let s_{i+1} be the unique
975 premise of s_i . By corollary 58 and since $[G_m] = \mathbf{0}$, there is an ordinal $\delta < \theta$ such that
976 $[K_m[\nu X^\delta.K_m/X]] = \mathbf{0}$. We set $f_{i+1}(H[G/X]) = K_m[\nu X^\delta.K_m/X]$ and for every other
977 formula occurrence L of s_{i+1} , we set $f_{i+1}(L) = f_i(L)$.
- 978 ■ Suppose that the rule applied to s_i is a cut on the formula occurrence G . By Lemma 59,
979 either $[G] = \mathbf{0}$ or $[G^\perp] = \mathbf{0}$, suppose wlog that $[G] = \mathbf{0}$. We set s_{i+1} to be the premise of
980 s_i containing G , $f_{i+1}(G) \equiv \overline{G}$ and for every other formula occurrence L of s_{i+1} , we set
981 $f_{i+1}(L) \equiv f_i(L)$.
- 982 ■ If the rule applied to s_i is the rule (τ) with a principal formula $G = \varphi_\alpha$, then $\alpha \in \text{Dom}(\tau)$
983 and $f_i(G) = \psi_\alpha$ where $\psi^* = \varphi$. Hence $[f_i(G)] = \tau(\alpha)$. By construction $[f_i(G)] = \mathbf{0}$, hence
984 $\tau(\alpha) = \mathbf{0}$ and $[\tau(\alpha)_{\alpha.i}] = \mathbf{0}$. We set s_{i+1} to be the unique premise of s_i .

985 Since π is a valid pre-proof, its branch γ must contain a valid thread $t = F_0 F_1 \dots$. Let
986 $\nu X.\varphi$ be the minimal formula of t and $i_0 i_1 \dots$ be the sequence of indices where $\nu X.\varphi$ gets
987 unfolded. By construction, for all $k > 0$ one has $f_{i_k}(F_{i_k}) = \nu X^{\theta_k}.G_k$ and the sequence of
988 ordinals $(\theta_k)_k$ is strictly decreasing, which contradicts the well-foundedness of ordinals. ◀

989 We finally prove Proposition 36, generalized as follows:

990 ► **Proposition 60.** *Let φ_α be a pre-formula occurrence compatible with τ and containing no*
991 *ν binders, no \top and no $\mathbf{1}$ subformulas. Let \mathcal{E} be an environment such that for all $\beta \notin \text{Dom}(\tau)$,*
992 *$\mathcal{E}(X)(\beta) = \mathbf{0}$. We have $[\varphi_\alpha]^\mathcal{E} = \mathbf{0}$.*

993 **Proof.** The proof is by induction on φ .

- 994 ■ The cases when $\varphi = \mathbf{0}$ or \perp are trivial.
- 995 ■ If $\varphi = X$, then $[X_\alpha]^\mathcal{E} = \mathcal{E}(X)(\alpha) = \mathbf{0}$ by hypothesis on \mathcal{E} and since $\alpha \notin \text{Dom}(\tau)$ by
996 compatibility with τ .
- 997 ■ If $\varphi = \xi \wp \psi$, where $\wp \in \{\oplus, \wp\}$, then $[(\xi \wp \psi)_\alpha]^\mathcal{E} = [\xi_{\alpha.l}]^\mathcal{E} \vee [\psi_{\alpha.r}]^\mathcal{E}$. Since $(\xi \wp \psi)_\alpha$
998 is compatible with τ , one has $\alpha.l \notin \text{Dom}(\tau)$ and $\alpha.r \notin \text{Dom}(\tau)$. Indeed, if a formula
999 is compatible with a truncation τ , then τ cannot truncate a son of \oplus or a \wp node.
1000 We can thus apply our induction hypothesis, obtaining $[\xi_{\alpha.l}]^\mathcal{E} = [\psi_{\alpha.r}]^\mathcal{E} = \mathbf{0}$, hence
1001 $[(\xi \wp \psi)_\alpha]^\mathcal{E} = \mathbf{0}$.
- 1002 ■ If $\varphi = \xi \wp \psi$, where $\wp \in \{\&, \otimes\}$, then $[(\xi \wp \psi)_\alpha]^\mathcal{E} = [\xi_{\alpha.l}]^\mathcal{E} \wedge [\psi_{\alpha.r}]^\mathcal{E}$. Since $(\xi \wp \psi)_\alpha$
1003 is compatible with τ , one has $\alpha.l \notin \text{Dom}(\tau)$ or $\alpha.r \notin \text{Dom}(\tau)$. Indeed, if a formula is
1004 compatible with a truncation τ , then τ cannot truncate both sons of a $\&$ or a \otimes node.
1005 We conclude by induction as before on the subformula that is not truncated, and which
1006 is thus still compatible with τ .

1007 ■ If $\varphi = \mu X.\lambda\psi$, then $[\mu X.B]^{\mathcal{E}} = \text{lfp}(f)(\tau)$ where f is as in the definition 33. By Cousot's
1008 theorem [9], $[(\mu X.B)_{\alpha}]^{\mathcal{E}} = \bigvee_{\delta < \lambda} \varphi^{\delta}(\bigwedge E)(\alpha)$. We show by an easy transfinite induction
1009 that for all $\delta < \lambda$ and $\beta \notin \text{Dom}(\tau)$, we have $\varphi^{\delta}(\bigwedge E)(\beta) = \mathbf{0}$. This concludes the proof.
1010 ◀

