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EULER-LAGRANGE EQUATION FOR A DELAY VARIATIONAL PROBLEM

JOËL BLOT AND MAMADOU I. KONÉ

ABSTRACT. We establish Euler-Lagrange equations for a problem of Calculus of variations where the unknown variable contains a term of delay on a segment.

Key words: Euler-Lagrange, Nemystkii operator.

MSC2010-AMS: .

1. INTRODUCTION

We consider the following problem of Calculus of Variations

$$(P) \begin{cases} \text{Minimize} & J(x) := \int_0^T F(t, x_t, x'(t)) dt \\ \text{when} & x \in C^0([-r, T], \mathbb{R}^n) \\ & x|_{[0, T]} \in C^1([0, T], \mathbb{R}^n) \\ & x_0 = \psi, x(T) = \zeta. \end{cases}$$

where $r, T \in (0, +\infty)$, $r < T$, $F : [0, T] \times C^0([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a functional, $\psi \in C^0([-r, 0], \mathbb{R}^n)$, $\zeta \in \mathbb{R}^n$, and $x_t(\theta) := x(t + \theta)$ when $\theta \in [-r, 0]$ and $t \in [0, T]$. C^0 denotes the continuity and C^1 denotes the continuous differentiability.

The aim of this paper is to establish a first-order necessary condition of optimality which is analogous to the Euler-Lagrange equation of the variational problem without delay.

Now we describe the contents of the paper.

2. NOTATION AND RECALL

Let E be a finite-dimensional real normed vector space and $a < b$ be two real numbers. $BV([a, b], E)$ denotes the space of the bounded variation functions from $[a, b]$ into E , [9]. $NBV([a, b], E)$ denotes the spaces of the functions in $BV([a, b], E)$ which are right-continuous on $[a, b)$ and which are equal to 0 at a . $\mathbb{M}_n(\mathbb{R})$ denotes the space of the real $n \times n$ matrices. When $f \in BV([a, b], E)$, the total variation of f on $[a, b]$ is $V_a^b(f)$ which is defined as the supremum of the nonnegative numbers $\sum_{i=0}^k \|f(t_i) - f(t_{i+1})\|$ on the finite lists $(t_i)_{0 \leq i \leq k}$ such that $a = t_0 < \dots < t_k = b$. On $NBV([a, b], E)$ we consider the norm $\|f\|_{BV} := V_a^b(f)$. When X and Y are normed vector space, $\mathfrak{L}(X, Y)$ is the space of the linear continuous mappings from X into Y ; when $\Lambda \in \mathfrak{L}(X, Y)$, we set $\|\Lambda\|_{\mathfrak{L}} := \sup\{\|\Lambda(x)\|_Y : x \in X, \|x\|_X \leq 1\}$. The dual space of \mathbb{R}^n is $\mathbb{R}^{n*} = \mathfrak{L}(\mathbb{R}^n, \mathbb{R})$. When X is a Banach space, the space $C^0([a, b], X)$ is endowed with the norm $\|f\|_{\infty, [a, b]} := \sup_{t \in I} \|f(t)\|$. $B([a, b], E)$ is the Banach space of the bounded functions from $[a, b]$ into E . $BC_L^0([a, b], E)$

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(respectively $BC_R^0([a, b], E)$) is the space of the bounded left-continuous (respectively bounded right-continuous) from $[a, b]$ into E . $R([a, b], E)$ is the space of the regulated functions from $[a, b]$ into E . $RI([a, b], E)$ is the space of the Riemann integrable functions from $[a, b]$ into E .

