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# A STABILITY APPROACH FOR SOLVING MULTIDIMENSIONAL QUADRATIC BSDEs

June 29, 2016

Jonathan Harter,<sup>1</sup> Adrien Richou<sup>2</sup>

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## Abstract

We establish an existence and uniqueness result for a class of multidimensional quadratic backward stochastic differential equations (BSDE). This class is characterized by constraints on some uniform a priori estimate on solutions of a sequence of approximated BSDEs. We also present effective examples of applications. Our approach relies on the strategy developed by Briand and Elie in [Stochastic Process. Appl. **123** 2921–2939] concerning scalar quadratic BSDEs. This manuscript is only a working paper.

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## 1 INTRODUCTION

**Backward Stochastic Differential Equations** Backward stochastic differential equations (BSDEs) have been first introduced in a linear version by Bismut [Bis73], but since the early nineties and the seminal work of Pardoux and Peng [PP90], there has been an increasing interest for these equations due to their wide range of applications in stochastic control, in finance or in the theory of partial differential equations. Let us recall that, solving a BSDE consists in finding an adapted pair of processes  $(Y, Z)$ , where  $Y$  is a  $\mathbf{R}^d$ -valued continuous process and  $Z$  is a  $\mathbf{R}^{d \times k}$ -valued progressively measurable process, satisfying the equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad a.s. \quad (1.1)$$

where  $W$  is a  $k$ -dimensional Brownian motion with filtration  $\mathcal{F}$ ,  $\xi$  is a  $\mathcal{F}_T$ -measurable random variable called the terminal condition, and  $f$  is a (possibly random) function called the generator. Since the seminal paper of Pardoux and Peng [PP90] that gives an existence and uniqueness result for BSDEs with a Lipschitz generator, a huge amount of paper deal with extensions and applications. In particular, the class of BSDE, with generators of quadratic growth with respect to the variable  $z$ , has received a lot of attention in recent years. Concerning the scalar case, i.e.  $d = 1$ , existence and uniqueness of solutions for quadratic BSDEs has been first proved by Kobylanski in [Kob00]. Since then, many authors worked on this question: when the terminal condition is bounded, we refer to [Kob00, Tev08, BE13] and [BH06, BEK13, DHR11].

In this paper we will focus on existence and uniqueness results for quadratic BSDEs in the multidimensional setting, i.e.  $d > 1$ . Let us remark that, in addition to his intrinsic mathematical interest, this question is important due to many applications of such equations. We can mention e.g. following applications: nonzero-sum risk-sensitive stochastic differential games in [EKH03, HT16], financial market equilibrium problems for several interacting agents in [ET15, FDR11, Fre14, BLDR15], financial price-impact models in [KP16, KP16], principal

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agent contracting problems with competitive interacting agents in [EP16], stochastic equilibria problems in incomplete financial markets [KXŽ15, XŽ16] or existence of martingales on curved spaces with a prescribed terminal condition [Dar95].

Let us note that moving from the scalar framework to the multidimensional one is quite challenging since tools used when  $d = 1$ , like monotone convergence or Girsanov transform, can no longer be used when  $d > 1$ . Moreover, Frei and dos Reis provide in [FDR11] an example of multidimensional quadratic BSDE with a bounded terminal condition and a very simple generator such that there is no solution to the equation. This informative counterexample show that it is hopeless to try to obtain a direct generalization of the Kobylanski existence and uniqueness theorem in the multidimensional framework or a direct extension of the Pardoux and Peng existence and uniqueness theorem for locally-Lipschitz generators. Nevertheless, we can find in the literature several papers that deal with special cases of multidimensional quadratic BSDEs. We give now a really brief summary of all these papers, up to our knowledge.

First of all, a quite general result was obtain by Tevzadze in [Tev08], when the bounded terminal condition is small enough, by using a fixed-point argument and the theory of BMO martingales. Some generalizations with somewhat more general terminal conditions are considered in [Fre14, KP16]. In [CN15], Cheridito and Nam treat some quadratic BSDEs with very specific generators. Before these papers, Darling was already able to construct a martingale on a manifold with a prescribed terminal condition by solving a multidimensional quadratic BSDE (see [Dar95]). Its proof relies on a stability result obtained by coupling arguments. Recently, the so-called quadratic diagonal case has been considered by Hu and Tang in [HT16]. To be more precise, they assume that the  $n$ th line of the generator has only a quadratic growth with respect to the  $n$ th line of  $Z$ . This type of assumption allows authors to use Girsanov transforms. Some little bit more general assumptions are treated by Jamneshan, Kupper and Luo in [JKL14] (see also [LT15]). Finally, in the very recent paper [XŽ16], Xing and Žitković obtained a general result in a Markovian setting with weak regularity assumptions on the generator and the terminal condition. Instead of assuming some specific hypotheses on the generator, they suppose the existence of a so called Liapounov function which allows to obtain a uniform a priori estimate on some sequence  $(Y^n, Z^n)$  of approximations of  $(Y, Z)$ . Their approach relies on analytic methods. We refer to this paper for references on analytic and PDE methods for solving systems of quadratic semilinear parabolic PDEs.

**Our approach** Our approach for solving multidimensional quadratic BSDEs relies on the theory of BMO martingales and stability results as in [BE13]. To get more into the details about our strategy, let us recall the sketch of the proof used by Briand and Elie in [BE13]. The generator  $f$  is assumed to be locally Lipschitz and, to simplify, we assume that it depends only on  $z$ . First of all, they consider the following approximated BSDE

$$Y_t^M = \xi + \int_t^T f(\rho^M(Z_s^M))ds - \int_t^T Z_s^M dW_s, \quad 0 \leq t \leq T, \quad a.s.$$

where  $\rho^M$  is a projection on the centered Euclidean ball of radius  $M$ . This new BSDE has a Lipschitz generator, so existence and uniqueness of  $(Y^M, Z^M)$  is obvious. Now, if we assume that  $\xi$  is Malliavin differentiable with a bounded Malliavin derivative, they show that  $Z^M$  is bounded uniformly with respect to  $M$ . Thus,  $(Y^M, Z^M) = (Y, Z)$  for  $M$  large enough. Importantly, the uniform bound on  $Z^M$  is obtain thanks to a uniform (with respect to  $M$ ) a priori estimate on the BMO norm of the martingale  $\int_0^T Z^M dW_s$ . Subsequently, they extend their existence and uniqueness result for a general bounded terminal condition:  $\xi$  is approximated by a sequence  $(\xi^n)_{n \in \mathbb{N}}$  of bounded terminal conditions with bounded Malliavin derivatives and they consider  $(Y^n, Z^n)$  the solution of the following BSDE

$$Y_t^n = \xi^n + \int_t^T f(Z_s^n)ds - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T, \quad a.s..$$

By using a stability result for quadratic BSDEs, they show that  $(Y^n, Z^n)$  is a Cauchy sequence that converges to the solution of the initial BSDE (1.1). Once again, the stability result used by Briand and Elie relies on a uniform (with respect to  $n$ ) a priori estimate on the BMO norm of the martingale  $\int_0^T Z^n dW_s$ .

The aim of this paper is to adapt this approach in our multidimensional setting. In the first approximation step, we are able to show that  $Z^M$  is bounded uniformly with respect to  $M$  if we have a small enough uniform (with respect to  $M$ ) a priori estimate on the BMO norm of the martingale  $\int_0^T Z^M dW_s$ . But, contrarily to the scalar case, it is not possible to show that we have an a priori estimate on the BMO norm of the martingale  $\int_0^T Z^M dW_s$  under general quadratic assumptions on the generator (let us recall the counterexample provides by Frei and dos Reis in

[FDR11]). So, this a priori estimate on the BMO norm of the martingale  $\int_0^\cdot Z^M dW_s$  becomes in our paper an a priori assumption and this assumption has to be verified on a case-by-case basis according to the BSDE structure. In the second approximation step, we are facing the same issue: we are able to show the existence and uniqueness of a solution to (1.1) by using a stability result if we have a small enough uniform (with respect to  $n$ ) a priori estimate on the BMO norm of the martingale  $\int_0^\cdot Z^n dW_s$ , and this a priori estimate becomes, once again, an assumption that has to be verified on a case-by-case basis according to the BSDE structure. Let us emphasize that the estimate on the boundedness of  $Z^M$  and the stability result used in the second step come from an adaptation of results obtained by Delbaen and Tang in [DT08].

To show the interest of these theoretical results, we have to find now some frameworks for which we are able to obtain estimates on the BMO norm of martingales  $\int_0^\cdot Z^M dW_s$  and  $\int_0^\cdot Z^n dW_s$ . This is the purpose of Section 4 where results of [Tev08, Dar95] are revisited. Let us note that one interest of our strategy comes from the fact that we obtain these estimates by very simple calculations that allows to easily get new results: for example, we are able to extend the result of Tevzadze when the generator satisfies a kind of monotone assumption with respect to  $y$ . Moreover, we can remark that obtaining such estimates is strongly related to finding a so-called Liapounov function in [XŽ16]. Result on the boundedness of  $Z$  is also interesting in itself since it allows to consider the initial quadratic BSDE (1.1) as a simple Lipschitz one which gives access to numerous results on Lipschitz BSDEs: numerical approximation schemes, differentiability, stability, and so on.

**Structure of the paper** In the rest of the introduction, we introduce notations, the framework and general assumptions. Section 2 contains some general results about SDEs and linear BSDEs adapted from [DT08]. Section 3 is devoted to our main results: stability properties, existence and uniqueness theorems for multidimensional quadratic BSDEs. Finally, some applications of previous theoretical results are treated in Section 4.

### § 1.1. Notations.

- ◇ Let  $T > 0$ .  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbf{P})$  will be a complete probability space and  $\mathcal{F}_t$  is a Brownian filtration satisfying the usual conditions. In particular every càdlàg process has a continuous version. Every Brownian motion will be considered relatively to this filtered probability space. A  $k$ -dimensional Brownian motion  $W = (W^i)_{1 \leq i \leq k}$  is a process with values in  $\mathbf{R}^k$  and with independent Brownian components. Almost every process will be defined on a finite horizon  $[0, T]$ , either we will precise it explicitly. The stochastic integral of an adapted process  $H$  will be denoted by  $H \star W$ , and the Euclidean quadratic variation by  $\langle \cdot, \cdot \rangle$ . The Dolean-Dade exponential of a continuous real local martingale  $M$  is denoted by

$$\mathcal{E}(M) := \exp\left(M - \frac{1}{2}\langle M, M \rangle\right).$$

- ◇ *Linear notions* – For every  $d, k$ , we denote by  $(E_{ij})_{1 \leq i \leq d, 1 \leq j \leq k}$  the canonical basis of  $\mathcal{M}_{dk}(\mathbf{R})$ . On each  $\mathbf{R}^p$ , the scalar product will be simply denoted by a dot, including the canonical scalar product on  $\mathcal{M}_{dk}(\mathbf{R})$ :

$$M.N = \sum_{1 \leq i \leq d, 1 \leq j \leq k} M_{i,j} N_{i,j}.$$

For  $A \in \mathcal{M}_{dk}(\mathbf{R})$ ,  $A^{(\cdot, p)}$  will be the column  $p \in \{1, \dots, k\}$  of  $A$ , and  $A^{(l, \cdot)}$  the line  $l \in \{1, \dots, d\}$ . If  $B \in \mathcal{L}(\mathbf{R}^{d \times k}, \mathbf{R}^d)$ , the space of all linear maps from  $\mathbf{R}^{d \times k}$  to  $\mathbf{R}^d$ , we can see  $B$  as a matrix of size  $(dk) \times d$  and we write for  $p \in \{1, \dots, k\}$ ,  $B_t^{(\cdot, p; \cdot)}$  the  $d \times d$  matrix formed by the columns indexed by  $(1, p), (2, p), \dots, (d, p)$  of  $B$ , in the basis

$$(E_{11}, \dots, E_{1p}, \dots, E_{1k}, E_{21}, \dots, E_{2p}, \dots, E_{2k}, \dots, E_{dk}).$$

If  $A$  and  $B$  are two processes with values in  $\mathcal{M}_{dk}(\mathbf{R})$  and  $\mathbf{R}^k$ , the quadratic variation  $\langle A, B \rangle$  is the  $\mathbf{R}^d$  vector

$$\left( \sum_{l=1}^d \langle A^{il}, B^l \rangle \right)_{i=1}^d$$

in order that the integration by part formula  $d(AB) = dA.B + A.dB + d\langle A, B \rangle$  be in force. We can also define the

covariation of two  $(A, B) \in \mathcal{M}_{dk}(\mathbf{R}) \times \mathcal{M}_{kd'}(\mathbf{R})$  by

$$\left( \sum_{l=1}^d \langle A^{il}, B^{lj} \rangle \right)_{i,j=1}^{d,d'}.$$

- ◇ *Functional spaces* – In a general way, Euclidean norms will be denoted by  $|\cdot|$  while relatively to  $\omega$  and  $t$  by  $\|\cdot\|$ . For a  $\mathcal{F}$ -adapted continuous process  $Y$  with values in  $\mathbf{R}^d$  and  $1 \leq p \leq \infty$ , let us define

$$\|Y\|_{\mathcal{S}^p} = \mathbf{E} \left( \sup_{0 \leq s \leq T} |Y_s|^p \right)^{1/p}, \quad \|Y\|_{\mathcal{S}^\infty} = \text{esssup}_{0 \leq s \leq T} |Y_s|.$$

For  $Z$ , a random variable with values in  $\mathbf{R}^d$ , we define

$$\|Z\|_{L^p} = \mathbf{E}(|Z|^p)^{1/p}.$$

A continuous martingale  $M$  with values in  $\mathbf{R}^d$  is in  $\mathcal{H}^p(\mathbf{R}^d)$ , or only  $\mathcal{H}^p$  when it is not necessary to specify the state space, if  $\sqrt{\langle M, M \rangle_T} \in L^p$ . And we define the  $\mathcal{H}^p$  norm by

$$\|M\|_{\mathcal{H}^p} := \mathbf{E} \left( \langle M, M \rangle_T^{p/2} \right)^{1/p} < \infty.$$

A martingale  $M = (M_t)_{0 \leq t \leq T}$  is said to be *BMO* (bounded in mean oscillation) if there exists a constant  $C \geq 0$  such that for every stopping time  $0 \leq \tau \leq T$ :

$$\mathbf{E}((M_T - M_\tau)^2 | \mathcal{F}_\tau) \leq C^2 \text{ a.s.}$$

The best constant  $C$  is called the *BMO* norm of  $M$ , denoted by  $\|M\|_{BMO}$ . In particular, the local martingale  $Z \star W = \int_0^\cdot Z_s dW_s$  is *BMO* if there exists a constant  $C \geq 0$  such that

$$\forall \tau, \quad \mathbf{E} \left( \int_\tau^T |Z_s|^2 ds \middle| \mathcal{F}_\tau \right) \leq C^2 \text{ a.s.}$$

For more details about BMO martingales, we can refer to [Kaz94]. For  $k \geq 1$ ,  $\mathcal{C}_b^\infty(\mathbf{R}^k)$  is the set of all  $\mathcal{C}^\infty$  functions with values in  $\mathbf{R}$  defined on  $\mathbf{R}^k$ , which have bounded derivatives.

- ◇ *Inequalities* – *BDG* inequalities claim that  $\|\cdot\|_{\mathcal{S}^p}$  and  $\|\cdot\|_{\mathcal{H}^p}$  are equivalent on martingale spaces with two universal constants denoted  $C'_p, C_p$ . It means that for all continuous local martingales  $M$  vanishing at 0,

$$\|M\|_{\mathcal{H}^p} \leq C_p \|M\|_{\mathcal{S}^p}$$

and

$$\|M\|_{\mathcal{S}^p} \leq C'_p \|M\|_{\mathcal{H}^p}.$$

We can choose universal constants valid for every dimension. In [MR16], Marinelli and Röckner consider the most general case; martingale with values in a separable Hilbert space. In particular, the upper constant  $C'$  (Proposition 2.1 and Proposition 3.1) defined below is valid for all dimensions:

$$C'_p = \begin{cases} \left( \frac{p}{p-1} \right)^{\frac{p}{2}} \left( \frac{p(p-1)}{2} \right)^2 & \text{if } p > 2, \\ 4\sqrt{\frac{2}{p}} & \text{if } p < 2, \\ 4 & \text{if } p = 2. \end{cases}$$

The first two cases are shown in [MR16], and the third one is the constant obtained in the scalar case which still valid in the general case. In the following every BDG inequality should be understood with this choice of  $C'$ . The Doob maximal inequality claims that for every  $\mathbf{R}^d$ -valued martingale  $M$  and  $p > 1$ ,

$$\|M\|_{\mathcal{S}^p} \leq \frac{p}{p-1} \|M_T\|_{L^p}.$$

And for  $p = \infty$ ,

$$\|M\|_{\mathcal{S}^\infty} \leq \|M_T\|_{L^\infty}.$$

If  $p \in ]1, \infty[$ , we will denote by  $p^*$  the conjugated exponent of  $p$ , they are linked by  $\frac{1}{p} + \frac{1}{p^*} = 1$ . We say that a process  $L = (L_t)_{0 \leq t \leq T}$  with values in  $\mathbf{R}^d$  satisfies a reverse Hölder inequality for some integer  $1 \leq p < \infty$  if there exists some constant  $K_p$  such that for every stopping time  $0 \leq \tau \leq T$  a.s.,

$$\mathbf{E}(|L_T|^p | \mathcal{F}_\tau) \leq K_p |L_\tau|^p \text{ a.s.}$$

The energy inequality (see [Kaz94]) tells us that for every  $BMO$  process  $Z$  and every integer  $n \geq 1$ , we have

$$\mathbf{E}(\langle Z \star W \rangle_T^n) \leq n! \|Z \star W\|_{BMO}^{2n}. \quad (1.2)$$

A conditional version of this inequality is the following: for all  $t \in [0, T]$ ,

$$\mathbf{E}(\langle Z \star W \rangle_T - \langle Z \star W \rangle_t)^n | \mathcal{F}_t) \leq n! \|Z \star W\|_{BMO}^{2n}.$$

Consequently,

$$BMO \subset \left( \bigcap_{p \geq 1} \mathcal{H}^p \right).$$

We recall also the so-called Fefferman inequality: for  $X \in \mathcal{H}^1$  and  $Y \in BMO$ ,

$$\mathbf{E} \left( \int_0^T |d\langle X, Y \rangle_s| \right) \leq \|X\|_{\mathcal{H}^1} \|Y\|_{BMO}.$$

This inequality yields the following technical lemma (see [BB88] and [DT08]) for more details.

