# A variational approach to the inviscous limit of fractional conservation laws

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# A VARIATIONAL APPROACH TO THE INVISCOUS LIMIT OF FRACTIONAL CONSERVATION LAWS

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Abstract. We are concerned with a control problem related to the vanishing *fractional* viscosity approximation to scalar conservation laws. We investigate the $\Gamma$-convergence of the control cost functional, as the viscosity coefficient tends to zero.

## 1. Introduction

1.1. **Optimal control for conservation laws.** We are concerned with the scalar one-dimensional conservation law

$$\partial_t u + \partial_x f(u) = 0 \tag{1.1}$$

where the time variable $t$ runs on a given interval $[0, T]$, the space variable $x$ runs, for the sake of simplicity, on a one dimensional torus $\mathbb{T}$, and $u = u(t, x)$. Even if the initial datum $u(0) = u(0, \cdot)$ is smooth, the flow (1.1) may develop singularities, so that in general no classical smooth solutions exist. On the other hand, if $f$ is nonlinear, there are in general infinitely many weak solutions to the Cauchy problem associated to (1.1). Existence and uniqueness of the solution are then recovered by imposing the so-called entropy condition. A celebrated result by Kruzhkov states the uniqueness of the entropy solution to the Cauchy problem associated to (1.1). Such an entropic solution can be obtained as limit of various approximations of the flow (1.1); namely the entropy solution is the relevant one. We refer to [5, 11, 12] for general theory of conservation laws.

In particular, see [6], the entropy solution to (1.1) can be recovered as the limit as $\varepsilon \to 0$ of solutions to

$$u_t + \partial_x f(u) = -\frac{\varepsilon}{2}(-\partial_{xx})^s u \tag{1.2}$$

where $1/2 < s \leq 1$, $(-\partial_{xx})^s$ denotes the $s$-th power of the negative Laplacian. In this paper a more general variational approach to the above problem will be addressed. Indeed (1.2) is a model for nonlinear transport-diffusion phenomena in a media allowing long-range correlations. The action of an external field $E$ on the system modifies (1.2) to

$$\partial_t u + \partial_x f(u) = -\frac{\varepsilon}{2}(-\partial_{xx})^s u + (-\partial_{xx})^{s/2} E \tag{1.3}$$

while the work done by the external field $E$ equals $\frac{1}{2}\|E\|^2_{L_2}$. A control problem is then naturally introduced by defining, for $\varepsilon > 0$, the functional (see Section 2 for a more precise definition)

$$I_\varepsilon(u) := \inf_E \frac{\varepsilon^{-1}}{2}\|E\|^2_{L_2(dt,dx)} \tag{1.4}$$

where the infimum is carried over the fields $E$ such that (1.3) holds. We then investigate the variational convergence of $I_\varepsilon$.

1.2. **A Statistical Mechanics interpretation.** Roughly speaking, (1.2) can be interpreted as the *typical* evolution behavior of a density (e.g. a density of charge) in a media with long-range correlations. Then (1.3) describes the same density when a random fluctuation $E$ is introduced, while $\|E\|^2_{L_2}$ gives a weight to how much unlikely the fluctuation is. Thus, $I_\varepsilon(u)$ quantifies how unlikely is to observe a density $u$, when the typical behavior is given by (1.2). By the end, one may interpret $I_\varepsilon$ as a free energy of the system, and thus one would be interested to understand the typical behavior of the infima of $I_\varepsilon$ over good sets, as $\varepsilon \to 0$.

Indeed the $\varepsilon \to 0$ limit corresponds, in this Statistical Mechanics' description, to a hydrodynamical limit, so that the variational limit of $I_\varepsilon$ should play the role of a free energy for the limiting (macroscopic) system. While this picture has been investigated in several models, see [13], and connections of microscopic, mesoscopic and macroscopic descriptions have been sometimes established rigorously, most of the literature concerns diffusive systems, where the limiting macroscopic behavior is of parabolic type. A major open problem concerns fluctuations of systems with an hydrodynamics given by non-linear transport evolutions as (1.1). No result at all concerning nonlocal (that is, long-range) fluctuations of hyperbolic systems is known to the authors.

It is well known that, in order to investigate the limit of infima of a sequence $I_\varepsilon$ of functional, the notion of $\Gamma$-convergence, see Section 2.2, is the relevant one. In [1] the $\Gamma$-convergence of $I_\varepsilon$ is investigated in the "local" case $s = 1$ corresponding to short-range correlations. In [9] such results are rigorously connected to the description of large deviations for stochastic PDEs modeling stochastic particles systems. In both papers, two different scalings are considered, corresponding to the $\Gamma$-limits of $\varepsilon I_\varepsilon$ and of $I_\varepsilon$. In this paper we only address the latter problem, the $\Gamma$-limit of $I_\varepsilon$. Indeed, only in this latter scaling the vanishing diffusive term $\varepsilon(-\partial_{xx})^s u$ is expected to play a different role than the standard Laplacian.

The main results here established are the following. A functional $I\colon L_p([0,T]\times\mathbb{T}) \to [0,+\infty]$ is introduced, see (2.2). $I(u)$ is set to be $+\infty$ if $u$ is not a weak solution to (1.1), while if $u$ solves (1.1), $I(u)$ quantifies how much the entropic condition of (1.1) is violated by $u$. In Theorem 2.5-(i) we prove that if $u_\varepsilon \to u$ in $L_p$, then $\underline{\lim}\, I_\varepsilon(u_\varepsilon) \geq I(u)$, a so-called $\Gamma$-liminf inequality. In Theorem 2.5-(ii), we prove that $I_\varepsilon$ is an equicoercive sequence in the strong $L_p$ topology. The two results imply that

if $\mathcal{C}$ is closed in $L_p$, then

$$\varliminf_{\varepsilon} \inf_{u\in\mathcal{C}} I_\varepsilon(u) \geq \inf_{u\in\mathcal{C}} I(u).$$

In order to characterize the $\Gamma$-limit of $I_\varepsilon$, one would need to establish a $\Gamma$-limsup inequality. This step is missing even in the local case $s = 1$, mainly because of open issues concerning chain-rules for non BV fields, see [1, 7]. We thus do not tackle the problem here, but rather give a qualitative hint that may suggest $I$ to be the $\Gamma$-limit (and not just an upper bound of the $\Gamma$-limit ) of $I_\varepsilon$. Indeed, in Theorem 2.7 we explicitly calculate the quasipotential of $I_\varepsilon$, a proper way to describe the long time asymptotic of a functional, and prove it to be independent of $\varepsilon$ and equal to the quasipotential of $I$. To put it shortly, the $\varepsilon \to 0$ limit and $T \to +\infty$ limit commute, if one assumes $I$ to be the $\Gamma$-limit of $I_\varepsilon$.

The functional $I$ was already introduced in [1]. The result in this paper then asserts that the limiting fluctuations are the same if $1/2 < s < 1$ or $s = 1$. $I$ thus appears to be a solid candidate as the proper generalization of the functional introduced by Jensen and Varadhan in a stochastic particles setting (see e.g. [14] for a summary of their results).

1.3. **Outline and generalizations.** Beyond considering the non-local case $s < 1$, two further technical difficulties are addressed in this paper with respect to [1]. First, we allow unbounded densities $u \in L_p([0, T] \times \mathbb{T})$ (while $u$ was a priori restricted to take values in $[0, 1]$ in [1]), and we fix an initial datum $u_0$ (whereas no initial condition was given in [1]). However, since we only achieve $L_2$ Hölder a priori bounds on $u$, we need $f$ to be uniformly Lipschitz (to make integrals meaningful) and $p < 2$ (to assure a needed uniform integrability in the topology considered).

From a technical point of view, the key proofs are achieved by heavily using the Caffarelli-Silvestre [4] representation of the fractional Laplacian operator, as opposed on the $s = 1$ case where of course only local evaluations were needed.

The paper is organized as follows. In Section 2 the main results are stated. In Section 3 some useful properties of the fractional Laplacian are recalled. In Section 4 we establish the basic estimates needed in Section 5, where the main results about $\Gamma$-convergence are proved. Section 6 is devoted to the proof of Theorem 2.7, characterizing the quasipotential.

We remark that we tried to keep the setting as readable as possible. Some generalizations are possible by using the same techniques of the paper. First, one can flawlessly change the torus $\mathbb{T}$ with the real line $\mathbb{R}$. Secondly, one may study the problem in higher dimensions: all the proofs go through but Theorem 2.5-(ii) which needs to be addressed by the means of averaging lemmas in this case, see [11, Chap. 5], and thus requires stronger hypotheses on $f$.

## 2. Preliminaries and main results

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one dimensional torus and let $\langle \cdot, \cdot \rangle$ denote duality in $L_2(\mathbb{T})$. Hereafter $T > 0$ is fixed, $\partial_t$ denotes derivative with respect to the time variable $t \in [0,T]$, $\partial_x$ derivative with respect to the space variable $x \in \mathbb{T}$.

