On the Diverio-Trapani Conjecture

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Abstract

The aim of this work is to study the conjecture on the ampleness of jet bundles raised by Diverio and Trapani, and also obtain some effective estimates related to this problem.

1 Introduction

In recent years, an important technique in studying hyperbolicity-related problems is *invariant* jet differentials $E_{k,m}T^*_X$ introduced by J.-P. Demailly, which can be seen as a generalization to higher order of symmetric differentials, but invariant under the reparametrization. If one deal with positivity for jet bundles of the complete intersection of hypersurfaces in $\mathbb{P}^N$, as was proved in [Div08], one cannot expect to achieve this in lower order jet differentials if the codimension of subvariety is small:

Theorem 1.1. *(Diverio) Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of hypersurfaces of any degree in $\mathbb{P}^N$. Then*

$$H^0(X, E^{GG}_{k,m}T^*_X) = 0$$

for all $m \geq 1$ and $1 \leq k < \dim(X)/\mathrm{codim}(X)$.

On the other hand, as a generalization of the Debarre conjecture, in [DT10], Simone Diverio and Stefano Trapani raised the following conjecture:

Conjecture 1.1. *Let $X \subset \mathbb{P}^N$ be the complete intersection of c general hypersurfaces of sufficiently high degree. Then, $E_{k,m}T^*_X$ is ample provided that $k \geq \frac{N}{c} - 1$, and therefore X is hyperbolic.*

In this paper, based mainly on the recent important breakthrough made in [BD15] and [Bro16] on the Debarre and Kobayashi conjectures, we prove the following theorem:

Main Theorem 1. *Let X be a projective manifold of dimension n endowed with a very ample line bundle A. Let $H \subset X$ be the complete intersection of c general hypersurfaces in $|H^0\big(X, \mathcal{O}_X(dA)\big)|$. Then for any positive integer $k \geq \frac{n}{c} - 1$, H has almost k-jet ampleness (see Definition 2.1 below) provided that $d \geq 2c(\lceil \frac{n}{c} \rceil)^{n+c+2}n^{n+c}$. In particular, H is Kobayashi hyperbolic.*

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Remark 1.1. *Since our definition for almost* 1*-jet ampleness coincides with ampleness of cotangent bundle, then our Main Theorem 1 integrates both the Kobayashi and Debarre conjectures, with some (non-optimal) effective estimates.*

2 Technical Preliminaries and Lemmas

In this section, we will recall some definitions and related results.

2.1 Invariant Jet Differentials

Let (X, V) be a directed manifold, i.e. a pair where X is a complex manifold and $V \subset T_X$ a holomorphic subbundle of the tangent bundle. One defines $J_k V \to X$ to be the bundle of k-jets of germs of parametrized curves in X, that is, the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$ which are tangent to V, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \leq j \leq k$, when computed in some local coordinate system of X near x. From now no, if not specially mentioned, we always assume that $V = T_X$. The projection map $p_k : J_k T_X \to X$ is simply taken to be $[f] \mapsto f(0)$. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on an open set $\Omega \subset X$, the elements $[f]$ of any fiber $J_{k,x}, x \in \Omega$, can be seen as $\mathbb{C}^n$-valued maps

$$f = (f_1, \ldots, f_n) : (\mathbb{C}, 0) \to \Omega \subset \mathbb{C}^n,$$

and they are completetely determined by their Taylor expansion of order k at $t = 0$:

$$f(t) = x + tf'(0) + \frac{t^2}{2!}f''(0) + \ldots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1}).$$

In these coordinates, the fiber $J_{k,x}$ can thus be identified with the set of k-tuples of vectors

$$(\xi_1, \ldots, \xi_k) = \big(f'(0), f''(0), \ldots, f^{(k)}(0)\big) \in \mathbb{C}^n.$$

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \to \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \quad a_1 \in \mathbb{C}^*, a_j \in \mathbb{C}, j > 2,$$

in which the composition law is taken modulo terms t_j of degree $j > k$. Then $\mathbb{G}_k$ is a k-dimensional nilpotent complex Lie group, which admits a natural fiberwise right action on $J_k T_X$. The action consists of reparametrizing k-jets of maps $f : (\mathbb{C}, 0) \to X$ by a biholomorphic change of parameter $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ defined by $(f, \varphi) \mapsto f \circ \varphi$. The corresponding $\mathbb{C}^*$-action on k-jets is described in coordinates by

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

The group $\mathbb{G}_k$ acts on J_k naturally. If we denote by X_k the relative compactification of $J_k \setminus \{0\}/\mathbb{G}_k$ over X, as was proved in [Dem95] (ref. Theorem 6.8), X_k is indeed the Demailly-Semple tower defined in the next subsection.

Green-Griffiths introduced the vector bundle $E^{GG}_{k,m}V^*$ whose fibres are complex valued polynomials $Q(f', f'', \dots, f^{(k)})$ on the fibres of J_kV, of weighted degree m with respect to the $\mathbb{C}^*$-action, *i.e.*, $Q(\lambda f, \lambda^2 f'', \dots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \dots, f^{(k)})$, for all $\lambda \in \mathbb{C}^*$ and $(f', f'', \dots, f^{(k)}) \in J_kV$. One calls $E^{GG}_{k,m}V^*$ the bundle of jet differentials of order k and weighted degree m. However, we are more interested in the more geometric context introduced by J.-P. Demailly: the subbundle $E_{k,m}V^* \subset E^{GG}_{k,m}V^*$ which is a set of polynomial differential operators $Q(f, f'', \dots, f^{(k)})$ which are invariant under arbitrary changes of parametrization, that is, for any $\varphi \in \mathbb{G}_k$, we have

$$Q\big((f\circ\varphi)', (f\circ\varphi)'', \dots, (f\circ\varphi)^{(k)}\big) = \varphi'(0)^m Q(f', f'', \dots, f^{(k)}).$$

The bundle $E_{k,m}V^*$ is called the bundle of *invariant jet differentials* of order k and degree m. A very natural construction for invariant jet differentials is *Wronskians*. In [Bro16] Brotbek introduce a type of Wronskians induced by global sections in some linear system. We will recall briefly his constructions in Section 2.3.

2.2 Demailly-Semple Jet Bundles

Let X be a complex manifold of dimension n. If V is a subbundle of rank r, one constructs a tower of "Demailly-Semple k-jet bundles" $\pi_{k,k-1} : (X_k, V_k) \to (X_{k-1}, V_{k-1})$ that are $\mathbb{P}^{r-1}$-bundles, with $\dim X_k = n + k(r-1)$ and $\operatorname{rank}(V_k) = r$. For this, we take $(X_0, V_0) = (X, V)$, and for every $k \geq 1$, inductively we set $X_k := P(V_{k-1})$ and

$$V_k := (\pi_{k,k-1})^{-1}_* \mathcal{O}_{X_k}(-1) \subset T_{X_k},$$

where $\mathcal{O}_{X_k}(1)$ is the tautological line bundle on X_k, $\pi_{k,k-1} : X_k = P(V_{k-1}) \to X_{k-1}$ the natural projection and $(\pi_{k,k-1})_* = d\pi_{k,k-1} : T_{X_k} \to \pi^*_{k,k-1}T_{X_{k-1}}$ its differential. By composing the projections we get for all pairs of indices $0 \leq j \leq k$ natural morphisms

$$\pi_{k,j} : X_k \to X_j, \quad (\pi_{k,j})_* = (d\pi_{k,j})|_{V_k} : V_k \to (\pi_{k,j})^* V_j,$$

and for every k-tuple $\mathbf{a} = (a_1, ..., a_k) \in \mathbb{Z}^k$ we define

$$\mathcal{O}_{X_k}(\mathbf{a}) = \otimes_{1\leq j\leq k} \pi^*_{k,j}\mathcal{O}_{X_j}(a_j).$$

We have an inductively defined k-lifting for germs of holomorphic curves such that $f_{[k]} : (\mathbb{C}, 0) \to X_k$ is obtained as $f_{[k]}(t) = \big(f_{[k-1]}(t), [f'_{[k-1]}(t)]\big)$. We will need the following parametrizing theorem due to J.-P. Demailly (ref. Corollary 5.12 in [Dem95]):

Theorem 2.1. *Let (X, V) be a directed variety. For any $w_0 \in X_k$, there exists an open neighborhood U_{w_0} of w_0 and a family of germs of curves $(f_w)_{w\in U_{w_0}}$, tangent to V depending holomorphically on w such that*

$$(f_w)_{[k]}(0) = w \quad \text{and} \quad (f_w)'_{[k-1]}(0) \neq 0, \quad \forall w \in U_{w_0}.$$

In particular, $(f_w)'_{[k-1]}(0)$ gives a local trivialization of the tautological line bundle $\mathcal{O}_{X_k}(-1)$ on U_{w_0}.