LEMMA – 1.1 *Let  $m \geq 1$ . We consider  $X$  an adapted process and  $M$  a local martingale.*

(i) *If  $X \in \mathcal{S}^m$  and  $M \in BMO$  then  $X \star M \in \mathcal{H}^m$  and*

$$\|X \star M\|_{\mathcal{H}^m} \leq \sqrt{2} \|X\|_{\mathcal{S}^m} \|M\|_{BMO}.$$

(ii) *If  $X \in \mathcal{H}^m$  and  $M \in BMO$  then  $\langle X, M \rangle_T \in L^m$  and*

$$\|\langle X, M \rangle_T\|_{L^m} \leq \sqrt{2} m \|X\|_{\mathcal{H}^m} \|M\|_{BMO}.$$

To conclude this paragraph, let us show a technical proposition that will be useful in this paper.

PROPOSITION – 1.1 *Let us consider  $m \geq 1$  and a sequence of uniformly bounded in  $BMO$  local martingales  $(Z^n \star W)_{n \in \mathbf{N}}$ . We denote  $K = \sup_{n \in \mathbf{N}} \|Z^n \star W\|_{BMO} < \infty$  and assume that  $Z^n \star W$  converge in  $\mathcal{H}^m$  to a martingale  $Z \star W$ . Then  $Z \star W$  is  $BMO$  too and satisfies the same inequality  $\|Z \star W\|_{BMO} \leq K$ .*

**Proof.** Let us define by  $\mathcal{M}$  the measure  $d\mathcal{M} = d\mathbf{P} \otimes dx$ . Firstly we show that convergence in  $\mathcal{H}^m$  implies the convergence for the measure  $\mathcal{M}$ .

Indeed, if  $m \geq 2$ , the Jensen inequality gives us

$$\mathbf{E} \left( \int_0^T |Z_s^n - Z_s|^2 ds \right) \leq \|Z^n \star W - Z \star W\|_{\mathcal{H}^m}^2,$$

and thus we get the convergence in measure, since for all  $\varepsilon > 0$ , we have

$$\mathcal{M}(|Z^n - Z| > \varepsilon) \leq \frac{1}{\varepsilon^2} \|Z^n \star W - Z \star W\|_{\mathcal{H}^2}^2 \leq \frac{1}{\varepsilon^2} \|Z^n \star W - Z \star W\|_{\mathcal{H}^m}^2.$$

Moreover, if  $m < 2$ , we get

$$\mathcal{M}(|Z^n - Z| > \varepsilon) \leq \frac{1}{\varepsilon^m} \mathbf{E} \left( \int_0^T |Z_s^n - Z_s|^m ds \right) \leq \frac{T^{1-m/2}}{\varepsilon^m} \mathbf{E} \left( \left( \int_0^T |Z_s^n - Z_s|^2 ds \right)^{m/2} \right).$$

For both cases, we get convergence in measure for  $\mathcal{M}$ . Hence there exists a subsequence  $(n_k)_{k \in \mathbf{N}}$  such that

$$|Z^{n_k}|^2 \xrightarrow[k \rightarrow \infty]{} |Z|^2 \quad \mathcal{M} - \text{a.s.}$$

The Fatou lemma gives us, for all stopping time  $\tau$ ,

$$\int_{\tau}^T |Z_s|^2 ds \leq \liminf_{k \rightarrow \infty} \int_{\tau}^T |Z_s^{n_k}|^2 ds \quad \text{a.s.},$$

and then the conditional version for the second inequality gives us

$$\mathbf{E} \left( \int_{\tau}^T |Z_s|^2 ds \middle| \mathcal{F}_{\tau} \right) \leq \mathbf{E} \left( \liminf_{k \rightarrow \infty} \int_{\tau}^T |Z_s^{n_k}|^2 ds \middle| \mathcal{F}_{\tau} \right) \leq \liminf_{k \rightarrow \infty} \mathbf{E} \left( \int_{\tau}^T |Z_s^{n_k}|^2 ds \middle| \mathcal{F}_{\tau} \right) \leq K \quad \text{a.s.}$$

Finally we conclude that  $Z \star W$  is  $BMO$  with  $\|Z \star W\|_{BMO} \leq K$ .  $\diamond$

$\diamond$  *Sliceability* – For a process  $X$  and a stopping time  $\tau$  we denote by  ${}^{\tau}X$  the process started at time  $\tau$ , that is  ${}^{\tau}X = X - X^{\downarrow \tau}$  where  $X^{\downarrow \tau}$  is the process stopped at  $\tau$ . For two stopping times  $\tau \leq \sigma$  a.s., we denote by  ${}^{\tau}X^{\downarrow \sigma}$  the process started at  $\tau$  and stopped at  $\sigma$ :

$${}^{\tau}X^{\downarrow \sigma} = ({}^{\tau}X)^{\downarrow \sigma}.$$

Associativity property of the stochastic integral gives us

$${}^{\tau}(H \star W)^{\downarrow \sigma} = H \star {}^{\tau}W^{\downarrow \sigma}.$$

Between  $\tau$  and  $\sigma$ , the started stopped process is simply a translation of the stopped process: for all  $u$  such that  $\tau \leq u \leq \sigma$  a.s.,

$${}^{\tau}X_u^{\downarrow \sigma} = X_u - X_{\tau}.$$

This process is thus constant after  $\sigma$  and vanishes before  $\tau$ . Let us suppose that  $X$  is a  $BMO$  martingale. We say that  $X$  is  $\varepsilon$ -sliceable if there exists some subsequence  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$  a.s. of stopping times such that

$$\left\| {}^{T_n}(X)^{\downarrow T_{n+1}} \right\|_{BMO} \leq \varepsilon.$$

The set of all  $\varepsilon$ -sliceable processes will be denoted by  $BMO_{\varepsilon}$ . In [Sch96], Schachermayer proved that

$$\bigcap_{\varepsilon > 0} BMO_{\varepsilon} = \overline{\mathcal{H}^{\infty} BMO},$$

Moreover,  $BMO$  norm of a stopped started stochastic integral process  ${}^{\tau}Z \star W^{\downarrow \sigma}$  is given by the following proposition

PROPOSITION – 1.2

$$\left\| {}^{\tau}Z \star W^{\downarrow \sigma} \right\|_{BMO} = \text{esssup}_{\tau' \in \mathcal{T}_{\tau, \sigma}} \sup_{\tau' \in \mathcal{T}_{\tau, \sigma}} \mathbf{E} \left( \int_{\tau'}^{\sigma} |Z_s|^2 ds \middle| \mathcal{F}_{\tau'} \right),$$

$$\text{where } \mathcal{T}_{\tau, \sigma} = \{ \tau' \text{ stopping time} : \tau \leq \tau' \leq \sigma \text{ a.s.} \}.$$

A proof of this proposition is given in the appendix part.

$\diamond$  *Malliavin calculus* – We denote by

$$\mathcal{P} = \{ f((g_1 \star W)_T, \dots, (g_n \star W)_T) : f \in \mathcal{C}_b^{\infty}(\mathbf{R}^n), g_i \text{ adapted}, n \geq 1 \},$$

the set of all Wiener functions. For  $F \in \mathcal{P}$ , the Malliavin derivative of  $F$  is a progressively measurable process  $\mathbf{D}F \in L^2([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}, dx \otimes d\mathbf{P})$  defined by

$$\mathbf{D}_t F = \sum_{i=1}^n \partial_i f((g_1 \star W)_{t=T}, \dots, (g_n \star W)_{t=T}) g_i(t).$$

In particular  $\mathbf{D}((h \star W)_T) = h$  for all adapted process  $h$ . We define such a kind of Sobolev norm on  $\mathcal{P}$  with the

following formula

$$\|F\|_{1,2} = \left[ \mathbf{E} \left( |F|^2 \right) + \mathbf{E} \left( \|\mathbf{D}F\|_{L^2(\mathrm{d}x)}^2 \right) \right]^{1/2}.$$

We can prove that  $\mathbf{D}$  is closable, consequently it is possible to extend the definition of  $\mathbf{D}$  to  $\mathbb{D}^{1,2} = \overline{\mathcal{D}}^{1,2}$ . Besides,  $\mathbb{D}^{1,2}$  is dense in  $L^2(\Omega)$ . For considerations on Malliavin calculus we can refer to [Nua06]. Let us cite also a useful result: all Lipschitz functionals of random variables in  $\mathbb{D}^{1,2}(\mathbf{R}^d)$  are in  $\mathbb{D}^{1,2}(\mathbf{R}^d)$  (see [Nua06], Proposition 1.2.4).

PROPOSITION – 1.3 *Let  $\varphi : \mathbf{R}^d \rightarrow \mathbf{R}$ . We assume that there exists a constant  $K$  such that for all  $x, y \in \mathbf{R}^d$ ,*

$$|\varphi(x) - \varphi(y)| \leq K |x - y|.$$

*Let  $(F^1, \dots, F^d)$  a vector in  $\mathbb{D}^{1,2}(\mathbf{R}^d) \cap L^\infty(\Omega)$ . Then  $\varphi(F) \in \mathbb{D}^{1,2}(\mathbf{R}^d)$  and there exists a vector  $(G^1, \dots, G^d)$  such that*

$$\mathbf{D}\varphi(F) = \sum_{i=1}^d G^i \mathbf{D}F^i, \quad |G| \leq K.$$

§ 1.2. **Framework and first assumptions.** In this paper we consider the following quadratic BSDE on  $\mathbf{R}^d$ :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad 0 \leq t \leq T, \quad (1.3)$$

where  $f$  is a random function  $\Omega \times [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  called the *generator* of the BSDE such that for all  $(y, z) \in \mathbf{R}^d \times \mathbf{R}^{d \times k}$  and  $t \in [0, T]$ ,  $(f(t, y, z))_{0 \leq t \leq T}$  is progressively measurable,  $(Y, Z)$  is a process with values in  $\mathbf{R}^d \times \mathbf{R}^{d \times k}$  and  $\xi \in L^2(\mathcal{F}_T, \mathbf{R}^d)$ .

DEFINITION – 1.1 *A solution of BSDE (1.3) is a process  $(Y, Z) \in \mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  satisfying usual integrability conditions and solving initial BSDE:*

$$(i) \int_0^T \left( |f(s, Y_s, Z_s)|^2 + |Z_s|^2 \right) \mathrm{d}s < \infty \text{ a.s.}$$

$$(ii) Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad 0 \leq t \leq T, \text{ a.s.}$$

Locally Lipschitz assumptions on  $f$  are assumed.

(H) For all  $(y, y', z, z') \in (\mathbf{R}^d)^2 \times (\mathbf{R}^{d \times k})^2$ , we suppose that there exists  $(K_y, L_y, K_z, L_z) \in (\mathbf{R}^+)^4$  such that  $\mathbf{P}$  – a.s for all  $t \in [0, T]$ :

$$|f(t, y, z) - f(t, y', z)| \leq (K_y + L_y |z|^2) |y - y'|,$$

$$|f(t, y, z) - f(t, y, z')| \leq (K_z + L_z (|z| + |z'|)) |z - z'|.$$

Let us also introduce a *localisation* of  $f$  defined by  $f^M(t, y, z) = f(t, y, \rho^M(z))$  where  $\rho^M : \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^{d \times k}$  satisfies the following properties :

- $\rho^M$  is the identity on  $\mathcal{B}_{\mathbf{R}^{d \times k}}(0, M)$ ,
- the projection on  $\mathcal{B}_{\mathbf{R}^{d \times k}}(0, M+1)$  outside  $\mathcal{B}_{\mathbf{R}^{d \times k}}(0, M+1)$ ,
- $\rho^M$  is a  $\mathcal{C}^\infty$  function with  $|\nabla \rho^M(z)| \leq 1$ .

Thus  $f^M$  is a globally Lipschitz function with constants depending on  $M$ . Indeed we have for all  $(t, y, y', z, z') \in [0, T] \times (\mathbf{R}^d)^2 \times (\mathbf{R}^{d \times k})^2$ ,

$$\begin{aligned} |f^M(t, y, z) - f^M(t, y', z')| &\leq |f(t, y, \rho^M(z)) - f(t, y', \rho^M(z))| + |f(t, y', \rho^M(z)) - f(t, y', \rho^M(z'))| \\ &\leq (K_y + L_y |\rho^M(z)|^2) |y - y'| + (K_z + L_z (|\rho^M(z)| + |\rho^M(z')|)) |z - z'| \\ &\leq (K_y + L_y (M+1)^2) |y - y'| + (K_z + 2L_z (M+1)) |z - z'|. \end{aligned}$$



According to the classical Pardoux-Peng result in [PP90], there exists a unique solution  $(Y^M, Z^M) \in \mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  of the localized BSDE

$$Y_t^M = \xi + \int_t^T f^M(s, Y_s^M, Z_s^M) ds - \int_t^T Z_s^M dW_s, \quad 0 \leq t \leq T. \quad (1.4)$$

## 2 GENERALITIES ABOUT SDES AND LINEAR BSDES

**§ 2.1. Representation of the solution in the linear case.** We investigate here the following linear BSDE

$$U_t = \zeta + \int_t^T (A_s U_s + B_s V_s + f_s) ds - \int_t^T V_s dW_s, \quad 0 \leq t \leq T, \quad (2.1)$$

where  $\zeta \in L^2(\mathcal{F}_T, \mathbf{R}^d)$ ,  $f \in L^2(\Omega \times [0, T])$  and  $A, B, f$  are three bounded processes with values in  $\mathcal{L}(\mathbf{R}^d, \mathbf{R}^d)$ ,  $\mathcal{L}(\mathbf{R}^{d \times k}, \mathbf{R}^d)$  and  $\mathbf{R}^d$ . For the linear case we have an explicit formulation of the solution. Let us begin to recall the classical scalar formula which comes from the Girsanov Theorem.

REMARK – 2.1 (ONE-DIMENSIONAL CASE ( $d = 1$ )) *It is well-known that the solution of (2.1) is given by the formula*

$$U_t = \mathbf{E} \left( S_t^{-1} S_T \zeta + \int_t^T S_t^{-1} S_s f_s ds \middle| \mathcal{F}_t \right), \quad 0 \leq t \leq T,$$

where

$$S_t = \exp \left( \int_0^t B_s dW_s - \frac{1}{2} \int_0^t |B_s|^2 ds + \int_0^t A_s ds \right) = \mathcal{E}(B \star W)_t \exp \left( \int_0^t A_s ds \right).$$

*In particular, the case  $A = 0$  a.s. is an easy consequence of the Girsanov theorem:  $U$  becomes an expectation with respect to an other probability measure.*

To extend this last formula in the general case we define a process  $S$  as the unique strong solution of

$$dS_t = \sum_{p=1}^k S_t B_t^{(:,p,:)} dW_t^p + S_t A_t dt, \quad S_0 = I_{d \times d}.$$

The case  $B = 0$  a.s. is treated in [DT08], we deduce easily the following generalization.

PROPOSITION – 2.1 (FORMULA FOR  $U$ )

(i) *The process  $S$  is invertible for all  $t \in [0, T]$  and  $S^{-1}$  is the solution of*

$$dS_t^{-1} = \left[ \left( \sum_{p=1}^k \left( B_t^{(:,p,:)} \right)^2 - A_t \right) dt - \sum_{p=1}^k B_t^{(:,p,:)} dW_t^p \right] S_t^{-1}, \quad S_0^{-1} = I_{d \times d}.$$

(ii) *The BSDE (2.1) has a unique solution  $(U, V)$  in  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$ , and  $U$  is given by:*

$$U_t = \mathbf{E} \left( S_t^{-1} S_T \zeta + \int_t^T S_t^{-1} S_s f_s ds \middle| \mathcal{F}_t \right). \quad (2.2)$$

Proof. Existence and uniqueness of a solution  $(U, V)$  in  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  is guaranteed by the Pardoux and Peng result in [PP90]. The solution  $(U, V)$  satisfies

$$U_t = \zeta + \int_t^T \left( A_s U_s + \sum_{p=1}^k B_s^{(:,p,:)} V_s^{(:,p)} + f_s \right) ds - \sum_{p=1}^k \int_t^T V_s^{(:,p)} dW_s^p.$$

The Itô formula gives the invertibility of  $S$  and the formula for  $S^{-1}$  on the one hand. On the other hand:

$$d(S_t U_t) = -S_t f_t dt + \sum_{p=1}^k \left( S_t B_t^{(:,p,:)} U_t + S_t V_t^{(:,p)} \right) dW_t^p,$$

and thus we get, for all  $t \in [0, T]$ ,

$$S_t U_t = S_T \zeta + \int_t^T S_s f_s ds - \int_t^T \sum_{p=1}^k \left( S_s B_s^{(:,p,:)} U_s + S_s V_s^{(:,p)} \right) dW_s^p.$$

By taking the conditional expectation  $S_t U_t = \mathbf{E} \left( S_T \zeta + \int_t^T S_s f_s ds \middle| \mathcal{F}_t \right)$ . Adaptability and invertibility of  $S$  give the result.  $\diamond$

**§ 2.2. A result about SDEs.** We consider a SDE on  $\mathbf{R}^{d \times d}$  of the form

$$X_t = X_0 + \int_0^t F(s, X_s) ds + \sum_{p=1}^k \int_0^t G^p(s, X_s) dW_s^p, \quad (2.3)$$

where  $F : \Omega \times [0, T] \times \mathbf{R}^{d \times d} \rightarrow \mathbf{R}^{d \times d}$  and for all  $p \in \{1, \dots, k\}$ ,  $G^p : \Omega \times [0, T] \times \mathbf{R}^{d \times d} \rightarrow \mathbf{R}^{d \times d}$  are progressively measurable functions. We start by recalling a result of Delbaen and Tang (see [DT08], Theorem 2.1.) about existence and uniqueness of a solution to the equation (2.3), under BMO assumptions.