Let $C\big([0,T]; H^{-1}(\mathbb{T})\big)$ be endowed with its natural metric

$$\text{distance}(u,v) := \sup\big\{\langle u(t) - v(t), \varphi\rangle,\, t \in [0,T],\, \|\varphi\|^2_{L_2(\mathbb{T})} + \|\partial_x \varphi\|^2_{L_2(\mathbb{T})} \le 1\big\} \quad (2.1)$$

Fix once and for all $p \in [1,2[$, and let $\mathcal{X} := C\big([0,T]; H^{-1}(\mathbb{T})\big) \cap L_p([0,T]\times\mathbb{T})$ be endowed with the refinement of the $C\big([0,T]; H^{-1}(\mathbb{T})\big)$ and the *strong* $L_p$ metrics.

Let moreover $H_s(\mathbb{T})$, $\dot H_s(\mathbb{T})$ be the fractional Sobolev space and the homogeneous fractional Sobolev space of exponent $s > 0$. $H_{-s}(\mathbb{T})$ and $\dot H_{-s}(\mathbb{T})$ denotes their dual spaces. Notice that with this notation $\dot H_{-s} = \big\{(-\partial_{xx})^s h,\, h \in \dot H_s(\mathbb{T})\big\}$. We finally introduce the space $\mathcal{H} = L_2\big([0,T]; \dot H_s(\mathbb{T})\big)$ and its dual $\mathcal{H}^* = L_2\big([0,T]; \dot H_{-s}(\mathbb{T})\big)$. We use the standard notation for the norms, for instance $\|g\|^2_{\mathcal{H}^*} = \int_0^T \big|(-\partial_{xx})^{s/2} h(t)\big|^2\, dt$ if $g(t) = (-\partial_{xx})^s h(t)$ for a.e. $t$, while $\|g\|_{\mathcal{H}^*} = +\infty$ if $g \notin \mathcal{H}^*$.

2.1. **Fractional parabolic cost functional.** We assume the flux $f$ to be bounded and Lipschitz, the initial datum $u_0 \in L_2(\mathbb{T})$, and the exponent $s > 1/2$. For $\varepsilon > 0$ the functional $I_\varepsilon \colon \mathcal{X} \to [0,+\infty]$ is defined as

$$I_\varepsilon(u) := \begin{cases} \frac{\varepsilon^{-1}}{2}\big\|\partial_t u + \partial_x f(u) + \frac{\varepsilon}{2}(-\partial_{xx})^s u\big\|^2_{\mathcal{H}^*} & \text{if } u \in \mathcal{H} \cap C([0,T]; L_2(\mathbb{T})) \\ & \text{and } u(0,x) = u_0(x) \\ +\infty & \text{otherwise} \end{cases} \quad (2.2)$$

The following proposition provides a characterization of $I_\varepsilon$ as the cost functional of the optimal control problem introduced in Section 1.

**Remark 2.1.** *If $u \in \mathcal{X}$ is such that $I_\varepsilon(u) < \infty$, then there exists a unique $\Phi \equiv \Phi_u \in \mathcal{H}$ such that the equation*

$$\partial_t u + \partial_x f(u) + \frac{\varepsilon}{2}(-\partial_{xx})^s u = (-\partial_{xx})^s \Phi \quad (2.3)$$

*holds weakly, when checked against test functions in $C^\infty([0,T]\times\mathbb{T})$. Moreover*

$$I_\varepsilon(u) := \frac{\varepsilon^{-1}}{2}\|\Phi\|^2_{\mathcal{H}} \quad (2.4)$$

The next proposition states that $I_\varepsilon$ is a *good* functional, namely that its sublevel sets are compact, a standard requirement for cost functionals. In particular it states that the condition $u \in \mathcal{H} \cap C([0,T]; L_2(\mathbb{T}))$ is the natural one to impose in the definition of the domain of $I_\varepsilon$, see (2.2). For instance, one would not in general have a lower-semicontinuous functional if higher regularity would be required on $u$, while the representation in Remark 2.1 would not hold for weaker regularity or indeed if $s < 1/2$.

**Proposition 2.2.** *$I_\varepsilon$ is a coercive lower-semicontinuous functional on $\mathcal{X}$.*

2.2. **Γ-convergence.** As well known, a most useful notion of variational convergence is the Γ-convergence which, together with some compactness estimates, implies convergence of the minima. Recall that a sequence $(F_\varepsilon)$ of functionals $F_\varepsilon\colon \mathcal{X} \to [0,+\infty]$ is *equicoercive* on $\mathcal{X}$ iff for each $M>0$ there exists a compact set $K_M$ such that $\overline{\lim}_{\varepsilon\downarrow 0}\{x\in\mathcal{X} \,:\, F_\varepsilon(x)\le M\}\subset K_M$. We briefly recall the basic definitions of the Γ-convergence theory, see e.g. [3]. Given $x\in\mathcal{X}$ we define

$$\begin{aligned}\big(\Gamma\text{–}\underline{\lim_{\varepsilon\to 0}} F_\varepsilon\big)(x) &:= \inf\big\{\underline{\lim_{\varepsilon\to 0}} F_\varepsilon(x^\varepsilon),\, \{x^\varepsilon\}\subset X \,:\, x^\varepsilon\to x\big\}\\ \big(\Gamma\text{–}\overline{\lim_{\varepsilon\to 0}} F_\varepsilon\big)(x) &:= \inf\big\{\overline{\lim_{\varepsilon\to 0}} F_\varepsilon(x^\varepsilon),\, \{x^\varepsilon\}\subset X \,:\, x^\varepsilon\to x\big\}\end{aligned}$$

Whenever $\Gamma\text{–}\underline{\lim_{\varepsilon}} F_\varepsilon = \Gamma\text{–}\overline{\lim_{\varepsilon}} F_\varepsilon = F$ we say that $F_\varepsilon$ Γ-converges to $F$ in $\mathcal{X}$. Equivalently, $F_\varepsilon$ Γ-converges to $F$ iff for each $x\in\mathcal{X}$:

- for any sequence $x^\varepsilon\to x$ we have $\underline{\lim}_\varepsilon F_\varepsilon(x^\varepsilon)\ge F(x)$ (Γ-*liminf inequality*);
- there exists a sequence $x^\varepsilon\to x$ such that $\overline{\lim}_\varepsilon F_\varepsilon(x^\varepsilon)\le F(x)$ (Γ-*limsup inequality*).

Equicoercivity and Γ-convergence of a sequence $(F_\varepsilon)$ imply an upper bound of infima over open sets, and a lower bound of infima over closed sets, see e.g. [3, Prop. 1.18], and therefore it is the relevant notion of variational convergence in the control setting introduced in (2.3).

2.3. **Solutions to scalar conservation law.** In order to describe the candidate Γ-limit of $I_\varepsilon$, further preliminaries are introduced in this section.

An element $u\in\mathcal{X}$ is a *weak solution* to (1.1) with initial condition $u_0\in L_2(\mathbb{T})$ iff for each $\varphi\in C^\infty_{\rm c}\big([0,T[\times\mathbb{T}\big)$ it satisfies

$$-\int_0^t \langle u(r),\partial_t\varphi(r)\rangle - \langle f(u(r)),\partial_x\varphi(r)\rangle\,dr - \langle u_0,\varphi(0)\rangle = 0$$

We denote by $C^2_{\rm b}(\mathbb{R})$ the set of twice differentiable functions with bounded second derivative. A function $\eta\in C^2_{\rm b}(\mathbb{R})$ is called an *entropy* and its *conjugated entropy flux* $q\in C^2_{\rm b}(\mathbb{R})$ is defined, up to an additive constant, by

$$q(w) := \int^w dv\,\eta'(v)f'(v)$$

For a weak solution $u$ to (1.1), for an entropy – entropy flux pair $(\eta,q)$, the *η-entropy production* is the distribution $\wp_{\eta,u}$ acting on $C^\infty_{\rm c}\big(]0,T[\times\mathbb{T}\big)$ as

$$\wp_{\eta,u}(\varphi) := -\int_0^T \langle \eta(u(r)),\partial_t\varphi(r)\rangle + \langle q(u(r)),\partial_x\varphi(r)\,\rangle\,dr \tag{2.5}$$

The next proposition introduces a suitable class of solutions to (1.1). Its proof is given in [1, Prop. 2.3], by adapting [7, Prop. 3.1]. We denote by $M\big(]0,T[\times\mathbb{T}\times\mathbb{R}\big)$ the

set of Radon measures on $]0,T[\times\mathbb{T}\times\mathbb{R}$. In the following, for $\varrho\in M\big(]0,T[\times\mathbb{T}\times\mathbb{R}\big)$ we denote by $\varrho^{\pm}$ the positive and negative part of $\varrho$.