By Theorem 6.8 in [Dem95], we have the following isomorphism between Demailly-Semple jet bundles and invariant jet differentials:

Theorem 2.2. *Let (X,V) be a directed variety. The direct image sheaf*

$$(\pi_{0,k})_*\mathcal{O}_{X_k}(m) \simeq E_{k,m}V^* \tag{2.1}$$

can be identified with the sheaf of holomorphic sections of $E_{k,m}V^*$. In particular, for any line bundle L, we have the following isomorphism induced by $(\pi_{0,k})_*$:

$$H^0\big(X_k, \mathcal{O}_{X_k}(m)\otimes\pi_{0,k}^*L\big) \xrightarrow{(\pi_{0,k})_*} H^0(X, E_{k,m}V^*\otimes L). \tag{2.2}$$

Therefore, with the notations in Theorem 2.1, for any given local jet differential $P \in E_{k,m}V^*(U)$, the image under the isomorphism (2.1) is the section $\sigma_P \in \mathcal{O}_{X_k}(m)(\pi_{0,k}^{-1}(U)\cap U_{w_0})$ defined by

$$\sigma_P(w) := P(f'_w, f''_w, \ldots, f_w^{(k)})\big((f_w)'_{[k-1]}(0)\big)^{-m}. \tag{2.3}$$

We denote by $X_k^{\mathrm{reg}} \subset X_k$ the set of elements $f_{[k]}(0)$ in X_k which can be reached by all regular germs of curves f, and $X_k^{\mathrm{sing}} := X_k \setminus X_k^{\mathrm{reg}}$.

The general philosophy of the theory of (invariant) jet differentials is that their global sections with values in an anti-ample divisor provide algebraic differential equations which every entire curve must satisfy, which is an application of Ahlfors-Schwarz lemma. The following *Fundamental Vanishing Theorem* shows the obstructions to the existence of entire curves:

Theorem 2.3. *(Demailly, Green-Griffiths, Siu-Yeung) Let (X,V) be a directed projective variety and $f:\mathbb{C}\to(X,V)$ an entire curve tangent to V. Then for every global section $P\in H^0\big(X, E_{k,m}V^*\otimes\mathcal{O}(-A)\big)$ where A is an ample divisor of X, one has $P(f',f'',\ldots,f^{(k)})=0$. In other word, if we denote by s the unique section in $H^0\big(X_k,\mathcal{O}_{X_k}(m)\otimes\pi_{0,k}^*(-A)\big)$ corresponding to P induced by the isomorphism (2.2), and $Z(s)\subset X_k$ the base locus of this section, then $f_{[k]}(\mathbb{C})\subset Z(s)$.*

Now we state the following definition which describes the positivity of the the invariant jet bundles:

Definition 2.1. *Let X be a projective manifold. We say that X has* almost k-jet ampleness *if and only if there exists a c-tuple of positive integers $(a_1,\ldots,a_k)$ such that $\mathcal{O}_{X_k}(a_1,\ldots,a_k)$ is big and its augmented base locus satisfies the condition*

$$\mathbb{B}_+\big(\mathcal{O}_{X_k}(a_1,\ldots,a_k)\big)\subset X_k^{\mathrm{sing}}.$$

By applying the Fundamental Vanishing Theorem 2.3, we can quickly conclude that, if X has almost k-jet ampleness, then its Demailly-Semple locus (ref. [DR13] for the definition) is an empty set, and thus X is Kobayashi hyperbolic.

2.3 Brotbek's Wronskians Operator

Let X be an n-dimensional compact complex manifold. If $(z_1, \dots, z_n)$ are local holomorphic coordinates on an open set $\Omega \subset X$, then since J_k is a locally trivial holomorphic fiber bundle, we have the homeomorphism

$$J_k|_\Omega \sim \Omega \times \mathbb{C}^n,$$

which is given by $[f] \mapsto \big(f(0), f'(0), \dots, f^{(k)}(0)\big)$.

For any holomorphic function $g \in \mathcal{O}(\Omega)$, and $1 \leq j \leq k$, there exists an induced holomorphic function $d_\Omega^{[j]}(g)$ on $\mathcal{O}(p_k^{-1}(\Omega))$, defined by

$$d_\Omega^{[j]}(g)\big(f'(0), f''(0), \dots, f^{(k)}(0)\big) := (g \circ f)^{(j)}(0),$$

and we will construct the Wronskian of any $(k+1)$ holomorphic functions $g_0, \dots, g_k \in \mathcal{O}(\Omega)$ by

$$W_\Omega(g_0, \dots, g_k) := \begin{vmatrix} d_\Omega^{[0]}(g_0) & \cdots & d_\Omega^{[0]}(g_k) \\ \vdots & \ddots & \vdots \\ d_\Omega^{[k]}(g_0) & \cdots & d_\Omega^{[k]}(g_k) \end{vmatrix} \in \mathcal{O}\big(p_k^{-1}(\Omega)\big).$$

An important fact for the Wronskian is that, it is invariant under the $\mathbb{G}_k$ action (ref. Proposition 2.2 in [Bro16]):

Lemma 2.1. *With the notation as above, $W_\Omega(g_0, \dots, g_k) \in E_{k,k'}T_X^*(\Omega)$, where $k' := \frac{k(k+1)}{2}$.*

Thus one has a natural construction for the invariant jet differentials: let L be any holomorphic line bundle on X, for any $s_0, \dots, s_k \in H^0(X, L)$, if we choose a local trivialization of L above Ω, we define

$$W_\Omega(s_0, \dots, s_k) := W_\Omega(s_{0,\Omega}, \dots, s_{k,\Omega}) \in E_{k,k'}T_X^*(\Omega),$$

and if gluing together, we have the global section (ref. Proposition 2.3 [Bro16]):

Theorem 2.4. *For any $s_0, \dots, s_k \in H^0(X, L)$, the locally defined jet differential equations $W_\Omega(s_0, \dots, s_k)$ glue together into a global section*

$$W(s_0, \dots, s_k) \in H^0(X, E_{k,k'}T_X^* \otimes L^{k+1}).$$

We will denote by $\omega(s_0, \dots, s_k) \in H^0(X_k, \mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{k+1})$ the inverse image of $W(s_0, \dots, s_k)$ under the isomorphism (2.2).

Now let

$$\mathbb{W}(X_k, L) := \mathrm{Span}\{\omega(s_0, \dots, s_n) | s_0, \dots, s_n \in H^0(X, L)\} \subset H^0\big(X_k, \mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^*(L^{k+1})\big)$$

be the associated sublinear system of $H^0\big(X_k, \mathcal{O}_{X_k}(\frac{k(k+1)}{2}) \otimes \pi_{0,k}^*(L^{k+1})\big)$. One defines $\mathfrak{w}(X_k, L)$ to be the base ideal of the linear system $\mathbb{W}(X_k, L)$.