**PROPOSITION – 2.2** *Let  $m \geq 1$ . We suppose that there are two non-negative adapted processes  $\alpha$  and  $\beta$  such that*

(i) *(Regularity)  $F(t, 0) = 0$ ,  $G(t, 0) = 0$  and for all  $(x_1, x_2, t) \in (\mathbf{R}^d)^2 \times [0, T]$ ,*

$$|F(t, x_1) - F(t, x_2)| \leq \alpha_t |x_1 - x_2| \text{ a.s.},$$

$$\sum_{p=1}^k |G^p(t, x_1) - G^p(t, x_2)|^2 \leq \beta_t^2 |x_1 - x_2|^2 \text{ a.s.}$$

(ii) *(Sliceability)  $\sqrt{\alpha} \star W$ ,  $\beta \star W$  are BMO and respectively  $\varepsilon_1$ ,  $\varepsilon_2$  sliceable with the condition*

$$2m\varepsilon_1^2 + \sqrt{2}\varepsilon_2 C'_m < 1.$$

*Then there exists a solution  $X \in \mathcal{S}^m(\mathbf{R}^d)$  to the equation (2.3) and a constant  $K_{m, \varepsilon_1, \varepsilon_2}$  such that*

$$\|X\|_{\mathcal{S}^m} \leq K_{m, \varepsilon_1, \varepsilon_2} \|X_0\|_{L^m}.$$

A proof can be found in the appendix for reader convenience. From this last proposition we can deduce the following corollary (see [DT08], Corollary 2.1)

**COROLLARY – 2.1** *Let  $m \geq 1$ . We suppose that there are two non-negative adapted processes  $\alpha$  and  $\beta$  such that*

(i) *(Regularity)  $F(t, 0) = 0$ ,  $G(t, 0) = 0$  and for all  $(x_1, x_2, t) \in (\mathbf{R}^d)^2 \times [0, T]$ ,*

$$|F(t, x_1) - F(t, x_2)| \leq \alpha_t |x_1 - x_2| \text{ a.s.},$$

$$\sum_{p=1}^k |G^p(t, x_1) - G^p(t, x_2)|^2 \leq \beta_t^2 |x_1 - x_2|^2 \text{ a.s.}$$

(ii) *(Sliceability)  $\sqrt{\alpha} \star W$ ,  $\beta \star W$  are BMO and respectively  $\varepsilon_1$ ,  $\varepsilon_2$  sliceable with the condition*

$$2m\varepsilon_1^2 + \sqrt{2}\varepsilon_2 C'_m < 1.$$

For  $t \in [0, T]$ , let  $X^{t, I_d}$  the unique solution defined on  $[t, T]$  of the SDE (2.3) such that  $X_t^{t, I_d} = I_d$ . Then  $X^{t, I_d}$  is in  $\mathcal{S}^m(\mathbf{R}^d)$  and satisfies for a constant  $K_m$  depending only on  $C'_m$ ,  $m$ ,  $k$  and  $\varepsilon_1, \varepsilon_2$ :

$$\mathbf{E} \left( \sup_{t \leq s \leq T} |X_s^{t, I_d}|^m \middle| \mathcal{F}_t \right) \leq K_{m, \varepsilon_1, \varepsilon_2}^m.$$

In particular, if  $X$  is an invertible solution to the equation (2.3), we get the reverse Hölder inequality

$$\mathbf{E} \left( \sup_{t \leq s \leq T} |X_t^{-1} X_s|^m \middle| \mathcal{F}_t \right) \leq K_{m, \varepsilon_1, \varepsilon_2}^m.$$

Proof. We can use Proposition 2.2. For all  $t \in [0, T]$  and all event  $\mathbb{A} \in \mathcal{F}_t$ ,

$$\|X^{t, I_d} \times \mathbb{1}_{\mathbb{A}}\|_{\mathcal{S}^m([t, T])} \leq K_{m, \varepsilon_1, \varepsilon_2} \|I_d \times \mathbb{1}_{\mathbb{A}}\|_{L^m}.$$

We get, for all  $t \in [0, T]$ ,

$$\mathbf{E} \left( \sup_{t \leq s \leq T} |X_s^{t, I_d} \times \mathbb{1}_{\mathbb{A}}|^m \right) \leq K_{m, \varepsilon_1, \varepsilon_2}^m \mathbf{E}(|\mathbb{1}_{\mathbb{A}}|^m),$$

and we have

$$\mathbf{E} \left( \sup_{t \leq s \leq T} |X_s^{t, I_d}|^m \times \mathbb{1}_{\mathbb{A}} \right) \leq K_{m, \varepsilon_1, \varepsilon_2}^m \mathbf{E}(\mathbb{1}_{\mathbb{A}}).$$

Definition of conditional expectation gives the result.  $\diamond$

**§ 2.3. Estimates for the first component of the solution to (2.1).** We come back to the linear BSDE (2.1), and we will be able to deduce  $\mathcal{S}^q$ -estimations for  $U$  with  $q$  large enough, including  $q = \infty$ .

PROPOSITION – 2.3 *Let  $m \geq 1$ . We assume that  $B$  and  $A$  are adapted, bounded in absolute value respectively by two real processes  $\beta$  and  $\alpha$  such that:  $\sqrt{\alpha} \star W, \beta \star W$  are BMO and respectively  $\varepsilon_1, \varepsilon_2$  sliceable with the condition*

$$2m\varepsilon_1^2 + \sqrt{2}\varepsilon_2 C'_m < 1.$$

Then

(i) If  $\zeta \in L^\infty$  and  $f \in \mathcal{S}^\infty$ , then  $U \in \mathcal{S}^\infty(\mathbf{R}^d)$  and

$$\|U\|_{\mathcal{S}^\infty} \leq K_{m, \varepsilon_1, \varepsilon_2} (\|\zeta\|_{L^\infty} + T \|f\|_{\mathcal{S}^\infty}),$$

(ii) Let us assume that  $m > 1$ . If  $\zeta \in L^\infty, \sqrt{|f|} \star W \in \text{BMO}$ , then  $U \in \mathcal{S}^\infty$  and

$$\|U\|_{\mathcal{S}^\infty} \leq (m^*)! K_{m, \varepsilon_1, \varepsilon_2} \left( \|\zeta\|_{L^\infty} + \left\| \sqrt{|f|} \star W \right\|_{\text{BMO}}^2 \right).$$

(iii) Let us assume that  $m > 1$ . For all  $q > m^* = \frac{m}{m-1}$ , if  $\left( \zeta, \int_0^T |f_s| ds \right) \in L^q \times L^q$ , then  $U \in \mathcal{S}^q(\mathbf{R}^d)$  and

$$\|U\|_{\mathcal{S}^q}^q \leq 2^{q-1} K_{m, \varepsilon_1, \varepsilon_2}^q \left( \frac{q}{q-m^*} \right)^{q/m^*} \left( \|\zeta\|_{L^q}^q + \left\| \int_0^T |f_s| ds \right\|_{L^q}^q \right).$$

In the following we will denote simply  $\mathcal{K}_{q, m, \varepsilon_1, \varepsilon_2} = 2^{q-1} K_{m, \varepsilon_1, \varepsilon_2}^q \left( \frac{q}{q-m^*} \right)^{q/m^*}$ .

Proof. By using Proposition 2.1, the Cauchy-Schwartz inequality applied to (2.2) gives us, for all  $t \in [0, T]$ :

$$|U_t| \leq \mathbf{E}(|S_t^{-1} S_T| |\zeta| | \mathcal{F}_t) + \mathbf{E} \left( \int_t^T |S_t^{-1} S_s| |f_s| ds \middle| \mathcal{F}_t \right),$$

with

$$dS_t = \sum_{p=1}^k S_t B_t^{(:,p,:)} dW_t^p + S_t A_t dt, \quad S_0 = I_{d \times d}.$$

If  $S$  satisfies a reverse Hölder inequality, we will be able to bound the right member. And  $S$  is the solution of a SDE on  $\mathbf{R}^{d \times d}$  for which we can use Corollary 2.1 with, for all  $1 \leq p \leq k$  and  $(x, y) \in (\mathbf{R}^{d \times d})^2$ ,  $G^p(s, x) = x B_s^{(:,p,:)}$ ,  $F(s, y) = y A_s$ . Let us note that  $|B^{(:,p,:)}| \leq |B|$  for all  $p \in \{1, \dots, k\}$ . Thus there exists a constant  $K_{m, \varepsilon_1, \varepsilon_2}$  such that:

$$\mathbf{E} \left( \sup_{t \leq s \leq T} |S_t^{-1} S_s|^m \right) \leq K_{m, \varepsilon_1, \varepsilon_2}^m.$$

◇ If  $\zeta \in L^\infty$  and  $f \in \mathcal{S}^\infty$ , with the Hölder inequality we have

$$\begin{aligned} |U_t| &\leq \|\zeta\|_{L^\infty} K_{m, \varepsilon_1, \varepsilon_2} + \|f\|_{\mathcal{S}^\infty} \mathbf{E} \left( (T-t) \sup_{t \leq s \leq T} |S_t^{-1} S_s|^m \middle| \mathcal{F}_t \right)^{1/m} \\ &\leq K_{m, \varepsilon_1, \varepsilon_2} (\|\zeta\|_{L^\infty} + T \|f\|_{\mathcal{S}^\infty}). \end{aligned}$$

◇ Let us consider  $m > 1$ , and assume that  $\zeta \in L^\infty$ ,  $\sqrt{|f|} \star W$  is *BMO*. Then with the Hölder and energy inequalities

$$\begin{aligned} |U_t| &\leq K_{m, \varepsilon_1, \varepsilon_2} \|\zeta\|_{L^\infty} + K_{m, \varepsilon_1, \varepsilon_2} \mathbf{E} \left( \left( \int_t^T |f_s| ds \right)^{m^*} \middle| \mathcal{F}_t \right)^{1/m^*} \\ &\leq (m^*)! K_{m, \varepsilon_1, \varepsilon_2} \left( \|\zeta\|_{L^\infty} + \left\| \sqrt{|f|} \star W \right\|_{BMO}^2 \right) \end{aligned}$$

◇ Let us consider  $m > 1$  and  $q > m^*$ . We get, for all  $t \in [0, T]$ ,

$$\begin{aligned} |U_t|^q &\leq 2^{q-1} \left( \mathbf{E}(|S_t^{-1} S_T| |\zeta| | \mathcal{F}_t)^q + \mathbf{E} \left( \int_t^T |S_t^{-1} S_s| |f_s| ds \middle| \mathcal{F}_t \right)^q \right) \\ &\leq 2^{q-1} \mathbf{E} \left( |S_t^{-1} S_T|^m \middle| \mathcal{F}_t \right)^{q/m} \mathbf{E} \left( |\zeta|^{m^*} \middle| \mathcal{F}_t \right)^{q/m^*} \\ &\quad + 2^{q-1} \mathbf{E} \left( \sup_{t \leq s \leq T} |S_t^{-1} S_s|^m \middle| \mathcal{F}_t \right)^{q/m} \mathbf{E} \left( \left( \int_t^T |f_s| ds \right)^{m^*} \middle| \mathcal{F}_t \right)^{q/m^*} \\ &\leq 2^{q-1} K_{m, \varepsilon_1, \varepsilon_2}^q \left( \mathbf{E} \left( |\zeta|^{m^*} \middle| \mathcal{F}_t \right)^{q/m^*} + \mathbf{E} \left( \left( \int_0^T |f_s| ds \right)^{m^*} \middle| \mathcal{F}_t \right)^{q/m^*} \right). \end{aligned}$$

The processes  $M_t = \mathbf{E} \left( |\zeta|^{m^*} \middle| \mathcal{F}_t \right)$  and  $N_t = \mathbf{E} \left( \left( \int_0^T |f_s| ds \right)^{m^*} \middle| \mathcal{F}_t \right)$  are two martingales with terminal values, respectively,  $|\zeta|^{m^*}$  and  $\left( \int_0^T |f_s| ds \right)^{m^*}$ . Hence the Doob maximal inequality gives us, if  $q > m^*$ ,

$$\mathbf{E} \left( \sup_{0 \leq t \leq T} |M_t|^{q/m^*} \right) = \|M\|_{\mathcal{S}^{q/m^*}}^{q/m^*} \leq \left( \frac{q}{q-m^*} \right)^{q/m^*} \|M_T\|_{L^{q/m^*}}^{q/m^*} = \left( \frac{q}{q-m^*} \right)^{q/m^*} \|\zeta\|_{L^q}^q,$$

and

$$\mathbf{E} \left( \sup_{0 \leq t \leq T} |N_t|^{q/m^*} \right) \leq \left( \frac{q}{q-m^*} \right)^{q/m^*} \left\| \int_0^T |f_s| ds \right\|_{L^q}^q.$$

So we obtain the announced result

$$\|U\|_{\mathcal{S}^q}^q \leq 2^{q-1} K_{m, \varepsilon_1, \varepsilon_2}^q \left( \frac{q}{q-m^*} \right)^{q/m^*} \left( \|\zeta\|_{L^q}^q + \left\| \int_0^T |f_s| ds \right\|_{L^q}^q \right).$$

◇

COROLLARY – 2.2 (AFFINE UPPER BOUND) *Let  $m \geq 1$ . Let us consider  $A$  and  $B$  adapted, bounded in absolute value respectively by two real processes  $\alpha$  and  $\beta$  of the form*

$$\alpha_s = K + L\mathcal{A}_s, \quad \beta_s = K' + L'\mathcal{B}_s,$$

*with  $(K, L, K', L') \in (\mathbf{R}^+)^4$ ,  $\mathcal{A}, \mathcal{B}$  two positive real processes such that  $\sqrt{\mathcal{A}} \star W, \mathcal{B} \star W$  are BMO with the condition*

$$mL \left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}^2 + L' \left\| \mathcal{B} \star W \right\|_{BMO} C'_m < \frac{1}{2}.$$

*We have the following estimates, with constants  $K_m, \mathcal{K}_{q,m}$  depending only on  $m, q, K_y, K_z, L_y, L_z$  and the BMO norms  $\left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}, \left\| \mathcal{B} \star W \right\|_{BMO}$ :*

(i) *If  $\zeta \in L^\infty$  and  $f \in \mathcal{S}^\infty$ , then  $U \in \mathcal{S}^\infty(\mathbf{R}^d)$  and*

$$\|U\|_{\mathcal{S}^\infty} \leq K_m (\|\zeta\|_{L^\infty} + T \|f\|_{\mathcal{S}^\infty}),$$

(ii) *Let us assume that  $m > 1$ . If  $\zeta \in L^\infty, \sqrt{|f|} \star W \in BMO$ , then  $U \in \mathcal{S}^\infty$  and*

$$\|U\|_{\mathcal{S}^\infty} \leq (m^*)! K_m \left( \|\zeta\|_{L^\infty} + \left\| \sqrt{|f|} \star W \right\|_{BMO}^2 \right).$$

(iii) *Let us assume that  $m > 1$ . For all  $q > m^* = \frac{m}{m-1}$ , if  $\left( \zeta, \int_0^T |f_s| ds \right) \in L^q \times L^q$ , then  $U \in \mathcal{S}^q(\mathbf{R}^d)$  and*

$$\|U\|_{\mathcal{S}^q}^q \leq 2^{q-1} K_m^q \left( \frac{q}{q-m^*} \right)^{q/m^*} \left( \|\zeta\|_{L^q}^q + \left\| \int_0^T |f_s| ds \right\|_{L^q}^q \right).$$

*In the following we will denote simply  $\mathcal{K}_{q,m}^q \left( \frac{q}{q-m^*} \right)^{q/m^*}$ .*

Proof. We obtain easily estimates about BMO-norms of  $\sqrt{\alpha} \star W$  and  $\beta \star W$ , since for every stopping time  $0 \leq \tau \leq T$ :

$$\begin{aligned} \mathbf{E} \left( \int_\tau^T \beta_s^2 ds \middle| \mathcal{F}_\tau \right) &= \mathbf{E} \left( \int_\tau^T (K' + L'\mathcal{B}_s)^2 ds \middle| \mathcal{F}_\tau \right) \\ &\leq 2 \left( (K')^2 T + (L')^2 \mathbf{E} \left( \int_\tau^T \mathcal{B}_s^2 ds \middle| \mathcal{F}_\tau \right) \right), \end{aligned}$$

$$\mathbf{E} \left( \int_\tau^T \alpha_s ds \middle| \mathcal{F}_\tau \right) \leq KT + L \mathbf{E} \left( \int_\tau^T \mathcal{A}_s ds \middle| \mathcal{F}_\tau \right).$$

By taking essential supremum in the set of all stopping times, we get

$$\|\sqrt{\alpha} \star W\|_{BMO} \leq \sqrt{KT} + \sqrt{L} \|\sqrt{\mathcal{A}} \star W\|_{BMO}, \quad \|\beta \star W\|_{BMO} \leq \sqrt{2} \left( K' \sqrt{T} + L' \|\mathcal{B} \star W\|_{BMO} \right),$$

and it follows that  $\sqrt{\alpha} \star W, \beta \star W$  are BMO.

Let us show that  $\sqrt{\alpha} \star W$  and  $\beta \star W$  are respectively  $\varepsilon_1$  and  $\varepsilon_2$  sliceable with  $2m\varepsilon_1^2 + \sqrt{2}\varepsilon_2 C'_m < 1$ . We consider the following uniform sequence of stopping times

$$T_j = j \frac{T}{N}, \quad j \in \{0, \dots, N\},$$

and a parameter  $\eta > 0$ . The Young inequality gives the following estimation: for all  $i \in \{0, \dots, N\}$ ,

$$\mathbf{E} \left( \int_{T_i}^{T_{i+1}} \beta_s^2 ds \middle| \mathcal{F}_{T_i} \right) \leq 2 \left[ K'^2 \left( 1 + \frac{1}{\eta} \right) \frac{T}{N} + L'^2 (1 + \eta) \mathbf{E} \left( \int_{T_i}^{T_{i+1}} \mathcal{B}_s^2 ds \middle| \mathcal{F}_{T_i} \right) \right],$$

and

$$\mathbf{E} \left( \int_{T_i}^{T_{i+1}} \alpha_s ds \middle| \mathcal{F}_{T_i} \right) \leq \frac{KT}{N} + L \mathbf{E} \left( \int_{T_i}^{T_{i+1}} \mathcal{A}_s ds \middle| \mathcal{F}_{T_i} \right).$$

Proposition 1.2 yields

$$\begin{aligned} \left\| T_n \beta \star W \right\|_{BMO}^{T_{n+1}} &= \operatorname{esssup} \sup_{\tau \in \mathcal{T}^n} \left( \mathbf{E} \left( \int_{\tau}^{T_{n+1}} \beta_s^2 ds \middle| \mathcal{F}_{\tau} \right) \right)^{1/2} \\ &\leq \sqrt{2} \left( \operatorname{esssup} \sup_{\tau \in \mathcal{T}^n} \left( K'^2 \frac{T}{N} \left( 1 + \frac{1}{\eta} \right) + L'^2 (1 + \eta) \mathbf{E} \left( \int_{\tau}^{T_{i+1}} \mathcal{B}_s^2 ds \middle| \mathcal{F}_{\tau} \right) \right)^{1/2} \right). \end{aligned}$$

Taking  $\eta = \frac{1}{\sqrt{N}}$ , we get

$$\left\| T_n (\beta \star W) \right\|_{BMO}^{T_{n+1}} \leq \sqrt{2} \left( K' \sqrt{\frac{T}{N}} \sqrt{1 + \sqrt{N}} + L' \sqrt{1 + \frac{1}{\sqrt{N}}} \|\mathcal{B} \star W\|_{BMO} \right), \quad (2.4)$$

and

$$\left\| T_n (\sqrt{\alpha} \star W) \right\|_{BMO}^{T_{n+1}} \leq \sqrt{K \frac{T}{N} + \sqrt{L}} \left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}. \quad (2.5)$$

By taking  $N$  large enough, we get  $2m\epsilon_1^2 + \sqrt{2}\epsilon_2 C'_m < 1$  since the following upper bound holds true

$$2mL \left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}^2 + 2L' \|\mathcal{B} \star W\|_{BMO} C'_m < 1.$$

◇

**REMARK – 2.2** In the inequalities (2.4) and (2.5), we used that  $\left\| T_i \mathcal{B} \star W \right\|_{BMO}^{T_{i+1}} \leq \|\mathcal{B} \star W\|_{BMO}$  and  $\left\| T_i \sqrt{\mathcal{A}} \star W \right\|_{BMO}^{T_{i+1}} \leq \left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}$ . We can easily obtain a more general result by replacing the following assumption:  $\mathcal{A}, \mathcal{B}$  are two positive real processes such that  $\sqrt{\mathcal{A}} \star W, \mathcal{B} \star W$  are BMO with the condition

$$2mL \left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}^2 + 2L' \|\mathcal{B} \star W\|_{BMO} C'_m < 1,$$

by the new one:  $\mathcal{A}, \mathcal{B}$  are two positive real processes such that  $\sqrt{\mathcal{A}} \star W, \mathcal{B} \star W$  are in  $BMO_{\epsilon_1}$  and  $BMO_{\epsilon_2}$  with the condition

$$2mL\epsilon_1^2 + 2L'\epsilon_2 C'_m < 1.$$

**REMARK – 2.3** We have not mentioned the dependance of the constants with respect to  $\left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}$  and  $\|\mathcal{B} \star W\|_{BMO}$  in the notations, since in the following, under our assumptions it will be not. But we will precise it explicitly when it is important.