**Proposition 2.3.** *Let $u\in\mathcal{X}$ be a weak solution to (1.1). The following statements are equivalent:*

(i) *for each entropy $\eta$, the $\eta$-entropy production $\wp_{\eta,u}$ can be extended to a Radon measure on $]0,T[\times\mathbb{T}$;*
(ii) *there exists $\varrho_u(dv,dt,dx)\in M\big(\mathbb{R}\times]0,T[\times\mathbb{T}\big)$, such that for any entropy $\eta$ and $\varphi\in C_{\mathrm{c}}^{\infty}\big(]0,T[\times\mathbb{T}\big)$*

$$\wp_{\eta,u}(\varphi)=\int\varrho_u(dv,dt,dx)\,\eta''(v)\varphi(t,x). \tag{2.6}$$

A weak solution $u\in\mathcal{X}$ that satisfies the equivalent conditions in Proposition 2.3 is called an *entropy-measure solution* to (1.1). We denote by $\mathcal{E}_{u_0}$ the set of entropy-measure solutions to (1.1) satisfying the initial condition $u(0)=u_0$.

A weak solution $u\in\mathcal{X}$ to (1.1) is called an *entropic solution* iff for each convex entropy $\eta$ the inequality $\wp_{\eta,u}\le 0$ holds. In particular entropic solutions are entropy-measure solutions such that $\varrho_u$ is a negative measure. It is well known, see e.g. [11], that there exists a unique entropic solution $\bar u\in C\left([0,T];L_2(\mathbb{T})\right)$ to (1.1) such that $\bar u(0)=u_0$.

2.4. **Fractional hyperbolic entropy cost of non-entropic solutions.** Recall that for $u\in\mathcal{E}_{u_0}$, $\varrho_u$ denotes its entropy production measure as defined in Proposition 2.3, while $\varrho^+$ is the positive part of $\varrho$. Define $I\colon\mathcal{X}\to[0,+\infty]$ by

$$I(u):=\begin{cases}\varrho_u^+(\mathbb{R}\times[0,T]\times\mathbb{T}) & \text{if } u\in\mathcal{E}_{u_0}\\ +\infty & \text{otherwise}\end{cases} \tag{2.7}$$

namely $I(u)$ is the total variation of the positive part of the entropy production of entropy-measure weak solutions to (1.1). The following proposition is proved in [1, Prop. 2.6].

**Proposition 2.4.** *The functional $I$ is lower semicontinuous on $\mathcal{X}$ and $I(u)=0$ iff $u$ is an entropic solution to (1.1).*

*Assume that there is no interval on $\mathbb{R}$ such that $f$ is affine on such an interval. Then $I$ is coercive on $\mathcal{X}$.*

The following theorem is the main result of this paper.

**Theorem 2.5.** (i) *The sequence of functionals $\{I_\varepsilon\}$ satisfies the $\Gamma$-liminf inequality $\Gamma\text{-}\underline{\lim}_\varepsilon I_\varepsilon\ge I$ on $\mathcal{X}$.*

(ii) *Assume that there is no interval on $\mathbb{R}$ such that $f$ is affine on such an interval. Then the sequence of functionals $\{I_\varepsilon\}$ is equicoercive on $\mathcal{X}$.*

Note that Theorem 2.5 implies that the Γ-liminf inequality holds even in weaker topologies. For instance, if $u_\varepsilon \to u$ in the sense of distributions, then still one has $\underline{\lim}_\varepsilon I_\varepsilon(u_\varepsilon) \geq I(u)$. The Γ-liminf inequality also implies some stability results for the fractional viscous approximation to conservation laws, as shown in the next corollary.

**Corollary 2.6.** *Let $E_\varepsilon \in L_2([0,T]\times\mathbb{T})$ be such that $\lim_\varepsilon \varepsilon^{-1}\|E_\varepsilon\|^2_{L_2([0,T]\times\mathbb{T})} = 0$. Then the solution $u_\varepsilon$ to (1.3) converges to the entropic solution of (1.1).*

One may prove that the above corollary is sharp, in the sense that, if $f$ is non-affine, for all $\delta > 0$ there exists a sequence $E_\varepsilon$ such that $\varepsilon^{-1}\|E_\varepsilon\|_{L_2([0,T]\times\mathbb{T})} \leq \delta$, but $u_\varepsilon$ converges to a solution to (1.1) which is not entropic.

2.5. **Quasipotential.** The functionals $I_\varepsilon$ and $I$ as well as the space $\mathcal{X}$ introduced above depend on the time horizon $T$. In this section we introduce in the notation the dependence on this parameter, so that these objects will be denoted by $I_{\varepsilon,T}$, $I_T$ and $\mathcal{X}_T$.

Let $\int_{\mathbb{T}} u_0(x)\,dx = m \in \mathbb{R}$, then $I_{\varepsilon,T}(u) = I_T(u) = +\infty$ unless $\int_{\mathbb{T}} u(t,x)\,dx = m$ for each $t \in [0,T]$. In addition, it is easy to see that the constant profile $w(x) \equiv m$ is "globally attractive" for $I_{\varepsilon,T}$, $I_T$, in the sense that if $u \in C([0,+\infty[, L_2(\mathbb{T}))$ is such that $I_{\varepsilon,T}(u)$ or $I_T(u)$ are bounded uniformly in $T$, then $u$ will stay most of the time close to $m$. We are thus interested in calculating the so-called quasipotential of the above functionals starting at $m$.

More precisely, let $m \in \mathbb{R}$ define $V_\varepsilon, V\colon \mathbb{R}\times L_2(\mathbb{T}) \to [0,+\infty]$ as

$$V_\varepsilon(m;w) := \inf_{T>0} \inf_{\substack{u\in\mathcal{X}_T \\ u(t=0)\equiv m,\, u(t=T)=w}} I_{\varepsilon,T}(u) \tag{2.8}$$

$$V(m;w) := \inf_{T>0} \inf_{\substack{u\in\mathcal{X}_T \\ u(t=0)\equiv m,\, u(t=T)=w}} I_T(u)$$

Note that the definition of $V_\varepsilon(m;w)$ and $V(m;w)$ also makes sense out of $L_2(\mathbb{T})$, but in view of (4.2) it is easily seen that $V_\varepsilon(m;w) = V(m;w) = +\infty$ if $w \notin L_2(\mathbb{T})$.

The following theorem gives an explicit characterization of $V_\varepsilon$.

**Theorem 2.7.** *It holds*

$$V_\varepsilon(m;w) = \begin{cases} \frac{1}{2}\|w-m\|^2_{L_2(\mathbb{T})} & \text{if } \int_{\mathbb{T}} w(x)\,dx = m \\ +\infty & \text{otherwise} \end{cases}$$

$V$ has been calculated in [2], where it is shown that it enjoys the same explicit representation as $V_\varepsilon$ in Theorem 2.7 above.

## 3. Local realization of the fractional Laplacian

It is well known that one can see the operator $(-\partial_{xx})^{1/2}$ on $\mathbb{R}$ by considering it as the Dirichlet to Neumann operator associated to the harmonic extension in the halfspace $\mathbb{R}\times\mathbb{R}^+$, paying the price to add a new variable. In [4], Caffarelli and Silvestre proved that this is also possible for any power $s\in]0,1[$ of the Laplacian. In this section we shortly recall such a realization when $\mathbb{R}$ is replaced by $\mathbb{T}$, together with a representation of the bilinear form $(\varphi,\psi)\mapsto\langle(-\partial_{xx})^s\varphi,\psi\rangle$, that will come useful later.

Hereafter in this paper, we denote by $\nabla$ the gradient operator $\nabla=(\partial_x,\partial_y)$ on $\mathbb{T}\times\mathbb{R}^+$. Given $u\in H^s(\mathbb{T})$, we let $(x,y)\in\mathbb{T}\times\mathbb{R}^+\mapsto\bar{u}(x,y)\in\mathbb{R}$ be the unique solution to

$$\begin{cases}\nabla\cdot(y^{1-2s}\nabla\bar{u})=0 & \text{on } \mathbb{T}\times\mathbb{R}^+\\ \bar{u}=u & \text{on } \mathbb{T}\times\{y=0\}\end{cases}\tag{3.1}$$

such that

$$\|\bar{u}\|^2_{\dot{H}^1(\mathbb{T}\times\mathbb{R}^+,y^{1-2s})}:=\int_{\mathbb{T}\times\mathbb{R}^+}y^{1-2s}|\nabla\bar{u}(x,y)|^2\,dx\,dy\tag{3.2}$$

is finite. $\bar{u}$ is called the $s$-harmonic extension of $u$. The following theorem is proved in [4].