Theorem 2.1. *Let $L \in C^0([0, T], \mathfrak{L}(C^0([-r, 0], \mathbb{R}^n), \mathbb{R}))$. Then there exists a mapping $g : [0, T] \times [-r, 0] \rightarrow \mathbb{R}^{n*}$ which satisfies the following properties.*

- (i) $\forall t \in [0, T], g(t, \cdot) \in NBV([-r, 0], \mathbb{R}^{n*})$.
- (ii) $\forall t \in [0, T], \|g(t, \cdot)\|_{BV} = \|L(t)\|_{\mathfrak{L}}$.
- (iii) $[t \mapsto g(t, \cdot)] \in C^0([0, T], NBV([-r, 0], \mathbb{R}^{n*}))$.
- (iv) $\forall \varphi \in C^0([-r, 0], \mathbb{R}^n), \forall t \in [0, T],$
 $L(t)(\varphi) = \int_{-r}^0 d_\theta g(t, \theta) \varphi(\theta) = \sum_{k=1}^n \int_{-r}^0 dg^j k[t](\theta) \varphi^k(\theta),$
where $[g^k[t]]_{1 \leq k \leq n} = g(t, \cdot)$.
- (v) g is Lebesgue measurable on $[0, T] \times [-r, 0]$.
- (vi) g is Riemann integrable on $[0, T] \times [-r, 0]$.

This theorem is proven in [6] (Theorem 4.1) in the case where $NBV([-r, 0], \mathbb{M}_n(\mathbb{R}))$ is the space of the functions in $BV([-r, 0], \mathbb{M}_n(\mathbb{R}))$ which are left-continuous on $(0, T]$ and equal to 0 at T . The modifications to do to adapt the proof to the case of the present paper are clear.

We need the two following results to study the Nemytskii (or superposition) operators.

Proposition 2.2. *Let \mathcal{E}, \mathcal{F} be two metric spaces, and $\Phi \in C^0(\mathcal{E}, \mathcal{F})$. $\mathcal{P}_c(\mathcal{E})$ denotes the set of the compacts subsets of \mathcal{E} . Then we have:*

$$\forall K \in \mathcal{P}_c(\mathcal{E}), \forall \epsilon > 0, \exists \delta^\epsilon > 0, \forall x \in K, \forall z \in \mathcal{E}, d(x, z) \leq \delta^\epsilon \implies d(\Phi(x), \Phi(z)) \leq \epsilon.$$

This result is established in [11], p. 355. It permits to compensate the lack for compact neighborhood of compact subset in non locally compact metric spaces, for instance in infinite-dimensional normed spaces. In [5] and in [4] we have called it "Lemma of Heine-Schwartz".

Theorem 2.3. *Let \mathcal{E}, \mathcal{F} be two metric spaces, A be a nonempty compact metric space, and $\Phi : A \times \mathcal{E} \rightarrow \mathcal{F}$ be a mapping. Then the two following assertions are equivalent.*

- (i) $\Phi \in C^0(A \times \mathcal{E}, \mathcal{F})$.
- (ii) $N_\Phi \in C^0(C^0(A, \mathcal{E}), C^0(A, \mathcal{F}))$ where $N_\Phi(u) := [a \mapsto \Phi(a, u(a))]$.

This result is established in [4] (Lemma 8.10).

We need to use the following classical Lemma of Dubois-Reymond.

Theorem 2.4. *let $\alpha < \beta$ be two real numbers, $a \in C^0([\alpha, \beta], \mathbb{R}^{n*})$ and $b \in C^0([\alpha, \beta], \mathbb{R}^{n*})$. We assume that, for all $h \in C^1([\alpha, \beta], \mathbb{R}^n)$ such that $h(\alpha) = h(\beta) = 0$, we have $\int_\alpha^\beta (a(t) \cdot h(t) + b(t) \cdot h'(t)) dt = 0$. Then we have $b \in C^1([\alpha, \beta], \mathbb{R}^{n*})$ and $b' = a$.*

This result is proven in [1] (p. 60) when $n = 1$.

3. THE MAIN RESULT

We consider the following assumptions on F .

- (A1) $F \in C^0([0, T] \times C^0([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n, \mathbb{R})$.
(A2) For all $(t, \varphi, v) \in [0, T] \times C^0([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n$, the partial Fréchet differential with respect to the second (function) variable, $D_2F(t, \varphi, v)$, exists and $D_2F \in C^0([0, T] \times C^0([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n, \mathfrak{L}(C^0([-r, 0], \mathbb{R}^n), \mathbb{R}))$.
(A3) For all $(t, \varphi, v) \in [0, T] \times C^0([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n$, the partial Fréchet differential with respect to the third (vector) variable, $D_3F(t, \varphi, v)$, exists and $D_3F \in C^0([0, T] \times C^0([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n, \mathfrak{L}(\mathbb{R}^n, \mathbb{R}))$.