### 3 STABILITY, EXISTENCE AND UNIQUENESS RESULTS FOR GENERAL MULTIDIMENSIONAL QUADRATIC BSDES

The  $\delta Z$  control step is treated as in [BE13] by Briand and Elie. The  $\delta Y$  control step should be changed for the multidimensional case: this step requires a BMO control assumption and the previous results concerning linear BSDEs.

**§ 3.1. Stability result.** With a linearization tool we can prove a stability lemma for the BSDE (1.3). Let us consider two solutions of (1.3) in  $\mathbf{R}^d \times \mathbf{R}^{d \times k}$ ,  $(Y^1, Z^1), (Y^2, Z^2)$ , with terminal conditions  $\xi^1$  and  $\xi^2$  and generators

respectively  $f_1$  and  $f_2$ . We assume that  $f_1, f_2$  satisfies the usual conditions **(H)**.

$$\begin{aligned} Y_t^1 &= \xi^1 + \int_t^T f_1(s, Y_s^1, Z_s^1) ds - \int_t^T Z_s^1 dW_s, \quad 0 \leq t \leq T, \\ Y_t^2 &= \xi^2 + \int_t^T f_2(s, Y_s^2, Z_s^2) ds - \int_t^T Z_s^2 dW_s, \quad 0 \leq t \leq T. \end{aligned}$$

Let us denote

$$\begin{aligned} \delta Y_s &= Y_s^1 - Y_s^2, \quad \delta Z_s = Z_s^1 - Z_s^2, \quad \delta F_s = f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2), \\ \delta f_s &= f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2) \quad \text{and} \quad \delta \xi = \xi^1 - \xi^2. \end{aligned}$$

The process  $(\delta Y, \delta Z)$  solves the BSDE

$$\delta Y_t = \delta \xi + \int_t^T \delta F_s ds - \int_t^T \delta Z_s dW_s, \quad 0 \leq t \leq T. \quad (3.1)$$

LEMMA – 3.1 (STABILITY RESULT) *Let  $m > 1$  and  $p > \frac{m^*}{2}$  and let us suppose that*

- (i)  $2mL_y \|Z^1 \star W\|_{BMO}^2 + 2L_z (\|Z^1 \star W\|_{BMO} + \|Z^2 \star W\|_{BMO}) C'_m < 1$ ,
- (ii)  $(\xi_1, \xi_2) \in (L^{2p})^2$ ,
- (iii)  $\int_0^T |\delta f_s| ds \in L^{2p}$ ,

then there exists a constant  $\widetilde{K}_p (\|Z^1 \star W\|_{BMO}, \|Z^2 \star W\|_{BMO})$  (depending only on  $p, K_y, L_y, K_z, L_z, T$  and the BMO norms of  $Z^1 \star W, Z^2 \star W$ ) such that

$$\|\delta Y\|_{\mathcal{L}^{2p}}^p + \|\delta Z \star W\|_{\mathcal{H}^p}^p \leq \widetilde{K}_p (\|Z^1 \star W\|_{BMO}, \|Z^2 \star W\|_{BMO}) \left( \|\delta \xi\|_{L^{2p}}^p + \left\| \int_0^T |\delta f_s| ds \right\|_{L^{2p}}^p \right).$$

Proof. By using a linearization tool, we can rewrite (3.1) as

$$\delta Y = \delta \xi + \int_t^T (A_s \delta Y_s + B_s (\delta Z_s) + \delta f_s) ds - \int_t^T \delta Z_s dW_s,$$

where

◇  $B$  is a  $\mathcal{L}(\mathbf{R}^{d \times k}, \mathbf{R}^d)$  process defined by blocks with

$$B_s = \begin{cases} \left( \frac{f_1^1(s, Y_s^2, Z_s^1) - f_1^1(s, Y_s^2, Z_s^2)}{|\delta Z_s|^2} \delta Z_s \quad \dots \quad \frac{f_1^d(s, Y_s^2, Z_s^1) - f_1^d(s, Y_s^2, Z_s^2)}{|\delta Z_s|^2} \delta Z_s \right) & \text{if } \delta Z_s \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

since we can see a linear map in  $\mathcal{L}(\mathbf{R}^{d \times k}, \mathbf{R}^d)$  as a matrix of size  $d \times (dk)$  (see the introduction). For all  $i$ , the  $i$ -block is thus a  $\mathbf{R}^{d \times k}$  vector.

◇  $A$  is a  $\mathcal{L}(\mathbf{R}^d, \mathbf{R}^d)$ -process defined by

$$A_s = \begin{cases} \frac{f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1)}{|\delta Y_s|^2} (\mathbf{T} \delta Y_s) & \text{if } \delta Y_s \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let us compute the term  $B_s(\delta Z_s)$  on the basis  $(E_{11}, \dots, E_{1k}, E_{21}, \dots, E_{2k}, \dots, E_{dk})$ :

$$\begin{aligned} B_s(\delta Z_s) &= \sum_{(p,p') \in \{1, \dots, d\} \times \{1, \dots, k\}} \delta Z_s^{p,p'} B_s(E_{pp'}) \\ &= \left( \sum_{(p,p') \in \{1, \dots, d\} \times \{1, \dots, k\}} \delta Z_s^{p,p'} \frac{f_1^i(s, Y_s^2, Z_s^1) - f_1^i(s, Y_s^2, Z_s^2)}{|\delta Z_s|^2} \delta Z_s^{p,p'} \right)_{1 \leq i \leq d} \\ &= f_2(s, Y_s^2, Z_s^1) - f_2(s, Y_s^2, Z_s^2). \end{aligned}$$

In the following  $B(\delta Z)$  will be simply denoted  $B\delta Z$ . Assumption **(H)** on  $f_1$  and  $f_2$  gives the following inequalities:

$$\begin{aligned} |B_s| &= \sum_{i=1}^d \left| \frac{f_1^i(s, Y_s^2, Z_s^1) - f_1^i(s, Y_s^2, Z_s^2)}{|\delta Z_s|^2} \delta Z_s \right| \leq K_z + L_z (|Z_s^1| + |Z_s^2|), \\ |A_s| &\leq K_y + L_y |Z_s^1|^2. \end{aligned}$$

**Step 1 – Control of  $\delta Y$ .**  $A$  and  $B$  are bounded in absolute value respectively by two real processes  $\alpha$  and  $\beta$  defined by

$$\alpha = K_y + L_y |Z^1|^2, \quad \beta = K_z + L_z (|Z^1| + |Z^2|),$$

and  $(\delta Y, \delta Z)$  solves a linear BSDE of the form (2.1) with  $\delta f$  instead of  $f$ . We can apply Corollary 2.2, (iii) with

$$\mathcal{B} = |Z^1| + |Z^2|, \quad \mathcal{A} = |Z^1|^2, \quad L' = L_z, \quad K = K_y, \quad K' = K_z, \quad \text{and} \quad L = L_y,$$

which gives, for all  $q > 1$  such that  $q > m^*$ ,

$$\|\delta Y\|_{\mathcal{S}^q}^q \leq \mathcal{K}_{q,m,\varepsilon_1,\varepsilon_2} \left( \|\delta \xi\|_{L^q}^q + \left\| \int_0^T |\delta f_s| ds \right\|_{L^q}^q \right). \quad (3.2)$$

**Step 2 – Control of  $\delta Z$ .** The Itô formula applied to  $|\delta Y|^2$  gives us

$$\begin{aligned} \int_0^T |\delta Z_s|^2 ds &= |\delta \xi|^2 - |\delta Y_0|^2 - 2 \int_0^T \delta Y_s \cdot (\delta Z_s dW_s) + 2 \int_0^T (\delta Y \cdot \delta F)_s ds \\ &\leq |\delta \xi|^2 - 2 \int_0^T \delta Y_s \cdot (\delta Z_s dW_s) + 2 \int_0^T (\delta Y \cdot \delta F)_s ds. \end{aligned} \quad (3.3)$$

Recalling assumption **(H)** we have

$$|\delta F_s| = |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)| \leq (K_y + L_y |Z_s^1|^2) |\delta Y_s| + (K_z + L_z (|Z_s^1| + |Z_s^2|)) |\delta Z_s| + |\delta f_s|.$$

With the Cauchy-Schwarz and Young inequalities, we get

$$\begin{aligned} 2 \int_0^T (\delta Y \cdot \delta F)_s ds &\leq 2 \int_0^T |\delta Y_s| |\delta F_s| ds \\ &\leq 2 \int_0^T \left[ (K_y + L_y |Z_s^1|^2) |\delta Y_s|^2 + (K_z + L_z (|Z_s^1| + |Z_s^2|)) |\delta Y_s| |\delta Z_s| + |\delta f_s| |\delta Y_s| \right] ds \\ &\leq 2 \left( \sup_{0 \leq s \leq T} |\delta Y_s|^2 \right) \int_0^T \left[ K_y + L_y |Z_s^1|^2 + (K_z + L_z (|Z_s^1| + |Z_s^2|))^2 + \frac{1}{2} \right] ds + \frac{1}{2} \int_0^T |\delta Z_s|^2 ds + \left( \int_0^T |\delta f_s| ds \right)^2. \end{aligned}$$

By using this last inequality in (3.3) we obtain

$$\begin{aligned} \frac{1}{2} \int_0^T |\delta Z_s|^2 ds &\leq |\delta \xi|^2 - 2 \int_0^T \delta Y_s \cdot (\delta Z_s dW_s) \\ &\quad + 2 \left( \sup_{0 \leq s \leq T} |\delta Y_s|^2 \right) \int_0^T \left[ K_y + L_y |Z_s^1|^2 + (K_z + L_z (|Z_s^1| + |Z_s^2|))^2 + \frac{1}{2} \right] ds + \left( \int_0^T |\delta f_s| ds \right)^2. \end{aligned}$$



Thus, for all  $p \geq 1$ , there exists a constant  $K$  depending only on  $p$  such that

$$\begin{aligned} \|\delta Z \star W\|_{\mathcal{H}^p}^p &\leq K \left[ \|\delta \xi\|_{L^p}^p + \mathbf{E} \left( \left( \sup_{0 \leq t \leq T} \left| \int_0^t \delta Y \cdot (\delta Z_s dW_s) \right| \right)^{p/2} \right) \right. \\ &\left. + \mathbf{E} \left( \left( \sup_{0 \leq s \leq T} |\delta Y_s|^2 \int_0^T \left[ K_y + L_y |Z_s^1|^2 + \frac{1}{2} (K_z + L_z (|Z_s^1| + |Z_s^2|))^2 + 1 \right] ds \right)^{p/2} \right) + \mathbf{E} \left( \left( \int_0^T |\delta f_s| ds \right)^p \right) \right]. \end{aligned}$$

In the following we keep the notation  $K$  for all constants appearing in the upper bounds. Since  $\left( \int_0^\cdot (\delta Y \delta Z)_s \cdot dW_s \right)$  is a local martingale vanishing at 0, then, according to the *BDG* inequalities, we get for all  $p \geq 1$ :

$$\mathbf{E} \left( \left( \sup_{0 \leq t \leq T} \left| \int_0^t (\delta Y \delta Z)_s \cdot dW_s \right| \right)^{p/2} \right) = \left\| \int_0^\cdot (\delta Y \delta Z)_s \cdot dW_s \right\|_{\mathcal{H}^{p/2}}^{p/2} \leq (C'_{p/2})^{p/2} \left\| \int_0^\cdot \delta Y \cdot (\delta Z_s dW_s) \right\|_{\mathcal{H}^{p/2}}^{p/2}.$$

Since we have

$$\left\| \int_0^\cdot \delta Y \cdot (\delta Z_s dW_s) \right\|_{\mathcal{H}^{p/2}}^{p/2} = \mathbf{E} \left( \left( \sum_{k=1}^d \int_0^T (\delta Y_s \cdot \delta Z_s^{(\cdot, k)})^2 ds \right)^{p/4} \right) \leq \mathbf{E} \left( \left( \sup_{0 \leq s \leq T} |\delta Y_s|^2 \times \int_0^T |\delta Z_s|^2 ds \right)^{p/4} \right),$$

then the Cauchy-Schwartz inequality gives us

$$\mathbf{E} \left( \left( \sup_{0 \leq t \leq T} \left| \int_0^t (\delta Y \delta Z)_s \cdot dW_s \right| \right)^{p/2} \right) \leq (C'_{p/2})^{p/2} \|\delta Z \star W\|_{\mathcal{H}^p}^{p/2} \|\delta Y\|_{\mathcal{S}^p}^{p/2}.$$

Moreover we obtain with the Cauchy-Schwarz and Young inequalities:

$$\begin{aligned} &\|\delta Z \star W\|_{\mathcal{H}^p}^p \\ &\leq K \left[ \|\delta \xi\|_{L^p}^p + \|\delta Z \star W\|_{\mathcal{H}^p}^{p/2} \|\delta Y\|_{\mathcal{S}^p}^{p/2} + \right. \\ &\quad \left. \|\delta Y\|_{\mathcal{S}^{2p}}^p \mathbf{E} \left( \left( \int_0^T [1 + |Z_s^1|^2 + |Z_s^2|^2] ds \right)^p \right)^{1/2} + \mathbf{E} \left( \left( \int_0^T |\delta f_s| ds \right)^p \right) \right] \\ &\leq K \left[ \|\delta \xi\|_{L^p}^p + \|\delta Y\|_{\mathcal{S}^p}^p + \|\delta Y\|_{\mathcal{S}^{2p}}^p \mathbf{E} \left( \left( \int_0^T [1 + |Z_s^1|^2 + |Z_s^2|^2] ds \right)^p \right)^{1/2} + \mathbf{E} \left( \left( \int_0^T |\delta f_s| ds \right)^p \right) \right] \\ &\quad + \frac{1}{2} \|\delta Z \star W\|_{\mathcal{H}^p}^p. \end{aligned}$$

The energy inequality allows us to bound  $\mathbf{E} \left( \left( \int_0^T [1 + |Z_s^1|^2 + |Z_s^2|^2] ds \right)^p \right)$  by

$$K \left( 1 + \|Z^1 \star W\|_{BMO}^{2p} + \|Z^2 \star W\|_{BMO}^{2p} \right),$$

which is finite recalling (i). Finally, for all  $p \geq 1$ , there exists some constant  $K$  (which depends only on  $p, K_y, L_y, K_z, L_z, T$  and the BMO norms of  $Z^1 \star W, Z^2 \star W$ ) such that

$$\|\delta Z \star W\|_{\mathcal{H}^p}^p \leq K \left( \|\delta \xi\|_{L^p}^p + \|\delta Y\|_{\mathcal{S}^{2p}}^p + \left\| \int_0^T |\delta f_s| ds \right\|_{L^p}^p \right). \quad (3.4)$$

**Step 3 – Stability.** Considering  $p > \frac{m^*}{2}$  and combining (3.2) where  $q = 2p$  with (3.4), we obtain existence of a constant  $\widetilde{K}_p$  ( $\|Z^1 \star W\|_{BMO}, \|Z^2 \star W\|_{BMO}$ ) which depends only on  $p, K_y, L_y, K_z, L_z, T, K$  and the BMO norms of

$Z^1 \star W, Z^2 \star W$  such that

$$\|\delta Y\|_{\mathcal{L}^{2p}}^p + \|\delta Z \star W\|_{\mathcal{H}^p}^p \leq \widetilde{K}_p (\|Z^1 \star W\|_{BMO}, \|Z^2 \star W\|_{BMO}) \left( \|\delta \xi\|_{L^{2p}}^p + \left\| \int_0^T |\delta f_s| ds \right\|_{L^{2p}}^p \right).$$

◇

REMARK – 3.1 We can obtain also a symmetrized version of (i) by changing the linearization step in the proof. For example we can commute  $Z^1$  and  $Z^2$  and (i) becomes

$$mL_y \|Z^2 \star W\|_{BMO}^2 + L_z (\|Z^1 \star W\|_{BMO} + \|Z^2 \star W\|_{BMO}) C'_m < \frac{1}{2}.$$

But we can also note that  $\delta F_s = \widetilde{A}_s \delta Y_s + \widetilde{B}_s \delta Z_s + \delta f_s$ , where

$$\widetilde{A}_s \delta Y_s = \frac{1}{2} (f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^1) + f^1(s, Y_s^1, Z_s^2) - f^1(s, Y_s^2, Z_s^2)),$$

$$\widetilde{B}_s \delta Z_s = \frac{1}{2} (f^1(s, Y_s^2, Z_s^1) - f^1(s, Y_s^2, Z_s^2) + f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^1, Z_s^2)),$$

and symmetric estimates for  $\widetilde{A}$  and  $\widetilde{B}$  hold true:

$$|\widetilde{A}_s| \leq K_y + \frac{L_y}{2} (|Z_s^1|^2 + |Z_s^2|^2), \quad |\widetilde{B}_s| \leq K_z + L_z (|Z_s^1| + |Z_s^2|).$$

Then (i) becomes

$$mL_y (\|Z^1 \star W\|_{BMO}^2 + \|Z^2 \star W\|_{BMO}^2) + L_z C'_m (\|Z^1 \star W\|_{BMO} + \|Z^2 \star W\|_{BMO}) < 1.$$

Let us denote for all  $m > 1$  by  $\mathcal{L}_{BMO}^m$  the set of all  $BMO$  processes for which an *a priori* upper bound on this norm is in force:

$$\mathcal{L}_{BMO}^m = \left\{ Z, \mathbf{R}^{d \times k} \text{-valued } BMO \text{ process} \left/ mL_y \|Z \star W\|_{BMO}^2 + L_z C'_m \|Z \star W\|_{BMO} < \frac{1}{2} \right. \right\}.$$

If  $L_y \neq 0$  we have:

$$\mathcal{L}_{BMO}^m = \left\{ Z, \mathbf{R}^{d \times k} \text{-valued } BMO \text{ process} \left/ \|Z^M \star W\|_{BMO} < \frac{-L_z C'_m + \sqrt{2mL_y + (L_z C'_m)^2}}{2mL_y} \right. \right\},$$

and if  $L_y = 0$  we can assume that  $L_z \neq 0$  (otherwise we obtain the classical Lipschitz framework) and we have

$$\mathcal{L}_{BMO}^m = \left\{ Z, \mathbf{R}^{d \times k} \text{-valued } BMO \text{ process} \left/ \|Z^M \star W\|_{BMO} < \frac{1}{2L_z C'_m} \right. \right\}.$$

We denote by  $\mathbb{B}^m(L_y, L_z)$ :

$$\mathbb{B}^m(L_y, L_z) = \begin{cases} \frac{-L_z C'_m + \sqrt{2mL_y + (L_z C'_m)^2}}{2mL_y} & \text{if } L_y \neq 0, \\ \frac{1}{2L_z C'_m} & \text{if } L_y = 0. \end{cases}$$

We keep the notation  $\widetilde{K} (\|Z^1 \star W\|_{BMO}, \|Z^2 \star W\|_{BMO})$  for the constant appearing in the following stability result.