By the definition, if A is any line bundle on X, and $s \in H^0(X, L)$, we have

$$W(s \cdot s_0, \dots, s \cdot s_k) = s^{k+1} W(s_0, \dots, s_k) \in H^0(X, E_{k,k'}T_X^* \otimes L^{k+1} \otimes A^{k+1}).$$

Thus if L is very ample we have

$$\mathfrak{w}(X_k, L) \subset \mathfrak{w}(X_k, L^2) \subset \ldots \subset \mathfrak{w}(X_k, L^m) \subset \ldots.$$

Then the Noetherian property shows that this increasing sequence stabilizes for some $\mathfrak{w}(X_k, L^{m_\infty(X_k,L)})$, and the asymptotic ideal sheaf is denoted by $\mathfrak{w}_\infty(X_k, L)$. An important property for $\mathfrak{w}(X_k, L)$ is the following (ref. Lemma 2.4 in [Bro16]):

Lemma 2.2. *If L generates k-jets at every point of X, that is, for any $x \in X$,*

$$H^0(X, L) \to L \otimes \mathcal{O}_{X,x}/m_{X,x}^{k+1}$$

is surjective, then

$$\mathrm{Supp}(\mathcal{O}_{X_k}/\mathfrak{w}(X_k, L)) \subset X_k^{\mathrm{sing}}.$$

In order to obtain the effectivity in the hyperbolicity-problems, we need to determine the constant $m_\infty(X_k, L)$. In [Den16], we obtained

Theorem 2.5. *If L generates k-jets at each point of X, then $m_\infty(X_k, L) = 1$. In particular, if L is known to be only very ample, we have $m_\infty(X_k, L) = k$.*

As was proved in [Bro16], $\mathfrak{w}_\infty(X_k, L)$ does not depend on the very ample line bundle L and is called the *asymptotic Wronskian ideal sheaf*, denoted by $\mathfrak{w}_\infty(X_k)$. As was also shown in Lemma 2.6 of [Bro16], $\mathfrak{w}_\infty(X_k)$ behaves well under restriction, that is, for any directed variety (Y, V_Y) with $Y \subset X$ and $V_Y \subset V_X|_Y$, under the induced inclusion $Y_k \subset X_k$ one has

$$\mathfrak{w}_\infty(X_k)|_{Y_k} = \mathfrak{w}_\infty(Y_k).$$

2.4 Blow-ups of the Wronskian Ideal Sheaf

This subsection are mainly borrowed from [Bro16]. We will state some important results without proof, and the readers who are interested in the details are encouraged to refer Section 2.4 of [Bro16].

Since the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(X_k)$ is an obstruction to the positivity of $\mathcal{O}_{X_k}(1)$, we would like to blow up this sheaf.

For any directed manifold (X, V), we denote by

$$\hat{X}_k := \mathrm{Bl}_{\mathfrak{w}_\infty(X_k)}(X_k) \to X_k$$

the blow-up of X_k along $\mathfrak{w}_\infty(X_k)$, and F the effective Cartier divisor on $\hat{X}_k$ such that

$$\mathcal{O}_{\hat{X}_k}(-F) = \nu_k^{-1}\mathfrak{w}_\infty(X_k).$$

Take a very ample line bundle L on X, for any $m \geq 0$, and any $s_0, \ldots, s_k \in H^0(X, L^m)$, there exists

$$\hat{\omega}(s_0, \ldots, s_k) \in H^0\Big(\hat{X}_k, \nu_k^*\big(\mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{m(k+1)}\big) \otimes \mathcal{O}_{\hat{X}_k}(-F)\Big),$$

such that

$$\nu_k^*\omega(s_0,\ldots,s_k) = s_F \cdot \hat{\omega}(s_0,\ldots,s_k).$$

Here $s_F \in H^0(\hat{X}_k, F)$ is the tautological section. Then by Theorem 2.5, for any $\hat{w} \in \hat{X}_k$ and any $m \geq k$, there exists $s_0,\ldots,s_k \in H^0(X, L^m)$ such that

$$\hat{\omega}(s_0,\ldots,s_k)(\hat{w}) \neq 0.$$

The blow-ups is fonctorial thanks to the fact that the asymptotic Wronskian ideal sheaf behaves well under restriction. Namely, if $(Y, V_Y) \subset (X, V_X)$ is a sub-directed variety, then $\hat{Y}_k$ is the strict transform of Y_k in X_k under the blowing-up morphism $\nu_k : \hat{X}_k \to X_k$. This fonctorial property also holds for families (ref. Proposition 2.7 in [Bro16]):

Theorem 2.6. *Let $\mathscr{X} \xrightarrow{\rho} T$ be a smooth and projective morphism between non-singular varieties. We denote by $\mathscr{X}_k^{\mathrm{rel}}$ the k-th Demailly-Semple tower of the relative directed variety $(\mathscr{X}, T_{\mathscr{X}/T})$. Take $\nu_k : \hat{\mathscr{X}}_k^{\mathrm{rel}} \to \mathscr{X}_k^{\mathrm{rel}}$ to be the blow-ups of the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(\mathscr{X}_k^{\mathrm{rel}})$. Then for any $t_0 \in T$ writing $X_{t_0} := \rho^{-1}(t_0)$, we have*

$$\nu_k^{-1}(X_{t_0,k}) = \hat{X}_{t_0,k}.$$

3 Proof of the Main Theorem

3.1 Families of Complete Intersections of Fermat-type Hypersurfaces

Let X be a projective manifold of dimension n endowed with a very ample line bundle A. We first construct a family of complete intersection subvarieties in X cut out by certain Fermat-type hypersurfaces. For an integer $N \geq n$, we fix $N+1$ sections in general position $\tau_0,\ldots,\tau_N \in H^0(X, A)$. By "general position" we mean that the hypersurfaces $\{\tau_i = 0\}_{i=0,\ldots,N}$ are all smooth and irreducibles ones, and they are simple normal crossing. For any $1 \leq c \leq n-1$, and two c-tuples of positive integers $\boldsymbol{\epsilon} = (\epsilon_1,\ldots,\epsilon_c), \boldsymbol{\delta} = (\delta_1,\ldots,\delta_c)$, we construct the family $\mathcal{X}$ as follows: For any $p = 1,\ldots,c$, set $\mathbb{I}^p := \{I = (i_0,\ldots,i_N) || I| = \delta_p\}$ and $\mathbf{a}^p := \Big(a_I^p \in H^0\big(X, \mathcal{O}_X(\epsilon_p A)\big)\Big)_{|I|=\delta_p}$. For the positive integers r and k fixed later according to our needs, we define the bihomogenous section of $\mathcal{O}_X\Big(\big(\epsilon_p + (r+k)\delta_p\big)A\Big)$ over X by

$$\boldsymbol{F}^p(\mathbf{a}^p)(x) : x \mapsto \sum_{|I|=\delta_p} a_I^p(x)\boldsymbol{\tau}(x)^{(r+k)I},$$

where $\mathbf{a}^p$ varies in the parameter space $S_p := \bigoplus_{I\in\mathbb{I}^p} H^0\big(X, \mathcal{O}_X(\epsilon_p A)\big)$.

We then consider the family $\overline{\mathcal{X}} \subset S_1 \times \ldots \times S_c \times X$ of complete intersection varieties in X defined by those sections:

$$\overline{\mathcal{X}} := \{(\mathbf{a}^1,\ldots,\mathbf{a}^c, x) \in S_1 \times \ldots \times S_c \times X | \boldsymbol{F}^1(\mathbf{a}^1)(x) = \ldots = \boldsymbol{F}^c(\mathbf{a}^c)(x) = 0\}. \tag{3.1}$$

We know that there is a non-empty Zariski open set $S \subset S_1 \times \ldots \times S_c$ parametrizing smooth varieties and we will work on $\mathcal{X} := q_1^{-1}(S) \cap \overline{\mathcal{X}}$, where q_1 is the natural projection from $S_1 \times \ldots \times$

$S_c \times X$ to $S_1 \times \ldots \times S_c$. Set $\mathcal{X}_k$ to be the k-th Demailly-Semple tower of the relative tangent bundle $(\mathcal{X}, T_{\mathcal{X}/S})$, and $\hat{\mathcal{X}}_k$ the blowing-up of the asmptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(\mathcal{X}_k)$, and we would like to construct a regular morphism from $\hat{\mathcal{X}}_k$ (after shrinking a bit) to a suitable generically finite-to-one family and to "pull-back" the positivity from the parameter space of this family. First we begin with a technique lemma:

Lemma 3.1. *Let Ω be an open subset of X on which A can be trivialized. Take $I = (i_0, \ldots, i_N) \in \mathbb{I}^p$ with any $1 \leq p \leq k$, for any $0 \leq j \leq k$, there exists a $\mathbb{C}$-linear map*

$$d^{[j]}_{I,\Omega} : H^0(X, A^\epsilon) \to \mathcal{O}\big(p_k^{-1}(\Omega)\big)$$

such that for any $a \in H^0(X, A^\epsilon)$, $d^{[j]}_{\Omega}(a\boldsymbol{\tau}^{(r+k)I}) = \tau_\Omega^{rI} d^{[j]}_{I,\Omega}(a)$.