**Theorem 3.1.** *There exists a constant $c_s>0$ depending only on $s$, such that for every $u\in H^s(\mathbb{T})$*

$$(-\partial_{xx})^s v=-c_s\lim_{y\downarrow 0}y^{1-2s}\partial_y\bar{u}$$

*where the equality holds in the distributional sense.*

In addition, it can be proved [10] that if $\tilde{u}\in\dot{H}^1(\mathbb{T}\times\mathbb{R}^+,y^{1-2s})$, then $\tilde{u}$ can be traced at $y=0$. By an integration by parts, it is then easy to verify that for $u\in H^s(\mathbb{T})$

$$\begin{aligned}\|u\|^2_{\dot{H}^s(\mathbb{T})}&=c_s\|\bar{u}\|^2_{\dot{H}^1(\mathbb{T}\times\mathbb{R}^+,y^{1-2s})}\\&=c_s\inf\left\{\|\tilde{u}\|^2_{\dot{H}^1(\mathbb{T}\times\mathbb{R}^+,y^{1-2s})}\,;\,\tilde{u}\in\dot{H}^1(\mathbb{T}\times\mathbb{R}^+,y^{1-2s}),\,\tilde{u}(x,0)=u(x)\right\}\end{aligned}\tag{3.3}$$

The following remark is obtained by multiplying (3.1) by a test function $\tilde{\varphi}$, integrating the equation by parts on $\mathbb{T}\times[\delta,+\infty[$, and passing to the limit $\delta\to 0$ thanks to Theorem 3.1.

**Remark 3.2.** *Let $u\in H^s(\mathbb{T})$ and $\varphi\in C^\infty(\mathbb{T})$. Then*

$$\langle(-\partial_{xx})^s u,\varphi\rangle=c_s\int_{\mathbb{T}\times\mathbb{R}^+}y^{1-2s}\nabla\bar{u}(x,y)\,\cdot\,\nabla\tilde{\varphi}(x,y)\,dx\,dy$$

*where $\tilde{\varphi}$ is any smooth, compactly supported function on $\mathbb{T}\times\mathbb{R}^+$ such that $\tilde{\varphi}(x,0)=\varphi(x)$.*

## 4. Regularity and a priori bounds

*Proof of Remark 2.1.* Assume that $I_\varepsilon(u) < +\infty$. Then Riesz representation theorem for the dual spaces $\mathcal{H}$, $\mathcal{H}^*$ implies the identity (2.3) when the left and right hand sides are seen as elements of $\mathcal{H}^*$. Since $f$ is Lipschitz and bounded and $s > 1/2$, $\partial_x f(u) \in L_2([0,T]; H^{s-1}(\mathbb{T})) \subset \mathcal{H}$; moreover $(-\partial_{xx})^s u \in \mathcal{H}^*$, thus all of the right hand side terms of the equation are separately in $\mathcal{H}^*$ as well, and the equation holds weakly when each single term is checked against test functions in $\mathcal{H}$. In particular, it holds weakly against smooth functions. □

**Lemma 4.1.** *There exists a constant $C_\varepsilon > 0$ such that for all $u$ with $I_\varepsilon(u) < +\infty$*

$$\varepsilon \|u\|^2_{\mathcal{H}} \le 2\|u_0\|^2_{L_2(\mathbb{T})} + 4I_\varepsilon(u) \tag{4.1}$$

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L_2(\mathbb{T})} \le 2\|u_0\|^2_{L_2(\mathbb{T})} + 4I_\varepsilon(u) \tag{4.2}$$

$$\|\partial_t u\|^2_{L_2([0,T];H^{-s}(\mathbb{T}))} \le C_\varepsilon \big[\|u_0\|^2_{L_2(\mathbb{T})} + I_\varepsilon(u)\big] \tag{4.3}$$

$$\|u\|^2_{H^{1/2}([0,T];L_2(\mathbb{T}))} \le C_\varepsilon \big[\|u_0\|^2_{L_2(\mathbb{T})} + I_\varepsilon(u)\big] \tag{4.4}$$

*Proof.* By Remark 2.1, (2.3) reads

$$\begin{aligned} &\langle u(t), \varphi(t)\rangle - \langle u_0, \varphi(0)\rangle - \int_0^t \big[\langle u(r), \partial_t\varphi(r)\rangle + \langle f(u(r)), \partial_x\varphi(r)\rangle\big]\, dr \\ &\qquad + \frac{\varepsilon}{2}\int_0^t \langle (-\partial_{xx})^{s/2} u(r), (-\partial_{xx})^{s/2}\varphi(r)\rangle\, dr \\ &= \int_0^t \langle -(\partial_{xx})^{s/2}\Phi(r), (-\partial_{xx})^{s/2}\varphi(r)\rangle\, dr \end{aligned} \tag{4.5}$$

for any $t \in [0,T]$, $\varphi \in C^\infty([0,T]\times\mathbb{T})$. Note that since $f$ is Lipschitz and bounded, $f(u) \in L_2([0,T]; H^s(\mathbb{T}))$. Therefore, by a density argument it is easy to see that (4.5) holds for any $\varphi \in L_2([0,T]; H^s(\mathbb{T}))$ such that $\partial_t\varphi$, $\partial_x\varphi \in \mathcal{H}^*$. Recalling that $\partial_t u \in \mathcal{H}^*$, that $u \in L_2([0,T]; H^s(\mathbb{T}))$, and thus, as $s > 1/2$, $\partial_x u \in L_2([0,T], \dot{H}^{s-1}(\mathbb{T})) \subset \mathcal{H}^*$, the choice $\varphi = u$ is allowed in (4.5). In view of $u \in C([0,T]; L_2(\mathbb{T}))$, with the same duality argument it is now immediate to verify that integration by parts are allowed in (4.5) with $\varphi = u$. Since $\langle f(u), \partial_x u\rangle = 0$, by Cauchy-Schwarz inequality and (2.4)

$$\begin{aligned} &\frac{1}{2}\|u(t)\|^2_{L_2(\mathbb{T})} - \frac{1}{2}\|u_0\|^2_{L_2(\mathbb{T})} + \frac{\varepsilon}{2}\|u\|^2_{\mathcal{H}} \\ &= \int_0^t \langle -(\partial_{xx})^{s/2}\Phi(r), (-\partial_{xx})^{s/2} u(r)\rangle\, dr \le \|u\|_{\mathcal{H}}\, \|\Phi\|_{\mathcal{H}} \\ &\le \frac{\varepsilon}{4}\|u\|^2_{\mathcal{H}} + 2I_\varepsilon(u) \end{aligned}$$

Passing to the supremum in $t$ one gets (4.1), (4.2). (4.3) is then obtained by (2.3) and (4.1), (4.2). (4.4) follows from the fact that $H^{1/2}([0,T]; L_2(\mathbb{T}))$ is the Hilbert

interpolation of parameter $1/2$ between $H^1([0,T];H^{-s}(\mathbb{T}))$ and $L_2([0,T];H^s(\mathbb{T}))$, while (4.1)-(4.3) grant the bounds in these latter spaces. □

*Proof of Proposition 2.2: lower semicontinuity.* Let $v^n$ be a sequence converging to $v$ in $\mathcal{X}$, such that $I_\varepsilon(v^n)$ is bounded uniformly in $n$. We need to show $\underline{\lim}_n I_\varepsilon(v^n) \geq I_\varepsilon(v)$. By the uniform bound on $I_\varepsilon(v^n)$, (4.1) and the lower semicontinuity of the Hilbert norm, we have that $v \in L_2([0,T];H^s(\mathbb{T}))$. Still by Lemma 4.1, (4.2)-(4.3) and the embedding of $H^{1/2}([0,T];L_2(\mathbb{T}))$ in $C([0,T];L_2(\mathbb{T}))$, we obtain that $v \in C([0,T];L_2(\mathbb{T}))$.

Note that if $I_\varepsilon(u) < +\infty$, for $\Phi$ as in Remark 2.1

$$\begin{aligned} I_\varepsilon(u) =&\varepsilon^{-1}\sup_{\phi\in\mathcal{H}}\langle -\partial_{xx}\Phi,\phi\rangle_{\mathcal{H}^*,\mathcal{H}} - \frac{1}{2}\|\phi\|_{\mathcal{H}}^2 \\ =&\varepsilon^{-1}\sup_{\varphi\in C^\infty([0,T]\times\mathbb{T})} \langle u(T),\varphi(T)\rangle - \langle u_0,\varphi(0)\rangle \\ &- \int_0^T \big[\langle u(r),\partial_t\varphi(r)\rangle + \langle f(u(r)),\partial_x\varphi(r)\rangle\big]\,dr \\ &+ \frac{\varepsilon}{2}\int_0^T \langle u(r),(-\partial_{xx})^s\varphi(r)\rangle\,dr - \frac{1}{2}\int_0^T \big\|(-\partial_{xx})^{s/2}\varphi(r)\big\|_{L_2(\mathbb{T})}^2\,dr \end{aligned} \tag{4.6}$$

while the latter sup in the above formula equals $+\infty$ if $u \in C([0,T];L_2(\mathbb{T}))\cap\mathcal{H}$ but $\partial_t u + \partial_x f + \frac{\varepsilon}{2}(-\partial_{xx})^s u \notin \mathcal{H}^*$. Thus (4.6) represents the restriction of $I_\varepsilon$ to $C([0,T];L_2(\mathbb{T}))\cap\mathcal{H}$ as a supremum of continuous functions on $\mathcal{X}$. Since, as remarked at the beginning of the proof, one can indeed restrict to the case $v_n$, $v \in C([0,T];L_2(\mathbb{T}))\cap\mathcal{H}$, the lower semicontinuity follows. □