Under (A2), for $L(t) := D_2F(t, x_t, x'(t))$, from Theorem 2.1 we know that there exists a mapping η which represents L . We extend η to a mapping $\eta^1 : [0, T+r] \times [-r, 0] \rightarrow \mathbb{M}_n(\mathbb{R})$ by setting

$$\eta^1(t, \theta) := \begin{cases} \eta(t, \theta) & \text{if } (t, \theta) \in [0, T] \times [-r, 0] \\ \eta(T, \theta) & \text{if } (t, \theta) \in (T, T+r] \times [-r, 0]. \end{cases}$$

Theorem 3.1. *Under (A1, A2, A3) let x be a local solution of the problem (P). We define $q : [0, T] \rightarrow \mathfrak{L}(\mathbb{R}^n, \mathbb{R})$ by setting*

$$q(t) := \begin{cases} \int_{-T}^0 g(t - \xi, \xi) d\xi & \text{if } t \in [0, T-r] \\ \int_t^T g(t, \xi - t) d\xi & \text{if } t \in (T-r, T]. \end{cases}$$

Then the function $[t \mapsto D_3F(t, x_t, x'(t)) - q(t)]$ is of class C^1 on $[0, T-r]$ and is of class C^1 on $[T-r, T]$, and we have

$$\frac{d}{dt}[D_3F(t, x_t, x'(t)) - q(t)] = g(t, 0).$$

Comments

4. A FUNCTION SPACE AND OPERATORS

We define the following function space

$$\mathfrak{X} := \{x \in C^0([-r, T], \mathbb{R}^n) : x|_{[0, T]} \in C^1([0, T], \mathbb{R}^n)\}. \quad (4.1)$$

On \mathfrak{X} we consider the following norm

$$\|x\|_{\mathfrak{X}} := \sup_{-r \leq t \leq T} \|x(t)\| + \sup_{0 \leq t \leq T} \|x'(t)\|. \quad (4.2)$$

Lemma 4.1. *$(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ is a Banach space.*

Proof. We can also write $\|x\|_{\mathfrak{X}} = \|x\|_{\infty, [-r, T]} + \|x'\|_{\infty, [0, T]}$. Since $\|\cdot\|_{\infty, [-r, T]}$ and $\|\cdot\|_{\infty, [0, T]}$ are norms, $\|\cdot\|_{\mathfrak{X}}$ is a norm. We consider the space $C^1([0, T], \mathbb{R}^n)$ endowed with the norm $\|x\|_{C^1, [0, T]} := \|x\|_{\infty, [0, T]} + \|x'\|_{\infty, [0, T]}$. We know that $(C^1([0, T], \mathbb{R}^n), \|\cdot\|_{C^1, [0, T]})$ is a Banach space, ref. Let $(x^k)_{k \in \mathbb{N}}$ be a Cauchy sequence in $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$. Since $(x^k|_{[0, T]})_{k \in \mathbb{N}}$ is also a Cauchy sequence in $C^1([0, T], \mathbb{R}^n)$ there exists $u \in C^1([0, T], \mathbb{R}^n)$ such that $\lim_{k \rightarrow +\infty} \|x^k|_{[0, T]} - u\|_{C^1, [0, T]} = 0$. Since $(x^k)_{k \in \mathbb{N}}$ is also a Cauchy sequence in the Banach space $(C^0([-r, T], \mathbb{R}^n), \|\cdot\|_{\infty, [-r, T]})$ there exists $v \in C^0([-r, T], \mathbb{R}^n)$ such that $\lim_{k \rightarrow +\infty} \|x^k - v\|_{\infty, [-r, T]} = 0$. Since $\|\cdot\|_{\infty, [0, T]} \leq \|\cdot\|_{C^1, [0, T]}$ we have $\lim_{k \rightarrow +\infty} \|x^k|_{[0, T]} - u\|_{\infty, [0, T]} = 0$, and since $\|\cdot\|_{\infty, [0, T]} \leq \|\cdot\|_{\infty, [-r, T]}$ we have $\lim_{k \rightarrow +\infty} \|x^k|_{[0, T]} - v|_{[0, T]}\|_{\infty, [0, T]} = 0$. Using the uniqueness of the limit we obtain $v|_{[0, T]} = u$. Therefore we have $v \in \mathfrak{X}$ and from the inequality $\|x^k - u\|_{\mathfrak{X}} \leq \|x^k - v\|_{\infty, [-r, T]} + \|x^k - v\|_{C^1, [0, T]}$ we obtain $\lim_{k \rightarrow +\infty} \|x^k - u\|_{\mathfrak{X}} = 0$. \square

