# Bayesian 3D X-ray Computed Tomography image reconstruction with a Scaled Gaussian Mixture prior model

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## Abstract

In order to improve quality of 3D X-ray tomography reconstruction for Non Destructive Testing (NDT), we investigate in this paper hierarchical Bayesian methods. In NDT, useful prior information on the volume like the limited number of materials or the presence of homogeneous area can be included in the iterative reconstruction algorithms. In hierarchical Bayesian methods, not only the volume is estimated thanks to the prior model of the volume but also the hyper parameters of this prior. This additional complexity in the reconstruction methods when applied to great volumes (form $512^3$ to $8192^3$ voxels) involves an increasing computational cost. To reduce it, the hierarchical Bayesian methods investigated in this paper result to an algorithm acceleration thanks to the Variational Bayesian Approximation (VBA) and hardware acceleration thanks to projection and back-projection operators parrallelized on many core processors like GPU. In this paper, we will consider a Student-t prior on the gradient of the image implemented in a hierarchical way. Operators $H$ (forward or projection) and $H^t$ (adjoint or back-projection) implemented in multi-GPU have been used in this study.

## Computed Tomography

X-ray computed tomography (X-ray CT) is a technology that uses computer-processed X-ray to produce tomographic images of specific areas of the scanned object, allowing the user to see what is inside it without cutting it open. When considering a practical problem, we discretize $f(x,y)$ in pixels and put all the pixels in a vector $\boldsymbol{f}$ and put all the data $g(\phi,\boldsymbol{r})$ for different angles $\phi$ in a vector $\boldsymbol{g}$, then we obtain:

$$\boldsymbol{g} = \boldsymbol{Hf} + \epsilon \quad (1)$$

where $\epsilon$ represents the error, and $\boldsymbol{H}$ is the projection operator in which the element $\boldsymbol{H}_{ij}$ represent the length of the ray $i$ in the pixel $j$.

## Bayesian Approach

The main objective is to infer on $\boldsymbol{f}$ given the data $\boldsymbol{g}$ assuming the forward model $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \epsilon$.

Bayes rule:

$$p(\boldsymbol{f} \mid \boldsymbol{g}) = \frac{p(\boldsymbol{g} \mid \boldsymbol{f})p(\boldsymbol{f})}{p(\boldsymbol{g})} \propto p(\boldsymbol{g} \mid \boldsymbol{f})p(\boldsymbol{f}) \quad (2)$$

Mathematical Model:

(a) Projection: $\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f}$

(b) Back-projection: $\boldsymbol{f} = \boldsymbol{H}^t\boldsymbol{g}$

(c) Filtered Back-projection: $\boldsymbol{f} = \boldsymbol{H}^t(\boldsymbol{H}^t\boldsymbol{H})^{-1}\boldsymbol{g}$

Markov Model (Prior law):

In the Markov model, the value of $f_j$ has a relation with the values of its neighbors pixels. The Markov model for a Gaussian model is:

$$p(\boldsymbol{f}) \propto \exp\left\{-\gamma\|\boldsymbol{D}\boldsymbol{f}\|^2\right\} \quad (3)$$

◇ We have considered different prior laws with Markov model: Gaussian law, Generalized Gaussian law, Cauchy law, Huber law.

Likelihood:

When supposing the noise $\epsilon$ follow a Gaussian law, we have:

$$p(\boldsymbol{g} \mid \boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2\right\} \quad (4)$$

Analytical solution of Gauss-Markov model:

With the Prior law and the Likelihood, we can estimate the reconstruction with the maximum a posterior method and the solution is as follow:

$$\widehat{\boldsymbol{f}} = \left(\boldsymbol{H}^t\boldsymbol{H} + \lambda\boldsymbol{D}^t\boldsymbol{D}\right)^{-1}\boldsymbol{H}^t\boldsymbol{g} \quad (5)$$

## Unsupervised Bayesian

There are parameters $\boldsymbol{\theta} = (\sigma_\epsilon^2, \sigma_g^2)$ as well as $\beta$ which have to be fixed. We will use a Joint Posterior method.