COROLLARY – 3.1 (STABILITY RESULT (2)) Let  $m > 1$ ,  $p > \frac{m^*}{2}$  and let us suppose that

- (i)  $(\xi_1, \xi_2) \in (L^{2p})^2$ ,
- (ii)  $\int_0^T |\delta f_s| ds \in L^{2p}$ .

If  $Z^1$  and  $Z^2$  are in  $\mathcal{L}_{BMO}^m$ , then there exists a constant  $\widetilde{K}_p(\|Z^1 \star W\|_{BMO}, \|Z^2 \star W\|_{BMO})$  (depending only on  $p, K_y, L_y, K_z, L_z, T$  and the BMO norms of  $Z^1 \star W, Z^2 \star W$ ) such that

$$\|\delta Y\|_{\mathcal{S}^{2p}}^p + \|\delta Z \star W\|_{\mathcal{H}^p}^p \leq \widetilde{K}_p(\|Z^1 \star W\|_{BMO}, \|Z^2 \star W\|_{BMO}) \left( \|\delta \xi\|_{L^{2p}}^p + \left\| \int_0^T |\delta f_s| ds \right\|_{L^{2p}}^p \right).$$

**§ 3.2. Existence and uniqueness results when terminal condition and the generator have bounded Malliavin derivative.** Let us consider some assumptions on Malliavin derivatives of  $\xi$  and  $f$ .

**(Dxi,b)** Malliavin derivative of  $\xi$  is bounded. That is

$$\|\mathbf{D}\xi\|_{\mathcal{S}^\infty} = \sup_{0 \leq t \leq T} \|\mathbf{D}_t \xi\|_{L^\infty} < \infty.$$

**(Df,b)** (i) For all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ , we have

$$f(t, y, z) \in \mathbb{D}^{1,2}(\mathbf{R}^d), \quad \text{and} \quad \mathbf{E} \left( \int_0^T \int_0^T |\mathbf{D}_u f(s, y, z)| du ds \right) < \infty.$$

(ii) There exists  $C \geq 0$  such that for all  $(u, t, y, z) \in [0, T]^2 \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ ,

$$|\mathbf{D}_u f(t, y, z)| \leq C(1 + |z|^2) \text{ a.s.}$$

(iii) For all  $(u, t) \in [0, T]^2$ , there exists a random variable  $C_u(t)$  such that for all  $(y^1, z^1, y^2, z^2) \in (\mathbf{R}^d \times \mathbf{R}^{d \times k})^2$ ,

$$|\mathbf{D}_u f(t, y^1, z^1) - \mathbf{D}_u f(t, y^2, z^2)| \leq C_u(t)(|y^1 - y^2| + |z^1 - z^2|) \text{ a.s.}$$

We recall first that  $(Y^M, Z^M)$  is the unique solution of (1.4). For all  $m > 1$ , let us consider the following assumption:

**(BMO,m)** there exists a constant  $\mathbb{K}$  such that

- (i)  $mL_y \mathbb{K}^2 + L_z \mathbb{K} C'_m < \frac{1}{2}$ , or equivalently  $\mathbb{K} < \mathbb{B}^m(L_y, L_z)$ ,
- (ii)  $\sup_{M \in \mathbf{R}^+} \|Z^M \star W\|_{BMO} \leq \mathbb{K}$ .

**THEOREM – 3.1 (EXISTENCE AND UNIQUENESS (1))** Let  $m > 1$ . Under the main assumption **(H)**, the BMO estimation **(BMO,m)**, and the boundeness of the Malliavin derivatives of  $\xi$  and  $f$ , **(Dxi,b)**–**(Df,b)**, the quadratic BSDE (1.3) has a unique solution in  $\mathcal{S}^\infty(\mathbf{R}^d) \times (\mathcal{S}^\infty(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ .

To show this theorem we begin to prove the following proposition which gives an uniform  $\mathcal{S}^\infty$  estimates for  $Y^M$ . This is the keystone of our procedure.

**PROPOSITION – 3.1** Let  $m > 1$ . If assumptions **(H)**–**(BMO,m)**–**(Dxi,b)**–**(Df,b)** hold true then  $Z^M \in \mathcal{S}^\infty(\mathbf{R}^d)$  and

$$\sup_M \|Z^M\|_{\mathcal{S}^\infty} < \infty.$$

Proof.

**Malliavin differentiation.** We assume that  $f$  is continuously differentiable with respect to  $(y, z)$ . This assumption is not restrictive by considering a smooth regularization of  $f$ .

Recalling assumptions **(Dxi,b)** and **(Df,b)**, Proposition 5.3 in [EKPQ97] gives us that for all  $0 \leq u \leq t \leq T$ ,  $Y_t^M$  and  $Z_t^M$  are respectively in  $\mathbb{D}^{1,2}(\mathbf{R}^d)$  and  $\mathbb{D}^{1,2}(\mathbf{R}^{d \times k})$ . Moreover the process  $(\mathbf{D}_u Y^M, \mathbf{D}_u Z^M) = (\mathbf{D}_u Y_t^M, \mathbf{D}_u Z_t^M)_{0 \leq t \leq T}$  solves for all  $u$  the following linear BSDE in  $\mathbf{R}^{d \times k}$ :

$$\begin{aligned} \mathbf{D}_u Y_t^M &= \mathbf{D}_u \xi + \int_t^T \left( \nabla_y f^M(s, Y_s^M, Z_s^M) \mathbf{D}_u Y_s^M + \nabla_z f^M(s, Y_s^M, Z_s^M) \mathbf{D}_u Z_s^M \right. \\ &\quad \left. + (\mathbf{D}_u f^M)(s, Y_s^M, Z_s^M) \right) ds - \int_t^T \mathbf{D}_u Z_s^M dW_s, \end{aligned} \quad (3.5)$$

and  $(\mathbf{D}_t Y_t)_{0 \leq t \leq T}$  is a version of  $(Z_t)_{0 \leq t \leq T}$ .

Let us emphasize that BSDE (3.5) means that for each  $p \in \{1, \dots, k\}$ ,

$$\begin{aligned} \mathbf{D}_u^p Y_t^M &= \mathbf{D}_u^p \xi + \int_t^T \left( \nabla_y f^M(s, Y_s^M, Z_s^M) \mathbf{D}_u^p Y_s^M + \nabla_z f^M(s, Y_s^M, Z_s^M) \mathbf{D}_u^p Z_s^M \right. \\ &\quad \left. + (\mathbf{D}_u^p f^M)(s, Y_s^M, Z_s^M) \right) ds - \int_t^T \mathbf{D}_u^p Z_s^M dW_s, \end{aligned} \quad (3.6)$$

besides  $\mathbf{D}^p Y^M$  is a process with values in  $\mathbf{R}^d$  for each integer  $p$ .

**$\mathcal{S}^\infty$ -Estimation.** We are looking for an  $\mathcal{S}^\infty$ -estimate for  $\mathbf{D}_u Y^M$  for all  $u \in [0, T]$  applying results of section 2. Since  $|\nabla_z \rho^M(z)| \leq 1$ , we obtain the following inequalities by recalling the main assumption **(H)**,

$$\begin{aligned} |\nabla_y f^M(s, Y_s^M, Z_s^M)| &= |\nabla_y f(s, Y_s^M, \rho^M(Z_s^M))| \leq K_y + L_y |Z_s^M|^2, \\ |\nabla_z f^M(s, Y_s^M, Z_s^M)| &= |\nabla_z f(s, Y_s^M, \rho^M(Z_s^M))| \leq K_z + 2L_z |Z_s^M|. \end{aligned}$$

Let us consider the two positive processes  $\alpha^M$  and  $\beta^M$  defined below,

$$\alpha^M = K_y + L_y |Z^M|^2, \quad \beta^M = K_z + 2L_z |Z^M|.$$

For all  $p \in \{1, \dots, k\}$ , by recalling **(BMO,m)**, we can apply Corollary 2.2 (iii), to the BSDE (3.6) with the following constants and processes:

$$L = L_y, \quad K = K_y, \quad K' = K_z, \quad L' = 2L_z, \quad \mathcal{A} = |Z^M|^2, \quad \mathcal{B} = |Z^M|.$$

Thus, we obtain for all  $u \in [0, T]$

$$\begin{aligned} \|\mathbf{D}_u Y^M\|_{\mathcal{S}^\infty} &\leq \sum_{p=1}^k \|\mathbf{D}_u^p Y^M\|_{\mathcal{S}^\infty} \leq (m^*)! K_m \sum_{p=1}^k \left( \|\mathbf{D}_u^p \xi\|_{L^\infty} + \left\| \sqrt{|\mathbf{D}_u^p f^M(\cdot, Y^M, Z^M)|} \star W \right\|_{BMO}^2 \right) \\ &\leq k \times (m^*)! K_m \left( \|\mathbf{D}_u \xi\|_{L^\infty} + \left\| \sqrt{|\mathbf{D}_u f^M(\cdot, Y^M, Z^M)|} \star W \right\|_{BMO}^2 \right). \end{aligned}$$

Under the assumption **(Df,b)** together with **(BMO,m)**, the last term has an uniformly  $\mathcal{S}^\infty$ -upper bound with respect to  $M$  since, for all  $(u, t) \in [0, T]^2$ ,

$$\mathbf{E} \left( \int_t^T |\mathbf{D}_u f^M(s, Y_s^M, Z_s^M)| ds \middle| \mathcal{F}_t \right) \leq C \left( T + \|Z^M \star W\|_{BMO}^2 \right),$$

hence we deduce

$$\sup_M \left\| \sqrt{|\mathbf{D}_u f^M(\cdot, Y^M, Z^M)|} \star W \right\|_{BMO} \leq \sqrt{C} \left( \sqrt{T} + \sup_M \|Z^M \star W\|_{BMO} \right).$$

The last supremum is finite under assumption **(BMO,m)** and we obtain the announced result taking  $u = t$ .

We omitted the dependance with respect to  $\|Z^M \star W\|_{BMO}$  of the constant  $K_m$ , since  $K_m$  is equal (see the appendix part) to

$$\sum_{i=0}^{N-1} \left( \frac{1}{1 - 2mL_y \|Z^M \star W\|_{BMO}^2 - 2L_z \|Z^M \star W\|_{BMO} C'_m} \right)^i \leq \sum_{i=0}^{N-1} \left( \frac{1}{1 - 2mL_y \mathbb{K}^2 - 2L_z \mathbb{K} C'_m} \right)^i,$$

where  $N$  is an integer large enough and the uniform bound with respect to  $M$  follows.  $\diamond$

**Proof.** [of Theorem 3.1] Uniqueness is already shown with stability Lemma 3.1 since, for all  $p > 1$ ,  $\mathcal{S}^\infty(\mathbf{R}^d) \subset \mathcal{S}^p(\mathbf{R}^d)$  and

$$\mathcal{S}^\infty(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m \subset \mathcal{H}^p(\mathbf{R}^{d \times k})$$

for all  $p > 1$ . For existence, we can fix  $M^* > \sup_M \|Z^M\|_{\mathcal{S}^\infty}$  thanks to Proposition 3.1. Because of the assumptions about  $f^M$  we get

$$f^{M^*}(s, Y_s^{M^*}, Z_s^{M^*}) = f(s, Y_s^{M^*}, \rho^{M^*}(Z_s^{M^*})) = f(s, Y_s^{M^*}, Z_s^{M^*}).$$

$(Y^{M^*}, Z^{M^*})$  becomes a solution of the quadratic BSDE (1.3) in  $\mathcal{S}^\infty(\mathbf{R}^d) \times (\mathcal{S}^\infty(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ .  $\diamond$

**§ 3.3. Extension to general terminal values and generators.** The aim of this subsection is to relax the assumptions **(Dxi,b)** and **(Df,b)** with density arguments. To ensure the convergence, the keystone result will be the stability Lemma 3.1.

We are going to assume that we can see  $f$  as a deterministic function  $\mathbf{f}$  of a progressively measurable continuous process. Randomness will be contained into this process.

- (H')** (i) There exists a progressively measurable continuous process  $\alpha \in \mathcal{S}^\infty$  with values in  $\mathbf{R}^{d'}$ ,  $d' \geq 1$ , and a function  $\mathbf{f}: \mathbf{R}^{d'} \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  such that for all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ :

$$f(t, y, z) = \mathbf{f}(\alpha_t, y, z).$$

Besides, we assume that **(H)** is true for  $\mathbf{f}$ .

- (ii) There exists  $D \in \mathbf{R}^+$  such that for all  $(y, z) \in \mathbf{R}^d \times \mathbf{R}^{d \times k}$ ,  $(\beta, \beta') \in (\mathbf{R}^{d'})^2$ :

$$|\mathbf{f}(\beta, y, z) - \mathbf{f}(\beta', y, z)| \leq D(1 + |z|^2) |\beta - \beta'|. \quad (3.7)$$

**Notation.** For  $\eta \in L^2(\Omega, \mathcal{F}_T)$ ,  $\beta \in \mathcal{S}^\infty$  and  $M \in \mathbf{R}^+$ , we denote by  $(Y^{(M, \eta, \beta)}, Z^{(M, \eta, \beta)})$  the unique solution of the BSDE

$$Y_t^{(M, \eta, \beta)} = \eta + \int_t^T \mathbf{f}^M(\beta_s, Y_s^{(M, \eta, \beta)}, Z_s^{(M, \eta, \beta)}) ds - \int_t^T Z_s^{(M, \eta, \beta)} dW_s, \quad 0 \leq t \leq T, \quad (3.8)$$

where for all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$  and  $\alpha$  a process with values in  $(\mathbf{R}^{d'})$ ,  $\mathbf{f}^M(\alpha_t, y, z) = f^M(t, y, z)$ .

**(BMO2,m)** We assume that  $\xi \in L^{2m^*}$  and that there exists a constant  $\mathbb{K}$  such that

- (i)  $mL_y \mathbb{K}^2 + L_z \mathbb{K} C'_m < \frac{1}{2}$ ,
- (ii)  $\sup_{M \in \mathbf{R}^+} \sup_{\|\eta\|_{L^{2m^*}} \leq \|\xi\|_{L^{2m^*}}, \|\beta\|_{L^2(\Omega \times [0, T])} \leq \|\alpha\|_{L^2(\Omega \times [0, T])}} \|Z^{(M, \eta, \beta)} \star W\|_{BMO} \leq \mathbb{K}.$

**THEOREM – 3.2 (EXISTENCE AND UNIQUENESS (2))** *Let  $m > 1$ . Under the main assumption **(H')** and the BMO estimation **(BMO2,m)**, the quadratic BSDE (1.3) has a unique solution in  $\mathcal{S}^{2m^*}(\mathbf{R}^d) \times (\mathcal{H}^{m^*}(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ .*

Proof.

**Step 1– Approximations.** We can approach  $\xi$  with a sequence of random variables  $(\xi^n)_{n \in \mathbf{N}}$  such that for every  $n$ ,  $\xi^n$  has a bounded Malliavin derivative:

$$\|\mathbf{D}\xi^n\|_{\mathcal{S}^\infty} < \infty.$$

More precisely  $\xi^n$  can be chosen of the form  $\Phi^n(W_{t_1}, \dots, W_{t_n})$  where  $\Phi^n \in \mathcal{C}_b^\infty(\mathbf{R}^n)$ ,  $(t_1, \dots, t_n) \in [0, T]^n$  and  $\xi^n$  tends to  $\xi$  in every  $L^p$  for  $p \geq 1$  (see [Nua06], Exercise 1.1.7).

