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# The Jacobi Stochastic Volatility Model\*

Damien Akerer<sup>†</sup>      Damir Filipović<sup>‡</sup>      Sergio Pulido<sup>§</sup>

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## Abstract

We introduce a novel stochastic volatility model where the squared volatility of the asset return follows a Jacobi process. It contains the Heston model as a limit case. We show that the joint distribution of any finite sequence of log returns admits a Gram–Charlier A expansion in closed-form. We use this to derive closed-form series representations for option prices whose payoff is a function of the underlying asset price trajectory at finitely many time points. This includes European call, put, and digital options, forward start options, and forward start options on the underlying return. We derive sharp analytical and numerical bounds on the series truncation errors. We illustrate the performance by numerical examples, which show that our approach offers a viable alternative to Fourier transform techniques.

**Keywords:** Jacobi process, option pricing, polynomial model, stochastic volatility

**JEL Classification:** C32, G12, G13

## 1 Introduction

Stochastic volatility models for asset returns are popular among practitioners and academics because they can generate implied volatility surfaces that match option price data to a great extent. They resolve the shortcomings of the Black–Scholes model (Black and Scholes 1973), where the return has constant volatility. Among the the most widely used stochastic volatility models is the Heston model (Heston 1993), where the squared volatility of the return follows an affine square-root diffusion. European call and put option prices in the Heston model can be computed using Fourier transform techniques, which have their numerical strengths and limitations; see for instance (Carr and Madan 1999), (Bakshi and Madan 2000), (Duffie, Filipović, and Schachermayer 2003), (Fang and Oosterlee 2009), and (Chen and Joslin 2012).

We introduce a novel stochastic volatility model, henceforth the Jacobi model, where the squared volatility of the return follows a Jacobi process. The Jacobi model belongs to the class of polynomial

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models studied in (Eriksson and Pistorius 2011), (Cuchiero, Keller-Ressel, and Teichmann 2012), and (Filipović and Larsson 2015). In particular, the moments of the finite-dimensional distributions of the log price process are given in closed-form. Depending on the specification, the Jacobi model includes the Black–Scholes model as a special case and converges weakly in the path space to the Heston model. The joint distribution of any finite sequence of log returns admits an explicit multivariate Gram–Charlier A expansion, as defined in (Cramér 1945). Specifically, we show that it admits a density whose likelihood ratio function with respect to a multivariate Gaussian density lies in the corresponding Gaussian weighted  $L^2$  space. The likelihood ratio function can be expanded as a generalized Fourier series with respect to the corresponding orthonormal basis of Hermite polynomials. The Fourier coefficients are the Hermite moments of the sequence of log returns and thus given in closed-form. In contrast, for the Heston model the Gram–Charlier A series is known to diverge (see Remark 3.6 below).

As an important application we can price any exotic option whose payoff is a function of a finite sequence of log returns. If the payoff function lies in the Gaussian weighted  $L^2$  space, the option price can be written as a series in terms of the Fourier coefficients of the payoff function and the explicit Hermite moments of the corresponding log returns. We derive the Fourier coefficients of the payoff function in closed-form for European call, put, and digital options, forward start options, and forward start options on the underlying return. Consequently, the pricing of these options is extremely efficient and does not require any numerical integration. We derive detailed analytical and numerical error bounds for the truncation of the option price series. We find that the relative pricing errors become small within approximation orders that can be achieved in short CPU time. This is in contrast to the Heston model, for which the pricing of such options using Fourier transform techniques is cumbersome and creates numerical difficulties as reported in (Kruse and Nögel 2005), (Kahl and Jäckel 2005), and (Albrecher, Mayer, Schoutens, and Tistaert 2006). In view of these limitations of the Heston model, the Jacobi model also provides a viable alternative to approximate prices in the Heston model.

The Jacobi process, also known as Wright–Fisher diffusion, was originally used to model gene frequencies; see for instance (Karlin and Taylor 1981) and (Ethier and Kurtz 1986). More recently, the Jacobi process has also been used to model financial factors. For example, (Delbaen and Shirakawa 2002) model interest rates by the Jacobi process and study moment-based techniques for pricing bonds. In their framework, bond prices admit a series representation in terms of Jacobi polynomials. These polynomials constitute an orthonormal basis of eigenfunctions of the infinitesimal generator and the stationary beta distribution of the Jacobi process; additional properties of the Jacobi process can be found in (Mazet 1997) and (Demni and Zani 2009). The Jacobi process has been also applied recently to model stochastic correlation matrices in (Ahdida and Alfonsi 2013) and credit default swap indexes in (Bernis and Scotti 2016).

Gram–Charlier A expansions of option prices were pioneered by (Jarrow and Rudd 1982). They propose expansions of option prices that can be interpreted as corrections to the pricing biases of the Black–Scholes formula. They study density expansions for the law of underlying prices, not the log returns, and express them in terms of cumulants. Evidently, since convergence cannot be guaranteed in general, their study is based on strong assumptions that imply convergence. In subsequent work, (Corrado and Su 1996) and (Corrado and Su 1997) study Gram–Charlier A expansions of 4<sup>th</sup> order for options on the S&P 500 index. These expansions contain skewness and kurtosis adjustments to option prices and implied volatility with respect to the Black–Scholes formula. The skewness and kurtosis correction terms, which depend on the cumulants of 3<sup>rd</sup> and 4<sup>th</sup> order, are estimated from data. Due to the instability of the estimation procedure, higher order expansions are not studied. Similar studies on the biases of the Black–Scholes formula using Gram–Charlier A expansions

include (Backus, Foresi, and Wu 2004) and (Li and Melnikov 2012). More recently, (Drimus, Necula, and Farkas 2013) and (Necula, Drimus, and Farkas 2015) study related expansions with physicist Hermite polynomials instead of probabilist Hermite polynomials. In order to guarantee the convergence of the Gram–Charlier A expansion for a general class of diffusions, (Ait-Sahalia 2002) develop a technique based on a suitable change of measure. As pointed out in (Filipović, Mayerhofer, and Schneider 2013), in the affine and polynomial settings this change of measure usually destroys the polynomial property and the ability to calculate moments efficiently. More recently a similar study has been carried out by (Xiu 2014). Gram–Charlier A expansions, under a change of measure, are also mentioned in the work of (Madan and Milne 1994), and the subsequent studies of (Longstaff 1995), (Abken, Madan, and Ramamurtie 1996) and (Brenner and Eom 1997), where they use these moment expansions to test the martingale property with financial data and hence the validity of a given model.

Our paper is similar to (Filipović, Mayerhofer, and Schneider 2013) that provides a generic framework to perform density expansions using orthonormal polynomial basis in weighted  $L^2$  spaces for affine models. They show that a bilateral Gamma density weight works for the Heston model. However, that expansion is numerically more cumbersome than the Gram–Charlier A expansion because the orthonormal basis of polynomials has to be constructed using Gram–Schmidt orthogonalization. In a related paper (Heston and Rossi 2015) study polynomial expansions of prices in the Heston, Hull–White and Variance Gamma models using logistic weight functions.

The remainder of the paper is as follows. In Section 2 we introduce the Jacobi stochastic volatility model. In Section 3 we derive European option prices method based on Gram–Charlier A series. In Section 4 we provide analytical bounds for the truncation error in the option price approximations. In Section 5 we derive the joint distribution of any finite sequence of log returns, which forms the basis for exotic option pricing and contains the European options as special case. In Section 6 we give some numerical examples. In Section 7 we conclude. In Appendix A we recap the basic properties of polynomial processes that form the basis for all statements. All proofs are collected in Appendix B.

## 2 Model specification

We study a stochastic volatility model where the squared volatility follows a Jacobi process. Fix some real parameters  $0 \leq v_{min} < v_{max}$ , and define the quadratic function

$$Q(v) = \frac{(v - v_{min})(v_{max} - v)}{(\sqrt{v_{max}} - \sqrt{v_{min}})^2}.$$

Inspection shows that  $v \geq Q(v)$ , with equality if and only if  $v = \sqrt{v_{min}v_{max}}$ , and  $Q(v) \geq 0$  for all  $v \in [v_{min}, v_{max}]$ , see Figure 1 for an illustration.

We consider the diffusion process  $(V_t, X_t)$  given by

$$\begin{aligned} dV_t &= \kappa(\theta - V_t) dt + \sigma \sqrt{Q(V_t)} dW_{1t} \\ dX_t &= (r - \delta - V_t/2) dt + \rho \sqrt{Q(V_t)} dW_{1t} + \sqrt{V_t - \rho^2 Q(V_t)} dW_{2t} \end{aligned} \tag{1}$$

for real parameters  $\kappa \geq 0$ ,  $\theta \in [v_{min}, v_{max}]$ ,  $\sigma > 0$ , interest rate  $r$ , dividend yield  $\delta$ , and  $\rho \in [-1, 1]$ , and where  $W_{1t}$  and  $W_{2t}$  are independent standard Brownian motions on some filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ . The following theorem shows that  $(V_t, X_t)$  is well defined.

**Theorem 2.1.** *For any deterministic initial state  $(V_0, X_0) \in [v_{\min}, v_{\max}] \times \mathbb{R}$  there exists a unique solution  $(V_t, X_t)$  of (1) taking values in  $[v_{\min}, v_{\max}] \times \mathbb{R}$ . Moreover, given an initial condition  $(V_0, X_0) \in (v_{\min}, v_{\max}) \times \mathbb{R}$ ,  $(V_t, X_t)$  takes values in  $(v_{\min}, v_{\max}) \times \mathbb{R}$  if and only if*

$$\frac{\sigma^2(v_{\max} - v_{\min})}{(\sqrt{v_{\max}} - \sqrt{v_{\min}})^2} \leq 2\kappa \min\{v_{\max} - \theta, \theta - v_{\min}\}. \quad (2)$$

**Remark 2.2.** *Taking  $v_{\min} = 0$  and the limit as  $v_{\max} \rightarrow \infty$ , condition (2) coincides with the known condition that precludes the zero lower bound for the CIR process:  $\sigma^2 \leq 2\kappa\theta$ .*

We specify the price of a traded asset by  $S_t = e^{X_t}$ . Then  $\sqrt{V_t}$  is the stochastic volatility of the asset return,

$$d\langle X, X \rangle_t = V_t dt.$$

The cumulative dividend discounted price process  $e^{-(r-\delta)t} S_t$  is a martingale. In other words,  $\mathbb{Q}$  is a risk-neutral measure. The parameter  $\rho$  tunes the instantaneous correlation between the asset return and the squared volatility,

$$\frac{d\langle V, X \rangle_t}{\sqrt{d\langle V, V \rangle_t} \sqrt{d\langle X, X \rangle_t}} = \rho \sqrt{Q(V_t)/V_t}.$$

This correlation is equal to  $\rho$  if  $V_t = \sqrt{v_{\min} v_{\max}}$ , see Figure 1. In general, we have  $\sqrt{Q(V_t)/V_t} \leq 1$ . Empirical evidences suggest that  $\rho$  is negative when  $S_t$  is a stock price or index. This is commonly referred as the leverage effect, that is, an increase in volatility often goes along with a decrease in asset value.

Since the instantaneous squared volatility  $V_t$  follows a bounded Jacobi process on the interval  $[v_{\min}, v_{\max}]$ , we refer to (1) as the *Jacobi model*. For  $V_0 = \theta = v_{\max}$  we have constant volatility  $V_t = V_0$  for all  $t \geq 0$  and we obtain the Black-Scholes model

$$dX_t = (r - \delta - V_0/2) dt + \sqrt{V_0} dW_{2t}. \quad (3)$$

For  $v_{\min} = 0$  and the limit  $v_{\max} \rightarrow \infty$  we have  $Q(v) \rightarrow v$ , and we formally obtain the Heston model as limit case of (1),

$$\begin{aligned} dV_t &= \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dW_{1t} \\ dX_t &= (r - \delta - V_t/2) dt + \sqrt{V_t} \left( \rho dW_{1t} + \sqrt{(1 - \rho^2)} dW_{2t} \right). \end{aligned} \quad (4)$$

In fact, the Jacobi model (1) is robust with respect to perturbations, or mis-specifications, of the model parameters  $v_{\min}$ ,  $v_{\max}$  and initial state  $(V_0, X_0)$ . Specifically, the following theorem shows that the diffusion (1) is weakly continuous in the space of continuous paths with respect to  $v_{\min}$ ,  $v_{\max}$  and  $(V_0, X_0)$ . In particular, the Heston model (4) is indeed a limit case of our model (1).

Consider a sequence of parameters  $0 \leq v_{\min}^{(n)} < v_{\max}^{(n)}$  and deterministic initial states  $(V_0^{(n)}, X_0^{(n)}) \in [v_{\min}^{(n)}, v_{\max}^{(n)}] \times \mathbb{R}$  converging to  $0 \leq v_{\min} < v_{\max} \leq \infty$  and  $(V_0, X_0) \in [0, \infty) \times \mathbb{R}$  as  $n \rightarrow \infty$ , respectively. We denote by  $(V_t^{(n)}, X_t^{(n)})$  and  $(V_t, X_t)$  the respective solutions of (1), or (4) if  $v_{\max} = \infty$ . Here is our main convergence result.

**Theorem 2.3.** *The sequence of diffusions  $(V_t^{(n)}, X_t^{(n)})$  converges weakly in the path space to  $(V_t, X_t)$  as  $n \rightarrow \infty$ .*

As the discounted put option payoff function  $f_{put}(x) = e^{-rT} (e^k - e^x)^+$  is bounded and continuous on  $\mathbb{R}$ , it follows from the weak continuity stated in Theorem 2.3 that the put option prices based on  $(V_t^{(n)}, X_t^{(n)})$  converge to the put option price based on the limiting model  $(V_t, X_t)$  as  $n \rightarrow \infty$ . The put-call parity,  $\pi_{call} - \pi_{put} = e^{-rT+k} - S_0$ , then implies that also call option prices converge as  $n \rightarrow \infty$ . This carries over to more complex path-dependent options with bounded continuous payoff functional.

### 3 European option pricing

Henceforth we assume that  $(V_0, X_0) \in [v_{min}, v_{max}] \times \mathbb{R}$  is a deterministic initial state and fix a time horizon  $T > 0$ . We first establish some key properties of the distribution of  $X_T$ . Denote the quadratic variation of the second martingale component of  $X_t$  in (1) by

$$C_t = \int_0^t (V_s - \rho^2 Q(V_s)) ds. \quad (5)$$

The following theorem is a special case of Theorem 5.1 below.

**Theorem 3.1.** *Let  $\epsilon < 1/(2v_{max}T)$ . If  $C_T > 0$  then the distribution of  $X_T$  admits a density  $g_T(x)$  on  $\mathbb{R}$  that satisfies*

$$\int_{\mathbb{R}} e^{\epsilon x^2} g_T(x) dx < \infty. \quad (6)$$

*If*

$$\mathbb{E} [C_T^{-1/2-k}] < \infty \quad (7)$$

*for some  $k \in \mathbb{N}_0$  then  $g_T(x)$  and  $e^{\epsilon x^2} g_T(x)$  are uniformly bounded and  $g_T(x)$  is  $k$ -times continuously differentiable on  $\mathbb{R}$ . A sufficient condition for (7) to hold for any  $k \in \mathbb{N}$  is*

$$v_{min} > 0 \text{ and } \rho^2 < 1.^1 \quad (8)$$

The condition that  $\epsilon < 1/(2v_{max}T)$  is sharp for (6) to hold. Indeed, consider the Black–Scholes model (3) where  $V_0 = \theta = v_{max}$ . Then  $V_t = v_{max}$  for all  $t \geq 0$ , and  $X_T$  is Gaussian with variance  $C_T = v_{max}T$ . Hence the integral in (6) is infinite for any  $\epsilon \geq 1/(2v_{max}T)$ .

**Remark 3.2.** *If (6) holds for some  $\epsilon > 0$ , then the moment generating function*

$$\widehat{g_T}(z) = \int_{\mathbb{R}} e^{zx} g_T(x) dx \quad (9)$$

*extends to an entire function in  $z \in \mathbb{C}$ . But this does not imply that its cumulant generating function  $\kappa(z) = \log \widehat{g_T}(z)$  extends to an entire function in  $z \in \mathbb{C}$ . Indeed, a density whose moment generating function extends to an entire function on  $\mathbb{C}$  may have zeros, and thus may not admit a logarithm, on  $\mathbb{C}$ . An example is the density  $\mathbf{1}_{[0,1]}(x)$ . Whence a Gram–Charlier A expansion of  $\widehat{g_T}(z)$  does not necessarily converge in general. We will show, however, that a Gram–Charlier A approximation of the density  $g_T(x)$  does converge in an  $L^2$ -sense.*

Since any uniformly bounded and integrable function on  $\mathbb{R}$  is square integrable on  $\mathbb{R}$ , as an immediate consequence of Theorem 3.1 we have the following corollary.