Therefore, for any $I_0, \ldots, I_k \in \mathbb{I}^p$ and any $a_{I_0}, \ldots, a_{I_k} \in H^0(X, A^\epsilon)$ one can define

$$W_{\Omega, I_0, \ldots, I_k}(a_{I_0}, \ldots, a_{I_k}) := \begin{vmatrix} d^{[0]}_{I_0,\Omega}(a_{I_0}) & \cdots & d^{[0]}_{I_k,\Omega}(a_{I_k}) \\ \vdots & \ddots & \vdots \\ d^{[k]}_{I_0,\Omega}(a_{I_0}) & \cdots & d^{[k]}_{I_k,\Omega}(a_{I_k}) \end{vmatrix} \in \mathcal{O}\big(p_k^{-1}(\Omega)\big), \tag{3.2}$$

and by Lemma 3.1 we obtain

$$W_\Omega(a_{I_0}\boldsymbol{\tau}^{(r+k)I_0}, \ldots, a_{I_k}\boldsymbol{\tau}^{(r+k)I_k}) = \boldsymbol{\tau}_\Omega^{r(I_0+\ldots+I_k)} W_{\Omega, I_0, \ldots, I_k}(a_{I_0}, \ldots, a_{I_k}).$$

From Proposition 2.4 we deduce the following

Lemma 3.2. *For any $I_0, \ldots, I_k \in \mathbb{I}^p$ and any $a_{I_0}, \ldots, a_{I_k} \in H^0(X, A^\epsilon)$, the locally defined functions $W_{\Omega, I_0, \ldots, I_k}(a_{I_0}, \ldots, a_{I_k})$ can be glued together into a global section*

$$W_{I_0, \ldots, I_k}(a_{I_0}, \ldots, a_{I_k}) \in H^0(X, E_{k,k'}T^*_X \otimes A^{(k+1)(\epsilon + k\delta_p)})$$

such that

$$W(a_{I_0}\boldsymbol{\tau}^{(r+k)I_0}, \ldots, a_{I_k}\boldsymbol{\tau}^{(r+k)I_k}) = \boldsymbol{\tau}^{r(I_0+\ldots+I_k)} W_{I_0, \ldots, I_k}(a_{I_0}, \ldots, a_{I_k}).$$

We denote by

$$\omega_{I_0, \ldots, I_k}(a_{I_0}, \ldots, a_{I_k}) \in H^0\big(X_k, \mathcal{O}_{X_k}(\frac{k(k+1)}{2}) \otimes \pi^*_{0,k} A^{(k+1)(\epsilon + k\delta)}\big) \tag{3.3}$$

the inverse image of $W_{I_0, \ldots, I_k}(a_{I_0}, \ldots, a_{I_k})$ under the isomorphism (2.2).

Now for every $1 \leq p \leq c$ we construct a rational map given by the Wronskians

$$\begin{array}{rcl} \Phi^p : S_p \times X_k & \dashrightarrow & P(\Lambda^{k+1}\mathbb{C}^{\mathbb{I}^p}) \\ (\mathbf{a}, w) & \mapsto & ([\omega_{I_0, \ldots, I_k}(a_{I_0}, \ldots, a_{I_k})(w)])_{I_0, \ldots, I_k \in \mathbb{I}^p}, \end{array}$$

where $\mathbb{C}^{\mathbb{I}^p} := \oplus_{I \in \mathbb{I}^p} \mathbb{C} \simeq \mathbb{C}^{\binom{N+\delta_p}{\delta_p}}$.

Claim: Φ^p factors through the Plücker embedding

$$\mathrm{Pluc} : \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \hookrightarrow P(\Lambda^{k+1}\mathbb{C}^{\mathbb{I}^p}).$$

Proof: for any $1 \leq p \leq c$ and any $w_0 \in X_k$, by Theorem 2.1, one can find an open neighborhood U_{w_0} of w_0 with $U_{w_0} \subset \pi_{0,k}^{-1}(U)$, where $A|_U$ can be trivialized; and a family of germs of curves $(f_w)_{w\in U_{w_0}}$ depending holomorphically on w with $(f_w)_{[k]}(0) = w$. Then for any $\mathbf{a} = (a_I)_{I\in\mathbb{I}^p} \in S_p$ and any $0 \leq j \leq k$, we denote by

$$d^{[j]}_{\bullet,w_0}(\mathbf{a}, w) := \big(d^{[j]}_{I,U}(a_I)(f'_w, f''_w, \ldots, f^{(k)}_w)\big)_{I\in\mathbb{I}^p} \in \mathbb{C}^{\mathbb{I}^p},$$

and the local rational map

$$\begin{array}{rcl} \Phi^p_{w_0} : S_p \times U_{w_0} & \dashrightarrow & \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \\ (\mathbf{a}, w) & \mapsto & \mathrm{Span}\big(d^{[0]}_{\bullet,w_0}(\mathbf{a}, w), \ldots, d^{[k]}_{\bullet,w_0}(\mathbf{a}, w)\big). \end{array}$$

We will show that this definition does not depend on the choice of w_0. Indeed, by Definition 3.2 one has $\Phi^p = \mathrm{Pluc} \circ \Phi^p_{w_0}$, which shows that Φ^p factor through Pluc and we still denote by $\Phi^p : S_p \times X_k \dashrightarrow \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p})$ by abuse of notation. ■

Recall that $\hat{X}_k$ is denoted to be the blow-up $\nu_k : \hat{X}_k \to X_k$ of the asymptotic k-th Wronskian ideal sheaf $\mathfrak{w}_\infty(X_k)$, such that $\nu_k^{-1}\mathfrak{w}_\infty(X_k) = \mathcal{O}_{\hat{X}_k}(-F)$ for some effective cartier divisor F on $\hat{X}_k$. Then under certain conditions ν_k partially resolves the indeterminacy of Φ^p. To clarify this, we need to introduce some notations. For any $x \in X$, we set

$$N_x := \#\{j \in \{0, \ldots, N\} \big| \tau_j(x) \neq 0\} \text{ and } \mathbb{I}^p_x := \{I \in \mathbb{I}^p, |\tau^I(x) \neq 0\}.$$

Since the τ_j's are in general position, and $N \geq n$, we have $N_x \geq 1$ for all $x \in X$. Then we define

$$\Sigma := \{x \in X \big| N_x = 1\} \text{ and } X^\circ := X \setminus \Sigma.$$

Observe that if $N > n$, then $X^\circ = X$, and if $N = n$, then Σ is a finite set of points. We denote by $\hat{X}^\circ_k := (\pi_{0,k} \circ \nu_k)^{-1}(X^\circ)$. We have the following crucial lemma of resolution of indeterminacy due to Brotbek (ref. Proposition 3.8 in [Bro16]):

Lemma 3.3. *Suppose that*

$$N \geq n \geq 2, \quad k \geq 1, \quad \epsilon_p \geq m_\infty(X_k, A) = k \text{ and } \delta_p \geq n(k+1). \tag{★}$$

Then there exists a non-empty Zariski open subset $U_{\mathrm{def},p} \subset S_p$ such that the rational map $\Phi^p|_{U_{\mathrm{def},p}\times X_k} \dashrightarrow \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p})$ can be resolved into a regular morphism:

$$\begin{array}{ccc} U_{\mathrm{def},p} \times \hat{X}^\circ_k & & \\ {\scriptstyle \mathbb{1}\times\nu_k}\downarrow & \searrow^{\hat{\Phi}^p} & \\ U_{\mathrm{def},p} \times X^\circ_k & \overset{\Phi^p}{\dashrightarrow} & \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \end{array}$$

In Lemma 3.3, we use the result in Theorem 2.5 that $m_\infty(X_k, A) = k$.