Joint Posterior

$$p(\boldsymbol{f},\boldsymbol{\theta} \mid \boldsymbol{g},\theta_0) = \frac{p(\boldsymbol{g} \mid \boldsymbol{f},\theta_1)p(\boldsymbol{f} \mid \theta_2)p(\theta_1,\theta_2 \mid \theta_0)}{p(\boldsymbol{g} \mid \theta_0)} \quad (6)$$

JMAP Method

$$\left(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}}\right) = \underset{(\boldsymbol{f},\boldsymbol{\theta})}{\arg\max}\left\{p(\boldsymbol{f},\boldsymbol{\theta} \mid \boldsymbol{g},\theta_0)\right\} \quad (7)$$

## JMAP and VBA with Student-t Prior I

We will consider the hierarchical model which use Student-t distribution for modeling the distribution of sparse signals or images. For Student-t prior law, we have the property:

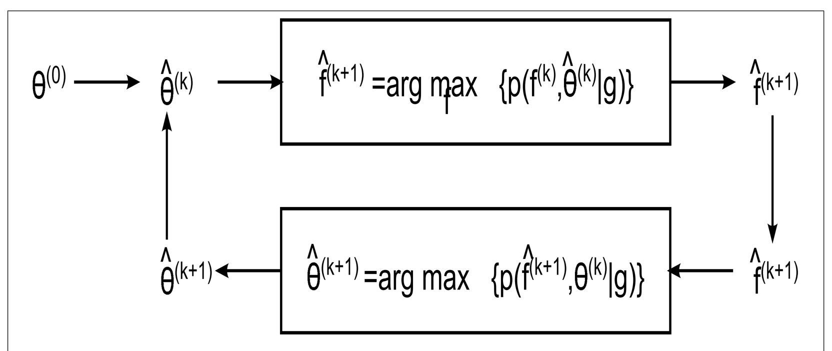
## JMAP and VBA with Student-t Prior II

For Student-t Prior Law:

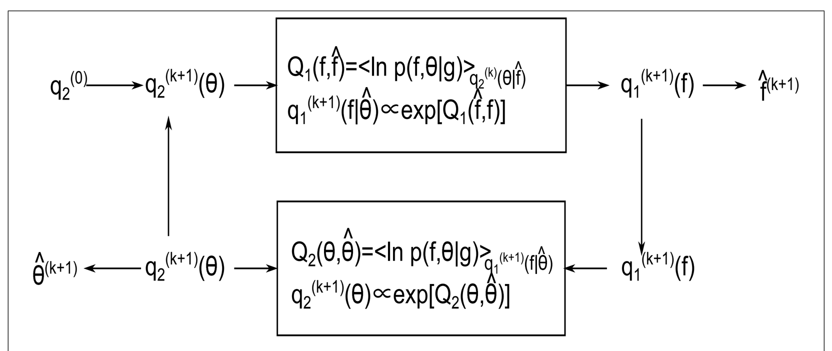
$$\mathcal{S}t\,(f_j) = \int_0^\infty \mathcal{N}(f_j \mid 0, \frac{1}{z_j})\,\mathcal{G}(z_j \mid \frac{\nu}{2},\frac{\nu}{2}) \quad (8)$$

Compare JMAP and VBA methods:

Both these two methods can solve a hirarchical problem. The difference is that the VBA method will consider not only the unknown parameters, but also the uncertainties of the unknowns.



(a) The alternate optimization of JMAP



(b) The alternate optimization of VBA

## Reconstruction Results

Implementation: We have used the synthetic volume "Shepp and Logan" ($256\times256\times256$ voxels), and we compare the image reconstructed after 200 iterations.