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<sup>1</sup>We conjecture that (7) also holds in case where  $v_{min} = 0$  as long as  $\rho^2 < 1$  and  $V_0$  is large enough. Indeed, note that  $C_T \geq (1 - \rho^2) \int_0^T V_t dt$ . However, explicit conditions on  $V_0$  are left for future research.

**Corollary 3.3.** Assume (7) holds for  $k = 0$ . Then

$$\int_{\mathbb{R}} \frac{g_T(x)^2}{w(x)} dx < \infty \quad (10)$$

for any Gaussian density  $w(x)$  with variance  $\sigma_w^2$  satisfying

$$\sigma_w^2 > \frac{v_{\max} T}{2}. \quad (11)$$

**Remark 3.4.** It follows from the proof that the statements of Theorem 3.1 also hold for the Heston model (4), with  $Q(v) = v$ . However, the Heston model does not satisfy (6) for any  $\epsilon > 0$ . Indeed, otherwise its moment generating function  $\widehat{g}_T(z)$  would extend to an entire function in  $z \in \mathbb{C}$ , as noted in Remark 3.2. But it is well known that  $\widehat{g}_T(z)$  becomes infinite for large enough  $z \in \mathbb{R}$ , see (Andersen and Piterbarg 2007). As a consequence, the Heston model does not satisfy (10) for any finite  $\sigma_w$ . Indeed, by the Cauchy-Schwarz inequality, (10) implies (6) for any  $\epsilon < 1/(4\sigma_w^2)$ .

We now compute the price at time  $t = 0$  of a European claim with discounted payoff  $f(X_T)$  at expiry date  $T > 0$ . We henceforth assume that (7) holds with  $k = 0$ , and we let  $w(x)$  be a Gaussian density with mean  $\mu_w$  and variance  $\sigma_w^2$  satisfying (11).

We define the weighted Lebesgue space

$$L_w^2 = \left\{ f(x) \mid \|f\|_w^2 = \int_{\mathbb{R}} f(x)^2 w(x) dx < \infty \right\},$$

which is a Hilbert space with scalar product

$$\langle f, g \rangle_w = \int_{\mathbb{R}} f(x)g(x) w(x) dx.$$

The space  $L_w^2$  admits the orthonormal basis of generalized Hermite polynomials  $H_n(x)$ ,  $n \geq 0$ , given by

$$H_n(x) = \frac{1}{\sqrt{n!}} \mathcal{H}_n \left( \frac{x - \mu_w}{\sigma_w} \right) \quad (12)$$

where  $\mathcal{H}_n(x)$  are the standard probabilist Hermite polynomials defined by

$$\mathcal{H}_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}},$$

see (Feller 1960, Section XVI.1). In particular,  $\deg H_n(x) = n$ , and  $\langle H_m, H_n \rangle_w = 1$  if  $m = n$  and zero otherwise.

Corollary 3.3 implies that the likelihood ratio function  $\ell(x) = g_T(x)/w(x)$  of the density  $g_T(x)$  of the log price  $X_T$  with respect to  $w(x)$  belongs to  $L_w^2$ . We henceforth assume that the discounted payoff function  $f(x) \in L_w^2$ . This hypothesis is satisfied for instance in the case of European call and put options. It implies that the price, denoted by  $\pi_f$ , is well defined and equals

$$\pi_f = \int_{\mathbb{R}} f(x)g_T(x) dx = \langle f, \ell \rangle_w = \sum_{n \geq 0} f_n \ell_n, \quad (13)$$

for the Fourier coefficients

$$f_n = \langle f, H_n \rangle_w, \quad (14)$$



and the *Hermite moments*

$$\ell_n = \langle \ell, H_n \rangle_w = \int_{\mathbb{R}} H_n(x) g_T(x) dx. \quad (15)$$

Note that  $\ell_n$  is a linear combination of moments of  $X_T$ , and thus given in closed-form, as we shall see in Theorem A.2 in Appendix A. This is a consequence of the polynomial property of the diffusion  $(V_t, X_t)$ .

We approximate the price  $\pi_f$  by truncating the series in (13) at some order  $N \geq 1$  and write

$$\pi_f^{(N)} = \sum_{n=0}^N f_n \ell_n. \quad (16)$$

While  $\pi_f^{(N)} \rightarrow \pi_f$  as  $N \rightarrow \infty$  in general, in some cases the proxy is exact as the following lemma states.

**Lemma 3.5.** *If  $f(x)$  is a polynomial then  $\pi_f^{(N)} = \pi_f$  for all  $N \geq \deg f(x)$ .*

*Proof.* By orthogonality of the basis polynomials  $H_n(x)$  we have that  $f_n = 0$  for  $n > \deg f(x)$ .  $\square$

In general, the computation of the proxy  $\pi_f^{(N)}$  in (16) boils down to a numerical integration over the real line,

$$\pi_f^{(N)} = \sum_{n=0}^N \langle f, \ell_n H_n \rangle_w = \int_{\mathbb{R}} f(x) g_T^{(N)}(x) dx, \quad (17)$$

where the *Gram–Charlier A approximation*

$$g_T^{(N)}(x) = \sum_{n=0}^N \ell_n H_n(x) w(x)$$

serves as a proxy for the density  $g_T(x)$ , which is in closed-form by Theorem A.2. In fact, we readily see from Lemma 3.5 that  $g_T^{(N)}(x)$  integrates to one and converges to  $g_T(x)$  in  $L^2_{1/w}$  as  $N \rightarrow \infty$ . Hence, in the Jacobi stochastic volatility model, we formally have  $L^2_{1/w}$ -convergence of the Gram–Charlier A series of the density of the log price  $X_T$ .

**Remark 3.6.** *In view of Remark 3.2, in the Heston model  $g_T^{(N)}(x)$  does not converge in  $L^2_{1/w}$ .*

In specific cases, we find recursive formulas for the Fourier coefficients  $f_n$  in (14), and no numerical integration is needed, as is the case for call and put options. Similar recursion relations of the Fourier coefficients for the Physicist Hermite polynomial basis can be found in (Drimus, Necula, and Farkas 2013).

**Theorem 3.7.** *Consider the discounted payoff function for a call option with log strike  $k$ ,*

$$f(x) = e^{-rT} (e^x - e^k)^+. \quad (18)$$

*Its Fourier coefficients  $f_n$  in (14) are given by*

$$\begin{aligned} f_0 &= e^{-rT+\mu_w} I_0 \left( \frac{k-\mu_w}{\sigma_w}; \sigma_w \right) - e^{-rT+k} \Phi \left( \frac{\mu_w - k}{\sigma_w} \right); \\ f_n &= e^{-rT+\mu_w} \frac{1}{\sqrt{n!}} \sigma_w I_{n-1} \left( \frac{k-\mu_w}{\sigma_w}; \sigma_w \right), \quad n \geq 1. \end{aligned} \quad (19)$$



The functions  $I_n(\mu; \nu)$  are defined recursively by

$$\begin{aligned} I_0(\mu; \nu) &= e^{\frac{\nu^2}{2}} \Phi(\nu - \mu); \\ I_n(\mu; \nu) &= \mathcal{H}_{n-1}(\mu) e^{\nu\mu} \phi(\mu) + \nu I_{n-1}(\mu; \nu), \quad n \geq 1, \end{aligned} \quad (20)$$

where  $\mathcal{H}_n(x)$  are the standard Hermite polynomials,  $\Phi(x)$  denotes the standard Gaussian distribution function, and  $\phi(x)$  its density.

**Remark 3.8.** The Fourier coefficients of a put option with discounted payoff  $\tilde{f}(x) = e^{-rT} (e^k - e^x)^+$  can be obtained from the put-call parity. More precisely, in this case

$$\begin{aligned} \tilde{f}_n &= \langle \tilde{f}, H_n \rangle_w \\ &= f_n - \left( \frac{1}{\sqrt{n!}} e^{-rT+\mu_w} I_n(-\infty; \sigma_w) - e^{-rT+k} \mathbf{1}_{\{n=0\}} \right) \\ &= f_n - \frac{\sigma_w^n}{\sqrt{n!}} e^{-rT+\mu_w+\sigma_w^2/2} + e^{-rT+k} \mathbf{1}_{\{n=0\}}, \end{aligned}$$

with  $f_n$  as in (19). Here we used the recursive equations (20) to deduce that

$$I_n(-\infty; \sigma_w) = \sigma_w I_{n-1}(-\infty; \sigma_w) = \sigma_w^n e^{\sigma_w^2/2}.$$

Alternatively, a similar argument as in the proof of Theorem 3.7 shows that

$$\begin{aligned} \tilde{f}_0 &= e^{-rT+k} \Phi\left(\frac{k - \mu_w}{\sigma_w}\right) - e^{-rT+\mu_w} J_0\left(\frac{k - \mu_w}{\sigma_w}; \sigma_w\right) \\ \tilde{f}_n &= -e^{-rT+\mu_w} \frac{1}{\sqrt{n!}} \sigma_w J_{n-1}\left(\frac{k - \mu_w}{\sigma_w}; \sigma_w\right), \quad n \geq 1, \end{aligned}$$

where the functions  $J_n(\mu; \nu)$  are defined recursively by

$$\begin{aligned} J_0(\mu; \nu) &= e^{\frac{\nu^2}{2}} \Phi(\mu - \nu); \\ J_n(\mu; \nu) &= -\mathcal{H}_{n-1}(\mu) e^{\nu\mu} \phi(\mu) + \nu J_{n-1}(\mu; \nu), \quad n \geq 1. \end{aligned}$$

**Remark 3.9.** If  $\mu_w = X_0 + rT - \sigma_w^2/2$ , then  $f_0$  is the Black-Scholes call option price with volatility  $\sigma_{BS} = \sigma_w/\sqrt{T}$ . That is,

$$f_0 = e^{X_0} \Phi(d_1) - e^{-rT+k} \Phi(d_2)$$

with  $d_{1,2} = \frac{X_0 - k + rT \pm \sigma_{BS}^2 T/2}{\sigma_{BS} \sqrt{T}}$ .

**Remark 3.10.** Using (19) and (20), we could write a more explicit expression for the Fourier coefficients  $f_n$  for  $n \geq 1$

$$\begin{aligned} f_n &= e^{-rT+k} \phi\left(\frac{k - \mu_w}{\sigma_w}\right) \frac{1}{\sqrt{n!}} \sum_{m=0}^{n-2} \mathcal{H}_m\left(\frac{k - \mu_w}{\sigma_w}\right) \sigma_w^{n-1-m} \\ &\quad + e^{-rT+\mu_w+\frac{\sigma_w^2}{2}} \Phi\left(\sigma_w - \frac{k - \mu_w}{\sigma_w}\right) \frac{\sigma_w^n}{\sqrt{n!}}. \end{aligned} \quad (21)$$

A similar formula can be found in (Heston and Rossi 2015, Appendix A)

For digital options, the formulas for the Fourier coefficients  $f_n$  are explicit.

**Theorem 3.11.** Consider the discounted payoff function for a digital option of the form

$$f(x) = e^{-rT} \mathbf{1}_{[k, \infty)}(x).$$

Its Fourier coefficients  $f_n$  are given by

$$\begin{aligned} f_0 &= e^{-rT} \Phi\left(\frac{\mu_w - k}{\sigma_w}\right); \\ f_n &= \frac{e^{-rT}}{\sqrt{n!}} \mathcal{H}_{n-1}\left(\frac{k - \mu_w}{\sigma_w}\right) \phi\left(\frac{k - \mu_w}{\sigma_w}\right), \quad n \geq 1, \end{aligned} \quad (22)$$

where  $\mathcal{H}_n(x)$  are the standard Hermite polynomials,  $\Phi(x)$  denotes the standard Gaussian distribution function, and  $\phi(x)$  its density.

**Remark 3.12.** For a generic digital option with payoff of the form  $f(x) = \mathbf{1}_{[k_1, k_2)}(x)$  we can derive its Fourier coefficients using the previous theorem as  $\mathbf{1}_{[k_1, k_2)}(x) = \mathbf{1}_{[k_1, \infty)}(x) - \mathbf{1}_{[k_2, \infty)}(x)$ .

An alternative dual expression of the price  $\pi_f$  in (13) is given by the Fourier integral

$$\pi_f = \frac{1}{2\pi} \int_{\mu + i\mathbb{R}} \widehat{f}(z) \widehat{g_T}(z) dz, \quad (23)$$

where  $\widehat{f}(z)$  and  $\widehat{g_T}(z)$  denote the moment generating functions given by (9), respectively. Here  $\mu \in \mathbb{R}$  is some appropriate dampening parameter such that  $e^{-\mu x} f(x)$  and  $e^{\mu x} g_T(x)$  are Lebesgue integrable and square integrable on  $\mathbb{R}$ . Indeed, Lebesgue integrability implies that  $\widehat{f}(z)$  and  $\widehat{g_T}(z)$  are well defined for  $z \in \mu + i\mathbb{R}$  through (9). Square integrability and the Plancherel Theorem then yield the representation (23). For example, for the European call option (18) we have  $\mu > 1$  and  $\widehat{f}(z) = e^{-rT+k(1-z)}/(z(z-1))$ .

Option pricing via (23) is the approach taken in the Heston model (4), for which there exists a closed-form expression for  $\widehat{g_T}(z)$ . It is given in terms of the solution of a Riccati equation. The computation of  $\pi_f$  boils down to the numerical integration of (23) along with the numerical solution of a Riccati equation for every argument  $z \in \mu + i\mathbb{R}$  that is needed for the integration. The Heston model (which entails  $v_{max} \rightarrow \infty$ ) does not adhere to the series representation (13) that is based on condition (10), see Remark 3.4.

The Jacobi model, on the other hand, does not admit a closed-form expression for  $\widehat{g_T}(z)$ . But the Fourier coefficients  $\ell_n$  are readily available as shown in Theorem A.2. In conjunction with Theorem 3.7, the (truncated) series representation (13) thus provides a valuable alternative to the (numerical) Fourier integral approach (23) for option pricing.

We will show in Section 6 that the numerical performance of the Jacobi model for European call option pricing is comparable to the Heston model. Moreover, our solution method can be applied to any discounted payoff function  $f(x) \in L_w^2$ . The computation of the price at most boils down to the numerical integration of (17). This includes functions  $f(x)$  that do not necessarily admit closed-form moment generating function  $\widehat{f}(z)$  as is required in the Heston model approach. Additionally, as we will see in Section 5, the density approximation technique in the Jacobi model can be used to price path dependent options which could be a cumbersome task using Fourier transform techniques in the Heston model.

**Remark 3.13.** The proposed option pricing approach in the Jacobi model can also be applied to price spread options of assets with stochastic correlation. More precisely, suppose that  $X_1$  and  $X_2$

are the log prices of two assets whose dynamics with respect to a risk-neutral measure are

$$\begin{aligned} dX_{1t} &= (r - \sigma_1^2/2) dt + \sigma_1 dW_{1t} \\ dX_{2t} &= (r - \sigma_2^2/2) dt + \sigma_2(\rho_t dW_{1t} + \sqrt{1 - \rho_t^2} dW_{2t}) \\ d\rho_t &= \kappa(\theta - \rho_t) dt + \sigma\sqrt{1 - \rho_t^2} dW_{2t} \end{aligned}$$

for real parameters  $\kappa \geq 0$ ,  $\theta \in [-1, 1]$ ,  $\sigma, \sigma_1, \sigma_2 \in (0, \infty)$ , interest rate  $r$ , and where  $W_{1t}$  and  $W_{2t}$  are independent standard Brownian motions on some filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ . This is a Black-Scholes-Merton model of two assets with stochastic correlation  $\rho_t \in [-1, 1]$ . In this framework, we can write the price a spread option with payoff  $(e^{X_{2T}} - Ke^{X_{1T}})^+$  at maturity  $T$  as

$$e^{-rT} \mathbb{E}[(e^{X_{2T}} - Ke^{X_{1T}})^+] = e^{X_{1,0}} \widehat{\mathbb{E}}[(e^{X_T} - K)^+], \quad (24)$$

with  $dX_t = dX_{2t} - dX_{1t}$ ,  $X_0 = X_{2,0} - X_{1,0}$ , and where  $\widehat{\mathbb{E}}$  denotes the expectation with respect to the probability measure  $\widehat{\mathbb{Q}}$  defined by  $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = e^{X_{1T} - rT - X_{1,0}}$ . We have that

$$\begin{aligned} dX_t &= (\sigma_1\sigma_2\rho_t - \sigma_1^2/2 - \sigma_2^2/2) dt + (\sigma_2\rho_t - \sigma_1) d\widehat{W}_{1t} + \sigma_2\sqrt{1 - \rho_t^2} d\widehat{W}_{2t} \\ d\rho_t &= \kappa(\theta - \rho_t) dt + \sigma\sqrt{1 - \rho_t^2} d\widehat{W}_{2t} \end{aligned}$$

where  $\widehat{W}_{1t}$  and  $\widehat{W}_{2t}$  are independent standard Brownian motions with respect to the measure  $\widehat{\mathbb{Q}}$ . In particular, using similar arguments to those presented in Appendix A, it can be shown that the diffusion  $(X_t, \rho_t)$  has the polynomial property with respect to  $\widehat{\mathbb{Q}}$ . In addition, a similar argument as in the proof of Theorem 5.1 shows that the density function of  $X_T$  has suitable decaying properties. These properties guarantee that the Gram-Charlier A expansion techniques discussed in this section can be used to approximate the expectation in (24).<sup>2</sup>

The following result, which is a special case of Theorem 5.4, provides universal upper and lower bounds on the implied volatility of a European option with discounted payoff  $f(X_T)$  at  $T$  and price  $\pi_f$  in the Jacobi model. The implied volatility  $\sigma_{IV}$  is defined as the volatility parameter that renders the corresponding Black-Scholes option price equal to  $\pi_f$ .