3.2 Mapping to the Universal grassmannian

Set $U_{\text{def}} := U_{\text{def},1} \times \dots U_{\text{def},c} \bigcap S$. We suppose from now on that $N \geq n \geq 2$, $\epsilon_p \geq k \geq 1$ and that $\delta_p \geq n(k+1)$ for any $1 \leq p \leq c$. Then by Lemma 3.3 we get a regular morphism

$$\begin{array}{rcl} \Psi : U_{\text{def}} \times \hat{X}_k^\circ & \to & \mathbf{G}_{k+1}(\delta_1) \times \dots \times \mathbf{G}_{k+1}(\delta_c) \times \mathbb{P}^N \\ (\mathbf{a}, \xi) & \mapsto & \big(\hat{\Phi}^1(\mathbf{a}^1, \xi), \dots, \hat{\Phi}^c(\mathbf{a}^c, \xi), [\tau^r(\xi)]\big). \end{array}$$

Here $[\tau^r(\xi)] := \big[\tau_0^r\big(\pi_{0,k} \circ \nu_k(\xi)\big) : \dots : \tau_N^r\big(\pi_{0,k} \circ \nu_k(\xi)\big)\big]$, and we write $\mathbf{G}_{k+1}(\delta_p) := \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p})$ and $\mathbf{G} := \mathbf{G}_{k+1}(\delta_1) \times \dots \mathbf{G}_{k+1}(\delta_c)$ for brevity.

From now on we **always** assume that $(k+1)c \geq N$. Set $\mathscr{Y}$ be the *universal Grassmannian* defined by

$$\mathscr{Y} := \{(\Delta_1, \dots, \Delta_c, [z]) \in \mathbf{G} \times \mathbb{P}^N | \forall P \in \Delta_1, \dots, \Delta_c : P([z]) = 0\}.$$

If we denote by $p_1 : \mathscr{Y} \to \mathbf{G}$ the first projection map, p_1 is a generically finite-to-one (may not surjective) morphism. Set G^∞ to be the set of points in $\mathbf{G} := \mathbf{G}_{k+1}(\delta_1) \times \dots \mathbf{G}_{k+1}(\delta_c)$ such that the fiber of $p_1 : \mathscr{Y} \to \mathbf{G}$ is not a finite set.

We need to cover X by a natural stratification induced by the vanishing of the τ_j's. For any $J \subset \{0, \dots, N\}$ and $1 \leq p \leq c$ we define

$$\begin{array}{rl} X_J & := \{x \in X \big| \tau_j(x) = 0 \Leftrightarrow j \in J\}, \\ \mathbb{I}_J^p & := \{I \in \mathbb{I}^p \big| \text{Supp}(I) \subset \{0, \dots, N\} \setminus J\}, \\ \mathbb{P}_J & := \{[z] \in \mathbb{P}^N \big| z_j = 0 \text{ if } j \in J\}, \end{array}$$

$\hat{X}_{k,J} := (\pi_{0,k} \circ \nu_k)^{-1}(X_J)$ and $\hat{X}_{k,J}^\circ := \hat{X}_{k,J} \cap \hat{X}_k^\circ$. Set $\mathscr{Y}_J := \mathscr{Y} \cap \big(\mathbf{G} \times \mathbb{P}_J\big) \subset \mathbf{G} \times \mathbb{P}^N$, and G_J^∞ also the set of points in $\mathbf{G}$ such that the fiber of the first projection map $p_J : \mathscr{Y}_J \to \mathbf{G}$ is not a finite set.

Now set

$$U_{\text{def,p}}^\circ := U_{\text{def,p}} \cap \big\{\mathbf{a}^p \in S_p \big| \forall I \in \mathbb{I}^p,\ \{\boldsymbol{F}^p(\mathbf{a}^p)(x) = 0\} \cap \Sigma = \emptyset\big\} \text{ and } U_{\text{def}}^\circ := U_{\text{def},1}^\circ \times \dots \times U_{\text{def,c}}^\circ.$$

Since Σ is a finite set, $U_{\text{def,p}}^\circ$ is a non-empty Zariski open subset of $U_{\text{def,p}}$. Consider the universal family of codimension c smooth varieties $\mathscr{H} := (U_{\text{def}}^\circ \times X) \cap \overline{\mathcal{X}}$, then

$$\mathscr{H} \cap \{U_{\text{def}}^\circ \times \Sigma\} = \emptyset. \tag{3.4}$$

We denote by $\mathscr{H}_k^{\text{rel}}$ the k-th Demailly-Semple tower of the relative directed variety $(\mathscr{H}, T_{\mathscr{H}/U_{\text{def}}^\circ})$. If $\hat{\mathscr{H}}_k^{\text{rel}}$ is obtained by the blow-ups of the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(\mathscr{H}_k^{\text{rel}})$, then by the arguments in Section 2.4 we have

$$(\mathbb{1} \times \nu_k)^{-1}(\mathscr{H}_k^{\text{rel}}) = \hat{\mathscr{H}}_k^{\text{rel}}.$$

Moreover for any $\mathbf{a} \in U_{\text{def}}^\circ$, if we denote by $H_{\mathbf{a},k} := \mathscr{H}_k^{\text{rel}} \cap (\{\mathbf{a}\} \times X_k)$ and $\hat{H}_{\mathbf{a},k} := \hat{\mathscr{H}}_k^{\text{rel}} \cap (\{\mathbf{a}\} \times \hat{X}_k)$, then $\nu_k|_{\hat{H}_{\mathbf{a},k}} : \hat{H}_{\mathbf{a},k} \to H_{\mathbf{a},k}$ is indeed the blow-ups of the Wronskian ideal sheaf $\mathfrak{w}_\infty(H_{\mathbf{a},k})$. By (3.4), $\Psi|_{\hat{\mathscr{H}}_k^{\text{rel}}}$ is a regular morphsim. Set

$$\mathscr{H}_{k,J}^{\text{rel}} := \mathscr{H}_k^{\text{rel}} \cap (U_{\text{def}}^\circ \times \hat{X}_{k,J}),$$

and we have the following

Theorem 3.1. *For any $J \subset \{0,\dots,N\}$, when restricted to $\mathscr{H}^{\mathrm{rel}}_{k,J}$ the morphism Ψ factors through $\mathscr{Y}_J$:*

$$\Psi|_{\hat{\mathscr{H}}^{\mathrm{rel}}_{k,J}} : \hat{\mathscr{H}}^{\mathrm{rel}}_{k,J} \to \mathscr{Y}_J \subset \mathbf{G} \times P_J.$$

Proof. Since when restricted to $U^\circ_{\mathrm{def}} \times \hat{X}^\circ_{k,J}$, Ψ factors through $\mathbf{G} \times \mathbb{P}_J$. Thus it suffices to prove that $\Psi|_{\hat{\mathscr{H}}^{\mathrm{rel}}_k}$ factors through $\mathscr{Y}$. By Lemma 3.3, it suffices to prove that the rational map

$$\begin{aligned} \tilde{\Psi} : U_{\mathrm{def}} \times X_k &\dashrightarrow \mathbf{G} \times \mathbb{P}^N && (3.5)\\ (\mathbf{a}, w) &\mapsto \big(\Phi^1(\mathbf{a}^1, w), \dots, \Phi^c(\mathbf{a}^c, w), [\tau^r(w)]\big) && (3.6) \end{aligned}$$

factor through $\mathscr{Y}$ when restricted to $\mathscr{H}^{\mathrm{rel}}_k$. Take any $(\mathbf{a}, w_0) \in \mathscr{H}^{\mathrm{rel}}_k$ outside the indeterminacy of $\tilde{\Psi}$, and by Lemma 2.1 one can find a germ of curve $f : (\mathbb{C}, 0) \to \big(X, x := \pi_{0,k}(w_0)\big)$ with $f_{[k]}(0) = w_0$, and an open subset $U \subset X$ containing x such that $A|_U$ can be trivialized. Recall that $H_{\mathbf{a},k} := \mathscr{H}^{\mathrm{rel}}_k \cap (\{\mathbf{a}\} \times X_k)$, and it is the k-th Demailly-Semple tower of $H_{\mathbf{a}}$. Therefore, we have $\big(f(0), f'(0), \dots, f^{(k)}(0)\big) \in J_k H_{\mathbf{a}}$.