**Lemma 4.2.** *If $I_\varepsilon(u) < +\infty$, there exists a constant $C > 0$ depending only on $f$, $T$, and $s$ such that for all $r$, $t \in [0,T]$ and $\varphi \in H^1(\mathbb{T})$*

$$\big|\langle u(t)-u(r),\varphi\rangle\big| \leq C(1+I_\varepsilon(u))|t-r|^{1/2}\|\varphi\|_{\dot{H}^1(\mathbb{T})}$$

*Proof.* For $\varphi \in C^\infty(\mathbb{T})$, by (2.3), the Cauchy-Schwarz inequality and Sobolev embedding

$$\begin{aligned} \big|\langle u(t)-u(r),\varphi\rangle\big| =& \Big| \int_r^t \langle f(u(r')),\partial_x\varphi\rangle\,dr' + \frac{\varepsilon}{2}\int_r^t \langle(-\partial_{xx})^{s/2}u(r'),(-\partial_{xx})^{s/2}\varphi\rangle\,dr' \\ &- \int_r^t \langle(-\partial_{xx})^{s/2}\Phi(r'),(-\partial_{xx})^{s/2}\varphi\rangle\Big|\,dr' \\ \leq& |t-s|\big[\sup_{w\in\mathbb{R}}|f(w)|\big]\|\partial_x\varphi\|_{L_2(\mathbb{T})} + |t-s|^{1/2}\frac{\varepsilon}{2}\|u\|_{\mathcal{H}}\|\partial_x\varphi\|_{L_2(\mathbb{T})} \\ &+ |t-s|^{1/2}\|\Phi\|_{\mathcal{H}}\|\partial_x\varphi\|_{L_2(\mathbb{T})} \end{aligned}$$

The proof is concluded using (2.4) and (4.1). □

*Proof of Proposition 2.2: coercivity.* We want to prove that if a sequence $(v^n)$ is such that $I_\varepsilon(v^n)$ is uniformly bounded in $n$, then $(v^n)$ is precompact in $\mathcal{X}$. By Lemma 4.2, $(v^n)$ is precompact in $C([0,T];H^{-1}(\mathbb{T}))$. Let $v$ be any limit point of $(v^n)$. Up to passing to a subsequence, we can assume $v^n \to v$ in $C([0,T];H^{-1}(\mathbb{T}))$. By (4.2) $v \in L_2([0,T]\times\mathbb{T})$, and it is enough to prove that $v^n$ converges to $v$ strongly in $L_2$ to conclude.

By (4.1), $v^n$ stays bounded in $\mathcal{H}$ and thus it converges to $v$ weakly in $\mathcal{H}$. Let $\jmath$ be a smooth convolution kernel on $\mathbb{T}$, and let $*$ denote convolution in space. Then

$$\begin{aligned}\left\|v^n - v\right\|_{L_2([0,T]\times\mathbb{T})} \le{}& \left\|\jmath * v^n - \jmath * v\right\|_{L_2([0,T]\times\mathbb{T})} \\ &+ \left\|\jmath * v - v\right\|_{L_2([0,T]\times\mathbb{T})} + \left\|v^n - \jmath * v^n\right\|_{L_2([0,T]\times\mathbb{T})}\end{aligned} \tag{4.7}$$

The first term in the right hand side vanishes as $n \to +\infty$ by the convergence of $v^n$ in $C([0,T];H^{-1}(\mathbb{T}))$. The second term vanishes if we let $\jmath$ converge to the Dirac mass at 0. As for the third term, by Sobolev embedding, there exists $\imath \in \dot{H}^{-a}(\mathbb{T})$ for $a > 1/2$, such that

$$\partial_x \imath = \delta_0 - \jmath$$

in the distribution sense.Then

$$\begin{aligned}\left\|v^n - \jmath * v^n\right\|_{L_2([0,T]\times\mathbb{T})} &= \left\|(-\partial_{xx})^{1/2}\imath * v^n\right\|_{L_2([0,T]\times\mathbb{T})} \\ &= \left\|[(-\partial_{xx})^{\frac{1-s}{2}}\imath] * [(-\partial_{xx})^{s/2}v^n]\right\|_{L_2([0,T]\times\mathbb{T})} \\ &\le \sqrt{T}\left\|(-\partial_{xx})^{\frac{1-s}{2}}\imath\right\|_{L_1(\mathbb{T})}\left\|(-\partial_{xx})^{s/2}v^n\right\|_{L_2([0,T]\times\mathbb{R})} \\ &\le \sqrt{T}\,\|\imath\|_{\dot{H}^{1-s}}\|v^n\|_{\mathcal{H}}\end{aligned}$$

where in the third line we used Young inequality. By (4.1), $\|v^n\|_{\mathcal{H}}$ is bounded uniformly in $n$, while $\|\imath\|_{\dot{H}^{1-s}}$ vanishes as we let $\jmath$ converge to $\delta_0$, since $1-s<1/2$. Therefore all of the terms in the right hand side of (4.7) vanish, as we let $n \to +\infty$ first, and $\jmath \to \delta_0$ next. □

## 5. Equicoercivity and the Γ-liminf inequality

In this section we prove Theorem 2.5 and Corollary 2.6.

*Proof of Theorem 2.5-(i).* Let $u_\varepsilon$ be a sequence converging to $u$ in $\mathcal{X}$. We want to prove that $\underline{\lim}_\varepsilon I_\varepsilon(u_\varepsilon) \ge I(u)$. With no loss of generality, we assume that $I_\varepsilon(u_\varepsilon)$ is uniformly bounded. Thus, by the convergence in $C([0,T];H^{-1}(\mathbb{T}))$, one has $u(t=0) = \lim_\varepsilon u_\varepsilon(0) = u_0$.

Let $\vartheta \in C^2_{\mathrm{c}}(\mathbb{R}\times]0,T[\times\mathbb{T})$. Let $Q \in C^1_{\mathrm{c}}(\mathbb{R}\times]0,T[\times\mathbb{T})$ be defined (up to an additive function of $(t,x)$) by

$$Q'(v,t,x) = \int^v f'(w)\,\vartheta'(w,t,x)\,dw$$

where $\vartheta'$, $Q'$ denote derivatives with respect to the first variable. We will also denote $\vartheta_t$, $Q_t$ and $\vartheta_x$, $Q_x$ the partial derivatives with respect to the second and third arguments of $\vartheta$ and $Q$ respectively. By (4.6), for all smooth $\varphi \in C^\infty_c(]0,T[\times\mathbb{T})$ and $\tilde\varphi \in C^\infty_c(]0,T[\times\mathbb{T}\times\mathbb{R})$ such that $\tilde\varphi(t,x,0)=\varphi(t,x)$

$$\begin{aligned}
I_\varepsilon(u_\varepsilon) \geq & -\varepsilon^{-1}\int_0^T \left[\langle u_\varepsilon(r),\partial_t\varphi(r)\rangle + \langle f(u_\varepsilon(r)),\partial_x\varphi(r)\rangle\right] dr \\
& + \frac{1}{2}\int_0^T \langle (-\partial_{xx})^{s/2}u_\varepsilon(r), (-\partial_{xx})^{s/2}\varphi(r)\rangle\, dr \\
& - \frac{\varepsilon^{-1}}{2}\int_0^T \left\|(-\partial_{xx})^{s/2}\varphi(r)\right\|^2_{L_2(\mathbb{T})} dr \\
\geq & -\varepsilon^{-1}\int_0^T \left[\langle u_\varepsilon(r),\partial_t\varphi(r)\rangle + \langle f(u_\varepsilon(r)),\partial_x\varphi(r)\rangle\right] dr \\
& + \frac{1}{2}\int_0^T\int y^{1-2s}\nabla\bar u_\varepsilon(r,x,y)\,\cdot\,\nabla\tilde\varphi(r,x,y)\,dx\,dy\,dr \\
& - \frac{\varepsilon^{-1}}{2}\int_0^T\int y^{1-2s}\nabla\tilde\varphi(r,x,y)\,\cdot\,\nabla\tilde\varphi(r,x,y)\,dx\,dy\,dr
\end{aligned}$$

where the last inequality follows from (3.3).