**Theorem 3.14.** *Assume that the discounted payoff function  $f(\log(s))$  is non-affine and convex in  $s > 0$ . Then the implied volatility is well-defined and satisfies  $\sqrt{v_{min}} \leq \sigma_{IV} \leq \sqrt{v_{max}}$ .*

## 4 Analytical error bounds

In this section we discuss how to control analytically the error of the price approximation scheme (16). An alternative numerical approach will be discussed in Section 6.2. For a fixed level  $N$ , the error in the approximation is

$$\epsilon^{(N)} = \pi_f - \pi_f^{(N)} = \sum_{n=N+1}^{\infty} f_n \ell_n.$$

Cauchy-Schwarz inequality implies that

$$|\epsilon^{(N)}| \leq \left( \sum_{n=N+1}^{\infty} f_n^2 \right)^{\frac{1}{2}} \left( \sum_{n=N+1}^{\infty} \ell_n^2 \right)^{\frac{1}{2}}.$$

To control the approximation error it is therefore sufficient to obtain estimates of the Hermite moments  $\ell_n$  and the Fourier coefficients  $f_n$ .

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<sup>2</sup>We thank Peter Tankov for pointing out this fact.

#### 4.1 Estimates for the Hermite moments

The following proposition, together with Theorem A.2, describe the main analytical properties of the Hermite moments  $\ell_n$  as defined in (15).

**Proposition 4.1.** *Assume (7) holds with  $k = 0$  and let  $w(x)$  be a Gaussian density with mean  $\mu_w$  and variance  $\sigma_w^2$  satisfying (11). Then:*

1. *The Hermite moments  $\ell_n$  as defined in (15) can be expressed as*

$$\ell_n = (n!)^{1/2} \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{1}{m!(n-2m)!} \frac{1}{2^m} \mathbb{E} [(\alpha^2 - 1)^m \beta^{n-2m}], \quad (25)$$

where  $\lfloor \cdot \rfloor$  denotes the integer part and

$$\alpha = \frac{\sqrt{C_T}}{\sigma_w}; \beta = \frac{M_T - \mu_w}{\sigma_w}, \quad (26)$$

with  $C_T$  as in (5) and

$$M_T = X_0 + \int_0^T (r - \delta - V_t/2) dt + \frac{\rho}{\sigma} \left( V_T - V_0 - \int_0^T \kappa (\theta - V_t) dt \right). \quad (27)$$

2. *If  $\alpha^2 < 1$  a.s. then*

$$\ell_n = (n!)^{-1/2} \mathbb{E} \left[ (1 - \alpha^2)^{\frac{n}{2}} \mathcal{H}_n((1 - \alpha^2)^{-\frac{1}{2}} \beta) \right]. \quad (28)$$

Additionally, in this case for all  $k \in \mathbb{N}$

$$\ell_n^2 \leq \mathbb{E} \left[ \frac{k! K^2 e^{\frac{\beta^2(1+\alpha^2/\gamma)}{2}}}{(\alpha^2 n)^k \sqrt{1 - \alpha^2}} \right], \quad (29)$$

where  $K \leq 1.086435$  is a constant. Moreover, if  $\tilde{K}$  is a constant such that  $|\beta| \leq \tilde{K}$  a.s. and there exist constants  $\epsilon, \delta > 0$  such that  $0 < \epsilon \leq \alpha^2 \leq \delta < 1$ , then for all  $N \geq 1$

$$\sum_{n=N+1}^{\infty} \ell_n^2 \leq K^2 \exp(\tilde{K}^2/(2(1-\delta))) \frac{(1-\epsilon)^{N+1}}{\epsilon}. \quad (30)$$

3. *We have*

$$\|\ell\|_w^2 = \sum_{n=0}^{\infty} \ell_n^2 \leq \sigma_w \mathbb{E} \left[ \frac{1}{\sqrt{\Delta}} \exp \left( \frac{(M_T - \mu_w)^2}{4\sigma_w^2 \Delta} \right) \right] \mathbb{E} \left[ \frac{\exp \left( \frac{(M_T - \mu_w)^2}{4\sigma_w^2 \Delta} \right)}{\sqrt{C_T}} \right], \quad (31)$$

with  $\Delta = 1 - \alpha^2/2$ .

**Remark 4.2.** *If the right side of equation (29) is finite, it shows what is known as the superpolynomial decay of the Hermite moments  $\ell_n$ .*

The following proposition, together with Proposition 4.1 (see in particular (28), (29) and (31)), is useful to bound the error in the approximation. It allows us to bound the variable  $M_T$  defined in (27) in terms of functions that depend only on  $C_T$ , which is a random variable that takes values on the interval  $[T(1 - \rho^2)v_{\min}, v_{\max}T]$ .

**Proposition 4.3.** Suppose that  $|\rho| < 1$ . We have that

$$C_T \leq \int_0^T V_t dt \leq \frac{C_T}{1 - \rho^2}. \quad (32)$$

Moreover, if  $\rho^2(v_{max} + v_{min}) < (\sqrt{v_{max}} - \sqrt{v_{min}})^2$ , then

$$C_T \leq \int_0^T V_t dt \leq \frac{(\sqrt{v_{max}} - \sqrt{v_{min}})^2 C_T - \rho^2 T v_{min}(v_{max} + v_{min})}{((\sqrt{v_{max}} - \sqrt{v_{min}})^2 - \rho^2(v_{max} + v_{min}))}. \quad (33)$$

With the inequalities (32)–(33) we can bound  $M_T$  (and hence  $\beta$  as defined in (26)) from above and below by functions in  $C_T$  because

$$M_T = X_0 + \left(r - \delta - \frac{\rho\kappa\theta}{\sigma}\right) T + \frac{\rho}{\sigma} (V_T - V_0) + \left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right) \int_0^T V_t dt.$$

## 4.2 Estimates for the Fourier coefficients

The following proposition provides analytical estimates of the Fourier coefficients defined in (14) for a call option. These estimates could be used to obtain estimates for the tails

$$\sum_{n=N+1}^{\infty} f_n^2 = \|f\|_w^2 - \sum_{n=0}^N f_n^2.$$

**Remark 4.4.** In general, since we can always approximate the value  $\|f\|_w$  via numerical integration, if we can efficiently compute the Fourier coefficients (14) (e.g. Call, Put and Digital options) this error can always be efficiently calculated.

**Proposition 4.5.** If  $f(x) = e^{-rT}(e^x - e^k)^+$ , the Fourier coefficients  $f_n$  in (14) satisfy for  $n \geq 6$

$$\begin{aligned} |f_n| \leq & e^{-rT+k-\frac{(k-\mu_w)^2}{4\sigma_w^2}+\sigma_w^2/2} \frac{K\tilde{K}\sigma_w(2\pi)^{-1/2}}{\sqrt{n(n-1)}} \\ & + e^{-rT+\mu_w+\frac{\sigma_w^2}{2}} \Phi\left(\sigma_w - \frac{k-\mu_w}{\sigma_w}\right) \frac{\sigma_w^n}{\sqrt{n!}}, \end{aligned} \quad (34)$$

where  $K \leq 1.086435$  and  $\tilde{K} \leq \sqrt{3}$  are constants, and  $\Phi$  denotes the standard Gaussian distribution function.

**Remark 4.6.** We know that  $\sum_n f_n^2 = \|f\|_w^2 < \infty$ , in particular  $\lim_{n \rightarrow \infty} f_n = 0$ . The estimate (34) gives a more precise idea of the rate of decay of the Fourier coefficients  $f_n$ .

## 5 Exotic option pricing

Pricing exotic options with stochastic volatility models is a challenging task. We show that the price of an exotic option whose payoff is a function of a finite sequence of log returns admits a polynomial series representation in the Jacobi model.

Henceforth we assume that  $(V_0, X_0) \in [v_{min}, v_{max}] \times \mathbb{R}$  is a deterministic initial state. Consider time points  $0 \leq t_0 < t_1 < t_2 < \dots < t_n$  and denote the log returns  $Y_{t_i} = X_{t_i} - X_{t_{i-1}}$  for  $i = 1, \dots, n$ . The following theorem contains Theorem 3.1 as special case where  $n = 1$  and  $t_0 = 0$ .

**Theorem 5.1.** Let  $\epsilon_1, \dots, \epsilon_n \in \mathbb{R}$  be such that  $\epsilon_i < 1/(2v_{\max}(t_i - t_{i-1}))$  for  $i = 1, \dots, n$ . If  $C_{t_i} - C_{t_{i-1}} > 0$  for  $i = 1, \dots, n$  then the random vector  $(Y_{t_1}, \dots, Y_{t_n})$  admits a density  $g_{t_0, \dots, t_n}(y) = g_{t_0, \dots, t_n}(y_1, \dots, y_n)$  on  $\mathbb{R}^n$  satisfying

$$\int_{\mathbb{R}^n} e^{\sum_{i=1}^n \epsilon_i y_i^2} g_{t_0, \dots, t_n}(y) dy < \infty. \quad (35)$$

If

$$\mathbb{E} \left[ \prod_i (C_{t_i} - C_{t_{i-1}})^{-1/2 - \alpha_i} \right] < \infty \quad (36)$$

for all  $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$  with  $\sum_i \alpha_i \leq k \in \mathbb{N}_0$ , for some  $k \in \mathbb{N}_0$ , then  $g_{t_0, \dots, t_n}(y)$  and  $e^{\sum_{i=1}^n \epsilon_i y_i^2} g_{t_0, \dots, t_n}(y)$  are uniformly bounded and  $g_{t_0, \dots, t_n}(y)$  is  $k$ -times continuously differentiable on  $\mathbb{R}^n$ . Property (8) implies (36) for any  $k \in \mathbb{N}$ .

Since any uniformly bounded and integrable function on  $\mathbb{R}^n$  is square integrable on  $\mathbb{R}^n$ , as an immediate consequence we have the following corollary.

**Corollary 5.2.** Assume (36) holds for  $k = 0$ . Then

$$\int_{\mathbb{R}^n} \frac{g_{t_0, \dots, t_n}(y)^2}{\prod_{i=1}^n w_i(y_i)} dy < \infty \quad (37)$$

for all Gaussian densities  $w_i(y_i)$  with variances  $\sigma_{w_i}$  satisfying

$$\sigma_{w_i}^2 > \frac{v_{\max}(t_i - t_{i-1})}{2}, \quad i = 1, \dots, n. \quad (38)$$

**Remark 5.3.** Assume that  $t_0 = 0$ , then there is a one-to-one correspondence between the vector of log returns  $(Y_{t_1}, \dots, Y_{t_n})$  and the vector of log prices  $(X_{t_1}, \dots, X_{t_n})$ . Indeed,

$$X_{t_i} = X_0 + \sum_{j=1}^i Y_{t_j}.$$

Hence, a crucial consequence of Theorem 5.1 is that the finite-dimensional distributions of the process  $X_t$  admit densities with nice decay properties. More precisely, the density of  $(X_{t_1}, \dots, X_{t_n})$  is  $g_{t_0, \dots, t_n}(x_1 - X_0, \dots, x_n - x_{n-1})$ .

Suppose that the discounted payoff of an exotic option is of the form  $f(X_{t_1}, \dots, X_{t_n})$ . Let  $t_0 = 0$  and assume that (36) holds with  $k = 0$ . Set  $w(y) = w(y_1, \dots, y_n) = \prod_{i=1}^n w_i(y_i)$ , where  $w_i(y)$  is a Gaussian density with variance  $\sigma_{w_i}$  satisfying (38). Define

$$\tilde{f}(y) = \tilde{f}(y_1, \dots, y_n) = f(X_0 + y_1, X_0 + y_1 + y_2, \dots, X_0 + y_1 + \dots + y_n).$$

Then by similar arguments as in Section 3 the price of the option is

$$\pi_f = \mathbb{E}[f(X_{t_1}, \dots, X_{t_n})] = \sum_{m_1, \dots, m_n \geq 0} \tilde{f}_{m_1, \dots, m_n} \ell_{m_1, \dots, m_n}$$

where the Fourier coefficients  $\tilde{f}_{m_1, \dots, m_n}$  are given by

$$\tilde{f}_{m_1, \dots, m_n} = \langle \tilde{f}, H_{m_1, \dots, m_n} \rangle_w = \int_{\mathbb{R}^n} \tilde{f}(y) H_{m_1, \dots, m_n}(y) w(y) dy$$

with  $H_{m_1, \dots, m_n}(y_1, \dots, y_n) = \prod_{i=1}^n H_{m_i}(y_i)$  and where the Hermite moments are

$$\ell_{m_1, \dots, m_n} = \mathbb{E} \left[ \prod_{i=1}^n H_{m_i}(X_{t_i} - X_{t_{i-1}}) \right]. \quad (39)$$

In the expressions above we assume that for each  $i$ ,  $H_{m_i}(y_i)$  is a generalized Hermite polynomial associated to parameters  $\mu_{w_i}$  and  $\sigma_{w_i}$  satisfying (38). This observation highlights the potential advantages of our proposed density expansion algorithm over the traditional Fourier transform technique used in the affine case for pricing exotic options.

We provide some examples of exotic options on the asset with price  $S_t = e^{X_t}$  for which the above methodology applies. In all but the first example we set  $t_0 = 0$ .

- The payoff of a *forward start call option on the underlying return* between dates  $t$  and  $T$ , and with strike  $K$  is

$$\left( \frac{S_T}{S_t} - K \right)^+$$

and its discounted payoff function is given by

$$\tilde{f}(y) = e^{-rT} (e^{y_1} - K)^+$$

with the times  $t_0 = t$  and  $t_1 = T$ . In this case, the Fourier coefficients  $\tilde{f}_n$  coincide with those of a call option and, as we shall see in Theorem A.4 in Appendix A, the *forward* Hermite moments  $\ell_n^* = \mathbb{E}[H_n(X_{t_1} - X_{t_0})]$  can be computed efficiently.

- The payoff of a *forward start call option* with maturity  $T$ , strike fixing date  $t$  and proportional strike  $K$  is

$$(S_T - K S_t)^+$$

and its discounted payoff function is given by

$$\tilde{f}(y) = e^{-rT} (e^{X_0 + y_1 + y_2} - K e^{X_0 + y_1})^+$$

with the times  $t_1 = t$  and  $t_2 = T$ . In this case the Fourier coefficients have the form

$$\begin{aligned} \tilde{f}_{m_1, m_2} &= e^{X_0 - rT} \int_{\mathbb{R}^2} e^{y_1} H_{m_1}(y_1) w_1(y_1) (e^{y_2} - K)^+ H_{m_2}(y_2) w_2(y_2) dy_1 dy_2 \\ &= e^{X_0 - rT} f_{m_1}^{(0, -\infty)} f_{m_2}^{(0, \log K)} \\ &= f_{m_2}^{(0, \log K)} \frac{\sigma_w^{m_1}}{\sqrt{m_1!}} e^{X_0 - rT + \mu_{w_1} + \sigma_{w_1}^2/2}, \end{aligned} \quad (40)$$

where  $f_n^{(r, k)}$  denotes the Fourier coefficient of a call option for interest rate  $r$  and log strike  $k$  as in (19). Here we have used (19)–(20) to deduce that

$$f_{m_1}^{(0, -\infty)} = \frac{\sigma_w^{m_1}}{\sqrt{m_1!}} e^{\mu_{w_1} + \sigma_{w_1}^2/2}.$$

In particular no numerical integration is needed. Additionally, the Hermite moments

$$\ell_{m_1, m_2} = \mathbb{E}[H_{m_1}(Y_{t_1}) H_{m_2}(Y_{t_2})] \quad (41)$$

can be calculated efficiently as explained in Theorem A.4 in Appendix A.