Results (Middle slice obtained with different methods):

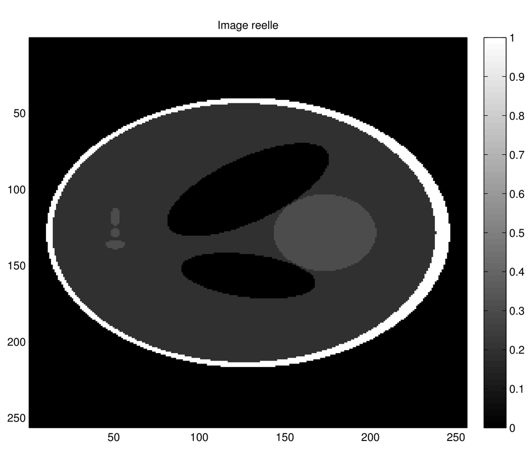


Image original

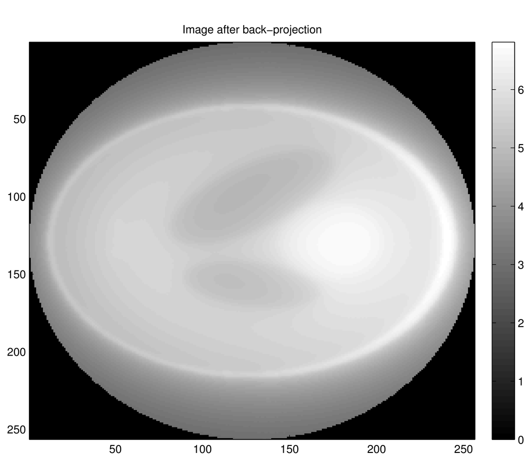


Image after BP
$\delta_f = 276.4415$ $\delta_g = 794.3637$

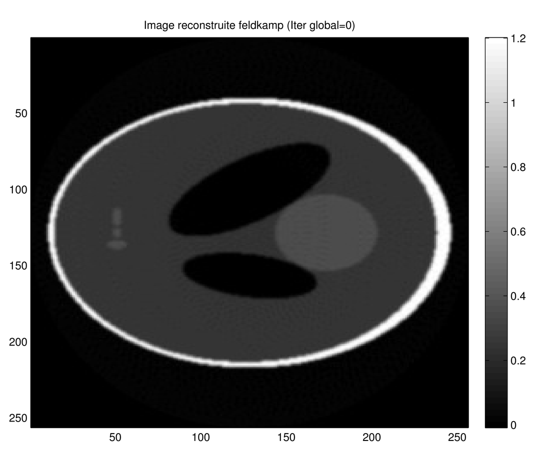
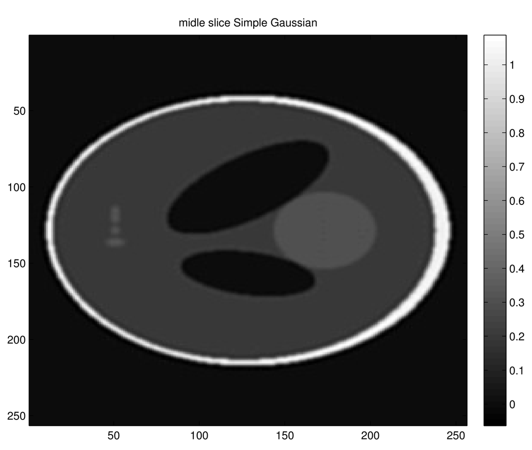
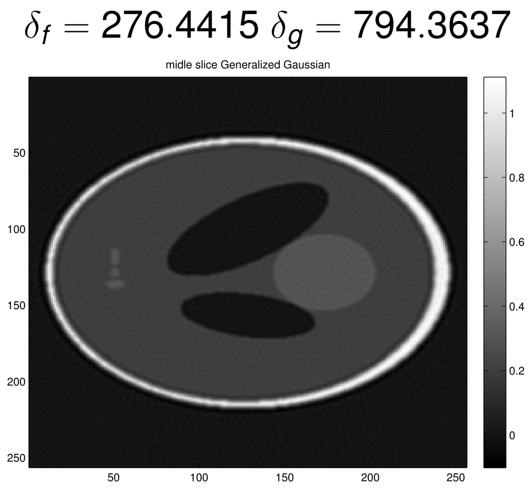