Let $\chi \in C^\infty_c(\mathbb{R}^+;[0,1])$, such that $\chi(0)=1$. By the same density argument used in the proof of Lemma 4.1, one can indeed plug $\varphi(t,x)=\varepsilon\vartheta'(u_\varepsilon(t,x),t,x)$ as a test function above. The key point is now the choice $\tilde\varphi(t,x,y)=\varepsilon\vartheta'(\bar u_\varepsilon(t,x,y),t,x)\chi(y)$, which is indeed an extension of $\varphi$, though not the $s$-harmonic one. Again reasoning as in Lemma 4.1, integrations by parts are allowed, so that

$$\begin{aligned}
I_\varepsilon(u_\varepsilon) \geq & -\int_0^T\int_{\mathbb{T}} \vartheta_t(u_\varepsilon(r,z),r,z) + Q_x(u_\varepsilon(r,z),r,z)\,dz\,dr \\
& + \frac{\varepsilon}{2}\int_0^T\int y^{1-2s}\chi(y)\vartheta''(\bar u_\varepsilon(r,z,y),r,z)\nabla\bar u_\varepsilon(r,x,y)\,\cdot\,\nabla\bar u_\varepsilon(r,x,y)\,dx\,dy\,dr \\
& + \frac{\varepsilon}{2}\int_0^T\int y^{1-2s}\begin{pmatrix}\vartheta'_x(\bar u_\varepsilon(r,z,y),r,z)\chi(y)\\ \vartheta'(\bar u_\varepsilon(r,z,y),r,z)\chi'(y)\end{pmatrix}\cdot\nabla\bar u_\varepsilon(r,x,y)\,dx\,dy\,dr \\
& - \frac{\varepsilon}{2}\int_0^T\int y^{1-2s}\chi(y)^2\vartheta''(\bar u_\varepsilon(r,z,y),r,z)^2\nabla\bar u_\varepsilon(r,x,y)\,\cdot\,\nabla\bar u_\varepsilon(r,x,y)\,dx\,dy\,dr \\
& - \frac{\varepsilon}{2}\int_0^T\int y^{1-2s}\vartheta''(\bar u_\varepsilon(r,z,y),r,z)\chi(y))\begin{pmatrix}\vartheta'_x(\bar u_\varepsilon(r,z,y),r,z)\chi(y)\\ \vartheta'(\bar u_\varepsilon(r,z,y),r,z)\chi'(y)\end{pmatrix}\cdot\nabla\bar u_\varepsilon(r,x,y)\,dx\,dy\,dr \\
& - \varepsilon\int_0^T\int y^{1-2s}\begin{pmatrix}\vartheta'_x(\bar u_\varepsilon(r,z,y),r,z)\chi(y)\\ \vartheta'(\bar u_\varepsilon(r,z,y),r,z)\chi'(y)\end{pmatrix}\cdot\begin{pmatrix}\vartheta'_x(\bar u_\varepsilon(r,z,y),r,z)\chi(y)\\ \vartheta'(\bar u_\varepsilon(r,z,y),r,z)\chi'(y)\end{pmatrix}dx\,dy\,dr
\end{aligned}$$

The main idea now is that the third, fifth and sixth lines vanish as $\varepsilon\to 0$, while the second and forth line compensate (each being order 1) for a suitable class of $\vartheta$.

Indeed, since $\vartheta$ and $\chi$ have bounded derivatives up to the second order, and $y^{1-2s}$ is integrable at 0, the Cauchy-Schwarz inequality yields for the third, fifth and sixth lines

$$\begin{aligned} I_\varepsilon(u_\varepsilon) \geq & -\int_0^T \int_{\mathbb{T}} \vartheta_t(u_\varepsilon(r,z),r,z) + Q_x(u_\varepsilon(r,z),r,z)\,dz\,dr \\ & + \frac{\varepsilon}{2}\int_0^T \int y^{1-2s}\big[\vartheta''(\bar u_\varepsilon(r,z,y),r,z)\chi(y) - \vartheta''(\bar u_\varepsilon(r,z,y),r,z)^2\chi(y)^2\big] \\ & \qquad \nabla \bar u_\varepsilon(r,x,y)\cdot\nabla\bar u_\varepsilon(r,x,y)\,dx\,dy\,dr \\ & + \varepsilon\, C\left[1+\|u_\varepsilon\|_{\mathcal{H}}\right] \end{aligned}$$

for a constant $C$ depending only on $\vartheta$, $\chi$, $T$ and $s$. By the bound (4.1), recalling that $I_\varepsilon(u_\varepsilon)$ is uniformly bounded, the last term vanishes as $\varepsilon \to 0$. Note moreover that if $0 \leq \vartheta'' \leq 1$ the second line in the last formula is positive. Recalling that $u_\varepsilon \to u$ strongly in $L_2([0,T]\times\mathbb{T})$, passing to the limit $\varepsilon \to 0$ and optimizing over smooth $\vartheta$ for which the inequality holds one thus obtains

$$\varliminf_{\varepsilon\downarrow 0} I_\varepsilon(u_\varepsilon) \geq \sup_{\vartheta\,:\,0\leq\vartheta''\leq 1} -\int_0^T\int_{\mathbb{T}} \vartheta_t(u(r,z),r,z) + Q_x(u(r,z),r,z)\,dz\,dr$$

In [1, Formula (5.1)–(5.2)] it is proved that the supremum in the right hand side of the above formula equals $I(u)$, provided $u(0)=u_0$ (which we already know). □

**Lemma 5.1.** *Let $(u_\varepsilon)$ be a sequence in $\mathcal{X}$ such that $I_\varepsilon(u_\varepsilon)$ is bounded uniformly in $\varepsilon$. Let $(\eta,q)$ be an entropy-entropy flux pair, with $\eta$ bounded with bounded first and second derivatives. Recall that $\wp_{\eta,u_\varepsilon}$ is the distribution $\wp_{\eta,u_\varepsilon} := \partial_t\eta(u_\varepsilon) + \partial_x q(u_\varepsilon)$. Then $(\wp_{\eta,u_\varepsilon})$ is strongly compact in $H^{-1}([0,T]\times\mathbb{T})$.*

*Proof.* Let $\varphi \in C^\infty([0,T]\times\mathbb{T})$. Reasoning as in Lemma 4.1, one is allowed to test equation (2.3) against the test function $(t,x)\mapsto \eta'(u_\varepsilon(t,x))\varphi(t,x)$. Still by the same argument as in Lemma 4.1, integrations by parts are allowed so that (in the following, we denote $\Phi_\varepsilon \equiv \Phi_{u_\varepsilon}$).

$$\begin{aligned} \wp_{\eta,u_\varepsilon}(\varphi) &= \langle \eta(u_\varepsilon(T)),\varphi(T)\rangle - \langle\eta(u_\varepsilon(0)),\varphi(0)\rangle \\ &\quad - \int_0^T \big[\langle \eta(u_\varepsilon(t)),\partial_t\varphi(t)\rangle + \langle q(u_\varepsilon(t)),\partial_x\varphi(t)\rangle\big]\,dt \\ &= -\frac{\varepsilon}{2}\int_0^T \langle \eta'(u_\varepsilon(t))\varphi(t),(-\partial_{xx})^s u_\varepsilon(t)\rangle\,dt \\ &\quad + \int_0^T \langle \eta'(u_\varepsilon(t))\varphi(t),(-\partial_{xx})^s\Phi_\varepsilon(t)\rangle\,dt \end{aligned}$$

Recall that for $v\in H^s(\mathbb{T})$, we denoted $\bar v$ its $s$-harmonic extension of $v$ to $\mathbb{T}\times\mathbb{R}^+$. Note that $\eta'(\bar u_\varepsilon)\bar\varphi$ provides an extension (thought not $s$-harmonic) of $\eta'(u_\varepsilon)\varphi$. Therefore,

by Remark 3.2 applied with $\tilde{\varphi} = \eta'(\bar{u}_\varepsilon)\bar{\varphi}$, we get

$$\begin{aligned}\wp_{\eta,u_\varepsilon}(\varphi) = &-\frac{\varepsilon}{2}\int_0^T \int_{\mathbb{T}\times\mathbb{R}^+} y^{1-2s}\eta''(\bar{u}_\varepsilon(t,x,y))\,\bar{\varphi}(t,x,y)\nabla u_\varepsilon(t,x,y)\,\cdot\,\nabla u_\varepsilon(t,x,y)\,dx\,dy\,dt\\ &-\frac{\varepsilon}{2}\int_0^T \int_{\mathbb{T}\times\mathbb{R}^+} y^{1-2s}\eta'(\bar{u}_\varepsilon(t,x,y))\,\nabla\bar{\varphi}(t,x,y)\,\cdot\,\nabla u_\varepsilon(t,x,y)\,dx\,dy\,dt\\ &+\int_0^T \int_{\mathbb{T}\times\mathbb{R}^+} y^{1-2s}\eta''(\bar{u}_\varepsilon(t,x,y))\,\bar{\varphi}(t,x,y)\nabla u_\varepsilon(t,x,y)\,\cdot\,\nabla\Phi_\varepsilon(t,x,y)\,dx\,dy\,dt\\ &+\int_0^T \int_{\mathbb{T}\times\mathbb{R}^+} y^{1-2s}\eta'(\bar{u}_\varepsilon(t,x,y))\,\nabla\bar{\varphi}(t,x,y)\,\cdot\,\nabla\Phi_\varepsilon(t,x,y)\,dx\,dy\,dt\end{aligned}$$