We define the set

$$\mathfrak{A} := \{x \in \mathfrak{X} : x_0 = \psi, x(T) = \zeta\}. \quad (4.3)$$

Lemma 4.2. *\mathfrak{A} is a non empty closed affine subset of \mathfrak{X} and the unique vector subspace which is parallel to \mathfrak{A} is $\mathfrak{V} := \{h \in \mathfrak{X} : h_0 = 0, h(T) = 0\}$.*

Proof. Setting $y(t) := \psi(t)$ when $t \in [-r, 0]$ and $y(t) := \frac{t}{T}(\zeta - \psi(0)) + \psi(0)$, we see that $y \in \mathfrak{A}$ which proves that \mathfrak{A} is nonempty. From the inequalities $\|x(T)\| \leq \|x\|_{\mathfrak{X}}$ and $\|x|_{[-r, 0]}\| \leq \|x\|_{\mathfrak{X}}$, we obtain that \mathfrak{A} is closed in \mathfrak{X} . It is easy to verify that \mathfrak{A} is affine. The unique vector subspace of \mathfrak{X} is $\mathfrak{V} = \mathfrak{A} - u$ where $u \in \mathfrak{A}$. and we can easily verify the announced formula for \mathfrak{V} . \square

When $x \in C^0([-r, T], \mathbb{R}^n)$ we define

$$\underline{x} : [0, T] \rightarrow C^0([-r, 0], \mathbb{R}^n), \quad \underline{x}(t) := x_t. \quad (4.4)$$

Lemma 4.3. *When $x \in C^0([-r, T], \mathbb{R}^n)$ we have $\underline{x} \in C^0([0, T], C^0([-r, 0], \mathbb{R}^n))$.*

Proof. Using a Heine's theorem, since $[-r, T]$ is compact and x is continuous, x is uniformly continuous on $[-r, T]$, i.e.

$$\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall t, s \in [-r, T], |t - s| \leq \delta_\epsilon \implies \|x(t) - x(s)\| \leq \epsilon.$$

Let $\epsilon > 0$; if $t, s \in [0, T]$ are such that $|t - s| \leq \delta_\epsilon$ then, for all $\theta \in [-r, 0]$ we have $|(t + \theta) - (s + \theta)| \leq \delta_\epsilon$ which implies $\|x(t + \theta) - x(s + \theta)\| \leq \epsilon$, therefore $\|\underline{x}(t) - \underline{x}(s)\|_{\infty, [0, T]} \leq \epsilon$. \square

After Lemma 4.3 we can define the operator

$$\mathcal{S} : C^0([-r, T], \mathbb{R}^n) \rightarrow C^0([0, T], C^0([-r, 0], \mathbb{R}^n)), \quad \mathcal{S}(x) := \underline{x}. \quad (4.5)$$

Lemma 4.4. *\mathcal{S} is a linear continuous operator from $(C^0([-r, T], \mathbb{R}^n), \|\cdot\|_\infty)$ into $(C^0([0, T], C^0([-r, 0], \mathbb{R}^n)), \|\cdot\|_\infty)$. Setting $\mathcal{S}^1 := \mathcal{S}|_{\mathfrak{X}}$, \mathcal{S}^1 is a linear continuous operator from $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ into $(C^0([0, T], C^0([-r, 0], \mathbb{R}^n)), \|\cdot\|_\infty)$.*

Proof. The linearity of \mathcal{S} is clear. When $x \in C^0([-r, T], \mathbb{R}^n)$ we have $\|\mathcal{S}(x)\|_\infty = \sup_{0 \leq t \leq T} (\sup_{-r \leq \theta \leq 0} \|x(t + \theta)\|) = \sup_{-r \leq s \leq T} \|x(s)\| = \|x\|_{\infty, [-r, T]}$ which implies the continuity of \mathcal{S} .