Since $H_{\mathbf{a}}$ is defined by the equations

$$\begin{cases} \boldsymbol{F}^1(\mathbf{a}^1)(x) := \sum_{|I|=\delta_1} a^1_I(x)\boldsymbol{\tau}(x)^{(r+k)I} = 0, \\ \vdots \\ \boldsymbol{F}^c(\mathbf{a}^c)(x) := \sum_{|I|=\delta_c} a^c_I(x)\boldsymbol{\tau}(x)^{(r+k)I} = 0, \end{cases}$$

then $d^{[j]}_U \boldsymbol{F}^p(\mathbf{a}^p)(f', f'', \dots, f^{(k)}) = 0$ for any $1 \le p \le c$ and $0 \le j \le k$. By Lemma 3.1 we have $d^{[j]}_U \boldsymbol{F}^p(\mathbf{a}^p) = \tau^{rI}_U \cdot \sum_{|I|=\delta_p} d^{[j]}_{I,U}(a^p_I)$, and by the definition for $\tilde{\Psi}(\mathbf{a}, w)$ we conclude that $\tilde{\Psi}(\mathbf{a}, w_0) \in \mathscr{Y}$. □

To proceed further, we need an important technique lemma in [Bro16] as follows

Lemma 3.4. *Suppose that $\epsilon \ge m_\infty(X_k, A) = k$. For any $p = 1, \dots, c$ and any $\hat{w}_0 \in \hat{X}_k$, there exists an open neighborhood $\hat{U}_{\hat{w}_0} \subset \hat{X}_k$ of $\hat{w}_0$ satisfying the following. For any $I \in \mathbb{I}^p$ and $0 \le i \le k$ there exists a linear map*

$$g^p_{i,I} : H^0(X, A^\epsilon) \to \mathcal{O}(\hat{U}_{\hat{w}_0})$$

such that for any $(\mathbf{a}^p, \hat{w}) \in S_p \times \hat{U}_{\hat{w}_0}$, writting $g^p_{i,\bullet}(\mathbf{a}^p, \hat{w}) = \big(g^p_{i,I}(a^p_I)(\hat{w})\big)_{I \in \mathbb{I}^p} \in \mathbb{C}^{\mathbb{I}^p}$ one has

(1) The Plücker coordinates of $\hat{\Phi}^p(\mathbf{a}^p, \hat{w})$ are all vanishing if and only if

$$\dim \mathrm{Span}\big(g^p_{0,\bullet}(\mathbf{a}^p, \hat{w}), \dots, g^p_{k,\bullet}(\mathbf{a}^p, \hat{w})\big) < k + 1.$$

(2) If $\dim \mathrm{Span}\big(g^p_{0,\bullet}(\mathbf{a}^p, \hat{w}), \dots, g^p_{k,\bullet}(\mathbf{a}^p, \hat{w})\big) = k + 1$, then

$$\hat{\Phi}^p(\mathbf{a}^p, \hat{w}) = \mathrm{Span}\big(g^p_{0,\bullet}(\mathbf{a}^p, \hat{w}), \dots, g^p_{k,\bullet}(\mathbf{a}^p, \hat{w})\big) \in \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}).$$

(3) For any fixed w_0, and every $1 \leq p \leq c$, we define the linear map

$$\begin{aligned} \hat{\varphi}^p_{\hat{w}_0} : S_p &\to (\mathbb{C}^{\mathbb{I}^p})^{k+1} \\ \mathbf{a}^p &\mapsto \big(g^p_{0,\bullet}(\mathbf{a}^p, \hat{w}_0), \ldots, g^p_{k,\bullet}(\mathbf{a}^p, \hat{w}_0)\big). \end{aligned}$$

Set $x := \pi_{0,k} \circ \nu_k(\hat{w}_0)$ and $\rho^p_x : (\mathbb{C}^{\mathbb{I}^p})^{k+1} \to (\mathbb{C}^{\mathbb{I}^p_x})^{k+1}$ the natural projection map, then one has

$$\mathrm{rank}\rho^p_x \circ \hat{\varphi}^p_{\hat{w}_0} = (k+1)\#\mathbb{I}^p_x.$$

Here $\mathbb{I}^p_x := \{I \in \mathbb{I}^p | \boldsymbol{\tau}^I(x) \neq 0\}$.

Now we are ready to prove the following crucial lemma:

Lemma 3.5. *(Avoiding exceptional locus) For any $J \subset \{0, \ldots, N\}$, suppose that $c(k+1) \geq N \geq n \geq 2$. If $\delta_p \geq (n-1)(k+1)+1$ for any $p = 1, \ldots, c$, then there exists a non-empty Zariski open subset $U_J \subset U^\circ_{\mathrm{def}}$ such that*

$$\hat{\Phi}^{-1}(G^\infty_J) \cap (U_J \times \hat{X}^\circ_{k,J}) = \emptyset.$$

Here we define the map

$$\begin{aligned} \hat{\Phi} : U^\circ_{\mathrm{def}} \times \hat{X}^\circ_k &\to \mathbf{G}_{k+1}(\delta_1) \times \ldots \times \mathbf{G}_{k+1}(\delta_c) \\ (\mathbf{a}, \xi) &\mapsto \big(\hat{\Phi}^1(\mathbf{a}^1, \xi), \ldots, \hat{\Phi}^c(\mathbf{a}^c, \xi)\big), \end{aligned}$$

which is the composition $\pi_1 \circ \Psi$. Here $\pi_1 : \mathbf{G} \times \mathbb{P}^N \to \mathbf{G}$ is the first projection.

Proof. Fix any $\hat{w}_0 \in \hat{X}^\circ_k$, we set $x := \pi_{0,k} \circ \nu_k(\hat{w}_0)$. Then there exists a unique $J \subset \{0, \ldots, N\}$ such that $x \in X_J$, and we define the following analogues of $\mathscr{Y}$ parametrized by affine spaces

$$\widetilde{\mathscr{Y}} := \{\big(\alpha_{10}, \ldots, \alpha_{1k}, \ldots, \alpha_{c0} \ldots, \alpha_{ck}, [z]\big) \in \prod_{p=1}^c (\mathbb{C}^{\mathbb{I}^p})^{k+1} \times \mathbb{P}_J \big| \forall 1 \leq p \leq c, 0 \leq j \leq k, \alpha_{pi}([z]) = 0\},$$

$$\widetilde{\mathscr{Y}_x} := \{\big(\alpha_{10}, \ldots, \alpha_{1k}, \ldots, \alpha_{c0} \ldots, \alpha_{ck}, [z]\big) \in \prod_{p=1}^c (\mathbb{C}^{\mathbb{I}^p_J})^{k+1} \times \mathbb{P}_J \big| \forall 1 \leq p \leq c, 0 \leq j \leq k, \alpha_{pi}([z]) = 0\},$$

here we use the identification $\mathbb{C}^{\mathbb{I}^p} \approx H^0\big(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta_p)\big)$ and $\mathbb{C}^{\mathbb{I}^p_J} \approx H^0\big(\mathbb{P}_J, \mathcal{O}_{\mathbb{P}_J}(\delta_p)\big)$. By analogy with G^∞_J, we denote by $V^\infty_{1,J}$ (resp. $V^\infty_{2,J}$) the set of points in $\prod_{p=1}^c (\mathbb{C}^{\mathbb{I}^p})^{k+1}$ (resp. $\prod_{p=1}^c (\mathbb{C}^{\mathbb{I}^p_J})^{k+1}$) at which the fiber in $\widetilde{\mathscr{Y}}$ (resp. $\widetilde{\mathscr{Y}_x}$) is positive dimensional.

By Lemma 3.4 for every $1 \leq p \leq c$ there exists the linear map

$$\hat{\varphi}^p_{\hat{w}_0} : S_p \to (\mathbb{C}^{\mathbb{I}^p})^{k+1}$$

such that for any $\mathbf{a} \in U^\circ_{\mathrm{def}}$ we have

$$\hat{\Phi}(\mathbf{a}, \hat{w}_0) = \big([\hat{\varphi}^1_{\hat{w}_0}(\mathbf{a}^1)], \ldots, [\hat{\varphi}^c_{\hat{w}_0}(\mathbf{a}^c)]\big),$$

here $[\hat{\varphi}^p_{\hat{w}_0}(\mathbf{a}^p)] := \mathrm{Span}\big(g^p_{0,\bullet}(\mathbf{a}^p,\hat{w}_0),\ldots,g^p_{k,\bullet}(\mathbf{a}^p,\hat{w}_0)\big) \in \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p})$. Then we have

$$\hat{\Phi}^{-1}(G^\infty_J)\cap(U^\circ_{\mathrm{def}}\times\{\hat{w}_0\}) = \hat{\varphi}^{-1}_{\hat{w}_0}(V^\infty_{1,J})\cap U^\circ_{\mathrm{def}} = (\rho_x\circ\hat{\varphi}_{\hat{w}_0})^{-1}(V^\infty_{2,J})\cap U^\circ_{\mathrm{def}},$$