Since $\eta$ has bounded derivatives, recalling the variational definition of the $H^s$-norm (3.3), the Cauchy-Schwarz inequality yields for a constant $C$ depending only on $\eta$ and $T$

$$\begin{aligned}\left|\wp_{\eta,u_\varepsilon}(\varphi)\right| \leq &\varepsilon\, C\|\bar{\varphi}\|_\infty\,\|u_\varepsilon\|_{\mathcal{H}}^2 + \varepsilon\, C\,\|u_\varepsilon\|_{\mathcal{H}}\|\varphi\|_{\mathcal{H}}\\ &+ C\,\|\bar{\varphi}\|_\infty\|u_\varepsilon\|_{\mathcal{H}}\|\Phi\|_{\mathcal{H}} + C\,\|\varphi\|_{\mathcal{H}}\|\Phi\|_{\mathcal{H}}\end{aligned}$$

The maximum principle holds for the $s$-harmonic extension, $\|\bar{\varphi}\|_\infty = \|\varphi\|_\infty$, so that using (4.1), (2.4)

$$\left|\wp_{\eta,u_\varepsilon}(\varphi)\right| \leq C'\big(1 + I_\varepsilon(u_\varepsilon)\big)\big(\|\varphi\|_\infty + \sqrt{\varepsilon}\|\varphi\|_{\mathcal{H}}\big)$$

for a suitable constant $C'$ independent of $\varepsilon$. The last inequality implies that $\wp_{\eta,u_\varepsilon}$ can be written as the sum of two distributions $\wp_{\eta,u_\varepsilon} = \wp_\varepsilon^1 + \wp_\varepsilon^2$, where $\wp_\varepsilon^1$ is a finite measure on $[0,T]\times\mathbb{T}$ with total variation bounded uniformly in $\varepsilon$, while $\wp_\varepsilon^2$ has vanishing $\mathcal{H}^*$-norm. By Sobolev compact embedding, both $\wp_\varepsilon^1$ and $\wp_\varepsilon^2$ are compact in $H^{-1}([0,T]\times\mathbb{T})$, and thus $\wp_{\eta,u_\varepsilon}$ is. □

Before proving Theorem 2.5-(ii) we recall some standard facts concerning Young measures. Let $(u^\varepsilon)$ be a sequence in $\mathcal{X}$ such that $\|u_\varepsilon\|_{L_2([0,T]\times\mathbb{T})}$ is uniformly bounded. Then the sequence of Radon measures $\delta_{u_\varepsilon(t,x)}(d\lambda)dt\,dx$ over $\mathbb{R}\times[0,T]\times\mathbb{T}$ is compact in the weak* topology of Radon measures, and any limit point can be represented by a Young measure, namely a measurable map $[0,T]\times\mathbb{T}\ni(t,x)\mapsto\mu_{t,x}\in\mathcal{P}(\mathbb{R})$ such that, up to passing to subsequences

$$\lim_{\varepsilon\downarrow 0}\int_{[0,T]\times\mathbb{T}} F(u_\varepsilon(t,x))\varphi(t,x)\,dt\,dx = \int_{\mathbb{R}\times[0,T]\times\mathbb{T}} F(\lambda)\varphi(t,x)\,\mu_{t,x}(d\lambda)\;dt\,dx \tag{5.1}$$

for all continuous compactly supported test functions $F$ and $\varphi$. Moreover, $u_\varepsilon$ converges strongly in $L_r([0,T]\times\mathbb{T})$ for $r<2$ to a $u$ iff $\mu_{t,x} = \delta_{u(t,x)}$.

*Proof of Theorem 2.5-(ii).* Let $(u_\varepsilon)$ be a sequence in $\mathcal{X}$ such that $I_\varepsilon(u_\varepsilon)$ is bounded uniformly in $\varepsilon$. Lemma 4.2 and Ascoli-Arzela theorem imply that $u_\varepsilon$ is compact in $C([0,T];H^{-1}(\mathbb{T}))$. Therefore we need to prove that $u_\varepsilon$ is strongly compact in $L_p([0,T]\times\mathbb{T})$ to conclude the proof.

In view of the uniform (in $\varepsilon$) bound (4.2) for $u_\varepsilon$, there exists a Young measure $\mu$ such that (5.1) holds (up to passing to a suitable subsequence still labeled by $\varepsilon$). We need to prove that there exists $u \in L_2([0,T]\times\mathbb{T})$ such that $\mu_{t,x} = \delta_{u(t,x)}$ for a.e. $(t,x)$. To achieve this, we will follow a celebrated argument by Tartar, [12, Chap. 9]. However, since we here lack the usual $L_\infty$ bounds used in this approach, we shortly reproduce the argument adapted to our setting.

Following closely [12, Chap. 9], thanks to Lemma 5.1 one has for a.e. $(t,x)$

$$\int_{\mathbb{R}^2} [\eta_1(\xi)-\eta_1(\zeta)][q_2(\xi)-q_2(\zeta)] - [\eta_2(\xi)-\eta_2(\zeta)][q_1(\xi)-q_1(\zeta)]\mu_{(t,x)}(d\xi)\mu_{(t,x)}(d\lambda) = 0 \tag{5.2}$$

for $\eta_1$, $\eta_2$ two smooth bounded entropies with bounded derivatives, and $q_1$, $q_2$ their respective conjugated entropy fluxes. By a density argument, (5.2) is easily seen to hold for $\eta_1$, $\eta_2$ Lipschitz and uniformly bounded. Fix $M>0$ and take

$$\eta_1(v) = \begin{cases} v & \text{if } v\in[-M,M] \\ -M & \text{if } v<-M \\ M & \text{if } v>M \end{cases}$$

$$\eta_2(v) = \begin{cases} f(v) & \text{if } v\in[-M,M] \\ f(-M) & \text{if } v<-M \\ f(M) & \text{if } v>M \end{cases}$$

Thus (5.2) now reads

$$\int_{\mathbb{R}^2} \Big[ \Big( \int_\xi^\zeta f'(a)\mathbb{1}_{[-M,M]}(a)\,da \Big)^2 - \Big( \int_\xi^\zeta \mathbb{1}_{[-M,M]}(a)\,da \Big)\Big( \int_\xi^\zeta f'(a)^2\mathbb{1}_{[-M,M]}(a)\,da \Big) \Big] \mu_{(t,x)}(d\xi)\mu_{(t,x)}(d\lambda) = 0$$

By Cauchy-Schwarz inequality, the integrand in square brackets is always negative, vanishing iff $f'(a)\mathbb{1}_{[-M,M]}(a)$ is constant between $\xi$ and $\zeta$. Since we assumed that there is no interval in which $f$ is affine, this implies that the restriction of $\mu_{(t,x)}$ to $[-M,M]$ is a Dirac mass for a.e. $(t,x)$. Since $M$ is arbitrary, we conclude. □

*Proof of Corollary 2.6.* If we let $\Phi_\varepsilon$ be the solution to

$$(-\partial_{xx})^s \Phi_\varepsilon = (-\partial_{xx})^{s/2} E_\varepsilon$$

then

$$I_\varepsilon(u_\varepsilon) = \frac{\varepsilon^{-1}}{2}\|\Phi_\varepsilon\|^2_{\mathcal{H}} \le \frac{\varepsilon^{-1}}{2}\|E_\varepsilon\|^2_{L_2([0,T]\times\mathbb{T})}$$

Therefore $I_\varepsilon(u_\varepsilon)\to 0$, and by Theorem 2.5-(ii), up to passing to subsequences, $u_\varepsilon \to u$ in $\mathcal{X}$ for a suitable $u\in\mathcal{X}$. By Theorem 2.5-(i), $I(u) \le \underline{\lim}_\varepsilon I_\varepsilon(u_\varepsilon) = 0$. Thus $I(u)=0$ and by Proposition 2.4, $u$ is the (unique) entropic solution to (1.1). □

## 6. Quasipotential

In this section we prove Theorem 2.7. As in Section 2.5, here we append a subscript $T$ to the notation, to stress the dependence on $T$.

If $\int_{\mathbb{T}} w(x)\,dx \neq m$, then Theorem 2.7 follows from the conservation of the total mass of $L_2$-solutions to (2.3). Namely, if $u \in \mathcal{X}_T$ is such that $I_{\varepsilon;T}(u) < +\infty$, then $\int_{\mathbb{T}} u(t,x)\,dx = \int_{\mathbb{T}} u(0,x)\,dx$ for all $t \geq 0$, and thus the infimum in (2.8) equals $+\infty$.