The continuity of \mathcal{S}^1 results from the inequality $\|\cdot\|_{\infty, [0, T]} \leq \|\cdot\|_{\infty, [-r, T]} \leq \|\cdot\|_{\mathfrak{X}}$. \square

Now we consider the following operator

$$\mathcal{D} : \mathfrak{X} \rightarrow C^0([0, T], \mathbb{R}^n), \quad \mathcal{D}(x) := x'. \quad (4.6)$$

Lemma 4.5. *The operator \mathcal{D} is linear continuous from $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ into $(C^0([0, T], \mathbb{R}^n), \|\cdot\|_\infty)$.*

Proof. The linearity of \mathcal{D} is clear.

When $x \in \mathfrak{X}$, we have $\|\mathcal{D}(x)\|_{\infty, [0, T]} = \|x'\|_{\infty, [0, T]} \leq \|x\|_{\mathfrak{X}}$ which implies the continuity of \mathcal{D} . \square

When V and W are normed vector spaces we consider the operator

$$B : \mathfrak{L}(V, W) \times E \rightarrow W, B(L, y) := L \cdot y.$$

B is bilinear continuous, and when I is a compact interval of \mathbb{R} , we consider the Nemytskii operator defined on B

$$\left. \begin{aligned} N_B &: C^0(I, \mathfrak{L}(V, W)) \times C^0(I, V) \rightarrow C^0(I, W) \\ N_B(L, h) &:= [t \mapsto B(L(t), h(t)) = L(t) \cdot h(t)] \end{aligned} \right\} \quad (4.7)$$

where we have assimilated $C^0(I, \mathfrak{L}(V, W)) \times C^0(I, V)$ and $C^0(I, \mathfrak{L}(V, W)) \times V$. N_B is bilinear and the following inequality holds

$$\forall L \in C^0(I, \mathfrak{L}(V, W)), \forall h \in C^0(I, V), \|N_B(L, h)\|_\infty \leq \|L\|_\infty \cdot \|h\|_\infty. \quad (4.8)$$

This inequality shows that N_B is continuous and consequently it is of class C^1 .

5. THE DIFFERENTIABILITY OF THE CRITERION

First we establish a general result on the differentiability of the Nemytskii operators.

Lemma 5.1. *Let I be a compact interval of \mathbb{R} , V, W be two normed vector spaces, and $\Phi : I \times V \rightarrow W$ be a mapping. We assume that the following conditions are fulfilled.*

- (a) $\Phi \in C^0(I \times V, W)$.
- (b) For all $t \in I$, the partial Fréchet differential of Φ with respect to the second variable, $D_2\Phi(t, x)$, exists for all $x \in V$, and $D_2\Phi \in C^0(I \times V, \mathfrak{L}(V, W))$.

Then the operator N_Φ defined by $N_\Phi(v) := [t \mapsto \Phi(t, v(t))]$ is of class C^1 from $C^0(I, V)$ into $C^0(I, W)$, and we have $DN_\Phi(v) \cdot \delta v = [t \mapsto D_2\Phi(t, v(t)) \cdot \delta v(t)]$.