The pricing of forward-starting options in the Black–Scholes model is straightforward. Explicit analytical expressions have been provided in the Heston model by (Kruse and Nögel 2005). Note that a series of consecutive forward start options forms a cliquet option. In Section 6.6 we give a numerical illustration of the density approximation method to price forward start options.

- The payoff of an *Asian call option* with maturity  $T$ , discrete monitoring dates  $t_1 < \dots < t_n = T$ , and fixed strike  $K$  is

$$\left( \frac{1}{n} \sum_{i=1}^n S_{t_i} - K \right)^+$$

and its discounted payoff function is given by

$$\tilde{f}(y) = e^{-rT} \left( \frac{1}{n} \sum_{i=1}^n e^{X_0 + \sum_{j=1}^i y_j} - K \right)^+.$$

Similarly, the payoff of an *Asian call option with floating strike* is

$$\left( S_T - \frac{K}{n} \sum_{i=1}^n S_{t_i} \right)^+$$

and its discounted payoff function is given by

$$\tilde{f}(y) = e^{-rT} \left( e^{X_0 + \sum_{j=1}^n y_j} - \frac{K}{n} \sum_{i=1}^n e^{X_0 + \sum_{j=1}^i y_j} \right)^+.$$

The valuation of Asian options with continuously monitoring in the Black–Scholes model has been studied in (Rogers and Shi 1995) and (Yor 2001) among others.

- The payoff of a *reset call option* with maturity  $T$ , reset dates  $t_1 < \dots < t_n \leq T$  and initial strike  $K$  is

$$(S_T - \min(K, S_{t_1}, \dots, S_{t_n}))^+$$

and its discounted payoff function is given by

$$\tilde{f}(y) = e^{-rT} \left( e^{X_0 + \sum_{i=1}^{n+1} y_i} - \min(K, e^{X_0 + y_1}, \dots, e^{X_0 + \sum_{i=1}^n y_i}) \right)^+.$$

The pricing of reset options in the Black–Scholes model has been studied in (Liao and Wang 2003). The payoff of lookback options<sup>3</sup> with a finite number of observation dates can also be represented in terms of a finite sequence of log returns.

We now derive universal upper and lower bounds on the implied volatility for the exotic option with discounted payoff function  $f(X_{t_1}, \dots, X_{t_n})$  and time-0 price  $\pi_f$ . We denote by

$$dS_t^{\text{BS}} = S_t^{\text{BS}}(r - \delta) dt + S_t^{\text{BS}} \sigma_{\text{BS}} dB_t$$

the Black–Scholes price process with volatility  $\sigma_{\text{BS}} > 0$  where  $B_t$  is some Brownian motion independent of  $(W_{1t}, W_{2t})$ . For any  $t_{i-1} \leq t < t_i$  and given a realization  $X_{t_1}, \dots, X_{t_{i-1}}$ , the time- $t$  Black–Scholes price of the option is a function  $\pi_f^{\sigma_{\text{BS}}}(t, S_t)$  of  $t$  and the spot price  $S_t$  defined by

$$e^{-rt} \pi_f^{\sigma_{\text{BS}}}(t, s) = \mathbb{E} [f(X_{t_1}, \dots, X_{t_{i-1}}, \log S_{t_i}^{\text{BS}}, \dots, \log S_{t_n}^{\text{BS}}) \mid \mathcal{F}_t, S_t^{\text{BS}} = s],$$

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<sup>3</sup>(Andreasen 1998) approximate the prices of lookback options in Black–Scholes model

for  $s > 0$ . The implied volatility  $\sigma_{IV}$  is the volatility parameter  $\sigma_{BS}$  that renders the time-0 Black–Scholes option price  $\pi_f^{\sigma_{IV}}(0, S_0) = \pi_f$ . The following theorem provides bounds on the values that  $\sigma_{IV}$  may take.

**Theorem 5.4.** *Assume that the Black–Scholes option gamma is nonnegative,*

$$\frac{\partial^2 \pi_f^{\sigma_{BS}}(t, s)}{\partial s^2} \geq 0, \quad (42)$$

*for all  $t \in [t_{i-1}, t_i)$  and  $s > 0$ , for any  $\sigma_{BS} > 0$  and given any realization  $X_{t_1}, \dots, X_{t_{i-1}}$ ,  $i = 1, \dots, n$ . Assume that the inequality in (42) is strict for  $t = 0$  and  $s > 0$ . Then the implied volatility  $\sigma_{IV}$  is well-defined and satisfies*

$$\sqrt{v_{min}} \leq \sigma_{IV} \leq \sqrt{v_{max}}.$$

Theorem 5.4 applies in particular to the implied volatility of a forward start call option on the underlying return. This is in contrast to the Heston model for which the implied volatility explodes (except at the money) when the time to maturity of the underlying call option decreases to zero,  $T \rightarrow t$ , see (Jacquier and Roome 2015) for more details.

## 6 Numerical analysis

The price of a call option has a series representation in the Jacobi stochastic volatility model. We investigate the quality of the approximation obtained by truncating this series at a finite order  $N$ . This approach is shown to be reliable and efficiently implementable. Numerical experiments suggest that the order does not need to exceed  $N = 100$  to accurately approximate option prices for a large range of parameter values and option moneyness. A smaller order may actually be sufficient in practice. The empirical fit of the Jacobi model to a sample of S&P 500 implied volatility surface is shown to be equivalent to the Heston model fit. We also explore in this section the call price approximation error, the computational time, and the forward start call option approximation. Unless otherwise stated, we use the following specification for weighted space and drift parameters:  $\sigma_w = \sqrt{v_{max}T/2} + 10^{-4}$ ,  $\mu_w = \mathbb{E}[X_T]$ , and  $r = q = 0$ .

### 6.1 Option price approximation

Figure 2 displays the Hermite moments, the Fourier coefficients, and the option price approximation sequence for different volatility upper bounds. We observe that the  $\ell_n$  and  $f_n$  sequences oscillate and converge toward zero. The amplitudes of these oscillations negatively impact the speed at which the price series  $\pi_f^{(n)}$  converge. The behavior of the coefficients clearly depends on the parameter values. For example, in Figure 2 we see that the magnitude of the oscillations is directly impacted by the choice of  $v_{max}$ . In this case, the weighted space variance parameter  $\sigma_w$  also changes, which further impacts the approximation. However, the moments of the random variable  $X_T$  and the true option price  $\pi_f$  are actually not significantly affected by an increase in  $v_{max}$  beyond some level.

One way to think about the price approximation is to recall from Remark 3.9 that  $f_0 \ell_0$  is the Black–Scholes call option price with volatility  $\sigma_{BS} = \sigma_w / \sqrt{T}$ . The additional terms  $f_n \ell_n$  may then be understood as higher order adjustments for the true distribution mean, scale, and non-normality. This explains why the initial approximation largely overshoots the true option price when  $v_{max}$  is significantly larger than the expected variance.

The parameter  $\sigma_w^2$  may take any value strictly larger than  $v_{max}T/2$ . For the same reason as above, we conjecture that the price series  $\pi_f^{(n)}$  will in general converge faster when  $\sigma_w$  is closer

to the standard deviation of  $X_T$ . Indeed, the density approximation  $g_T^{(n)}$  is expected to be more accurate when the Gaussian density  $w$  is closer to the true density  $g_T$ . Figure 3 displays the density approximation and the corresponding price series of a call option for the Jacobi model with various valid choices of  $\sigma_w > \sqrt{v_{max}T/2}$ . Because the parameter  $v_{max}$  is large, the smallest  $\sigma_w$  offers the best price approximation for any given order  $n$ .

We also emphasize that the relative contribution of the increment  $f_n \ell_n$  to the price approximation depends on the moneyness. Far out of the money options with close to zero value will benefit more from expanding the option price to a higher order than deep in the money options, as can be seen in Figure 4.

## 6.2 Numerical error bounds

We numerically bound the pricing error  $\epsilon^{(N)} = \pi_f - \pi_f^{(N)}$  hereinbelow. From Section 4 we know that the pricing error can be bounded as follows

$$|\epsilon^{(N)}| \leq \left( \|f\|_w^2 - \sum_{n=0}^N f_n^2 \right)^{\frac{1}{2}} \left( \|\ell\|_w^2 - \sum_{n=0}^N \ell_n^2 \right)^{\frac{1}{2}}.$$

In order to provide the sharpest value possible for this illustration, we numerically approximate the  $L_w^2$ -norms of  $f$  and  $\ell$ . The norm  $\|f\|_w$  has an explicit expression that can be computed by means of quadrature

$$\|f\|_w^2 = \int_{\mathbb{R}} f(x)^2 w(x) dx \approx b_{quad}.$$

The norm  $\|\ell\|_w$  can be written

$$\|\ell\|_w^2 = \int_{\mathbb{R}} \frac{g_T(x)^2}{w(x)} dx = \mathbb{E} \left[ \frac{g_T(X_T)}{w(X_T)} \right] = \mathbb{E} \left[ \frac{\phi \left( X_T; \widetilde{M}_T, \sqrt{\widetilde{C}_T} \right)}{\phi(X_T, \mu_w, \sigma_w)} \right]$$

where  $\phi(x, \mu, \sigma)$  is the normal density function in  $x$  with mean  $\mu$  and variance  $\sigma^2$ , and the pair of random variables  $(\widetilde{M}_T, \widetilde{C}_T)$  is independent from  $X_T$  and has the same distribution as  $(M_T, C_T)$  defined in (27) and (5). We simulate  $N_{mc} = 10^6$  triplets and take the 99% confidence interval upper bound, that is

$$b_{mc} = \frac{1}{N_{mc}} \sum_{i=1}^{N_{mc}} \frac{\phi \left( X_T^{(i)}; \widetilde{M}_T^{(i)}, \sqrt{\widetilde{C}_T^{(i)}} \right)}{\phi \left( X_T^{(i)}, \mu_w, \sigma_w \right)} + \Phi^{-1}(0.99) \frac{\sigma_{mc}}{\sqrt{N_{mc}}}$$

where  $\sigma_{mc}$  is the standard deviation of the generated sample, and  $\Phi$  is the standard normal cumulative distribution function. We define the approximation of the absolute error bound  $|\epsilon^{(N)}|$  as follows

$$b^{(N)} = \left( b_{quad} - \sum_{n=0}^N f_n^2 \right)^{\frac{1}{2}} \left( b_{mc} - \sum_{n=0}^N \ell_n^2 \right)^{\frac{1}{2}}. \quad (43)$$

Figure 5 displays the approximate relative absolute error bound  $b^{(n)}/\pi_f^{(n)}$  on the left panel, and the absolute confidence interval on the right panel narrowed around the terminal price approximation for an at the money option. The error bound decreases rapidly with the order  $n$  such that at  $n = 100$  it is about 0.25% of the price approximation. This is a non-tight upper bound and the price approximation is likely to be more accurate than what the computed bound may suggest.

### 6.3 Implied volatility smile

Figure 6 displays the implied volatility smile for different  $v_{min}$  and  $v_{max}$  parameters, and for the Heston model whose dynamics are given by (4). We observe that the smile of the Jacobi model approaches the Heston smile when  $v_{min}$  is small and  $v_{max}$  is large. Somewhat surprisingly, a relatively small value for  $v_{max}$  seems to be sufficient for the two smiles to coincide for options around the money. Indeed, although the variance process has an unbounded support in the Heston model, the probability that it will visit values beyond some large threshold can be extremely small.

Figure 6 also illustrates how the implied volatility smile flattens when the variance support shrinks. In the limit, when this support converges to a singleton, we obtain the flat implied volatility smile of the Black–Scholes model. This shows that the Jacobi model lies between the Black–Scholes model and the Heston model and that the parameters  $v_{min}$  and  $v_{max}$  offer additional degrees of flexibility to model the volatility surface.

### 6.4 Calibration

We calibrate the Jacobi and Heston models to a sample of S&P 500 option prices. First we select all the call and put options available on January 2<sup>th</sup> 2014 with maturity in 1, 2, 3, or 4 weeks from the OptionMetrics database. With a linear regression we extract using the put-call parity the risk-free rate parameter  $r = 0.5676\%$  and the dividend yield parameter  $d = 1.1324\%$ . For each maturity 25 calls with Black–Scholes option delta ranging from 5% to 95% are selected to construct the data sample. We denote here  $\pi_{ij}$ ,  $\sigma_{ij}$ , and  $\nu_{ij}$  the  $j$ -th option price, implied volatility, and Black–Scholes option vega with maturity of  $i$  weeks. Similarly  $\hat{\pi}_{ij}$  and  $\hat{\sigma}_{ij}$  denote the model, Jacobi or Heston, option price and implied volatility. We calibrate the two models to the implied volatility surface by minimizing the following weighted root-mean-square-error (RMSE),

$$\sqrt{\frac{1}{100} \sum_{i=1}^4 \sum_{j=1}^{25} \left( \frac{\pi_{ij} - \hat{\pi}_{ij}}{\nu_{ij}} \right)^2}.$$

This criterion is a computationally efficient approximation of the implied volatility surface RMSE criterion which follows from observing that

$$\sigma_{ij} - \hat{\sigma}_{ij} \approx \frac{\pi_{ij} - \hat{\pi}_{ij}}{\nu_{ij}} \quad \text{when} \quad \pi_{ij} \approx \hat{\pi}_{ij}.$$

We first calibrate the Heston model and use its parameters as an initial guess to calibrate the Jacobi model along with  $v_{min} = 1e^{-4}$  and  $v_{max} = 1$ . Table 1 reports the fitted parameters and Figure 7 displays the corresponding implied volatility surfaces. We observe that the common parameter values are almost identical, with the exception of the vol-of-vol parameter. The fitted volatility support goes from 5% to 45% which seems reasonable given the observed surface. With two additional parameters, the Jacobi model is able to fit slightly better the implied volatility surface than the Heston model. Yet, the two models have the same difficulties in capturing the steep short-term skew.

The appropriate truncation order to accurately price options in the Jacobi model depends on the model parameters. Therefore, a genuine calibration procedure should also adapt the truncation order and avoid the evaluation of aberrant specifications.

### 6.5 Computational performance

The pricing approximation methodology may be divided into two tasks, the computation of the Fourier coefficients  $f_n$  and the computation of the Hermite moments  $\ell_n$ , both having roughly a

linear cost in  $n$ . The same Fourier coefficients can be used for different model parameters, and the same Hermite moments can be used to price multiple options. The computational burden decreases rapidly with the total number of options, the number of maturities, and the number of observations dates. For example, it takes on average 15 milliseconds to price an option with  $N = 100$  for a portfolio of 1000 calls equally split over 10 different maturities on a standard desktop computer with a 3.5 Ghz 64 bits CPU and implemented in the R programming language.

The Fourier coefficients can be computed efficiently thanks to the recursive scheme (19)-(20), as shown on the right panel of Figure 8. For example, it takes a couple of milliseconds to compute all the coefficients up to the order  $n = 100$ . The Fourier coefficients depend only on the weighted space parameters  $\mu_w$  and  $\sigma_w$ , and the payoff. The same  $f_n$  therefore applies to different models when the density  $g_T$  is expanded in the same weighted space.

The computation of the Hermite moments can also be divided into two main tasks, the construction of the matrix representation  $G_N$  and the computation of the matrix exponential. The most expensive task seems to be the construction of the matrix  $G_N$  which takes about 150 milliseconds on a standard desktop computer implemented in the scientific programming language R, see the left panel of Figure 8. However, this matrix is constructed only once, and an implementation in another programming language may perform better. We recall that the matrix  $G_N$  is of dimension  $(N + 1)(N + 2)/2$  and extremely sparse with at most 7 non-null elements per column. For the calculation of the matrix exponential, we remark that only the action of the matrix exponential is needed, that is the vector  $v_{n,T} = e^{G_n T} e_{\pi(0,n)}$ , which can be used to compute the Hermite moments for any initial state  $(X_0, V_0)$ . Specific numerical methods have been developed to perform these computations which can also be found in Physics and Biology, for examples see (Al-Mohy and Higham 2011) (Hochbruck and Lubich 1997) and references therein.