Since  $\alpha$  is adapted, we can approach this process with a sequence of sample processes  $\alpha^n$  of the form

$$\alpha_t^n = \sum_{i=0}^{p_n-1} \alpha_i^n \mathbb{1}_{[t_i^n, t_{i+1}^n]}(t),$$

where  $(t_i^n)_{i=0}^{p_n}$  is a sequence of subdivisions of  $[0, T]$ , with  $\sup_{0 \leq i \leq p_n-1} |t_{i+1}^n - t_i^n| \xrightarrow{n \rightarrow \infty} 0$ , and, for all  $0 \leq i \leq p_n - 1$ ,  $n \in \mathbf{N}$ ,  $\alpha_i^n$  is a  $\mathcal{F}_{t_i^n}$ -measurable bounded random variable. We have a convergence of this sequence to  $\alpha$  in  $L^2(\Omega \times [0, T])$ :

$$\mathbf{E} \left( \int_0^T |\alpha_s^n - \alpha_s|^2 ds \right) \xrightarrow{n \rightarrow \infty} 0.$$

We can assume in addition that for all  $n$  and for all  $0 \leq i \leq p_n$ ,  $\alpha_i^n$  has a bounded Malliavin derivative since this set is dense in  $L^2(\Omega)$ . It is obvious that for all  $0 \leq u \leq T$  and  $0 \leq t \leq T$ ,

$$\mathbf{D}_u \alpha_t^n = \sum_{i=0}^{p_n-1} \mathbf{D}_u \alpha_i^n \mathbb{1}_{[t_i^n, t_{i+1}^n]}(u).$$

According to Proposition 1.3 applied to  $\varphi = \mathbf{f}(\cdot, y, z)$ , there exists for all  $n \in \mathbf{N}$  and  $t \in [0, T]$  a bounded random variable  $\mathbf{G}$  such that

$$\mathbf{D}_t \mathbf{f}(\alpha_t^n, y, z) = \mathbf{G} \cdot \mathbf{D}_t \alpha_t^n, \quad \text{and} \quad |\mathbf{G}| \leq D(1 + |z|^2).$$

For each  $n \in \mathbf{N}$ :  $\xi^n$  satisfies **(Dxi,b)**,  $\mathbf{f}(\alpha^n, \dots)$  satisfies **(Df,b)** hence for all  $M \in \mathbf{R}^+$ ,  $\mathbf{f}^M(\alpha^n, \dots)$  too. And we can consider the localized approximated BSDE for all  $M \in \mathbf{R}^+$  and  $n \in \mathbf{N}$ :

$$Y_t^{M,n} = \xi^n + \int_t^T \mathbf{f}^M(\alpha_s^n, Y_s^{M,n}, Z_s^{M,n}) ds - \int_t^T Z_s^{M,n} dW_s, \quad 0 \leq t \leq T.$$

For each  $n \in \mathbf{N}$ , under **(H')**, **(Dxi,b)** for  $\xi^n$ , **(Df,b)** for  $\mathbf{f}^M(\alpha^n, \dots)$  and **(BMO2,m)**, we can apply Proposition 3.1 and we obtain that  $\sup_{M \in \mathbf{R}^+} \|Z^{M,n}\|_{\mathcal{S}^\infty} < \infty$ . The classical procedure of localization allows us to take  $M$  large enough and we get  $(Y^n, Z^n)$  solution in  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  of

$$Y_t^n = \xi^n + \int_t^T \mathbf{f}(\alpha_s^n, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T.$$

**Step 2– Application of the stability result.** We can assume that for all  $n$ ,  $\|\xi^n\|_{L^{2m^*}} \leq \|\xi\|_{L^{2m^*}}$ . If it is not true, we consider the sequence  $\widetilde{\xi}^n = \frac{\|\xi\|_{L^{m^*}}}{\|\xi\|_{L^{m^*}} + \|\xi^n - \xi\|_{L^{m^*}}} \xi^n$  instead of  $\xi^n$ . The same argument allows us to assume that

$$\|\alpha^n\|_{L^{2m^*}(\Omega \times [0, T])} \leq \|\alpha\|_{L^{2m^*}(\Omega \times [0, T])}.$$

Under **(BMO2,m)**, we have the estimate

$$mL_y \|Z^n \star W\|_{BMO}^2 + L_z \|Z^n \star W\|_{BMO} C'_m \leq mL_y \mathbb{K}^2 + L_z \mathbb{K} C'_m < \frac{1}{2}.$$

Hence, for all  $n_1, n_2 \in \mathbf{N}$ , we can use the stability Lemma 3.1 for  $p = m^*$  which gives us

$$\begin{aligned} & \|Y^{n_1} - Y^{n_2}\|_{\mathcal{S}^{2m^*}}^{m^*} + \|Z^{n_1} \star W - Z^{n_2} \star W\|_{\mathcal{H}^{m^*}}^{m^*} \\ & \leq \tilde{K}_{m^*} \left( \|\xi^{n_1} - \xi^{n_2}\|_{L^{2m^*}}^{m^*} + \mathbf{E} \left( \left( \int_0^T |\mathbf{f}(\alpha_t^{n_1}, Y_t^{n_2}, Z_t^{n_2}) - \mathbf{f}(\alpha_t^{n_2}, Y_t^{n_2}, Z_t^{n_2})| dt \right)^{2m^*} \right)^{1/2} \right). \end{aligned}$$

The constant  $\tilde{K}_{m^*}$  appearing does not depend on  $n$  under **(BMO2,m)**: this fact was already highlighted in the proof of Proposition 3.1 where an explicit formula for  $\tilde{K}_{m^*}$  is given.  $(\xi^n)_{n \in \mathbf{N}}$  is a Cauchy sequence in  $L^{2m^*}$ :

$$\|\xi^{n_1} - \xi^{n_2}\|_{L^{2m^*}} \xrightarrow{n_1, n_2 \rightarrow \infty} 0.$$

For the second term, we use the Hölder inequality:

$$\begin{aligned} & \mathbf{E} \left( \left( \int_0^T |\mathbf{f}(\alpha_t^{n_1}, Y_t^{n_2}, Z_t^{n_2}) - \mathbf{f}(\alpha_t^{n_2}, Y_t^{n_2}, Z_t^{n_2})| dt \right)^{2m^*} \right) \\ & \leq D^{2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^2) |\alpha_t^{n_1} - \alpha_t^{n_2}| dt \right)^{2m^*} \right) \\ & \leq D^{m^*} \|\alpha^{n_1} - \alpha^{n_2}\|_{\mathcal{S}^{4m^*}}^{2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^2) dt \right)^{4m^*} \right)^{1/2}. \end{aligned} \quad (3.9)$$

Since  $Z^n \star W \in BMO$ , and **(BMO2,m)** holds true, the last term is uniformly bounded with respect to  $n_2$  with the energy inequality. Besides, let us show that the first one tends to 0 when  $n_1, n_2$  go to infinity by using the dominated convergence theorem. We show

$$\mathbf{E} \left( \sup_{t \in [0, T]} |\alpha_t^n - \alpha_t|^{4m^*} \right) \xrightarrow{n \rightarrow \infty} 0.$$

For all  $t \in [0, T]$  and  $n \in \mathbf{N}$ ,

$$|\alpha_t^n - \alpha_t| \leq |\alpha_t^n| + |\alpha_t| \leq 2 \|\alpha\|_{\mathcal{S}^\infty} \text{ a.s.},$$

and by using the uniform continuity of  $\alpha$  on  $[0, T]$ , we get:

$$\sup_{t \in [0, T]} |\alpha_t^n - \alpha_t| \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}$$

And finally

$$\mathbf{E} \left( \left( \int_0^T |\mathbf{f}(\alpha_t^{n_1}, Y_t^{n_2}, Z_t^{n_2}) - \mathbf{f}(\alpha_t^{n_2}, Y_t^{n_2}, Z_t^{n_2})| dt \right)^{2m^*} \right) \xrightarrow{n_1, n_2 \rightarrow \infty} 0.$$

Consequently  $(Y^n, Z^n \star W)_{n \in \mathbf{N}}$  is a Cauchy sequence in  $\mathcal{S}^{2m^*}(\mathbf{R}^d) \times \mathcal{H}^{m^*}(\mathbf{R}^{d \times k})$ , hence the convergence in  $\mathcal{S}^{2m^*}(\mathbf{R}^d) \times \mathcal{H}^{m^*}(\mathbf{R}^{d \times k})$  to a process  $(Y, Z \star W)$ , and  $(Y, Z)$  solves the BSDE (1.3). Uniqueness follows from the stability lemma, since, according to Proposition 4.5, the upper bound for  $\|Z^n \star W\|_{BMO}$  holds true for  $\|Z \star W\|_{BMO}$ .  $\diamond$

**REMARK – 3.2** *If  $f$  is a deterministic function, then the assumption **(Df,b)** is not required: for example when we consider the problem of martingale in manifolds with prescribed terminal value.*

**REMARK – 3.3** *If we replace the inequality 3.7 by the new one: there exists  $\eta > 0$  such that for all  $(\beta, \beta', y, z) \in (\mathbf{R}^d)^2 \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ :*

$$|\mathbf{f}(\beta, y, z) - \mathbf{f}(\beta', y, z)| \leq D \left( 1 + |z|^{2-\eta} \right) |\beta - \beta'|,$$

$\alpha$  progressively measurable is enough, and does not need to be continuous. Let us show this property. We change

the inequality (3.9) by the following for all  $p > 1$ , by using the Hölder and Cauchy-Schwartz inequalities:

$$\begin{aligned}
& \mathbf{E} \left( \left( \int_0^T |\mathbf{f}(\alpha_t^{n_1}, Y_t^{n_2}, Z_t^{n_2}) - \mathbf{f}(\alpha_t^{n_2}, Y_t^{n_2}, Z_t^{n_2})| dt \right)^{2m^*} \right) \\
& \leq D^{2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^{2-\eta}) |\alpha_t^{n_1} - \alpha_t^{n_2}| dt \right)^{2m^*} \right) \\
& \leq D^{2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^{2-\eta})^p dt \right)^{2m^*/p} \times \left( \int_0^T |\alpha_t^{n_1} - \alpha_t^{n_2}|^{p^*} dt \right)^{2m^*/p^*} \right) \\
& \leq D^{2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^{2-\eta})^p dt \right)^{4m^*/p} \right)^{1/2} \mathbf{E} \left( \left( \int_0^T |\alpha_t^{n_1} - \alpha_t^{n_2}|^{p^*} dt \right)^{4m^*/p^*} \right)^{1/2} \\
& \leq D^{2m^*} T^{\frac{4m^*}{p^*} - 2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^{2-\eta})^p dt \right)^{4m^*} \right)^{1/2p} \|(\alpha^{n_1} - \alpha^{n_2}) \star W\|_{\mathcal{H}^{4m^*}}^{2m^*}.
\end{aligned}$$

With the energy inequality, the first term is uniformly bounded with respect to  $n_2$  under the assumption **(BMO2,m)** by choosing  $1 < p \leq \frac{2}{2-\eta}$ . The second one tends to zero when  $n_1, n_2$  go to infinity since the convergence in every  $\mathcal{H}^r$  for  $r > 1$  holds true.

## 4 APPLICATION OF EXISTENCE AND UNIQUENESS RESULTS TO MULTIDIMENSIONAL QUADRATIC BSDEs WITH SPECIAL STRUCTURE.

In this part we consider the initial quadratic BSDE (1.3):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

and the localized one (1.4)

$$Y_t^M = \xi + \int_t^T f^M(s, Y_s^M, Z_s^M) ds - \int_t^T Z_s^M dW_s, \quad 0 \leq t \leq T.$$

The aim of this section is to give explicit assumptions on the terminal condition and the generator to ensure that **(BMO2,m)** is fulfilled for one  $m > 1$ .

More precisely, let us consider a Lipschitz random generator  $g: [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$  and the following BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s. \quad (4.1)$$

The generator  $g$  will play the role of  $f^M$  in the localized BSDE (1.4), so  $g$  will always be assumed to be a Lipschitz function. In this section we will display different assumptions on  $\xi$  and  $g$  that implies **(BMO2,m)** for one  $m > 1$ .

**§ 4.1. An existence and uniqueness result for BSDEs with a small terminal condition.** In [Tev08], Tevzadze obtains a result of existence and uniqueness for multidimensional quadratic BSDEs when the terminal condition is small enough by using a contraction argument in  $\mathcal{S}^\infty \times BMO$ . We are able to deal with this kind of assumptions with our approach.

Let us assume in this subsection the following hypothesis.

- (HQ)** (i) there exists  $\gamma \in \mathbf{R}^+$  such that for all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ , we have  $|g(t, y, z)| \leq \gamma |z|^2$ ,  
(ii)  $4\gamma^2 \|\xi\|_{L^\infty}^2 \leq 1$ .

**PROPOSITION – 4.1** *Let us assume that **(HQ)** is in force. Then  $Z \star W \in BMO$ ,  $Y \in \mathcal{S}^\infty$  and we have following estimates:*



$$(i) \|Z \star W\|_{BMO}^2 \leq \frac{1}{2\gamma^2} \left( 1 - \sqrt{1 - 4\gamma^2 \|\xi\|_{L^\infty}^2} \right),$$

$$(ii) \|Y\|_{\mathcal{S}^\infty} \leq \|\xi\|_{L^\infty} + \gamma \|Z \star W\|_{BMO}^2.$$

We can note that the first upper bound does not depend on Lipschitz constants of  $g$ . So, for  $g = f^M$ , it will not depend on  $M$ .

Proof. The unique solution  $(Y, Z) \in \mathcal{S}^2 \times \mathcal{H}^2$  of (4.1) can be constructed with a Picard principle as in the seminal paper of Pardoux and Peng (see [PP90]). We consider a sequence  $(Y^n, Z^n)_{n \in \mathbf{N}}$  such that  $(Y^n, Z^n)_{n \in \mathbf{N}}$  tends to  $(Y, Z)$  in  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$ . This sequence is given by

$$Y_t^{n+1} = \xi + \int_t^T g(s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^{n+1} dW_s, \quad 0 \leq t \leq T, \quad (Y^0, Z^0) = (0, 0).$$

We will prove with an induction that: for all  $n \in \mathbf{N}$ ,  $Y^n \in \mathcal{S}^\infty$ ,  $Z^n \star W \in BMO$  and

$$\|Z^n \star W\|_{BMO}^2 \leq \frac{1}{2\gamma^2} \left( 1 - \sqrt{1 - 4\gamma^2 \|\xi\|_{L^\infty}^2} \right).$$

Loosely speaking the case  $n = 0$  is satisfied. Let us suppose that  $Y^n \in \mathcal{S}^\infty$  and  $Z^n \star W \in BMO$ . Then for all  $t \in [0, T]$ , under (HQ),

$$|Y_t^{n+1}| \leq \mathbf{E}(|\xi| | \mathcal{F}_t) + \gamma \times \mathbf{E} \left( \int_t^T |Z_s^n|^2 ds \middle| \mathcal{F}_t \right). \quad (4.2)$$

We get  $Y^{n+1} \in \mathcal{S}^\infty$  since  $\|Y^{n+1}\|_{\mathcal{S}^\infty} \leq \|\xi\|_{L^\infty} + \gamma \|Z^n \star W\|_{BMO}$ . Itô formula gives the following equality

$$|Y_t^{n+1}|^2 = |\xi|^2 + 2 \int_t^T Y_s^{n+1} \cdot g(s, Y_s^n, Z_s^n) ds - 2 \int_t^T Y_s^{n+1} \cdot (Z_s^{n+1} dW_s) - \int_t^T |Z_s^{n+1}|^2 ds.$$

By taking conditional expectation we get for every stopping time  $\tau$ :

$$\begin{aligned} |Y_\tau^{n+1}|^2 + \mathbf{E} \left( \int_\tau^T |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_\tau \right) &\leq |\xi|^2 + 2\mathbf{E} \left( \int_\tau^T Y_s^{n+1} \cdot g(s, Y_s^n, Z_s^n) ds \middle| \mathcal{F}_\tau \right) \\ &\leq \|\xi\|_{L^\infty}^2 + 2\|Y^{n+1}\|_{\mathcal{S}^\infty} \mathbf{E} \left( \int_\tau^T \gamma |Z_s^n|^2 ds \middle| \mathcal{F}_\tau \right) \\ &\leq \|\xi\|_{L^\infty}^2 + 2\gamma \|Y^{n+1}\|_{\mathcal{S}^\infty} \|Z^n \star W\|_{BMO}^2. \end{aligned}$$

And finally

$$\begin{aligned} \|Y^{n+1}\|_{\mathcal{S}^\infty}^2 + \mathbf{E} \left( \int_\tau^T |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_\tau \right) &\leq \|\xi\|_{L^\infty}^2 + 2\gamma \|Y^{n+1}\|_{\mathcal{S}^\infty} \|Z^n \star W\|_{BMO}^2 \\ &\leq \|\xi\|_{L^\infty}^2 + \|Y^{n+1}\|_{\mathcal{S}^\infty}^2 + \gamma^2 \|Z^n \star W\|_{BMO}^4. \end{aligned}$$

Thus  $Z^{n+1} \star W \in BMO$  and we have

$$\|Z^{n+1} \star W\|_{BMO}^2 \leq \|\xi\|_{L^\infty}^2 + \gamma^2 \|Z^n \star W\|_{BMO}^4.$$

Finally, it yields with the induction assumption that

$$\|Z^{n+1} \star W\|_{BMO}^2 \leq \frac{1}{2\gamma^2} \left( 1 - \sqrt{1 - 4\gamma^2 \|\xi\|_{L^\infty}^2} \right).$$

The induction is achieved. Now we can use Proposition 4.5 with  $K = \frac{1}{2\gamma^2} \left( 1 - \sqrt{1 - 4\gamma^2 \|\xi\|_{L^\infty}^2} \right)$ , since  $Z^n \star W$  tends to  $Z \star W$  in  $\mathcal{H}^2$ , we conclude  $\|Z \star W\|_{BMO}^2 \leq \frac{1}{2\gamma^2} \left( 1 - \sqrt{1 - 4\gamma^2 \|\xi\|_{L^\infty}^2} \right)$  and the final upper bound on  $\|Y\|_{\mathcal{S}^\infty}$ .  $\diamond$

We come back to the initial quadratic BSDE (1.3).

PROPOSITION – 4.2 *Let  $m > 1$ . Under **(H')**—**(HQ)**, and the following condition on  $\gamma$ :*

$$\frac{1}{\sqrt{2}\gamma} \left(1 - \sqrt{1 - 4\gamma^2 \|\xi\|_{L^\infty}^2}\right)^{\frac{1}{2}} < \mathbb{B}^m(L_y, L_z),$$

*the BSDE (1.3) has a unique solution in  $\mathcal{S}^{2m^*}(\mathbf{R}^d) \times (\mathcal{H}^{m^*}(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ . If in addition **(Dxi,b)** holds true, there exists an unique solution in  $\mathcal{S}^\infty(\mathbf{R}^d) \times (\mathcal{S}^\infty(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ .*

Proof. The proof of the proposition is a direct consequence of Theorem 3.2 together with Proposition 4.1, since the map

$$x \in \mathbf{R}^+ \mapsto \frac{1}{\sqrt{2}\gamma} \left(1 - \sqrt{1 - 4\gamma^2 x^2}\right)^{\frac{1}{2}}$$

is nondecreasing, the assumption **(BMO2,m)** is satisfied.  $\diamond$

**§ 4.2. An existence and uniqueness result for BSDEs with a monotone generator.** In this subsection we investigate the case where we have for  $g$  a kind of monotony assumption with respect to  $y$ .

**(HMon)** (i) There exists  $\mu > 0$  and  $\alpha, \gamma \geq 0$  such that for all  $(s, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$

$$y.g(s, y, z) \leq \alpha |y| - \mu |y|^2 + \gamma |y| |z|^2,$$

(ii)  $4\gamma^2 \|\xi\|_{L^\infty}^2 \leq 1$ .