Proof. Under our assumptions, from Theorem 2.3 the following assertions hold.

$$N_\Phi \in C^0(C^0(I, V), C^0(I, W)) \quad (5.1)$$

$$N_{D_2\Phi} \in C^0(C^0(I, V), C^0(I, \mathfrak{L}(V, W))). \quad (5.2)$$

We arbitrarily fix $v \in C^0(I, V)$. The set $K := \{(t, v(t)) : t \in I\}$ is compact as the image of a compact by a continuous mapping. Let $\epsilon > 0$; using Proposition 2.2 we have

$$\begin{cases} \exists \beta^\epsilon > 0, \forall t \in I, \forall s \in I, \forall y \in V, \\ |t - s| + \|v(t) - y\| \leq \beta^\epsilon \implies \|D_2\Phi(t, u(t)) - D_2\Phi(s, y)\| \leq \epsilon, \end{cases}$$

which implies

$$\exists \beta^\epsilon > 0, \forall t \in I, \forall y \in V, \|v(t) - y\| \leq \beta^\epsilon \implies \|D_2\Phi(t, u(t)) - D_2\Phi(t, y)\| \leq \epsilon,$$

Let $\delta v \in C^0(I, V)$ such that $\|\delta v\|_\infty \leq \beta^\epsilon$. For all $y \in]v(t), v(t) + \delta v(t)[= \{(1 - \lambda)v(t) + \lambda(v(t) + \delta v(t))\}$, we have $\|y\| \leq \|\delta v(t)\| \leq \beta^\epsilon$, and consequently $\|D_2\Phi(t, u(t)) - D_2\Phi(t, y)\| \leq \epsilon$. Using the mean value theorem ([1], Corollaire 1, p. 141), we have

$$\begin{cases} \|\Phi(t, v(t) + \delta v(t)) - \Phi(t, v(t)) - D_2\Phi(t, v(t)) \cdot \delta v(t)\| \leq \\ \sup_{y \in]v(t), v(t) + \delta v(t)[} \|D_2\Phi(t, v(t)) - D_2\Phi(t, y)\| \cdot \|\delta v(t)\| \leq \epsilon \|\delta v(t)\| \end{cases}$$

which implies, taking the supremum on the $t \in I$,

$$\|N_\Phi(v + \delta v) - N_\Phi(v) - N_B(N_{D_2\Phi}(v), \delta v)\|_\infty \leq \epsilon \|\delta v\|_\infty.$$

And so we have proven that N_Φ is Fréchet differentiable at v and

$$DN_\Phi(v) \cdot \delta v = N_B(N_{D_2\Phi}(v), \delta v).$$

When $v, v^1, \delta v \in C^0(I, V)$, using (4.8) we have

$$\begin{aligned} \|(DN_\Phi(v) - DN_\Phi(v^1)) \cdot \delta v\|_\infty &= \|N_B(N_{D_2\Phi}(v), \delta v) - N_B(N_{D_2\Phi}(v^1), \delta v)\|_\infty = \\ &\|N_B(N_{D_2\Phi}(v) - N_{D_2\Phi}(v^1), \delta v)\|_\infty \leq \|N_{D_2\Phi}(v) - N_{D_2\Phi}(v^1)\| \cdot \|\delta v\|_\infty, \end{aligned}$$

and taking the supremum on the $\delta v \in C^0(I, V)$ such that $\|\delta v\|_\infty \leq 1$ we obtain

$$\|DN_\Phi(v) - DN_\Phi(v^1)\|_\infty \leq \|N_{D_2\Phi}(v) - N_{D_2\Phi}(v^1)\|_\infty$$

and (5.2) implies the continuity of DN_{Φ} . \square

In different frameworks, similar results of differentiability of Nemytskii operators were proven in [4] (for almost periodic functions), in [5] (for bounded sequences), in [3] (for continuous functions which converge to zero at infinite).

From $F : [0, T] \times C^0([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}$ we define the following Nemytskii operator

$$\left. \begin{aligned} N_F &: C^0([0, T], C^0([-r, 0], \mathbb{R}^n)) \times C^0([0, T], \mathbb{R}^n) \rightarrow C^0([0, T], \mathbb{R}^n) \\ N_F(U, v) &:= [t \mapsto F(t, U(t), v(t))]. \end{aligned} \right\} \quad (5.3)$$