So hereafter in this section we assume $\int_{\mathbb{T}} w(x)\,dx = m$. Then the proof of Theorem 2.7 is a consequence of the following Lemmata. In fact from Lemma 6.1 one gets $V_\varepsilon(m,w) \geq \frac{1}{2}\|w-m\|^2_{L_2(\mathbb{T})}$, and from Lemma 6.1 and Lemma 6.2 one has $V_\varepsilon(m,w) \leq \frac{1}{2}\|w-m\|^2_{L_2(\mathbb{T})} + \gamma$ for each $\gamma > 0$.

**Lemma 6.1.** *Let $T > 0$ and $u \in \mathcal{X}_T$ be such that $I_{\varepsilon;T}(u) < +\infty$, $u(0,x) \equiv m$, $u(T,x) = w(x)$. Then*

$$I_{\varepsilon;T}(u) = \frac{1}{2}\|w-m\|^2_{L_2(\mathbb{T})} + \frac{\varepsilon^{-1}}{2}\Big\|\partial_t u + \partial_x f(u) - \frac{\varepsilon}{2}(-\partial_{xx})^s u\Big\|^2_{\mathcal{H}^*_T}$$

**Lemma 6.2.** *For each $\gamma > 0$, there exists $T > 0$ and $u \in \mathcal{X}_T$ such that $I_{\varepsilon;T}(u) < +\infty$, $u(0) \equiv m$, $u(T) = w$ and*

$$\frac{\varepsilon^{-1}}{2}\Big\|\partial_t u + \partial_x f(u) - \frac{\varepsilon}{2}(-\partial_{xx})^s u\Big\|^2_{\mathcal{H}^*_T} \leq \gamma \tag{6.1}$$

*Proof of Lemma 6.1.* Since $I_{\varepsilon;T}(u) < +\infty$, as observed in the proof of Remark 2.1, $u \in \mathcal{H}_T$, $\partial_t u \in \mathcal{H}^*_T$, $\Phi_u \in \mathcal{H}_T$, and $\partial_x f(u) \in L_2([0,T];\dot{H}^{s-1}(\mathbb{T})) \subset \mathcal{H}^*$ as $s > 1/2$. Therefore, by decomposition of Hilbert scalar products

$$\begin{aligned} I_{\varepsilon;T}(u) =& \frac{\varepsilon^{-1}}{2}\Big\|\partial_t u + \partial_x f(u) + \frac{\varepsilon}{2}(-\partial_{xx})^s u\Big\|^2_{\mathcal{H}^*_T} \\ =& \frac{\varepsilon^{-1}}{2}\Big\|\partial_t u + \partial_x f(u) - \frac{\varepsilon}{2}(-\partial_{xx})^s u\Big\|^2_{\mathcal{H}^*_T} \\ &+ \Big(\partial_t u, (-\partial_{xx})^s u\Big)_{\mathcal{H}^*_T} + \Big(\partial_x f(u), (-\partial_{xx})^s u\Big)_{\mathcal{H}^*_T} \end{aligned} \tag{6.2}$$

Now note that, by the same arguments as in Lemma 4.1, integration by parts are allowed and

$$\begin{aligned} \Big(\partial_x f(u), (-\partial_{xx})^s u\Big)_{\mathcal{H}^*_T} =& \Big(\partial_x f(u), u-m\Big)_{L_2([0,T]\times\mathbb{T})} \\ =& \int_0^T \int_{\mathbb{T}} \partial_x q(u(t,x))\,dx\,dt = 0 \end{aligned} \tag{6.3}$$

where $q \in C^1(\mathbb{R})$ is such that $q'(v) = (v-m)f'(v)$. On the other hand

$$\begin{aligned}\Big(\partial_t u, (-\partial_{xx})^s u\Big)_{\mathcal{H}^*_T} =& \Big(\partial_t u, u-m\Big)_{L_2([0,T]\times\mathbb{T})} \\ =& \int_0^T \int_{\mathbb{T}} (u(t,x)-m)\partial_t u(t,x)\,dx\,dt \\ =& \frac{1}{2}\int_0^T \int_{\mathbb{T}} \partial_t (u(t,x)-m)^2\,dx\,dt = \frac{1}{2}\|w-m\|^2_{L_2(\mathbb{T})}\end{aligned} \tag{6.4}$$

Patching (6.2), (6.3), (6.4) together, the result follows. □

*Proof of Lemma 6.2.* Let $v : [0,\infty[\times\mathbb{T} \to \mathbb{R}$ be the solution to (1.2) with initial datum $v(0,x) = w(-x)$, and for $T_1$, $T_2 > 0$ let $u \in \mathcal{X}_{T_1+T_2}$ be defined as

$$u(t,x) = \begin{cases} (1-\frac{t}{T_1})m + \frac{t}{T_1}v(T_2,-x) & \text{for } t \in [0,T_1] \\ v(T_1+T_2-t,-x) & \text{for } t \in [T_1, T_1+T_2] \end{cases}$$

Note that $u(0,x) = m$ and $u(T,x) = w(x)$ for $T = T_1 + T_2$, so that we are left with the proof of (6.1).

Since $u$ satisfies $\partial_t u + \partial_x f(u) - \frac{\varepsilon}{2}(-\partial_{xx})^s u = 0$ for $t \in [T_1, T_1+T_2]$, while calculations are explicit for $t \in [0,T_1]$

$$\begin{aligned}&\frac{\varepsilon^{-1}}{2}\big\|\partial_t u + \partial_x f(u) - \frac{\varepsilon}{2}(-\partial_{xx})^s u\big\|^2_{\mathcal{H}^*_{T_1+T_2}} \\ &\quad= \frac{\varepsilon^{-1}}{2}\big\|\partial_t u + \partial_x f(u) - \frac{\varepsilon}{2}(-\partial_{xx})^s u\big\|^2_{\mathcal{H}^*_{T_1}} \\ &\quad\le \frac{3\varepsilon^{-1}}{2}\big[\big\|\partial_t u\big\|^2_{\mathcal{H}^*_{T_1}} + \big\|\partial_x f(u)\big\|^2_{\mathcal{H}^*_{T_1}} + \big\|\frac{\varepsilon}{2}(-\partial_{xx})^s u\big\|^2_{\mathcal{H}^*_{T_1}}\big] \\ &\quad\le \frac{3\varepsilon^{-1}}{2}\big\|v(T_2)-m\big\|^2_{\dot H^{-s}(\mathbb{T})} + \frac{3\varepsilon^{-1}}{2}\big\|f'(u)\partial_x u\big\|^2_{\mathcal{H}^*_{T_1}} + \frac{3\varepsilon\,T_1}{16}\big\|v(T_2)\big\|^2_{\dot H^s(\mathbb{T})}\end{aligned} \tag{6.5}$$

Now note that if $\omega_1$, $\omega_2 \in \dot H^{-s}(\mathbb{T})$ are such that

$$\int_{\mathbb{T}} \omega_1(x)\,dx = 0 \qquad \int_{\mathbb{T}} \omega_1(x)\,\omega_2(x)dx = 0$$

then

$$\|\omega_1\|^2_{\dot H^{-s}(\mathbb{T})} \le \|\omega_1\|^2_{\dot H^s(\mathbb{T})} \tag{6.6}$$

$$\|\omega_1\,\omega_2\|^2_{\dot H^{-s}(\mathbb{T})} \le \|\omega_1\|^2_{\dot H^{-s}(\mathbb{T})}\,\|\omega_2\|^2_{L_\infty(\mathbb{T})} \tag{6.7}$$

Applying (6.6) to the first term of the last line of (6.5); applying (6.7) integrated over $t \in [0,T_1]$ to the second term in the last line of (6.5) with $\omega_1 = \partial_x u(t)$ and

$\omega_2 = f'(u(t))$, one gets

$$\begin{aligned}\frac{\varepsilon^{-1}}{2}\big\|\partial_t u + \partial_x f(u) - \frac{\varepsilon}{2}(-\partial_{xx})^s u\big\|^2_{\mathcal{H}^*_{T_1+T_2}} \\ \leq C\left(\|\partial_x v(T_2)\|^2_{\dot{H}^{-s}(\mathbb{T})} + \|v(T_2)\|^2_{\dot{H}^s(\mathbb{T})}\right) \leq 2\,C\,\|v(T_2)\|^2_{\dot{H}^s(\mathbb{T})}\end{aligned} \tag{6.8}$$

where $C$ is a constant depending only on $\varepsilon$, $T_1$, $f$, and we used $\|\partial_x v(T_2)\|_{\dot{H}^{-s}} \leq \|v(T_2)\|_{\dot{H}^s}$ as $s > 1/2$. Now note that by a standard parabolic estimate (indeed by (4.1) calculated for $I_{\varepsilon,T}(u) = 0$)

$$\int_0^{+\infty} \|v(t)\|^2_{\dot{H}^s(\mathbb{T})} \leq \|w\|^2_{L_2(\mathbb{T})} < +\infty$$

Therefore for each $\gamma > 0$, there exists $T_2$ large enough such that the rightest hand side of (6.8) is smaller than $\gamma$. □

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