## 6.6 Forward start call option

Figure 9 displays the doubly indexed Fourier coefficients  $\tilde{f}_{m_1, m_2}$  in (40) and Hermite coefficients  $\ell_{m_1, m_2}$  in (41) for the orders  $0 \leq m_1, m_2 \leq 30$ . In a similar way as for the simple call option, the coefficients oscillate on the bivariate plane. The oscillations are possibly more pronounced in certain directions than others but always flatten to zero as the cumulative order  $m_1 + m_2$  increases. We therefore consider on the third row of Figure 9 a price approximation of order  $n$  with the truncation defined as follow

$$\pi_{FS}^{(n)} = \sum_{m_1, m_2 \geq 0}^{m_1 + m_2 \leq n} \tilde{f}_{m_1, m_2} \ell_{m_1, m_2}.$$

The behavior of the price series is similar to the one for the vanilla call option. The main difference is that approximating the forward start call option is computationally more costly. A price approximation of order  $n$  indeed requires to compute  $n + 1$  actions of a vector on a matrix exponential. However, this per option cost also decreases when multiple options have to be approximated because the Hermite coefficients do not depend on the discounted payoff function. In addition, the bivariate Fourier coefficients are in this case simply the cross product of two univariate Fourier coefficient sequences.

## 7 Conclusion

The Jacobi model is a highly tractable and versatile stochastic volatility model. It contains the Heston stochastic volatility model as a limit case. The series approximation techniques based on the Gram–Charlier A expansions of the joint distributions of finite sequences of log returns allow

us to efficiently compute prices of options whose payoff depends on the underlying asset price at finitely many points. Compared to the Heston model, the Jacobi model offers additional flexibility to fit a large range of Black–Scholes implied volatility surfaces. Our numerical analysis shows that the series approximations of European call, put and digital option prices in the Jacobi model are computationally comparable to the widely used Fourier transform techniques for option pricing in the Heston model. The truncated series of prices, whose computation does not require any numerical integration, can be implemented efficiently and reliably up to orders that guarantee accurate approximations as shown by our numerical error analysis. The pricing of forward start options, which does not involve any numerical integration, is significantly simpler and faster than the iterative numerical integration method used in the Heston model. The minimal and maximal volatility parameters are universal bounds for the implied volatility and provide additional stability to the model. In particular, implied volatilities of forward start options in the Jacobi model do not experience the explosions observed in the Heston model.

## A Polynomial property

We denote the generator of the Jacobi model (1) by

$$\mathcal{G}f(v, x) = b(v)^\top \nabla f(v, x) + \frac{1}{2} \text{Tr}(a(v) \nabla^2 f(v, x)),$$

where the drift vector,  $b(v)$ , and the diffusion matrix,  $a(v)$ , are given by

$$b(v) = \begin{bmatrix} \kappa(\theta - v) \\ r - \delta - v/2 \end{bmatrix}, \quad a(v) = \begin{bmatrix} \sigma^2 Q(v) & \rho \sigma Q(v) \\ \rho \sigma Q(v) & v \end{bmatrix}. \quad (44)$$

The diffusion matrix function  $a(v)$  is continuous in the parameters  $v_{\min}$ ,  $v_{\max}$ . In particular, for  $v_{\min} = 0$  and  $v_{\max} \rightarrow \infty$ , we obtain

$$a(v) \rightarrow \begin{bmatrix} \sigma^2 v & \rho \sigma v \\ \rho \sigma v & v \end{bmatrix},$$

which corresponds to the generator of the Heston model (4).

We denote by  $\text{Pol}_n$  the vector space of polynomials in  $(v, x)$  of degree less than or equal to  $n$ . It then follows by inspection that the components of  $b(v)$  and  $a(v)$  lie in  $\text{Pol}_1$  and  $\text{Pol}_2$ , respectively. As a consequence, the generator  $\mathcal{G}$  of the Jacobi model (1), and of the Heston model (4), is polynomial. That is,  $\mathcal{G}$  maps any polynomial of degree  $n$  onto a polynomial of degree  $n$  or less,  $\mathcal{G} \text{Pol}_n \subset \text{Pol}_n$ , see also (Filipović and Larsson 2015, Lemma 2.2). From this we can easily calculate the moments of  $(V_T, X_T)$  as follows. For  $N \in \mathbb{N}$ , let  $M = (N+2)(N+1)/2$  be the dimension of the vector space  $\text{Pol}_N$ . Suppose that

$$h_1(v, x), \dots, h_M(v, x)$$

is a basis of  $\text{Pol}_N$ . Denote by  $G$  the matrix of the linear map  $\mathcal{G}$  restricted to  $\text{Pol}_N$  with respect to this basis. The following theorem gives an algorithm to calculate the moments.

**Theorem A.1.** *For any polynomial  $p(v, x) \in \text{Pol}_N$  and  $0 \leq t \leq T$  we have*

$$\mathbb{E}[p(V_T, X_T) | \mathcal{F}_t] = [h_1(V_t, X_t), \dots, h_M(V_t, X_t)] e^{(T-t)G} \vec{p}$$

where  $\vec{p} \in \mathbb{R}^M$  is the coordinate representation of the polynomial  $p(v, x)$  with respect to the basis  $h_1(v, x), \dots, h_M(v, x)$ . In particular, all moments of  $(V_T, X_T)$  are finite.

*Proof.* See (Filipović and Larsson 2015, Theorem 3.1). □

We now apply Theorem A.1 to describe more explicitly how the coefficients  $\ell_0, \dots, \ell_N$  in (15) can be efficiently computed for any fixed truncation order  $N \geq 1$ . We let  $\pi : \mathcal{E} \rightarrow \{1, \dots, M\}$  be an enumeration of the set of exponents

$$\mathcal{E} = \{(m, n) : m, n \geq 0; m + n \leq N\}.$$

The polynomials

$$h_{\pi(m, n)}(v, x) = v^m H_n(x), \quad (m, n) \in \mathcal{E} \quad (45)$$

then form a basis of  $\text{Pol}_N$ . In view of the elementary property

$$H'_n(x) = \frac{\sqrt{n}}{\sigma_w} H_{n-1}(x), \quad n \geq 1,$$



we obtain that the  $M \times M$ -matrix  $G$  representing  $\mathcal{G}$  on  $\text{Pol}_N$  has at most 7 nonzero elements in column  $\pi(m, n)$  with  $(m, n) \in \mathcal{E}$  given by

$$\begin{aligned}
G_{\pi(m-2, n), \pi(m, n)} &= -\frac{\sigma^2 m(m-1)v_{\max}v_{\min}}{2(\sqrt{v_{\max}} - \sqrt{v_{\min}})^2}, \quad m \geq 2; \\
G_{\pi(m-1, n-1), \pi(m, n)} &= -\frac{\sigma \rho m \sqrt{n} v_{\max} v_{\min}}{\sigma_w (\sqrt{v_{\max}} - \sqrt{v_{\min}})^2}, \quad m, n \geq 1; \\
G_{\pi(m-1, n), \pi(m, n)} &= \kappa \theta m + \frac{\sigma^2 m(m-1)(v_{\max} + v_{\min})}{2(\sqrt{v_{\max}} - \sqrt{v_{\min}})^2}, \quad m \geq 1; \\
G_{\pi(m, n-1), \pi(m, n)} &= \frac{(r - \delta)\sqrt{n}}{\sigma_w} + \frac{\sigma \rho m \sqrt{n}(v_{\max} + v_{\min})}{\sigma_w (\sqrt{v_{\max}} - \sqrt{v_{\min}})^2}, \quad n \geq 1; \\
G_{\pi(m+1, n-2), \pi(m, n)} &= \frac{\sqrt{n(n-1)}}{2\sigma_w^2}, \quad n \geq 2; \\
G_{\pi(m, n), \pi(m, n)} &= -\kappa m - \frac{\sigma^2 m(m-1)}{2(\sqrt{v_{\max}} - \sqrt{v_{\min}})^2} \\
G_{\pi(m+1, n-1), \pi(m, n)} &= -\frac{\sqrt{n}}{2\sigma_w} - \frac{\sigma \rho m \sqrt{n}}{\sigma_w (\sqrt{v_{\max}} - \sqrt{v_{\min}})^2}, \quad n \geq 1.
\end{aligned} \tag{46}$$

Theorem A.1 now implies the following result.

**Theorem A.2.** *The coefficients  $\ell_n$  are given by*

$$\ell_n = [h_1(V_0, X_0), \dots, h_M(V_0, X_0)] e^{TG} \mathbf{e}_{\pi(0, n)}, \quad 0 \leq n \leq N, \tag{47}$$

where  $\mathbf{e}_i$  is the  $i$ -th standard basis vector in  $\mathbb{R}^M$ . In particular,

$$\begin{aligned}
\ell_0 &= 1; \\
\ell_1 &= \frac{1}{\sigma_w} \left( (r - \delta)T - \frac{\theta}{2} \left( T + \frac{e^{-\kappa T} - 1}{\kappa} \right) + \frac{e^{-\kappa T} - 1}{2\kappa} V_0 + X_0 - \mu_w \right).
\end{aligned}$$

**Remark A.3.** *The choice of the basis polynomials  $h_{\pi(m, n)}$  in (45) is convenient for our purposes because: 1) each column of the  $M \times M$ -matrix  $G$  has at most seven nonzero entries. 2) The coefficients  $\ell_n$  in the expansion of prices (13), can be obtained directly from the action of  $e^{GnT}$  on  $\mathbf{e}_{\pi(0, n)}$  as specified in (47). In practice, it is more efficient to compute directly this action, rather than computing the matrix exponential  $e^{GnT}$  and then selecting the  $\pi(0, n)$ -column.*

We can extend the previous results to a multi-dimensional setting. The following theorem provides an efficient way to calculate multi-dimensional Hermite moments as defined in (39). More precisely, Hermite moments of the form  $\ell_{m_1, \dots, m_n} = \mathbb{E} [\prod_{i=1}^n H_{m_i}(X_{t_i} - X_{t_{i-1}})]$  where  $0 = t_0 < \dots < t_n < \infty$  and for each  $i$ ,  $H_{m_i}(y_i)$  is a generalized Hermite polynomial associated to the parameters  $\mu_{w_i}$  and  $\sigma_{w_i}$  satisfying (38).

Before stating the theorem we fix some notation. Set  $N = \sum_{i=1}^n m_i$  and  $M = (N+2)(N+1)/2$ . Let  $G_N^{(i)}$  be the  $M \times M$ -matrix representation of the infinitesimal generator  $\mathcal{G}$  associated to the generalized Hermite basis with parameters  $\mu_{w_i}$  and  $\sigma_{w_i}$  satisfying (38). We denote this basis in vector form as follows

$$h^{(i)}(v, x) = [h_1^{(i)}(v, x), \dots, h_M^{(i)}(v, x)], \tag{48}$$

with  $h_{\pi(m,n)}^{(i)}(v, x) = v^m H_n^{(i)}(x)$  and  $H_n^{(i)}$  the  $n$ -order generalized Hermite polynomial associated to the parameters  $\mu_{w_i}$  and  $\sigma_{w_i}$ . Define the  $M \times M$ -matrix  $A^{(k,l)}$  by

$$A_{i,j}^{(k,l)} = \begin{cases} H_n^{(l)}(0) & \text{if } i = \pi(m, k) \text{ and } j = \pi(m, n) \text{ for some } m, n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \quad (49)$$

**Theorem A.4.** *Suppose that  $0 \leq t_0 < \dots < t_n < \infty$ . Then for  $m_1, \dots, m_n \in \mathbb{N}_0$  the multi-dimensional Hermite moment as defined in (39) can be computed through*

$$\ell_{m_1, \dots, m_n} = h^{(1)}(V_0, 0)^\top e^{G_N^{(1)} \Delta t_1} A^{(m_1, 1)} \dots e^{G_N^{(n-1)} \Delta t_{n-1}} A^{(m_{n-1}, n-1)} e^{G_N^{(n)} \Delta t_n} e_{\pi(0, m_n)},$$

where  $A^{(m_i, i)}$  is the matrix defined in (49),  $h^{(1)}(v, x)$  is defined by (48) and  $\Delta t_i = t_i - t_{i-1}$ .

*Proof.* By an inductive argument it is sufficient to illustrate the case  $n = 2$ . In this case the result follows from the following chain of identities

$$\begin{aligned} \ell_{m_1, m_2} &= \mathbb{E}[H_{m_1}(X_{t_1} - X_{t_0}) \mathbb{E}_{t_1}[H_{m_2}(X_{t_2}) \mid X_{t_1} = 0]] \\ &= \mathbb{E} \left[ H_{m_1}(X_{t_1} - X_{t_0}) h^{(2)}(V_{t_1}, 0)^\top e^{G_N^{(2)} \Delta t_2} e_{\pi(0, m_2)} \right] \\ &= h^{(1)}(V_0, 0)^\top e^{G_N^{(1)} \Delta t_1} A^{(m_1, 1)} e^{G_N^{(2)} \Delta t_2} e_{\pi(0, m_2)}. \end{aligned}$$

□

## B Proofs

This appendix contains the proofs of all theorems and propositions in the main text.

### Proof of Theorem 2.1

As the system of SDEs (1) is not coupled, it is sufficient to show strong existence and path-wise uniqueness for solutions of the SDE

$$dV_t = \kappa(\theta - V_t) dt + \sigma \sqrt{Q(V_t)} dW_{1t}, \quad (50)$$

where  $\kappa \geq 0$ ,  $\theta \in [v_{\min}, v_{\max}]$ ,  $\sigma > 0$  and  $W_{1t}$  is a one-dimensional Brownian motion. Since the interval  $[0, 1]$  is an affine transformation of the unit ball in  $\mathbb{R}$ , weak existence can be deduced from (Larsson and Pulido 2015, Theorem 2.1). Path-wise uniqueness of solutions follows from (Yamada and Watanabe 1971, Theorem 1). Strong existence of solutions for the SDE (50) is a consequence of path-wise uniqueness and weak existence of solutions, see for instance (Yamada and Watanabe 1971, Corollary 1).

To conclude, Proposition 2.2 in (Larsson and Pulido 2015) shows that the interior of the interval  $[v_{\min}, v_{\max}]$  is invariant for  $V$  (i.e.  $V_t \in (v_{\min}, v_{\max})$ ) if and only if condition (2) holds.

### Proof of Theorem 2.3

The proof of Theorem 2.3 builds on the following four lemmas.

**Lemma B.1.** Suppose that  $Y$  and  $Y^{(n)}$ ,  $n \geq 1$ , are random variables in  $\mathbb{R}^d$  for which all moments exist. Assume further that

$$\lim_n \mathbb{E}[p(Y^{(n)})] = \mathbb{E}[p(Y)], \quad (51)$$

for any polynomial  $p(y)$  and that the distribution of  $Y$  is determined by its moments. Then the sequence  $Y^{(n)}$  converges weakly to  $Y$  as  $n \rightarrow \infty$ .