where we denote by

$$\begin{array}{rcl}\hat{\varphi}_{\hat{w}_0} : S_1\times\ldots S_c & \to & \prod_{p=1}^c(\mathbb{C}^{\mathbb{I}^p})^{k+1}\\ \mathbf{a}=(\mathbf{a}^1,\ldots,\mathbf{a}^c) & \mapsto & \big(\hat{\varphi}^1_{\hat{w}_0}(\mathbf{a}^1),\ldots,\hat{\varphi}^c_{\hat{w}_0}(\mathbf{a}^c)\big),\end{array}$$

and

$$\rho_x : \prod_{p=1}^c(\mathbb{C}^{\mathbb{I}^p})^{k+1}\to\prod_{p=1}^c(\mathbb{C}^{\mathbb{I}^p_x})^{k+1}$$

is the natural projection map. By the above notations we have $\mathbb{I}^p_J = \mathbb{I}^p_x$ for any $p=1,\ldots,c$. Since the linear map $\rho_x\circ\hat{\varphi}_{\hat{w}_0}$ is diagonal by blocks, by Lemma 3.4 we have

$$\mathrm{rank}\rho_x\circ\hat{\varphi}_{\hat{w}_0} = \sum_{p=1}^c\mathrm{rank}\rho^p_x\circ\hat{\varphi}^p_{\hat{w}_0} = \sum_{p=1}^c(k+1)\#\mathbb{I}^p_x.$$

Therefore

$$\begin{array}{rcl}\dim\big(\hat{\Phi}^{-1}(G^\infty_J)\cap(U^\circ_{\mathrm{def}}\times\{\hat{w}_0\})\big) & \leq & \dim\big((\rho_x\circ\hat{\varphi}_{\hat{w}_0})^{-1}(V^\infty_{2,J})\big)\\ & \leq & \dim(V^\infty_{2,J})+\dim\ker(\rho_x\circ\hat{\varphi}_{\hat{w}_0})\\ & \leq & \dim(V^\infty_{2,J})+\dim(S_1\times\ldots\times S_c)-\mathrm{rank}(\rho_x\circ\hat{\varphi}_{\hat{w}_0})\\ & = & \dim(V^\infty_{2,J})+\dim(S_1\times\ldots\times S_c)-\sum_{p=1}^c(k+1)\#\mathbb{I}^p_x.\end{array}$$

Since

$$\begin{array}{rcl}\dim(V^\infty_{2,J}) & = & \dim(\prod_{p=1}^c(\mathbb{C}^{\mathbb{I}^p_J})^{k+1})-\mathrm{codim}(V^\infty_{2,J},\prod_{p=1}^c(\mathbb{C}^{\mathbb{I}^p_J})^{k+1})\\ & = & \sum_{p=1}^c(k+1)\#\mathbb{I}^p_J-\mathrm{codim}(V^\infty_{2,J},\prod_{p=1}^c(\mathbb{C}^{\mathbb{I}^p_J})^{k+1})\\ & = & \sum_{p=1}^c(k+1)\#\mathbb{I}^p_x-\mathrm{codim}(V^\infty_{2,J},\prod_{p=1}^c(\mathbb{C}^{\mathbb{I}^p_J})^{k+1}),\end{array}$$

then we have

$$\dim\big(\hat{\Phi}^{-1}(G^\infty_J)\cap U^\circ_{\mathrm{def}}\times\{\hat{w}_0\}\big)\leq\dim(S_1\times\ldots\times S_c)-\mathrm{codim}(V^\infty_{2,J},\prod_{p=1}^c(\mathbb{C}^{\mathbb{I}^p_J})^{k+1}),$$

which yields

$$\dim\big(\hat{\Phi}^{-1}(G_J^\infty)\cap U_{\mathrm{def}}^\circ\times\hat{X}_{k,J}^\circ\big)\leq\dim(S_1\times\ldots\times S_c)-\mathrm{codim}(V_{2,J}^\infty,\prod_{p=1}^{c}(\mathbb{C}^{\mathbb{I}_J^p})^{k+1})+\dim\hat{X}_k.$$

By a result due to Olivier Benoist (ref. Corollary 3.2 in [BD15]), we have

$$\mathrm{codim}(V_{2,J}^\infty,\prod_{p=1}^{c}(\mathbb{C}^{\mathbb{I}_J^p})^{k+1})\geq\min_{1\leq j\leq c}\delta_j+1.$$

Therefore, if

$$\dim\hat{X}_k<\min_{1\leq j\leq c}\delta_j+1,\tag{♣}$$

$\hat{\Phi}^{-1}(G_J^\infty)$ doesn't dominate U_{def}° via the projection $U_{\mathrm{def}}^\circ\times\hat{X}_{k,J}^\circ\to U_{\mathrm{def}}^\circ$, and thus we can find a non-empty Zariski open subset $U_J\subset U_{\mathrm{def}}^\circ$ such that

$$\hat{\Phi}^{-1}(G_J^\infty)\cap(U_J\times\hat{X}_{k,J}^\circ)=\emptyset.$$

Thus if $\min\limits_{1\leq j\leq c}\delta_j\geq(n-1)(k+1)+1$, Condition ♣ is satisfied. □

3.3 Pull-back of the Positivity

For any c-tuple of positive integers $\mathbf{e}=(e_1,\ldots,e_c)$, we denote by

$$\mathcal{L}(\mathbf{e}):=\mathcal{O}_{\mathbf{G}_{k+1}(\delta_1)}(e_1)\boxtimes\ldots\boxtimes\mathcal{O}_{\mathbf{G}_{k+1}(\delta_c)}(e_c),$$

which is a very ample line bundle on $\mathbf{G}$. Since $p_J:\mathscr{Y}_J\to\mathbf{G}$ is a generically finite-to-one morphism, by the Nakamaye Theorem (ref. Theorem 10.3.5 in [Laz04]), the augmented base locus for $p_J^*\mathcal{L}(\mathbf{e})$ coincides with its Null locus

$$E_J:=\{y\in\mathscr{Y}|\dim_y\Big(p_J^{-1}\big(p_J(y)\big)\Big)>0\},$$

which is contained in $p_J^{-1}(G_J^\infty)$. Thus if $e_i\gg0$ for all $1\leq i\leq c$, we have

$$E_J=\mathrm{Bs}\big(p_J^*\mathcal{L}(\mathbf{e})\otimes q_J^*\mathcal{O}_{P_J}(-1)\big)\subset p_J^{-1}(G_J^\infty),\tag{3.7}$$

where $q_J:\mathscr{Y}_J\to\mathbb{P}_J$ is the second projection map. In [Den16], we can get an effective estimate for $\mathbf{e}$ such that the inclusive relation in (3.7) holds. Our theorem is the following:

Theorem 3.2. *With the notations as above, if we set* $b_i:=\frac{\prod_{j=1}^c\delta_j^{k+1}}{\delta_i}$, *then for any* $J\subset\{0,\ldots,N\}$ *we have*

$$\mathrm{Bs}\big(p_J^*\mathcal{L}(\mathbf{b})\otimes q_J^*\mathcal{O}_{P_J}(-1)\big)\subset p_J^{-1}(G_J^\infty).\tag{3.8}$$

By (3.3) we have

$$\Psi^*\big(\mathcal{L}(\mathbf{b})\boxtimes\mathcal{O}_{\mathbb{P}^N}(-1)\big)=\nu_k^*\big(\mathcal{O}_{X_k}(\sum_{i=1}^c b_ik')\otimes\pi_{0,k}^*A^{-q(\boldsymbol{\epsilon},\boldsymbol{\delta},r)}\big)\otimes\mathcal{O}_{\hat{X}_k}(-\sum_{i=1}^c b_iF). \tag{3.9}$$

Here we set $q(\boldsymbol{\epsilon},\boldsymbol{\delta},r):=r-\sum_{i=1}^c b_i(k+1)(\epsilon_i+k\delta_i)$. Observe that if we take

$$\sum_{i=1}^c b_i(k+1)(\epsilon_i+k\delta_i)<r, \tag{♠}$$

then the above bundle becomes an *invariant* k-jet differential with a negative twist, which enable us to apply *Fundamental Vanishing Theorem* to rule all the entire curves. More precisely, we have the following theorem:

Theorem 3.3. *On an n-dimensional smooth projective variety X, equipped with a very ample line bundle A. Let c be any integer satisfying $1\le c\le n-1$. If we take $k_0=\lceil\frac{n}{c}\rceil-1$ and $N=n$, then for any degrees $(d_1,\dots,d_c)\in(\mathbb{N})^c$ satisfying*

$$\exists\boldsymbol{\delta}_{\left(\delta_p\ge\delta_0:=n(k_0+1)\right)},\ \exists\boldsymbol{\epsilon}_{(\epsilon_p\ge k_0)},\ \exists r>\sum_{i=1}^c b_i(k_0+1)(\epsilon_i+k_0\delta_i),\ \ s.t.$$
$$d_p=\delta_p(r+k_0)+\epsilon_p\qquad(p=1,\dots,c),$$

the complete intersection $\mathbf{H}:=H_1\cap\dots H_c$ of general hypersurfaces $H_1\in|A^{d_1}|,\dots,H_c\in|A^{d_c}|$ has almost k-jet ampleness.