PROPOSITION – 4.3 *Let us assume that **(HMon)** is in force. Then  $Z \star W \in BMO$ ,  $Y \in \mathcal{S}^\infty$  and we have*

$$(i) \operatorname{esssup}_{t \in [0, T]} \sup \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s|^2 ds \middle| \mathcal{F}_t \right) \leq \frac{1}{2\gamma^2} \left(1 - \sqrt{1 - 4\gamma^2 A^2}\right),$$

$$(ii) \|Y\|_{\mathcal{S}^\infty} \leq \frac{1}{2\gamma} \left(1 - \sqrt{1 - 4\gamma^2 A^2}\right) + \sqrt{A^2 + \frac{1}{4\gamma^2} \left(1 - \sqrt{1 - 4\gamma^2 A^2}\right)^2}.$$

with  $A = \max \left( \|\xi\|_{L^\infty}, \frac{\alpha}{\mu} \right)$ .

Proof. The unique solution  $(Y, Z)$  of (4.1) can be constructed with a Picard principle. We consider a sequence  $(Y^n, Z^n)_{n \in \mathbf{N}}$  such that  $(Y^n, Z^n)_{n \in \mathbf{N}}$  tends to  $(Y, Z)$  in  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$ , with

$$Y_t^{n+1} = \xi + \int_t^T g(s, Y_s^{n+1}, Z_s^n) ds - \int_t^T Z_s^{n+1} dW_s, \quad 0 \leq t \leq T, \quad (Y^0, Z^0) = (0, 0), .$$

We can easily show that replacing  $Y^{n+1}$  by  $Y^n$  in the generator does not affect the convergence of the scheme since  $g$  is a Lipschitz function. Itô formula to the process  $e^{-\mu t} |Y_t^{n+1}|^2$  gives

$$\begin{aligned} e^{-\mu t} |Y_t^{n+1}|^2 &= e^{-\mu T} |\xi|^2 - \int_t^T \left( -\mu e^{-\mu s} |Y_s^{n+1}|^2 - 2e^{-\mu s} Y_s^{n+1} \cdot g(s, Y_s^{n+1}, Z_s^n) + |Z_s^{n+1}|^2 \right) ds \\ &\quad - 2 \int_t^T e^{-\mu s} Y_s^{n+1} \cdot (Z_s^{n+1} dW_s). \end{aligned}$$

Taking conditional expectation, and using assumption **(HMon)**, we get:

$$\begin{aligned} |Y_t^{n+1}|^2 &\leq e^{-\mu(T-t)} \|\xi\|_{L^\infty}^2 + \mathbf{E} \left( \int_t^T 2e^{-\mu(s-t)} \left( \alpha |Y_s^{n+1}| - \frac{\mu}{2} |Y_s^{n+1}|^2 + \gamma |Y_s^{n+1}| |Z_s^n|^2 \right) ds \middle| \mathcal{F}_t \right) \\ &\quad - \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_t \right). \end{aligned}$$

With the Young inequality we get the following estimate for all  $n$  and  $s \in [0, T]$ :

$$\alpha |Y_s^{n+1}| \leq \frac{\mu}{2} |Y_s^{n+1}|^2 + \frac{\alpha^2}{2\mu},$$

and thus

$$\begin{aligned} |Y_t^{n+1}|^2 &\leq e^{-\mu(T-t)} \|\xi\|_{L^\infty}^2 + \mathbf{E} \left( \int_t^T 2e^{-\mu(s-T)} \left( \frac{\alpha^2}{2\mu} + \gamma |Y_s^{n+1}| |Z_s^n|^2 \right) ds \middle| \mathcal{F}_t \right) \\ &\quad - \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_t \right) \\ &\leq \underbrace{e^{-\mu(T-t)} \|\xi\|_{L^\infty}^2 + \frac{\alpha^2}{\mu^2} (1 - e^{-\mu(T-t)})}_{\leq A^2} + 2\gamma \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Y_s^{n+1}| |Z_s^n|^2 ds \middle| \mathcal{F}_t \right) \\ &\quad - \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_t \right). \end{aligned}$$

Finally we get

$$|Y_t^{n+1}|^2 + \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_t \right) \leq A^2 + 2\gamma \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Y_s^{n+1}| |Z_s^n|^2 ds \middle| \mathcal{F}_t \right).$$

With the Itô formula applied to  $e^{Kt} |Y_t^{n+1}|^2$  with  $K$  large enough, we justify with classical inequalities since  $g$  is Lipschitz, and that for all  $n \in \mathbf{N}$ ,  $Y^{n+1} \in \mathcal{S}^\infty$ . And then

$$\begin{aligned} \|Y^{n+1}\|_{\mathcal{S}^\infty}^2 + \text{esssup}_{t \in [0, T]} \sup \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_t \right) \\ \leq A^2 + 2\gamma \|Y^{n+1}\|_{\mathcal{S}^\infty} \text{esssup}_{t \in [0, T]} \sup \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^n|^2 ds \middle| \mathcal{F}_t \right) \\ \leq A^2 + \|Y^{n+1}\|_{\mathcal{S}^\infty}^2 + \gamma^2 \text{esssup}_{t \in [0, T]} \sup \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^n|^2 ds \middle| \mathcal{F}_t \right). \end{aligned} \quad (4.3)$$

With an induction we show easily that for all  $n$ ,  $Z^n \star W \in BMO$ ,  $Y^n \in \mathcal{S}^\infty$  and

$$\text{esssup}_{t \in [0, T]} \sup \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^n|^2 ds \middle| \mathcal{F}_t \right) \leq \frac{1}{2\gamma^2} (1 - \sqrt{1 - 4\gamma^2 A^2}).$$

The inequality (4.3) gives

$$\|Y^{n+1}\|_{\mathcal{S}^\infty} \leq \gamma \text{esssup}_{t \in [0, T]} \sup \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^n|^2 ds \middle| \mathcal{F}_t \right) + \sqrt{A^2 + \gamma^2 \left( \text{esssup}_{t \in [0, T]} \sup \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^n|^2 ds \middle| \mathcal{F}_t \right) \right)^2}.$$

Letting  $n$  to infinity, and with the Proposition 4.5, we finally get

$$\text{esssup}_{t \in [0, T]} \sup \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s|^2 ds \middle| \mathcal{F}_t \right) \leq \frac{1}{2\gamma^2} (1 - \sqrt{1 - 4\gamma^2 A^2}).$$

We deduce

$$\|Y\|_{\mathcal{S}^\infty} \leq \frac{1}{2\gamma} (1 - \sqrt{1 - 4\gamma^2 A^2}) + \sqrt{A^2 + \frac{1}{4\gamma^2} (1 - \sqrt{1 - 4\gamma^2 A^2})^2}.$$

◇

REMARK – 4.1 *What about the BMO norm of  $Z$ ? — we can slice  $[0, T]$  with a uniform sequence  $(T_i)_{i=1}^N$  such that  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$  and for all  $i, h = |T_{i+1} - T_i| = \frac{T}{N}$ . The last inequality can be used for each started and stopped process  $T_i Z \star W \uparrow^{T_{i+1}}$ :*

$$\begin{aligned} \operatorname{esssup}_{T_i \leq t \leq T_{i+1}} \mathbf{E} \left( \int_t^{T_{i+1}} |Z_s|^2 ds \middle| \mathcal{F}_t \right) &\leq e^{\mu h} \left( \operatorname{esssup}_{T_i \leq t \leq T_{i+1}} \mathbf{E} \left( \int_t^{T_{i+1}} e^{-\mu(s-t)} |Z_s|^2 ds \middle| \mathcal{F}_t \right) \right) \\ &\leq \frac{e^{\mu h}}{2\gamma^2} \left( 1 - \sqrt{1 - 4\gamma^2 A^2} \right). \end{aligned}$$

Previous remark showed that  $Z \star W$  is  $\left( \frac{e^{\frac{1}{2}\mu h}}{\sqrt{2\gamma}} \left( 1 - \sqrt{1 - 4\gamma^2 A^2} \right)^{\frac{1}{2}} \right)$ -sliceable for each  $h > 0$ . According to Remark 2.2, we can enounce an other result of existence and uniqueness for the BSDE (1.3).

PROPOSITION – 4.4 *Let  $m > 1$ . Under  $(H')$ – $(HMon)$  and the following estimate on  $\gamma$ :*

$$\frac{1}{\sqrt{2\gamma}} \left( 1 - \sqrt{1 - 4\gamma^2 A^2} \right)^{\frac{1}{2}} < \mathbb{B}^m(L_y, L_z)$$

with  $A = \max \left( \|\xi\|_{L^\infty}, \frac{\alpha}{\mu} \right)$ , the quadratic BSDE (1.3) has a solution in  $\mathcal{S}^{2m^*}(\mathbf{R}^d) \times (\mathcal{H}^{m^*}(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ . If in addition  $(Dxi, b)$  holds true, there exists a unique solution in  $\mathcal{S}^\infty(\mathbf{R}^d) \times (\mathcal{S}^\infty(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ .

Proof. By using Remark 4.1, we obtain that assumption  $(BMO2, m)$  is fulfilled. Since the map  $x \in \mathbf{R}^+ \mapsto \frac{1}{\sqrt{2\gamma}} \left( 1 - \sqrt{1 - 4\gamma^2 x^2} \right)^{\frac{1}{2}}$  is nondecreasing, assumption  $(BMO2, m)$  is satisfied.  $\diamond$

**§ 4.3. Convexity assumption and martingales in manifolds with prescribed terminal condition.** Problem of finding martingales on a manifold with prescribed terminal value generated a huge literature. On the one hand with geometrical methods, Kendall in [Ken90] treat the case where the terminal value lies in a geodesic ball and is expressed as a functional of the Brownian motion. Kendall gives too a characterization of the uniqueness in terms of existence of a convex separative function: a convex function on the product which vanishes exactly on the diagonal. Besides, in [Ken92], Kendall proved that the following property *every couple of points are connected by a unique geodesic* is not sufficient to ensure existence of a separative convex function, which was conjectured by Émery. An approach by barycenters, of the martingale notion on a manifold, is used by Picard in [Pic94] for Brownian filtrations. Arnaudon in [Arn97] solved the problem in a complex analytic manifold having a convex geometry property for continuous filtrations. The main ideas are related to differentiable families of martingales. For all this results, a convex geometry property is assumed. The first approach by systems of BSDEs is shown by Darling in [Dar95].

A so-called linear connection structure is required to define martingales on a manifold  $\mathcal{M}$  in a intrinsic way. *A contrario* for the semimartingales a differential structure is enough. Definition of martingale can be rewritten with a system of coupled BSDEs having a quadratic growth. We begin to recall it. We can refer to [Eme89] for more details about stochastic calculus on manifolds.

Let us consider  $(\mathcal{M}, \nabla)$  a differential manifold equipped with a linear connection  $\nabla$ . This is equivalent to give ourself a Hessian notion or a covariant derivative. We say that a continuous process  $X$  is a semimartingale on  $M$  if for all  $F \in \mathcal{C}^2(\mathcal{M})$ ,  $F \circ X$  is a real semimartingale. Consistance of the definition is simply due to the Itô formula. We say that a continuous process  $Y$  is a (local)  $\nabla$ -martingale if for all  $F \in \mathcal{C}^2(\mathcal{M})$ ,

$$F(Y)_t - \frac{1}{2} \int_0^t \nabla dF(dY, dY)_s$$

is a real local martingale on  $[0, T]$ . Again it is not very hard to see with the Itô formula that this definition is equivalent to the Euclidean one in the flat case.  $\int_0^t \nabla dF(dY, dY)_s$  is a notation for the quadratic variation of  $Y$  with

respect to the  $(0,2)$ -tensor field  $\nabla dF$ . This notion is defined by considering a proper embedding  $(x_i)_{1 \leq i \leq d}$  into  $\mathbf{R}^d$  such that every bilinear form  $b$  can be written as  $b = b_{ij} dx^i \otimes dx^j$  (implicit summation). On the other hand we prove that

$$\int_0^\cdot b(dY, dY)_s := \int_0^\cdot b_{ij}(Y_s) d\langle Y^i, Y^j \rangle_s$$

does not depend on  $(x^i)_{1 \leq i \leq d}$ . It is well-known that for all  $m \in \mathcal{M}$ ,

$$(\nabla dF)_{ij}(m) = D_{ij}F(m) - \Gamma_{ij}^k(m) D_k F(m),$$

where  $\Gamma_{ij}^k(m)$  denote a  $\nabla$ -Christoffel symbol at the point  $m$ . The coefficients are symmetric with respect to  $i, j$ . Hence martingale property in the domain of a local chart is equivalent to the existence of a process  $Z$  such that  $(Y, Z)$  solves the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

with  $f : [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  defined by  $f(s, y, z) = \frac{1}{2} \left( \Gamma_{ij}^k(y) z^{(i,:)} \cdot z^{(j,:)} \right)_{1 \leq k \leq d}$ . It is an easy consequence of the representation theorem for the Brownian martingales and the definition applied to  $F = x^i$ . We consider in addition the following assumption

**(HGam)** there exists two constants  $L_y$  and  $L_z$  such that for all  $i, j, k$

$$\left| \Gamma_{ij}^k(y) - \Gamma_{ij}^k(y') \right| \leq 2L_y |y - y'|, \quad \left| \Gamma_{ij}^k(y) \right| \leq 2L_z.$$

For example **(HGam)** is in force if the domain of the chart is a compact set. It is also true if we choose an exponential chart. Without loss of generality we can suppose that  $\mathcal{M}$  has a global system of coordinates: all the Christoffel symbols will be computed in this system.

Under **(HGam)**, assumption **(H)** is in force: for all  $(y, y', z, z') \in (\mathbf{R}^d)^2 \times (\mathbf{R}^{d \times k})^2$ ,

$$|f(t, y, z) - f(t, y', z)| \leq L_y |z|^2 |y - y'|,$$

and with the symmetric property of the Christoffel symbols, we have

$$f(t, y, z') - f(t, y, z) = -\frac{1}{2} \sum_{i,j} \Gamma_{ij}^k(y) \left( z^{(i,:)} \cdot z^{(j,:)} - (z')^{(i,:)} \cdot (z')^{(j,:)} \right) = -\frac{1}{2} \sum_{i,j} \Gamma_{ij}^k(y) \left( z^{(i,:)} - (z')^{(i,:)} \right) \left( z^{(j,:)} + (z')^{(j,:)} \right),$$

$$|f(t, y, z) - f(t, y, z')| \leq L_z (|z| + |z'|) |z - z'|.$$

**Localization.** Let us consider for all  $M \in \mathbf{R}^+$  a localization  $\rho^M : \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^{d \times k}$ , and let us define a function  $h^M : \mathbf{R}^{d \times k} \rightarrow \mathbf{R}$  such that:

$$\rho^M(z) = \frac{z}{\sqrt{1 + h^M(z)}}.$$

We define  $h^M$  as a smooth map satisfying:

$$h^M(z) = \begin{cases} 0 & \text{if } |z| \leq M, \\ \left( \frac{|z|}{M+1} \right)^2 - 1 & \text{if } |z| \geq M+1. \end{cases}$$

As usual we denote by  $(Y^M, Z^M)$  the solution obtained by replacing  $f$  by  $f^M$ . This choice of  $\rho^M$  will be useful in the following computations.

Since  $f$  does not depend on time and random, assumption **(Df,b)** is not required here and we get the following proposition.

**PROPOSITION – 4.5** *Let  $m > 1$ . If **(HGam)** and **(BMO2,m)** hold true, there exists a unique  $\nabla$ -martingale in  $\mathcal{S}^{m*}(\mathbf{R}^d)$  with terminal value  $\xi$ , such that  $\mathbf{E} \left( \langle Y, Y \rangle_T^{m*/2} \right) < \infty$ . If in addition **(Dxi,b)** is in force, there exists a unique martingale in  $\mathcal{S}^\infty(\mathbf{R}^d)$  such that  $\langle Y, Y \rangle_T \in L^\infty$ .*

In [Dar95] Proposition 3.4., Darling shows that a convex geometry property gives uniform estimations for the BMO norm of  $Z^M \star W$ . We deal with this kind of assumption.

DEFINITION – 4.1 We say that a function  $F \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$  (seen as a function on  $\mathbf{R}^d$ ) is doubly convex on a set  $G \subset \mathbf{R}^d$  if for all  $y \in G$  and  $z \in \mathbf{R}^d$ ,

$$\min \{ \text{Hess } F(y)(z, z), \nabla \text{d}F(y)(z, z) \} \geq 0,$$

and, for  $\alpha > 0$ ,  $\alpha$ -strictly doubly convex on  $G$  if for all  $y \in G$  and  $z \in \mathbf{R}^d$ ,

$$\min \{ \text{Hess } F(y)(z, z), \nabla \text{d}F(y)(z, z) \} \geq \alpha |z|^2.$$

This property means that  $F$  is convex with respect to the flat connection, and, with respect to the connection  $\nabla$ .

THEOREM – 4.1 Let  $m > 1$ . Let us assume that:

- (i) there exists a function  $F^{dc} \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$ , such that  $G = (F^{dc})^{-1}(]-\infty, 0])$  is compact and  $\xi \in G$ ,
- (ii)  $F^{dc}$  is doubly convex on  $\mathcal{M}$ , and there exists  $\alpha > 0$  such that  $F^{dc}$  is  $\alpha$ -strictly doubly convex on  $G$  with:

$$\left( \sup_{(x,y) \in G^2} \{ F^{dc}(x) - F^{dc}(y) \} \right)^{1/2} \leq \sqrt{\frac{\alpha}{2}} \times \mathbb{B}^m(L_y, L_z),$$

- (iii) (**HGam**) holds true.

Then there exists a unique  $\nabla$ -martingale in  $\mathcal{S}^\infty(\mathbf{R}^d)$  with terminal value  $\xi$  such that  $\langle Y, Y \rangle_T \in L^\infty$ .