Lemma 5.2. *Under (A1, A2, A3), N_F is of class C^1 and for all $U, \delta U \in C^0([0, T], C^0([-r, 0], \mathbb{R}^n))$, for all $v, \delta v \in C^0([0, T], \mathbb{R}^n)$ we have*
 $DN_F(U, v) \cdot (\delta U, \delta v) = [t \mapsto D_2F(t, U(t), v(t)) \cdot \delta U(t) + D_3F(t, U(t), v(t)) \cdot \delta v(t)].$

Proof. it is a straightforward consequence of Lemma 5.1 with $V = C^0([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n$, $W = \mathbb{R}$, $\Phi = F$, and by using that the differential of $F(t, \cdot, \cdot)$ at $(U(t), v(t))$ applied to $(\delta U(t), \delta v(t))$ is equal to $D_2F(t, U(t), v(t)) \cdot \delta U(t) + D_3F(t, U(t), v(t)) \cdot \delta v(t)$. \square

Lemma 5.3. *Under (A1, A2, A3), $J \in C^1(\mathfrak{X}, \mathbb{R})$ and for all $x \in \mathfrak{A}$ and for all $h \in \mathfrak{V}$, we have*

$$DJ(x) \cdot h = \int_0^T (D_2F(t, x_t, x'(t)) \cdot h_t + D_3F(t, x_t, x'(t)) \cdot h'(t)) dt.$$

Proof. We introduce the operator $in : \mathfrak{X} \rightarrow C^0([-r, T], \mathbb{R}^n)$ by setting $in(x) := x$, and the functional $I : C^0([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ by setting $I(f) := \int_0^T f(t) dt$ the Riemann integral of f on $[0, T]$. The operator in is clearly linear and from the inequality $\|\cdot\|_{\infty, [-r, T]} \leq \|\cdot\|_{\mathfrak{X}}$, it is continuous. I is linear and by using the mean value theorem, it is continuous.

Note that $J = I \circ N_F \circ (\mathcal{S} \circ in, \mathcal{D})$. Since $in, \mathcal{S}, \mathcal{D}$ and I are linear continuous, they are of class C^1 , and so $(\mathcal{S} \circ in, \mathcal{D})$ is of class C^1 . Using Lemma 5.2, N_F is of class C^1 , and so J is of class C^1 as a composition of C^1 mappings. The calculation of DJ is a simple application of the Chain Rule :

$$\begin{aligned} DJ(x) \cdot h &= DI(N_F(\mathcal{S}(in(x), \mathcal{D}(x))) \cdot DN_F \mathcal{S}(in(x), \mathcal{D}(x))) \\ &\quad (D\mathcal{S}(in(x)) \cdot Din(x)h, D\mathcal{D}(x) \cdot h) \\ &= I(DN_F(\underline{x}, \underline{x}') \cdot (\underline{h}, h')) \\ &= \int_0^T (D_2F(t, x_t, x'(t)) \cdot h_t + D_3F(t, x_t, x'(t)) \cdot h'(t)) dt. \end{aligned}$$

\square

6. PROOF OF THE MAIN RESULT

Lemma 6.1. *For all $h \in \mathfrak{V}$, we have*

$$\int_0^T D_2F(t, x_t, x'(t)) \cdot h_t dt = \int_0^T g(t, 0) \cdot h(t) dt - \int_0^T q(t) \cdot h'(t) dt.$$

Proof. Using Proposition 3.2 in [6] and $g(t, -r) = 0$, we have, for all $t \in [0, T]$,

$$\begin{aligned} D_2F(t, x_t, x'(t)) \cdot h_t &= \int_{-r}^0 d_{\theta} g(t, \theta) \cdot h(t + \theta) \\ &= \int_{t-r}^t d_{\xi} g(t, \xi - t) \cdot h(\xi) \\ &= g(t, 0) \cdot h(t) - \int_{t-r}^t g(t, \xi - t) \cdot h'(\xi) d\xi, \end{aligned}$$

which implies

$$\int_0^T D_2 F(t, x_t, x'(t)) \cdot h_t dt = \int_0^T g(t, 0) \cdot h(t) dt - \int_0^T \int_{t-r}^t g(t, \xi - t) \cdot h'(\xi) d\xi dt. \quad (6.1)$$