*Proof.* Theorem 30.2 in (Billingsley 1995) proves this result for the case  $d = 1$ . Inspection shows that the proof is still valid for the general case.  $\square$

**Lemma B.2.** The moments of the finite-dimensional distributions of the diffusions  $(V_t^{(n)}, X_t^{(n)})$  converge to the respective moments of the finite-dimensional distributions of  $(V_t, X_t)$ . That is, for any  $0 \leq t_1 < \dots < t_d < \infty$  and for any polynomials  $p_1(v, x), \dots, p_d(v, x)$  we have

$$\lim_n \mathbb{E} \left[ \prod_{i=1}^d p_i(V_{t_i}^{(n)}, X_{t_i}^{(n)}) \right] = \mathbb{E} \left[ \prod_{i=1}^d p_i(V_{t_i}, X_{t_i}) \right]. \quad (52)$$

*Proof.* Let  $N = \sum_{i=1}^d \deg p_i$ . Throughout the proof we fix a basis

$$h_1(v, x), \dots, h_M(v, x)$$

of  $\text{Pol}_N$ , where  $M = \dim \text{Pol}_N$ , and for any polynomial  $p(v, x)$  we denote by  $\vec{p}$  its coordinates with respect to this basis. We denote by  $G$  and  $G^{(n)}$  the respective  $M \times M$ -matrix representations of the generators restricted to  $\text{Pol}_N$  of  $(V_t, X_t)$  and  $(V_t^{(n)}, X_t^{(n)})$ , respectively. We then define recursively the polynomials  $q_i(v, x)$  and  $q_i^{(n)}(v, x)$  for  $1 \leq i \leq d$  by

$$\begin{aligned} q_d(v, x) &= q_d^{(n)}(v, x) = p_d(v, x), \\ q_i(v, x) &= p_i(v, x)[h_1(v, x), \dots, h_M(v, x)] e^{(t_{i+1}-t_i)G} \overrightarrow{q_{i+1}}, \quad 1 \leq i < d, \\ q_i^{(n)}(v, x) &= p_i(v, x)[h_1(v, x), \dots, h_M(v, x)] e^{(t_{i+1}-t_i)G^{(n)}} \overrightarrow{q_{i+1}^{(n)}}, \quad 1 \leq i < d. \end{aligned}$$

A successive application of Theorem A.1 implies that

$$\begin{aligned} \mathbb{E} \left[ \prod_{i=1}^d p_i(V_{t_i}, X_{t_i}) \right] &= \mathbb{E} \left[ \prod_{i=1}^{d-1} p_i(V_{t_i}, X_{t_i}) \mathbb{E} [p_d(V_{t_d}, X_{t_d}) \mid \mathcal{F}_{t_{d-1}}] \right] \\ &= \dots = [h_1(V_0, X_0), \dots, h_M(V_0, X_0)] e^{t_1 G} \overrightarrow{q_1}. \end{aligned}$$

and similarly,

$$\mathbb{E} \left[ \prod_{i=1}^d p_i(V_{t_i}^{(n)}, X_{t_i}^{(n)}) \right] = [h_1(V_0^{(n)}, X_0^{(n)}), \dots, h_M(V_0^{(n)}, X_0^{(n)})] e^{t_1 G^{(n)}} \overrightarrow{q_1^{(n)}}.$$

We deduce from (44) that

$$\lim_n G^{(n)} = G. \quad (53)$$

Note that this is valid also for the limit case  $v_{\max} = \infty$ , that is,  $Q(v) = v - v_{\min}$ . This fact together with an inductive argument shows that

$$\lim_n \overrightarrow{q_1^{(n)}} = \overrightarrow{q_1}.$$

This combined with (53) proves (52).  $\square$

**Lemma B.3.** *The finite-dimensional distributions of  $(V_t, X_t)$  are determined by their moments.*

*Proof.* The proof of this result is contained in the proof of (Filipović and Larsson 2015, Theorem 4.2).  $\square$

**Lemma B.4.** *The family of diffusions  $(V_t^{(n)}, X_t^{(n)})$  is tight.*

*Proof.* Fix a time horizon  $N \in \mathbb{N}$ . We first observe that by (Karatzas and Shreve 1991, Problem V.3.15) there is a constant  $K$  independent of  $n$  such that

$$\mathbb{E}[\|(V_t^{(n)}, X_t^{(n)}) - (V_s^{(n)}, X_s^{(n)})\|^4] \leq K|t - s|^2, \quad 0 \leq s < t \leq N. \quad (54)$$

Now fix any positive  $\alpha < 1/4$ . Kolmogorov's continuity theorem (see (Revuz and Yor 1999, Theorem I.2.1)) implies that

$$\mathbb{E} \left[ \left( \sup_{0 \leq s < t \leq N} \frac{\|(V_t^{(n)}, X_t^{(n)}) - (V_s^{(n)}, X_s^{(n)})\|}{|t - s|^\alpha} \right)^4 \right] \leq J$$

for a finite constant  $J$  that is independent of  $n$ . The modulus of continuity

$$\Delta(\delta, n) = \sup \left\{ \|(V_t^{(n)}, X_t^{(n)}) - (V_s^{(n)}, X_s^{(n)})\| \mid 0 \leq s < t \leq N, |t - s| < \delta \right\}$$

thus satisfies

$$\mathbb{E}[\Delta(\delta, n)^4] \leq \delta^\alpha J.$$

Using Chebyshev's inequality we conclude that, for every  $\epsilon > 0$ ,

$$\mathbb{Q}[\Delta(\delta, n) > \epsilon] \leq \frac{\mathbb{E}[\Delta(\delta, n)^4]}{\epsilon^4} \leq \frac{\delta^\alpha J}{\epsilon^4},$$

and thus  $\sup_n \mathbb{Q}[\Delta(\delta, n) > \epsilon] \rightarrow 0$  as  $\delta \rightarrow 0$ . This together with the property that the initial states  $(V_0^{(n)}, X_0^{(n)})$  converge to  $(V_0, X_0)$  as  $n \rightarrow \infty$  proves the lemma, see (Rogers and Williams 2000, Theorem II.85.3).<sup>4</sup>  $\square$

**Remark B.5.** *Kolmogorov's continuity theorem (see (Revuz and Yor 1999, Theorem I.2.1)) and (54) imply that the paths of  $(V_t, X_t)$  are  $\alpha$ -Hölder continuous for any  $\alpha < 1/4$ .*

Lemmas B.1–B.3 imply that the finite-dimensional distributions of the diffusions  $(V_t^{(n)}, X_t^{(n)})$  converge weakly to those of  $(V_t, X_t)$  as  $n \rightarrow \infty$ . Theorem 2.3 thus follows from Lemma B.4 and (Rogers and Williams 2000, Lemma II.87.3).

## Proof of Theorem 3.7

We claim that the solution of the recursion (20) is given by

$$I_n(\mu; \nu) = \int_\mu^\infty \mathcal{H}_n(x) e^{\nu x} \phi(x) dx, \quad n \geq 0. \quad (55)$$

---

<sup>4</sup>The derivation of the tightness of  $(V_t^{(n)}, X_t^{(n)})$  from (54) is also stated without proof in (Rogers and Williams 2000, Theorem II.85.5). For the sake of completeness we give a short self-contained argument here.

Indeed, for  $n = 0$  the right hand side of (55) equals

$$\int_{\mu}^{\infty} \mathcal{H}_0(x) e^{\nu x} \phi(x) dx = e^{\frac{\nu^2}{2}} \int_{\mu-\nu}^{\infty} \phi(x) dx,$$

which is  $I_0(\mu; \nu)$ . For  $n \geq 1$ , we recall that the standard Hermite polynomials  $\mathcal{H}_n(x)$  satisfy

$$\mathcal{H}_n(x) = x\mathcal{H}_{n-1}(x) - \mathcal{H}'_{n-1}(x). \quad (56)$$

Integration by parts and (56) then show that

$$\begin{aligned} \int_{\mu}^{\infty} \mathcal{H}_n(x) e^{\nu x} \phi(x) dx &= \int_{\mu}^{\infty} \mathcal{H}_{n-1}(x) e^{\nu x} x \phi(x) dx - \int_{\mu}^{\infty} \mathcal{H}'_{n-1}(x) e^{\nu x} \phi(x) dx \\ &= -\mathcal{H}_{n-1}(x) e^{\nu x} \phi(x) \Big|_{\mu}^{\infty} + \int_{\mu}^{\infty} \mathcal{H}_{n-1}(x) \nu e^{\nu x} \phi(x) dx. \\ &= \mathcal{H}_{n-1}(\mu) e^{\nu \mu} \phi(\mu) + \nu \int_{\mu}^{\infty} \mathcal{H}_{n-1}(x) e^{\nu x} \phi(x) dx, \end{aligned}$$

which proves (55).

A change of variables, using (12) and (55), shows

$$\begin{aligned} f_n &= e^{-rT} \int_k^{\infty} (e^x - e^k) H_n(x) w(x) dx \\ &= e^{-rT} \int_{\frac{k-\mu_w}{\sigma_w}}^{\infty} (e^{\mu_w + \sigma_w z} - e^k) H_n(\mu_w + \sigma_w z) w(\mu_w + \sigma_w z) \sigma_w dz \\ &= e^{-rT} \frac{1}{\sqrt{n!}} \int_{\frac{k-\mu_w}{\sigma_w}}^{\infty} (e^{\mu_w + \sigma_w z} - e^k) \mathcal{H}_n(z) \phi(z) dz \\ &= e^{-rT + \mu_w} \frac{1}{\sqrt{n!}} I_n \left( \frac{k - \mu_w}{\sigma_w}; \sigma_w \right) - e^{-rT + k} \frac{1}{\sqrt{n!}} I_n \left( \frac{k - \mu_w}{\sigma_w}; 0 \right). \end{aligned}$$

Formulas (19) follow from the recursion formula (20).

### Proof of Theorem 3.11

As before, a change of variables, using (12) and (55), shows

$$\begin{aligned} f_n &= e^{-rT} \int_k^{\infty} H_n(x) w(x) dx \\ &= \frac{e^{-rT}}{\sqrt{n!}} \int_{\frac{k-\mu_w}{\sigma_w}}^{\infty} \mathcal{H}_n(z) \phi(z) dz \\ &= \frac{e^{-rT}}{\sqrt{n!}} I_n \left( \frac{k - \mu_w}{\sigma_w}; 0 \right). \end{aligned}$$

Formulas (22) follow directly from (20).

### Proof of Theorem 3.14

For any  $t < T$ , the time- $t$  Black–Scholes option price with volatility  $\sigma_{BS}$  as function of the spot price  $S_t = s$  satisfies

$$e^{-rt} \pi_f^{\sigma_{BS}}(t, s) = \mathbb{E}[f(\log(sY))]$$

where  $Y$  is some log-normal random variable independent of  $\mathcal{F}_t$ . By convexity of the payoff function we have that

$$\alpha f(\log(s_1 Y)) + (1 - \alpha) f(\log(s_2 Y)) \geq f(\log(s_3 Y))$$

with  $s_3 = \alpha s_1 + (1 - \alpha) s_2$ . Since  $f(\log(s))$  is non-affine, the previous inequality is strict with a strict positive probability. This implies that  $\pi_f^{\sigma_{BS}}(t, s)$  is strictly convex in  $s > 0$  and the theorem now follows from Theorem 5.4.

### Proof of Proposition 4.1

We use similar notation as in the proof of Theorem 5.1. In particular, with  $C_T$  as in (5) and  $M_T$  as in (27), we denote

$$G_T(x) = (2\pi C_T)^{-\frac{1}{2}} \exp\left(-\frac{(x - M_T)^2}{2C_T}\right)$$

the density of  $X_T$  conditional on  $\{V_t : t \in [0, T]\}$  and  $g_T(x) = \mathbb{E}[G_T(x)]$  the density of  $X_T$ .

1. We first recall that by Cramér's inequality for all  $n \geq 0$

$$e^{-(x-\mu_w)^2/4\sigma_w^2} |H_n(x)| = (n!)^{-1/2} e^{-(x-\mu_w)^2/4\sigma_w^2} \left| \mathcal{H}_n\left(\frac{x-\mu_w}{\sigma_w}\right) \right| \leq K, \quad (57)$$

with  $K \leq 1.086435$  a constant; see for instance (Hille 1926). On the other hand, as in the proof Theorem 5.1, since  $1/4\sigma_w^2 < 1/(2v_{max}T)$ ,

$$\mathbb{E} \left[ \int_{\mathbb{R}} e^{(x-\mu_w)^2/4\sigma_w^2} G_T(x) dx \right] < \infty.$$

Hence

$$\begin{aligned} \mathbb{E} \left[ \int_{\mathbb{R}} |H_n(x)| G_T(x) dx \right] &= \mathbb{E} \left[ \int_{\mathbb{R}} |H_n(x)| e^{-(x-\mu_w)^2/4\sigma_w^2} e^{(x-\mu_w)^2/4\sigma_w^2} G_T(x) dx \right] \\ &\leq K \mathbb{E} \left[ \int_{\mathbb{R}} e^{(x-\mu_w)^2/4\sigma_w^2} G_T(x) dx \right] < \infty, \end{aligned}$$

and we can use Fubini's theorem to deduce that

$$\ell_n = \int_{\mathbb{R}} H_n(x) g_T(x) dx = \mathbb{E} \left[ \int_{\mathbb{R}} H_n(x) G_T(x) dx \right] = \mathbb{E}[Y_n]. \quad (58)$$

We now analyze the term inside the expectation in (58). A change of variables shows

$$Y_n = \int_{\mathbb{R}} H_n(x) G_T(x) dx = (2\pi n!)^{-1/2} \int_{\mathbb{R}} \mathcal{H}_n(\alpha y + \beta) e^{-y^2/2} dy, \quad (59)$$

with  $\alpha = \frac{\sqrt{C_T}}{\sigma_w}$  and  $\beta = \frac{M_T - \mu_w}{\sigma_w}$ . Define

$$x_n = (2\pi)^{-1/2} \int_{\mathbb{R}} \mathcal{H}_n(\alpha y + \beta) e^{-y^2/2} dy,$$

so that

$$Y_n = \int_{\mathbb{R}} H_n(x) G_T(x) dx = (n!)^{-1/2} x_n. \quad (60)$$

An integration by parts argument using (56) and the identity

$$\mathcal{H}'_n(x) = n\mathcal{H}_{n-1}(x)$$

shows the following recursion formula

$$x_n = \beta x_{n-1} - (n-1)(1-\alpha^2)x_{n-2},$$

with  $x_0 = 1$  and  $x_1 = \beta$ . This recursion formula is closely related to the recursion formula of the Hermite polynomials which helps us deduce the following explicit expression

$$x_n = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(\alpha^2 - 1)^m}{m!(n-2m)!} \frac{\beta^{n-2m}}{2^m}. \quad (61)$$

By (58), (60), (61), we derive the expression for the Hermite moments  $l_n$  given by (25).

2. Suppose that  $\alpha^2 < 1$  a.s. We first recall that

$$\mathcal{H}_n(x) = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m!(n-2m)!} \frac{x^{n-2m}}{2^m}. \quad (62)$$

By (61) and (62) we have

$$\begin{aligned} x_n &= n!(1-\alpha^2)^{\frac{n}{2}} \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m!(n-2m)!} \frac{((1-\alpha^2)^{-\frac{1}{2}}\beta)^{n-2m}}{2^m} \\ &= (1-\alpha^2)^{\frac{n}{2}} \mathcal{H}_n((1-\alpha^2)^{-\frac{1}{2}}\beta). \end{aligned} \quad (63)$$

Formula (28) follows from (58), (60) and (63). Iteratively using integration by parts in (59) we conclude that for all  $k \geq 1$

$$Y_n = \frac{(2\pi(n+k)!)^{-1/2}}{\alpha^k \sqrt{(n+1)(n+2)\cdots(n+k)}} \int_{\mathbb{R}} \mathcal{H}_k(y) e^{-y^2/2} \mathcal{H}_{n+k}(\alpha y + \beta) dy.$$

Since  $(2\pi)^{-1/2} \int_{\mathbb{R}} \mathcal{H}_k(y)^2 e^{-y^2/2} dy = k!$ , Cauchy-Schwarz inequality implies that

$$|Y_n|^2 \leq \frac{(2\pi)^{-1/2} k!}{(\alpha^2 n)^k} \int_{\mathbb{R}} (n+k)!^{-1} \mathcal{H}_{n+k}^2(\alpha y + \beta) e^{-y^2/2} dy.$$

By (57) we obtain

$$|Y_n|^2 \leq \frac{(2\pi)^{-1/2} k! K^2}{(\alpha^2 n)^k} \int_{\mathbb{R}} e^{-y^2/2 + (\alpha y + \beta)^2/2} dy = \frac{k! K^2 e^{\frac{\beta^2(1+\alpha^2/\gamma)}{2}}}{(\alpha^2 n)^k \sqrt{\gamma}}, \quad (64)$$

with

$$\gamma = 1 - \alpha^2.$$

The inequality (29) follows from Jensen's inequality, (58) and (64). If  $\alpha^2 \leq \delta < 1$  with  $\delta$  a constant, since  $\tilde{K}$  is a constant such that  $|\beta| \leq \tilde{K}$ , by Cauchy-Schwarz inequality and (57)

$$\begin{aligned} \ell_n^2 &\leq \mathbb{E}[(1-\alpha^2)^n] \mathbb{E}[(n!^{-\frac{1}{2}} \mathcal{H}_n((1-\alpha^2)^{-\frac{1}{2}}\beta))^2] \\ &\leq \mathbb{E}[(1-\alpha^2)^n] K^2 \mathbb{E}[\exp(\beta^2/(2(1-\alpha^2)))] \\ &\leq K^2 \exp(\tilde{K}^2/(2(1-\delta))) \mathbb{E}[(1-\alpha^2)^n]. \end{aligned} \quad (65)$$

Hence, if in addition  $\alpha^2 \geq \epsilon > 0$ , the inequality (30) follows.



3. Observe that

$$\sum_{n=0}^{\infty} \ell_n^2 = \|\ell\|_w^2 = \int_{\mathbb{R}} \frac{g_T^2(x)}{w(x)} dx.$$

Additionally we have

$$\begin{aligned} \int_{\mathbb{R}} \frac{g_T^2(x)}{w(x)} dx &= \sqrt{2\pi}\sigma_w \int_{\mathbb{R}} e^{(x-\mu_w)^2/(2\sigma_w^2)} g_T^2(x) dx \\ &= \sqrt{2\pi}\sigma_w \int_{\mathbb{R}} (e^{x^2/4\sigma_w^2} g_T(x + \mu_w))^2 dx. \end{aligned}$$

Following the proof of Theorem 5.1 we have

$$\begin{aligned} \int_{\mathbb{R}} e^{x^2/4\sigma_w^2} g_T(x + \mu_w) dx &= \mathbb{E} \left[ \frac{1}{\sqrt{\Delta}} \exp \left( \frac{(M_T - \mu_w)^2}{4\sigma_w^2 \Delta} \right) \right] \\ e^{x^2/4\sigma_w^2} g_T(x + \mu_w) &\leq \mathbb{E} \left[ \frac{\exp \left( \frac{(M_T - \mu_w)^2}{4\sigma_w^2 \Delta} \right)}{\sqrt{2\pi C_T}} \right], \end{aligned}$$

where

$$\Delta = 1 - \frac{C_T}{2\sigma_w^2}.$$

This readily implies (31). Observe that  $\Delta > 0$  because by (11)  $\frac{C_T}{2\sigma_w^2} \leq \frac{v_{max}T}{2\sigma_w^2} < 1$ .