Proof. For  $F \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$ , the Itô formula with  $F$  seen as a function on  $\mathbf{R}^d$  gives for all stopping time  $\tau$ :

$$\mathbf{E}(F(\xi) - F(Y_\tau^M) | \mathcal{F}_\tau) = \mathbf{E} \left( \int_\tau^T \left( \frac{1}{2} \sum_{l=1}^k \text{Hess } F(Y_s^M) \left( Z_s^{M,(:,l)}, Z_s^{M,(:,l)} \right) - \text{d}F(Y_s^M) f^M(Y_s^M, Z_s^M) \right) \text{d}s \middle| \mathcal{F}_\tau \right).$$

The local martingale part is a martingale if  $F$  has bounded first derivative. By using the form of  $f$ , its formulation in the local chart and the formula between  $Z^M$  and  $\rho^M(Z^M)$ , we get

$$\begin{aligned} & \mathbf{E}(F(\xi) - F(Y_\tau^M) | \mathcal{F}_\tau) \\ &= \frac{1}{2} \mathbf{E} \left( \int_\tau^T \sum_{l=1}^k \left( h^M(Z_s^M) \text{Hess } F(Y_s^M) \left( \rho^M(Z_s^M)(:,l), \rho^M(Z_s^M)(:,l) \right) + \nabla \text{d}F(Y_s^M) \left( \rho^M(Z_s^M)(:,l), \rho^M(Z_s^M)(:,l) \right) \right) \text{d}s \middle| \mathcal{F}_\tau \right). \end{aligned} \tag{4.4}$$

Let us show that  $Y^M$  is in  $G$  almost surely. We reproduce the proof of Darling in [Dar95] to get this property. We know with (4.4) applied to  $F = F^{dc}$ , integrating between  $\tau$  and  $\sigma$  together with (ii), that for all  $\sigma \geq \tau$  a.s we have

$$\mathbf{E} \left( F^{dc}(Y_\sigma^M) \middle| \mathcal{F}_\tau \right) \geq F^{dc}(Y_\tau^M) \text{ a.s.}$$

Let us consider for all  $n \in \mathbf{N}$  the following sequence of stopping times:  $\sigma^n = \inf \left\{ u \geq \tau \mid F^{dc}(Y_u^M) \leq \frac{1}{n} \text{ a.s.} \right\}$ .

Each  $\sigma^n$  is finite almost surely if  $\xi \in G$ . Continuity of  $Y^M$  gives for all  $n \in \mathbf{N}$ ,  $F^{dc}(Y_{\sigma^n}^M) \leq \frac{1}{n}$  a.s. So we get for all stopping time  $\tau$ :

$$F^{dc}(Y_\tau^M) \leq \mathbf{E} \left( F^{dc}(Y_{\sigma^n}^M) \middle| \mathcal{F}_\tau \right) \leq \frac{1}{n} \text{ a.s.}$$

Consequently, for all  $t \in [0, T]$ ,  $\mathbf{P}(Y_t \in G) = 1$ . The  $\alpha$ -strictly doubly convexity on  $G$  gives us

$$\begin{aligned}
& \mathbf{E}\left(F^{dc}(\xi) - F^{dc}(Y_\tau^M) \middle| \mathcal{F}_\tau\right) \\
& \geq \frac{\alpha}{2} \mathbf{E}\left(\int_\tau^T \sum_{l=1}^k \left(h^M(Z_s^M) \left|\rho^M(Z_s^M)^{(:,l)}\right|^2 + \left|\rho^M(Z_s^M)^{(:,l)}\right|^2\right) ds \middle| \mathcal{F}_\tau\right) = \frac{\alpha}{2} \mathbf{E}\left(\int_\tau^T |Z_s^M|^2 ds \middle| \mathcal{F}_\tau\right). \tag{4.5}
\end{aligned}$$

Continuity of  $F^{dc}$  on  $G$  yields

$$\|Z^M \star W\|_{BMO} \leq \sqrt{\frac{2}{\alpha} \times \left( \sup_{(x,y) \in G \times G} \{F^{dc}(x) - F^{dc}(y)\} \right)}.$$

Assumption (ii) ensures assumption **(BMO,m)**. Since the terminal value is bounded (in  $G$ ), Theorem 3.2 together with Remark 3.2 gives the result.  $\diamond$

## 5 APPENDIX – TECHNICAL PROOFS

Proof. [of Proposition 1.2] Let us show the *BMO* property for a started stopped process. Let us consider a stopping time  $\tau'$  such that  $0 \leq \tau' \leq T$  a.s we have

$$\begin{aligned}
\mathbf{E}\left(\left\langle \tau Z \star W \right\rangle_T - \left\langle \tau Z \star W \right\rangle_{\tau'} \middle| \mathcal{F}_{\tau'}\right) &= \mathbf{E}\left(\left(\left\langle \tau Z \star W \right\rangle_\sigma - \left\langle \tau Z \star W \right\rangle_{\min(\tau', \sigma)}\right) \middle| \mathcal{F}_{\tau'}\right) \\
&= \mathbf{E}\left(\left(\left\langle Z \star W \right\rangle_\sigma - \left\langle Z \star W \right\rangle_{\tau'}\right) \mathbb{1}_{(0 \leq \tau' \leq \sigma)} \middle| \mathcal{F}_{\tau'}\right).
\end{aligned}$$

Since  $\tau Z \star W$  vanishes before  $\tau$  and  $\mathcal{F}_{\tau'} \subset \mathcal{F}_{\max(\tau', \tau)}$ , we get:

$$\begin{aligned}
\mathbf{E}\left(\left(\left\langle Z \star W \right\rangle_\sigma - \left\langle Z \star W \right\rangle_{\tau'}\right) \mathbb{1}_{(0 \leq \tau' \leq \sigma)} \middle| \mathcal{F}_{\tau'}\right) &= \mathbf{E}\left(\left(\left\langle Z \star W \right\rangle_\sigma - \left\langle Z \star W \right\rangle_{\max(\tau', \tau)}\right) \mathbb{1}_{(0 \leq \tau' \leq \sigma)} \middle| \mathcal{F}_{\tau'}\right) \\
&= \mathbf{E}\left(\mathbf{E}\left(\left(\left\langle Z \star W \right\rangle_\sigma - \left\langle Z \star W \right\rangle_{\max(\tau', \tau)}\right) \mathbb{1}_{(0 \leq \tau' \leq \sigma)} \middle| \mathcal{F}_{\max(\tau, \tau')}\right) \middle| \mathcal{F}_{\tau'}\right) \\
&\leq \text{esssup} \sup_{\tau' \in \mathcal{F}^{\tau, \sigma}} \mathbf{E}\left(\left\langle Z \star M \right\rangle_\sigma - \left\langle Z \star M \right\rangle_{\tau'} \middle| \mathcal{F}_{\tau'}\right).
\end{aligned}$$

Finally we have shown that

$$\text{esssup} \sup_{0 \leq \tau' \leq T} \mathbf{E}\left(\left\langle \tau Z \star W \right\rangle_T - \left\langle \tau Z \star W \right\rangle_{\tau'} \middle| \mathcal{F}_{\tau'}\right) \leq \text{esssup} \sup_{\tau' \in \mathcal{F}^{\tau, \sigma}} \mathbf{E}\left(\left\langle Z \star M \right\rangle_\sigma - \left\langle Z \star M \right\rangle_{\tau'} \middle| \mathcal{F}_{\tau'}\right),$$

and the inequality is obviously an equality.  $\diamond$

Proof. [of Proposition 2.2] We are going to use classical inequalities given by Lemma 1.1. Let us suppose existence

of a solution  $X$  for the equation (2.3). We have for all  $m \geq 1$ ,

$$\begin{aligned} \|X\|_{\mathcal{S}^m} &\leq \|X_0\|_{L^m} + \left\| \int_0^\cdot F(s, X_s) ds \right\|_{\mathcal{S}^m} + \left\| \sum_{p=1}^k \int_0^\cdot G^p(s, X_s) dW_s^p \right\|_{\mathcal{S}^m} \\ &\leq \|X_0\|_{L^m} + \mathbf{E} \left( \sup_{0 \leq u \leq T} \left( \int_0^u \alpha_s |X_s| ds \right)^m \right)^{1/m} + C'_m \mathbf{E} \left( \left( \sum_{p=1}^k \int_0^T |G^p(s, X_s)|^2 ds \right)^{m/2} \right)^{1/m} \\ &\leq \|X_0\|_{L^m} + \mathbf{E} \left( \left( \int_0^T \alpha_s |X_s| ds \right)^m \right)^{1/m} + C'_m \mathbf{E} \left( \left( \int_0^T \beta_s^2 |X_s|^2 ds \right)^{m/2} \right)^{1/m}. \end{aligned}$$

On the one hand, according to Lemma 1.1, we have

$$\begin{aligned} \mathbf{E} \left( \left( \int_0^T \alpha_s |X_s| ds \right)^m \right)^{1/m} &= \left\| \langle \sqrt{\alpha} \star W, (\sqrt{\alpha} |X|) \star W \rangle_T \right\|_{L^m} \\ &\leq \sqrt{2m} \|\sqrt{\alpha} \star W\|_{BMO} \|(\sqrt{\alpha} |X|) \star W\|_{\mathcal{H}^m} \\ &\leq 2m \|X\|_{\mathcal{S}^m} \|\sqrt{\alpha} \star W\|_{BMO}^2. \end{aligned}$$

On the other hand, we get for the last term

$$\mathbf{E} \left( \left( \int_0^T \beta_s^2 |X_s|^2 ds \right)^{m/2} \right)^{1/m} = \|(\beta X) \star W\|_{\mathcal{H}^m} = \|X \star (\beta \star W)\|_{\mathcal{H}^m} \leq \sqrt{2} \|X\|_{\mathcal{S}^m} \|\beta \star W\|_{BMO}.$$

Hence we obtain the following inequality

$$\|X\|_{\mathcal{S}^m} \left( 1 - 2m \|\sqrt{\alpha} \star W\|_{BMO}^2 - \sqrt{2} C'_m \|\beta \star W\|_{BMO} \right) \leq \|X_0\|_{L^m}. \quad (5.1)$$

The constant behind  $\|X\|_{\mathcal{S}^m}$  is not always positive, but we can use the sliceability assumption in order to construct piece by piece the process  $X$ , and on each piece the constant will be positive.

More precisely there exists a sequence of stopping times  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$  a.s such that for all  $i \in \{0, \dots, N-1\}$ :

$$\left\| \int_{T_i}^{T_{i+1}} \sqrt{\alpha} \star W \right\|_{BMO} \leq \varepsilon_1, \quad \left\| \int_{T_i}^{T_{i+1}} \beta \star W \right\|_{BMO} \leq \varepsilon_2.$$

The process  $X$  is equal to

$$X_t = \sum_{i=1}^{N-1} \tilde{X}_t^i \mathbb{1}_{[T_i, T_{i+1}[}(t)$$

where each  $\tilde{X}^i$  is the restriction of  $X$  to the stochastic interval  $[T_i, T_{i+1}[$ . By convention we extend  $\tilde{X}^i$  to  $[0, T]$  by zero outside  $[T_i, T_{i+1}[$ .  $\tilde{X}_i$  satisfies the following SDE:

$$\tilde{X}_t^i = \tilde{X}_{T_i}^{i-1} + \int_{T_i}^t F(s, \tilde{X}_s^i) ds + \sum_{p=1}^k \int_{T_i}^t G^p(s, \tilde{X}_s^i) dW_s^p, \quad t \in [T_i, T_{i+1}[, \quad \text{and} \quad \tilde{X}^{-1} = X_0.$$

For all  $i \in \{0, \dots, N-1\}$ , by considering above computations on each  $[T_i, T_{i+1}[$ , (5.1) becomes

$$\left\| \tilde{X}^i \right\|_{\mathcal{S}^m} \left( 1 - 2m\varepsilon_1^2 - \varepsilon_2 \sqrt{2} C'_m \right) \leq \left\| \tilde{X}_{T_i}^{i-1} \right\|_{L^m}.$$

Denoting by  $\mathbf{K}_{\varepsilon_1, \varepsilon_2}$  the constant

$$\mathbf{K}_{\varepsilon_1, \varepsilon_2} := \frac{1}{1 - 2m\varepsilon_1^2 - \varepsilon_2 \sqrt{2} C'_m} > 0,$$

$$\left\| \tilde{X}^i \right\|_{\mathcal{S}^m} \leq \mathbf{K}_{\varepsilon_1, \varepsilon_2}^i \|X^0\|_{L^m}.$$



Thus we obtain

$$\|X\|_{\mathcal{S}^m} \leq \sum_{i=0}^{N-1} \|\tilde{X}^i\|_{\mathcal{S}^m} \leq \left( \sum_{i=0}^{N-1} \mathbf{K}_{\varepsilon_1, \varepsilon_2}^i \right) \|X^0\|_{L^m}.$$

The result follows for  $K_{m, \varepsilon_1, \varepsilon_2} = \sum_{i=0}^{N-1} \mathbf{K}_{\varepsilon_1, \varepsilon_2}^i$ .

◇

## REFERENCES

- [Arn97] Marc Arnaudon. Differentiable and analytic families of continuous martingales in manifolds with connection. *Probability Theory and Related Fields*, 108(2):219–257, June 1997. 27
- [BB88] Rodrigo Bañuelos and Andrew Bennett. Paraproducts and commutators of martingale transforms. *Proceedings of the American Mathematical Society*, 103(4):1226–1234, 1988. 5
- [BE13] Philippe Briand and Romuald Elie. A simple constructive approach to quadratic BSDEs with or without delay. *Stochastic Processes and their Applications*, 123(8):2921–2939, August 2013. 1, 2, 13
- [BEK13] Pauline Barrieu and Nicole EL Karoui. Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. *The Annals of Probability*, 41(3B):1831–1863, May 2013. 1
- [BH06] Philippe Briand and Ying Hu. BSDE with quadratic growth and unbounded terminal value. *Probability Theory and Related Fields*, 136(4):604–618, April 2006. 1
- [Bis73] Jean-Michel Bismut. Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis and Applications*, 44(2):384–404, November 1973. 1
- [BLDR15] Jana Bielagk, Arnaud Lionnet, and Gonçalo Dos Reis. Equilibrium pricing under relative performance concerns. *arXiv.org*, November 2015. 1
- [CN15] Patrick Cheridito and Kihun Nam. Multidimensional quadratic and subquadratic BSDEs with special structure. . . . *An International Journal of Probability and . . .*, 87(5):871–884, April 2015. 2
- [Dar95] Richard W. R Darling. Constructing gamma-martingales with prescribed limit, using backwards SDE. *The Annals of Probability*, 23(3):1234–1261, 1995. 2, 3, 27, 29
- [DHR11] Freddy Delbaen, Ying Hu, and Adrien Richou. On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. *Annales de l’Institut Henri Poincaré. Probabilités et Statistiques*, 47(2):559–574, May 2011. 1
- [DT08] Freddy Delbaen and Shanjian Tang. Harmonic analysis of stochastic equations and backward stochastic differential equations. *Probability Theory and Related Fields*, 146(1-2):291–336, December 2008. 3, 5, 8, 9
- [EKH03] Nicole EL Karoui and Said Hamadène. BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations. *Stochastic Processes and their Applications*, 107(1):145–169, September 2003. 1
- [EKPQ97] Nicole EL Karoui, Serge Peng, and Marie-Claire Quenez. Backward Stochastic Differential Equations in Finance. *Mathematical Finance*, 7(1):1–71, January 1997. 19
- [Eme89] Michel Emery. *Stochastic calculus in manifolds*. Universitext. Springer-Verlag, Berlin, Berlin, Heidelberg, 1989. 27
- [EP16] Romuald Elie and Dylan Possamaï. Contracting theory with competitive interacting agents. *arXiv.org*, page arXiv:1605.08099, May 2016. 2
- [ET15] Gilles Edouard Espinosa and Nizar Touzi. Optimal investment under relative performance concerns. *Mathematical Finance*, 25(2):221–257, April 2015. 1
- [FDR11] Christoph Frei and Gonçalo Dos Reis. A financial market with interacting investors: does an equilibrium exist? *Mathematics and Financial Economics*, 4(3):161–182, February 2011. 1, 2, 3
- [Fre14] Christoph Frei. Splitting multidimensional BSDEs and finding local equilibria. *Stochastic Processes and their Applications*, 124(8):2654–2671, August 2014. 1, 2
- [HT16] Ying Hu and Shanjian Tang. Multi-dimensional backward stochastic differential equations of diagonally quadratic generators. *Stochastic Processes and their Applications*, 126(4):1066–1086, April 2016. 1, 2

- [JKL14] Asgar Jamneshan, Michael Kupper, and Peng Luo. Multidimensional quadratic BSDEs with separated generators. *arXiv.org*, December 2014. [2](#)
- [Kaz94] Norihiko Kazamaki. *Continuous exponential martingales and BMO*, volume 1579 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994. [4](#), [5](#)
- [Ken90] Wilfrid S Kendall. Probability, Convexity, and Harmonic Maps with Small Image I: Uniqueness and Fine Existence. *Proceedings of the London Mathematical Society*, s3-61(2):371–406, September 1990. [27](#)
- [Ken92] Wilfrid S Kendall. The Propeller: A Counterexample to a Conjectured Criterion for the Existence of Certain Convex Functions. *Journal of the London Mathematical Society. Second Series*, s2-46(2):364–374, October 1992. [27](#)
- [Kob00] Magdalena Kobylanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *Annals of Probability*, 2000. [1](#)
- [KP16] Dmitry Kramkov and Sergio Pulido. A system of quadratic BSDEs arising in a price impact model. *The Annals of Applied Probability*, 26(2):794–817, April 2016. [1](#), [2](#)
- [KXŽ15] Constantinos Kardaras, Hao Xing, and Gordan Žitković. Incomplete stochastic equilibria with exponential utilities close to Pareto optimality. *arXiv.org*, page arXiv:1505.07224, May 2015. [2](#)
- [LT15] Peng Luo and Ludovic Tangpi. Solvability of coupled FBSDEs with quadratic and superquadratic growth. *arXiv.org*, page arXiv:1505.01796, May 2015. [2](#)
- [MR16] Carlo Marinelli and Michael Röckner. On the maximal inequalities of Burkholder, Davis and Gundy. *Expositiones Mathematicae*, 34(1):1–26, 2016. [4](#)
- [Nua06] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006. [7](#), [21](#)
- [Pic94] Picard, Jean. Barycentres et martingales sur une variété. *Annales de l’IHP Probabilités et statistiques*, 1994. [27](#)
- [PP90] Etienne Pardoux and Serge Peng. Adapted solution of a backward stochastic differential equation. *Systems and Control Letters*, 14(1):55–61, January 1990. [1](#), [8](#), [24](#)
- [Sch96] Walter Schachermayer. A characterisation of the closure  $H_\infty$  in BMO. *Séminaire de Probabilités de Strasbourg*, 1996. [6](#)
- [Tev08] Revaz Tevzadze. Solvability of backward stochastic differential equations with quadratic growth. *Stochastic Processes and their Applications*, 118(3):503–515, March 2008. [1](#), [2](#), [3](#), [23](#)
- [XŽ16] Hao Xing and Gordan Žitković. A class of globally solvable Markovian quadratic BSDE systems and applications. *arXiv.org*, March 2016. [2](#), [3](#)