Proof. We will prove the theorem in several steps. First observe that, the choice for $(\boldsymbol{\epsilon},\boldsymbol{\delta},r,c,N,k)$ in the Theorem fufills the requirements in Condition ★ ♠ and ♣, and thus we are free to apply all the theorems above. Now we want to prove the following
Claim: Set $U_{\text{nef}}:=\cap_J U_J$. For any $\mathbf{a}\in U_{\text{nef}}$, the line bundle

$$\nu_k^*\big(\mathcal{O}_{X_k}(\sum_{i=1}^c b_ik')\otimes\pi_{0,k}^*A^{-q(\boldsymbol{\epsilon},\boldsymbol{\delta},r)}\big)\otimes\mathcal{O}_{\hat{X}_k}(-\sum_{i=1}^c b_iF)|_{\hat{H}_{\mathbf{a},k}}$$

is nef on $\hat{H}_{\mathbf{a},k}$. Recall that we denote by $q(\boldsymbol{\epsilon},\boldsymbol{\delta},r):=r-\sum_{i=1}^c b_i(k_0+1)(\epsilon_i+k_0\delta_i)>0$.
Proof: In order to prove nefness, it suffices to show that for any irreducible curve, its intersection with the line bundle is non-negative. For any fixed $\mathbf{a}\in U_{\text{nef}}$, and any irreducible curve $C\subset\hat{H}_{\mathbf{a},k}$, one can find the unique $J\subset\{0,\dots,N\}$ such that $\hat{X}_{k,J}\cap C=:C^\circ$ is a non-empty Zariski open subset of C, and thus $C^\circ\subset\hat{\mathscr{H}}_{k,J}$. Since Ψ factors through $\mathscr{Y}_J$ when restricted to $\hat{\mathscr{H}}_{k,J}$, $\Psi|_{C^\circ}$ also factors through $\mathscr{Y}_J$, and by the properness of $\mathscr{Y}_J$, $\Psi|_C$ factors through $\mathscr{Y}_J$ as well. By Lemma 3.5 and the definition of U_{nef}, we have

$$\Psi(C^\circ)\cap G_J^\infty=\emptyset.$$

From Theorem 3.2 we know that

$$\text{Bs}\big(p_J^*\mathcal{L}(\mathbf{b})\otimes q_J^*\mathcal{O}_{P_J}(-1)\big)\subset p_J^{-1}(G_J^\infty),$$

which yields

$$\Psi(C)\cdot\big(p_J^*\mathcal{L}(\mathbf{b})\otimes q_J^*\mathcal{O}_{P_J}(-1)\big)\geq 0.$$

From the relation (3.9) we obtain that

$$C\cdot\Big(\nu_k^*\big(\mathcal{O}_{X_k}(\sum_{i=1}^c b_ik')\otimes\pi_{0,k}^*A^{-q(\boldsymbol{\epsilon},\boldsymbol{\delta},r)}\big)\otimes\mathcal{O}_{\hat{X}_k}(-\sum_{i=1}^c b_iF)\Big)\geq 0,$$

which proves the claim. ■

By Proposition 6.16 in [Dem95], we can find an ample line bundle on $\hat{X}^k$ of the form

$$\tilde{A}:=\nu_k^*\big(\mathcal{O}_{X_k}(a_1,\ldots,a_k)\otimes\pi_{0,k}^*A^{a_0}\big)\otimes\mathcal{O}_{\hat{X}_k}(-F)$$

for some $a_0,\ldots,a_k\in\mathbb{N}$. Therefore, for any $m>a_0$, the line bundle

$$\nu_k^*\big(\mathcal{O}_{X_k}(a_1,\ldots,a_{k-1},a_k+\sum_{i=1}^c mb_ik')\otimes\pi_{0,k}^*A^{a_0-mq(\boldsymbol{\epsilon},\boldsymbol{\delta},r)}\big)\otimes\mathcal{O}_{\hat{X}_k}\big(-(\sum_{i=1}^c mb_i+1)F\big)|_{\hat{H}_{\mathbf{a},k}}$$

is ample for any $\mathbf{a}\in U_{\mathrm{nef}}$, and thus there exists $e_1,\ldots,e_k,q\in\mathbb{N}$ such that

$$\nu_k^*\mathcal{O}_{X_k}(e_1,\ldots,e_k)\otimes\mathcal{O}_{\hat{X}_k}(-qF)|_{\hat{H}_{\mathbf{a},k}}$$

is ample. By the openness property of ampleness, one has a non-empty Zariski open subset $U_{\mathrm{ample}}\subset\prod_{i=1}^c|A^{d_i}|$ such that for any $(H_1,\ldots,H_c)\in U_{\mathrm{ample}}$, their intersection $\mathbf{H}:=H_1\cap\ldots\cap H_c$ is a reduced smooth variety of codimension c in X, and the restriction of the line bundle $\nu_k^*\mathcal{O}_{X_k}(e_1,\ldots,e_k)\otimes\mathcal{O}_{\hat{X}_k}(-qF)|_{\hat{\mathbf{H}}_k}$ is ample (recall that $\hat{\mathbf{H}}_k$ is denoted to be the blow-up of $\mathbf{H}_k$ along $\mathfrak{w}_\infty(\mathbf{H}_k)$). Since the exceptional locus of the blow-up $\nu_k:\hat{X}_k\to X_k$ is contained in X_k^{sing}, then for the complete intersection $\mathbf{H}:=H_1\cap\ldots H_c$ of general hypersurfaces $H_1\in|A^{d_1}|,\ldots,H_c\in|A^{d_c}|$, the augmented base locus of the line bundle

$$\mathcal{O}_{\mathbf{H}_k}(e_1,\ldots,e_k)=\mathcal{O}_{X_k}(e_1,\ldots,e_k)|_{\mathbf{H}_k}$$

is contained in $X_k^{\mathrm{sing}}\cap\mathbf{H}_k$, and we conclude that H_k has almost k-jet ampleness by the fact that $X_k^{\mathrm{sing}}\cap\mathbf{H}_k=\mathbf{H}_k^{\mathrm{sing}}$. □

Now we make some effective estimates based on Theorem 3.3. If we take $d_0:=\delta_0\big(c(k_0+1)(k_0+\delta_0+k_0\delta_0-1)\delta_0^{c(k_0+1)-1}+k_0+1\big)+k_0$, then any $d\geq d_0$ has a decomposition

$$d=(t+k_0)\delta_0+\epsilon$$

with $k_0\leq\epsilon<\delta_0+k_0$ and $t\geq c(k_0+1)(k_0+\delta_0+k_0\delta_0-1)\delta_0^{c(k_0+1)-1}+1$, satisfying the conditions in Theorem 3.3. Therefore, the complete intersection $H_1\cap\ldots\cap H_c$ of general hypersurfaces $H_1,\ldots,H_c\in|A^d|$ with $d\geq d_0$ has almost k_0-jet ampleness. By Lemma 7.6 in [Dem95], if a complex manifold Y has k-jet ampleness, then it will also has l-jet ampleness for any $l\geq k$. A computation gives a rough estimate $d_0\leq 2c(\lceil\frac{n}{c}\rceil)^{n+c+2}n^{n+c}$, and this concludes the proof of our Main Theorem 1.

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