### Proof of Proposition 4.3

From the definition of  $C_T$  it is obvious that  $C_T \leq \int_0^T V_t dt$ . Since  $Q(v) \leq v$ , the inequality (32) follows. The inequality (33) is a consequence of

$$\begin{aligned} C_T &= \left( 1 - \frac{\rho^2(v_{max} + v_{min})}{(\sqrt{v_{max}} - \sqrt{v_{min}})^2} \right) \int_0^T V_t dt \\ &\quad + \frac{\rho^2}{(\sqrt{v_{max}} - \sqrt{v_{min}})^2} \int_0^T (v_{max}v_{min} + V_t^2) dt. \end{aligned}$$

### Proof of Proposition 4.5

By (21), (57) and the Cauchy-Schwarz inequality we have for  $n \geq 1$

$$\begin{aligned} |f_n| &\leq e^{-rT+k} \frac{\sigma_w \phi \left( \frac{k-\mu_w}{\sigma_w} \right)}{\sqrt{n(n-1)}} \sum_{m=0}^{n-2} \left| \mathcal{H}_m \left( \frac{k-\mu_w}{\sigma_w} \right) \right| m!^{-1/2} \frac{\sigma_w^{n-2-m}}{\sqrt{(n-2-m)!}} \binom{n-2}{m}^{-1/2} \\ &\quad + e^{-rT+\mu_w+\frac{\sigma_w^2}{2}} \Phi \left( \sigma_w - \frac{k-\mu_w}{\sigma_w} \right) \frac{\sigma_w^n}{\sqrt{n!}} \\ &\leq e^{-rT+k-\frac{(k-\mu_w)^2}{4\sigma_w^2}} \frac{K\sigma_w(2\pi)^{-1/2}}{\sqrt{n(n-1)}} \left( \sum_{m=0}^{n-2} \frac{(\sigma_w^2)^m}{m!} \right)^{1/2} \left( \sum_{m=0}^{n-2} \binom{n-2}{m}^{-1} \right)^{1/2} \\ &\quad + e^{-rT+\mu_w+\frac{\sigma_w^2}{2}} \Phi \left( \sigma_w - \frac{k-\mu_w}{\sigma_w} \right) \frac{\sigma_w^n}{\sqrt{n!}} \\ &\leq e^{-rT+k-\frac{(k-\mu_w)^2}{4\sigma_w^2}+\sigma_w^2/2} \frac{K\tilde{K}\sigma_w(2\pi)^{-1/2}}{\sqrt{n(n-1)}} + e^{-rT+\mu_w+\frac{\sigma_w^2}{2}} \Phi \left( \sigma_w - \frac{k-\mu_w}{\sigma_w} \right) \frac{\sigma_w^n}{\sqrt{n!}}, \end{aligned}$$

where  $\left(\sum_{m=0}^{n-2} \binom{n-2}{m}^{-1}\right)^{1/2} \leq \tilde{K}$ . Note that for  $n \geq 6$  and  $2 \leq m \leq n-4$ , we have  $\binom{n-2}{m} \geq \binom{n-2}{2} = \frac{(n-2)(n-3)}{2}$ , hence for  $n \geq 6$

$$2 \leq \sum_{m=0}^{n-2} \binom{n-2}{m}^{-1} \leq 2 + \frac{2}{n-2} + \frac{2(n-5)}{(n-2)(n-3)}.$$

This implies that  $\lim_{n \rightarrow \infty} \sum_{m=0}^{n-2} \binom{n-2}{m}^{-1} = 2$  and in particular we have the existence of  $\tilde{K}$ , e.g.  $\tilde{K} = \sqrt{3}$ .

### Proof of Theorem 5.1

In order to shorten the notation throughout the proof, given a stochastic process  $Z_t$  we set  $\Delta Z_{t_i} = Z_{t_i} - Z_{t_{i-1}}$ . From (1) we infer that the log price  $X_t$  can be written as

$$X_t = M_t + \int_0^t \sqrt{V_s - \rho^2 Q(V_s)} dW_{2s}$$

where as in (27) we denote

$$M_t = X_0 + \int_0^t (r - \delta - V_s/2) ds + \frac{\rho}{\sigma} \left( V_t - V_0 - \int_0^t \kappa(\theta - V_s) ds \right).$$

In particular the log returns  $Y_{t_i} = \Delta X_{t_i}$  have the form

$$Y_{t_i} = \Delta M_{t_i} + \int_{t_{i-1}}^{t_i} \sqrt{V_s - \rho^2 Q(V_s)} dW_{2s}.$$

Assume  $\Delta C_{t_i} > 0$  for  $i = 1, \dots, n$ . Motivated by (Broadie and Kaya 2006), we notice that in this case, conditional on  $\{V_t, t \in [0, T]\}$ , the random variable  $(Y_{t_1}, \dots, Y_{t_n})$  is Gaussian with mean vector

$$\mu = (\Delta M_{t_1}, \dots, \Delta M_{t_n})$$

and covariance matrix

$$\Sigma = \text{diag}(\Delta C_{t_1}, \dots, \Delta C_{t_n}).$$

Its density  $G_{t_0, \dots, t_n}(y) = G_{t_0, \dots, t_n}(y_1, \dots, y_n)$  has the form

$$G_{t_0, \dots, t_n}(y) = (2\pi)^{-n/2} \prod_i (\Delta C_{t_i})^{-1/2} \exp \left[ - \sum_i \frac{(y_i - \Delta M_{t_i})^2}{2\Delta C_{t_i}} \right].$$

Fubini's theorem implies that

$$g_{t_0, \dots, t_n}(y) = \mathbb{E}[G_{t_0, \dots, t_n}(y)]$$

is measurable and satisfies, for any bounded measurable function  $f(y)$

$$\mathbb{E}[f(Y_{t_1}, \dots, Y_{t_n})] = \mathbb{E} \left[ \int_{\mathbb{R}^n} f(y) G_{t_0, \dots, t_n}(y) dy \right] = \int_{\mathbb{R}^n} f(y) g_{t_0, \dots, t_n}(y) dy.$$

Hence the distribution of  $(Y_{t_1}, \dots, Y_{t_n})$  admits the density  $g_{t_0, \dots, t_n}(y)$  on  $\mathbb{R}^n$ . Observe that the mean vector and covariance matrix of  $G_{t_0, \dots, t_n}(y)$  admit the uniform bounds

$$|\mu_i| \leq K, \quad |\Sigma_{ii}| \leq v_{\max}(t_i - t_{i-1}),$$

for some finite constant  $K$ . Dominated convergence implies that  $g_{t_0, \dots, t_n}(y)$  is uniformly bounded and  $k$ -times continuously differentiable on  $\mathbb{R}^n$  if (36) holds.

We can assume without loss of generality that  $\epsilon_i > 0$  for  $i = 1, \dots, n$ . Define  $\Delta_i = 1 - 2\epsilon_i \Delta C_{t_i}$  and  $\delta_i = 1 - 2\epsilon_i v_{\max}(t_i - t_{i-1})$ . Then  $\delta_i \in (0, 1)$  and  $\Delta_i \geq \delta_i$ . Completing the square implies

$$\begin{aligned} e^{\sum_i \epsilon_i y_i^2} G_{t_0, \dots, t_n}(y) &= \prod_i (2\pi \Delta C_{t_i})^{-\frac{1}{2}} \exp \left[ \epsilon_i y_i^2 - \frac{(y_i - \Delta M_{t_i})^2}{2\Delta C_{t_i}} \right] \\ &= \prod_i (2\pi \Delta C_{t_i})^{-\frac{1}{2}} \exp \left[ -\frac{\Delta_i}{2\Delta C_{t_i}} \left( y_i - \frac{\Delta M_{t_i}}{\Delta_i} \right)^2 + \frac{\Delta M_{t_i}^2}{2\Delta C_{t_i}} \left( \frac{1}{\Delta_i} - 1 \right) \right] \\ &= \prod_i (2\pi \Delta C_{t_i})^{-\frac{1}{2}} \exp \left[ -\frac{\Delta_i}{2\Delta C_{t_i}} \left( y_i - \frac{\Delta M_{t_i}}{\Delta_i} \right)^2 + \frac{\epsilon_i \Delta M_{t_i}^2}{\Delta_i} \right]. \end{aligned} \quad (66)$$

Integration of (66) then gives

$$\int_{\mathbb{R}^n} e^{\sum_i \epsilon_i y_i^2} G_{t_0, \dots, t_n}(y) dy = \prod_i \frac{1}{\sqrt{\Delta_i}} \exp \left[ \frac{\epsilon_i \Delta M_{t_i}^2}{\Delta_i} \right] \leq \prod_i \frac{1}{\sqrt{\delta_i}} \exp \left[ \frac{\epsilon_i K^2}{\delta_i} \right].$$

Hence (6) follows by Fubini's theorem after taking expectation on both sides. We also derive from (66) that

$$e^{\sum_i \epsilon_i y_i^2} g_{t_0, \dots, t_n}(y) = \mathbb{E} \left[ e^{\sum_i \epsilon_i y_i^2} G_{t_0, \dots, t_n}(y) \right] \leq \mathbb{E} \left[ \prod_i (2\pi \Delta C_{t_i})^{-\frac{1}{2}} \prod_i \exp \left[ \frac{\epsilon_i K^2}{\delta_i} \right] \right].$$

Hence  $e^{\sum_i \epsilon_i y_i^2} g_{t_0, \dots, t_n}(y)$  is uniformly bounded and continuous on  $\mathbb{R}^n$  if (36) holds. In fact, for this to hold it is enough suppose that (36) holds with  $k = 0$ . Moreover, (8) implies that  $\Delta C_{t_i} \geq (t_i - t_{i-1})(1 - \rho^2)v_{\min} > 0$  and (36) follows.

## Proof of Theorem 5.4

We denote by  $\pi_{f,t}$  the time- $t$  price of the exotic option in the Jacobi model. For  $t \in (t_{i-1}, t_i)$  and given  $X_{t_1}, \dots, X_{t_{i-1}}$ , the Black-Scholes price function  $\pi_f^{\sigma_{BS}}(t, s)$  satisfies the following PDE

$$r\pi_f^{\sigma_{BS}}(t, s) = \frac{\partial \pi_f^{\sigma_{BS}}(t, s)}{\partial t} + (r - \delta)s \frac{\partial \pi_f^{\sigma_{BS}}(t, s)}{\partial s} + \frac{1}{2}\sigma_{BS}^2 s^2 \frac{\partial^2 \pi_f^{\sigma_{BS}}(t, s)}{\partial s^2} \quad (67)$$

and has terminal value satisfying  $\pi_f^{\sigma_{BS}}(T, S_T) = \pi_{f,T}$ . Write

$$\pi_{f,t}^{\sigma_{BS}} = \pi_f^{\sigma_{BS}}(t, S_t), \quad \Theta_{f,t}^{\sigma_{BS}} = -\frac{\partial \pi_f^{\sigma_{BS}}(t, S_t)}{\partial t}, \quad \Delta_{f,t}^{\sigma_{BS}} = \frac{\partial \pi_f^{\sigma_{BS}}(t, S_t)}{\partial s}, \quad \Gamma_{f,t}^{\sigma_{BS}} = \frac{\partial^2 \pi_f^{\sigma_{BS}}(t, S_t)}{\partial s^2}$$

and  $dN_t = \rho \sqrt{Q(V_t)} dW_{1t} + \sqrt{V_t - \rho^2 Q(V_t)} dW_{2t}$  for the martingale driving the asset return in (1) such that, using (67),

$$\begin{aligned} d(e^{-rt} \pi_{f,t}^{\sigma_{BS}}) &= e^{-rt} \left( -r\pi_{f,t}^{\sigma_{BS}} - \Theta_{f,t}^{\sigma_{BS}} + (r - \delta)S_t \Delta_{f,t}^{\sigma_{BS}} + \frac{1}{2}V_t S_t^2 \Gamma_{f,t}^{\sigma_{BS}} \right) dt + e^{-rt} \Delta_{f,t}^{\sigma_{BS}} S_t dN_t \\ &= \frac{1}{2}e^{-rt}(V_t - \sigma_{BS}^2)S_t^2 \Gamma_{f,t}^{\sigma_{BS}} dt + e^{-rt} \Delta_{f,t}^{\sigma_{BS}} S_t dN_t. \end{aligned}$$

Consider the self-financing portfolio with zero initial value, long one unit of the exotic option, and short  $\Delta_{f,t}^{\sigma_{BS}}$  units of the underlying asset. Let  $\Pi_t$  denote the time- $t$  value of this portfolio. Its discounted price dynamics then satisfies

$$\begin{aligned} d(e^{-rt}\Pi_t) &= d(e^{-rt}\pi_{f,t}) - \Delta_{f,t}^{\sigma_{BS}} (d(e^{-rt}S_t) + e^{-rt}S_t\delta dt) \\ &= d(e^{-rt}\pi_{f,t}) - \Delta_{f,t}^{\sigma_{BS}} e^{-rt}S_t dN_t \\ &= d(e^{-rt}\pi_{f,t}) - d(e^{-rt}\pi_{f,t}^{\sigma_{BS}}) + \frac{1}{2}e^{-rt}(V_t - \sigma_{BS}^2)S_t^2\Gamma_{f,t}^{\sigma_{BS}} dt. \end{aligned}$$

Integrating in  $t$  gives

$$e^{-rT}\Pi_T = -\pi_{f,0} + \pi_{f,0}^{\sigma_{BS}} + \frac{1}{2} \int_0^T e^{-rt}(V_t - \sigma_{BS}^2)S_t^2\Gamma_{f,t}^{\sigma_{BS}} dt \quad (68)$$

as  $\pi_{f,T} - \pi_{f,T}^{\sigma_{BS}} = 0$ . We now claim that the time-0 option price  $\pi_{f,0} = \pi_f$  lies between the Black-Scholes option prices for  $\sigma_{BS} = \sqrt{v_{min}}$  and  $\sigma_{BS} = \sqrt{v_{max}}$ ,

$$\pi_{f,0}^{\sqrt{v_{min}}} \leq \pi_f \leq \pi_{f,0}^{\sqrt{v_{max}}}. \quad (69)$$

Indeed, let  $\sigma_{BS} = \sqrt{v_{min}}$ . Because  $\Gamma_{f,t}^{BS} \geq 0$  by assumption, it follows from (68) that  $e^{-rT}\Pi_T \geq -\pi_{f,0} + \pi_{f,0}^{\sqrt{v_{min}}}$ . Absence of arbitrage implies that  $\Pi_T$  must not be bounded away from zero, hence  $-\pi_{f,0} + \pi_{f,0}^{\sqrt{v_{min}}} \leq 0$ . This proves the left inequality in (69). The right inequality follows similarly, whence the claim (69) is proved.

It remains to prove that the time-0 Black-Scholes price  $\pi_{f,0}^{\sigma_{BS}}$  is strictly increasing in  $\sigma_{BS}$ . Applying the same arguments as above to the two Black-Scholes models with volatilities  $\sigma_a > \sigma_b$  implies

$$e^{-rT}\Pi_T = -\pi_{f,0}^{\sigma_a} + \pi_{f,0}^{\sigma_b} + \frac{1}{2} \int_0^T e^{-rt}(\sigma_a^2 - \sigma_b^2)S_t\Gamma_{f,t}^{\sigma_b} dt > -\pi_{f,0}^{\sigma_a} + \pi_{f,0}^{\sigma_b},$$

where we used that  $\Gamma_{f,0}^{\sigma_b} > 0$ . Absence of arbitrage then implies that  $\pi_{f,0}^{\sigma_a} > \pi_{f,0}^{\sigma_b}$ , as desired. Hence the implied volatility  $\sigma_{IV}$  at  $t = 0$  is well defined and satisfies the bounds as claimed in the theorem.

	$\sqrt{\theta}$	$\kappa$	$\sigma$	$\rho$	$\sqrt{V_0}$	$\sqrt{V_{min}}$	$\sqrt{V_{max}}$	RMSE
Jacobi	0.3660	0.7507	1.0072	-0.6057	0.1178	0.0499	0.4476	0.8461
Heston	0.3655	0.7498	0.8573	-0.6047	0.1178			0.9447

Table 1: Calibrated parameters.