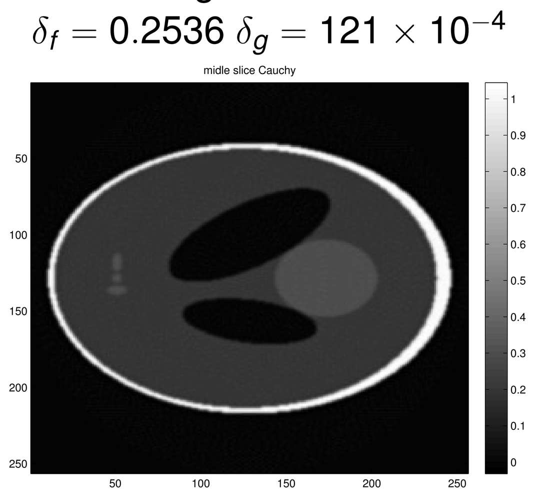
Image after BPF
$\delta_f = 0.2536$ $\delta_g = 121 \times 10^{-4}$



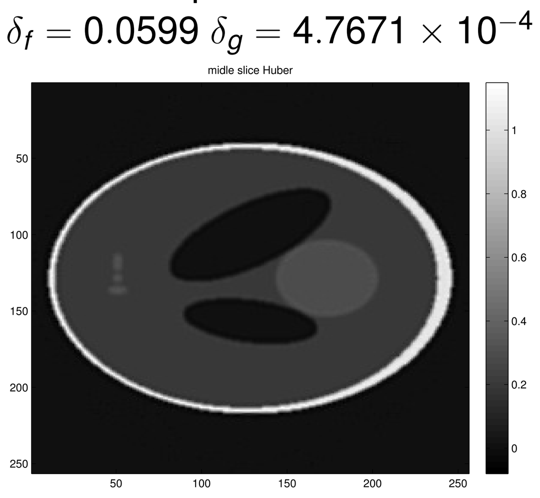
Simple Gaussian
$\delta_f = 0.0599$ $\delta_g = 4.7671 \times 10^{-4}$



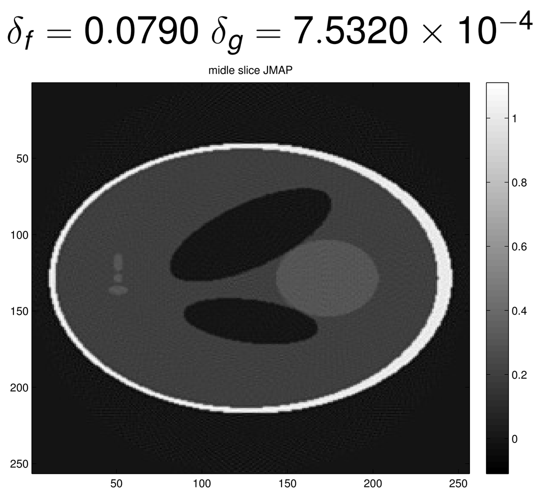
Generalized Gaussian
$\delta_f = 0.0790$ $\delta_g = 7.5320 \times 10^{-4}$



Cauchy
$\delta_f = 0.0496$ $\delta_g = 3.5128 \times 10^{-4}$



Huber
$\delta_f = 0.0516$ $\delta_g = 2.4340 \times 10^{-4}$



JMAP
$\delta_f = 0.0226$ $\delta_g = 3.4952 \times 10^{-5}$

◇ $\delta_f = \frac{\|\widehat{f}-f\|^2}{\|f\|^2}$ $\delta_g = \frac{\|\widehat{g}-g\|^2}{\|g\|^2}$

## Conclusion

◇ Both the MAP method with different prior laws and JMAP method can solve the reconstruction problem well.

◇ The JMAP method has a better property than the other methods. We can distinguish the different materials clearly.

◇ Perspectives: The projection and back-projection programmes have to be optimised to obtain the diagonals elements of $\boldsymbol{H}^t\boldsymbol{H}$ which are needed for the VBA method.

[1] China Scoloarship Council