We set $A := \{(t, \xi) : 0 \leq t \leq T, t - r \leq \xi \leq t\}$ and from the Fubini-Tonelli theorem we gave

$$\int_0^T \int_{t-r}^t g(t, \xi - t) \cdot h'(\xi) d\xi dt = \int \int_A f(t, \xi) d\xi dt \quad (6.2)$$

where $f(t, \xi) := g(t, \xi - t) \cdot h'(\xi)$. We also define

$$\begin{aligned} A_1 &:= \{(t, \xi) : T - r \leq \xi \leq T, \xi \leq t \leq T\} \\ A_2 &:= \{(t, \xi) : 0 \leq \xi \leq T - r, \xi \leq t \leq \xi + r\} \\ A_3 &:= \{(t, \xi) : -r \leq \xi \leq 0, 0 \leq t \leq \xi + r\}. \end{aligned}$$

We have $A = A_1 \cup A_2 \cap A_3$ and $A_i \cap A_j$ is Lebesgue negligible for all $(i, j) \in \{1, 2, 3\}^2$ such that $i \neq j$. Therefore we have the following relation.

$$\int \int_A f(t, \xi) d\xi dt = \int \int_{A_1} f(t, \xi) d\xi dt + \int \int_{A_2} f(t, \xi) d\xi dt + \int \int_{A_3} f(t, \xi) d\xi dt \quad (6.3)$$

Note that $A_1 = \{(t, \xi) : T - r \leq t \leq T, T - r \leq \xi \leq t\}$. Using the Fubini-Tonelli theorem, we obtain

$$\begin{aligned} \int \int_{A_1} f(t, \xi) d\xi dt &= \int_{T-r}^T \left(\int_{\xi}^T g(t, \xi - t) d\xi \right) dt \\ &= \int_{T-r}^T \left(\int_{\xi}^T g(t, \xi - t) dt \right) h'(\xi) d\xi \end{aligned}$$

which implies

$$\int \int_{A_1} f(t, \xi) d\xi dt = \int_{T-r}^T q(\xi) \cdot h'(\xi) d\xi. \quad (6.4)$$

Using the Fubini-Tonelli theorem we have

$$\int \int_{A_2} f(t, \xi) d\xi dt = \int_0^T \left(\int_{\xi}^{\xi+r} g(t, \xi - t) dt \right) d\xi$$

which implies

$$\int \int_{A_2} f(t, \xi) d\xi dt = \int_0^{T-r} q(\xi) \cdot h'(\xi) d\xi. \quad (6.5)$$

Using the Fubini-Tonelli theorem we obtain

$$\int \int_{A_3} f(t, \xi) d\xi dt = \int_{-r}^0 \left(\int_0^{\xi+r} g(t, \xi - t) dt \right) h'(\xi) d\xi$$

and since h' is equal to zero on $[-r, 0]$ we obtain

$$\int \int_{A_3} f(t, \xi) d\xi dt = 0. \quad (6.6)$$

Using (6.2), (6.3), (6.4), (6.5), (6.6), and the chasles relation we obtain

$$\int_0^T \left(\int_{t-r}^t g(t, \xi - t) \cdot h'(\xi) d\xi \right) dt = \int_0^T q(\xi) \cdot h'(\xi) d\xi.$$

Then from this last equality and from (6.1) we obtain the lemma. \square

From Lemma 5.3 and the previous lemma we have, for all $h \in \mathfrak{V}$ the following relation

$$\begin{aligned} 0 &= \int_0^T (D_3F(t, x_t, x'(t)) \cdot h_t + D_2F(t, x_t, x'(t)) \cdot h'(t)) dt \\ &= \int_0^T (g(t, 0) \cdot h(t) + (D_3F(t, x_t, x'(t)) - q(t)) \cdot h'(t)) dt. \end{aligned}$$

Using Theorem 2.4 we obtain, for all $t \in [0, T]$, that the function $[t \mapsto D_3F(t, x_t, x'(t)) - q(t)]$ is differentiable at t and we have $\frac{d}{dt}(D_3F(t, x_t, x'(t)) - q(t)) = g(t, 0)$.

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