The Jacobi and Heston models fitted parameters are reported as well as the implied volatility surface root-mean-squared-error (RMSE). The models were calibrated on a subset of S&P 500 options with maturity less than one month observed on January 2<sup>nd</sup> 2014.

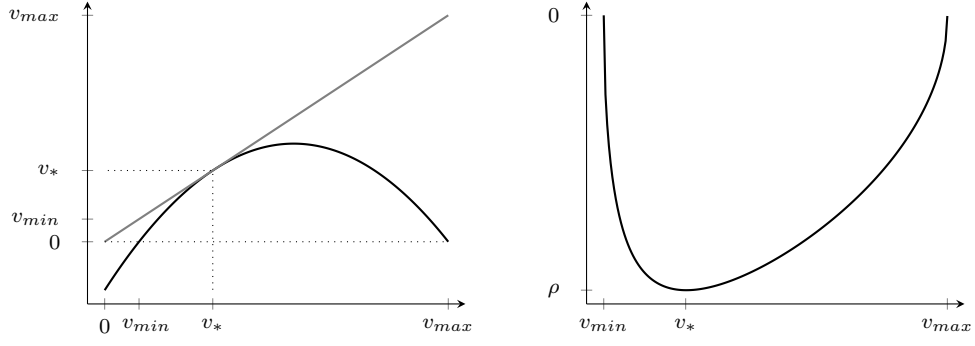


Figure 1: Variance and correlation.

The quadratic variation of the Jacobi model (black line) and of the Heston model (gray line) are displayed in the left panel as a function of the instantaneous variance. The right panel displays the instantaneous correlation between the processes  $X_t$  and  $V_t$  as a function of the instantaneous variance. We denote  $v_* = \sqrt{v_{min}v_{max}}$  and assumed that  $\rho < 0$ .

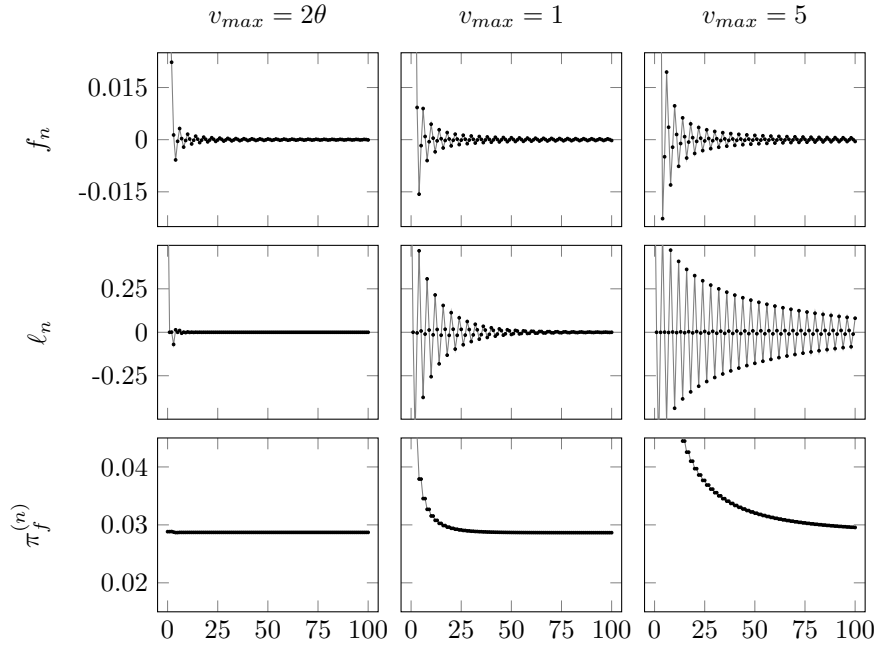


Figure 2: Coefficients and price approximation.

The Fourier coefficients (first row), the Hermite coefficients (second row), and the price expansion (third row) are displayed as a function of the order  $n$ . The parameters values are  $T = 1/12$ ,  $x_0 = k = 0$ ,  $\kappa = 0.5$ ,  $\theta = v_0 = (0.25)^2$ ,  $\sigma = 0.25$ ,  $v_{min} = (0.10)^2$ ,  $\rho = -0.5$ , and  $v_{max} \in \{0.3, 1, 5\}$ .

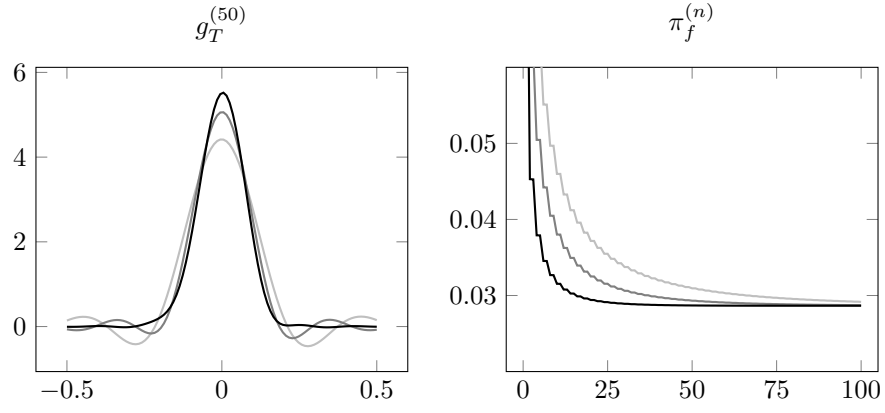


Figure 3: Impact of the weighted space variance parameter.

The left panel displays the expanded density of the process  $X_T$  truncated at the order  $n = 50$ , and the right panel displays the price expansion as a function of the order  $n$ . The different lines correspond to different weighted space variance parameter, that is  $\sigma_w = \nu$  (black line),  $\sigma_w = 1.5\nu$  (gray line), and  $\sigma_w = 2\nu$  (light gray line) with  $\nu = \sqrt{(v_{max}T/2)} + \epsilon$ . The parameters values are  $\epsilon = 10^{-4}$ ,  $T = 1/12$ ,  $x_0 = k = 0$ ,  $\kappa = 0.5$ ,  $\theta = v_0 = (0.25)^2$ ,  $\sigma = 0.25$ ,  $v_{min} = (0.10)^2$ ,  $\rho = -0.5$ , and  $v_{max} = 1$ .



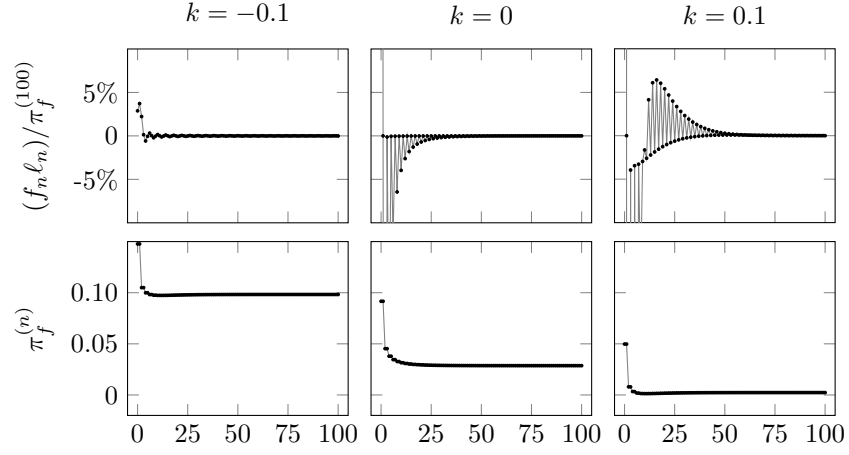


Figure 4: Relative increment contribution.

The contribution of an increment is defined as the product of the Fourier and Hermite coefficients normalized by the price approximation of order  $n = 100$ . The contribution to the final price (first row), and the price expansion (second row) are displayed as a function of the order  $n$ . The parameters values are  $T = 1/12$ ,  $x_0 = 0$ ,  $\kappa = 0.5$ ,  $\theta = v_0 = (0.25)^2$ ,  $\sigma = 0.25$ ,  $v_{min} = (0.10)^2$ ,  $\rho = -0.5$ , and  $v_{max} = 1$ . Note that the option with strike  $k = 0.10$  is a far out of the money option with approximate value of 23.3 basis points.

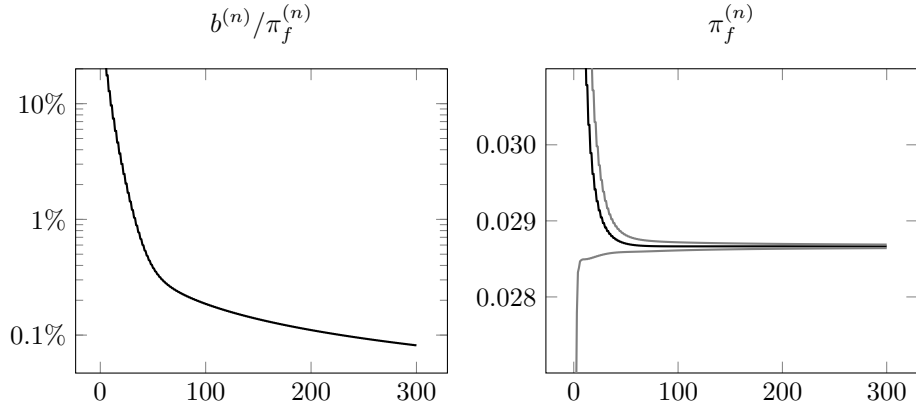


Figure 5: Relative and Absolute error bound.

The left panel displays the relative error bound  $b^{(n)}/\pi_f^{(n)}$  in percentage as a function of  $n$ . The right panel displays the price approximation convergence (black line) and an absolute confidence interval (gray lines). The parameters values are  $T = 1/12$ ,  $x_0 = k = 0$ ,  $\kappa = 0.5$ ,  $\theta = v_0 = (0.25)^2$ ,  $\sigma = 0.25$ ,  $v_{min} = (0.10)^2$ ,  $v_{max} = 1$ , and  $\rho = -0.5$ .

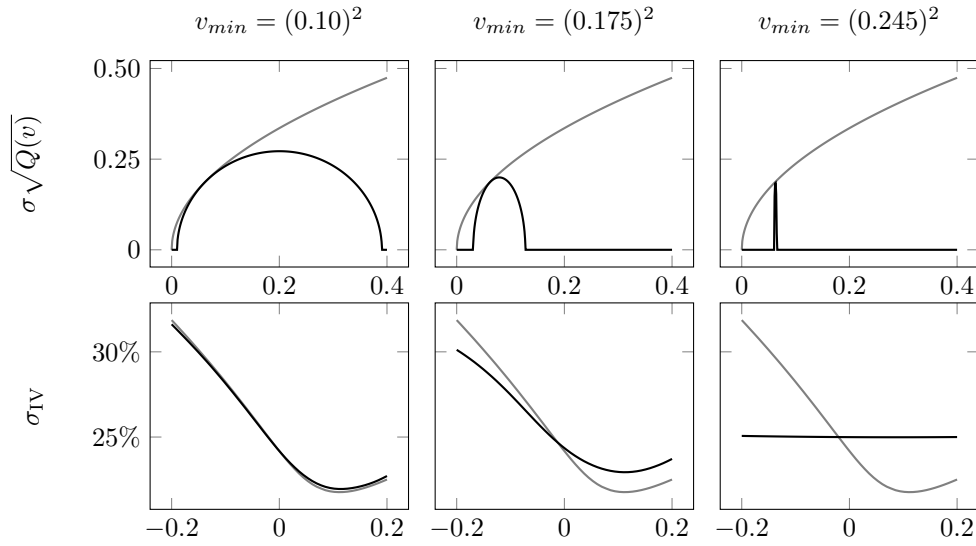


Figure 6: Implied volatility smile: from Heston to Black-Scholes.

The first row displays the variance process' diffusion function in the Jacobi model (black line) and in the Heston model (gray line). The second row displays the implied volatility as a function of the log strike  $k$  in the Jacobi model (black line) and in the Heston model (gray line). The parameters values are  $T = 1/12$ ,  $x_0 = 0$ ,  $\kappa = 0.5$ ,  $\theta = v_0 = (0.25)^2$ ,  $\sigma = 0.75$ ,  $\rho = -0.5$ , and  $v_{max} = \theta^2/v_{min}$  such that  $v_* = \theta$ .

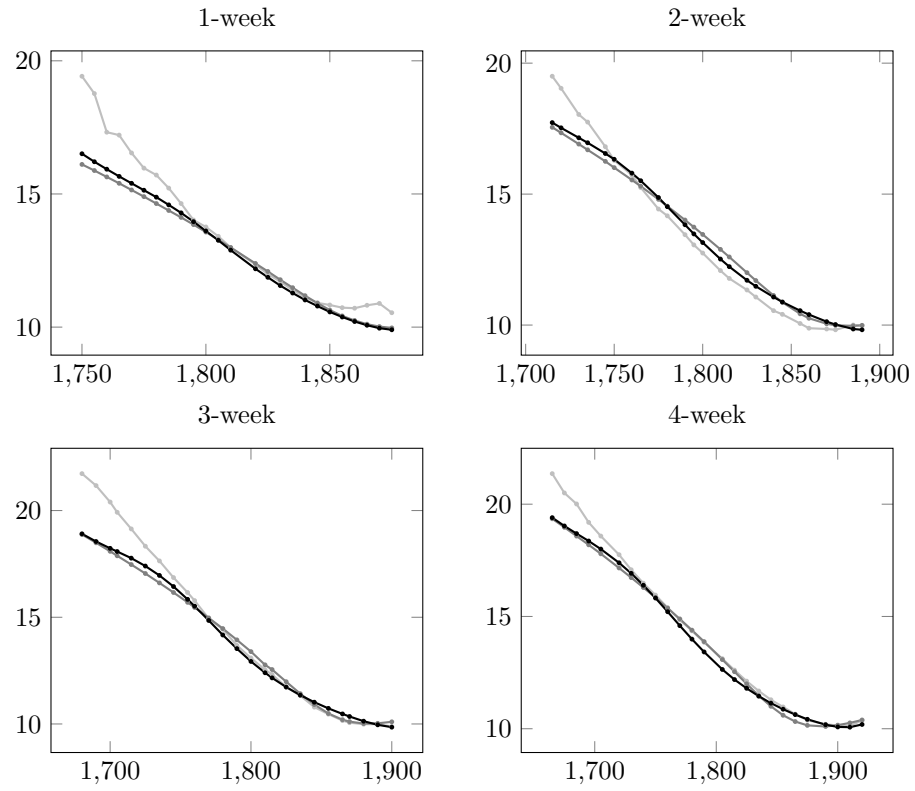


Figure 7: Implied volatility fits.

The true (light-gray), the Heston (gray), and the Jacobi (black) implied volatility surfaces are displayed for each maturity as a function of the strike price. The data sample is a subset of S&P 500 options with maturity less than one month observed on January 2<sup>nd</sup> 2014.

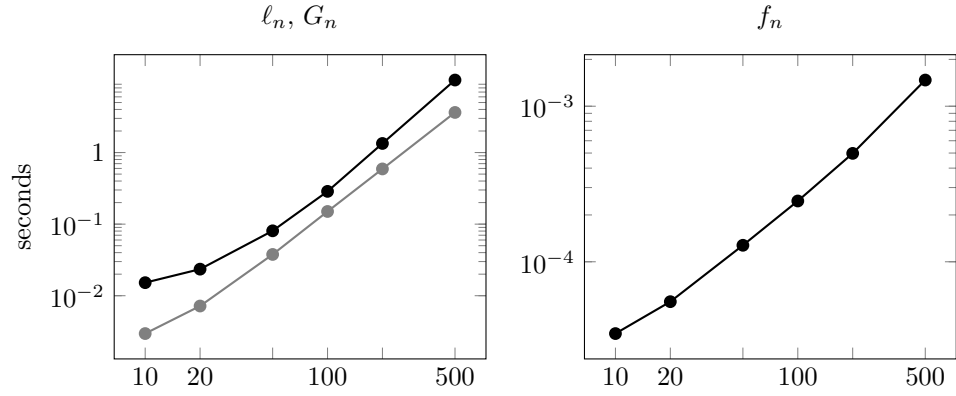


Figure 8: Computational performance.

The left panel displays the computing time to derive the Hermite moments  $\ell_n$  (black line) and the matrix  $G_n$  (gray line) as a function of the order  $n$ . The right panel displays the same relation for the Fourier coefficients  $f_n$  (black line). The calculations have been run on a standard personal computer with a 3.5 Ghz 64 bits CPU and implemented in the R programming language.

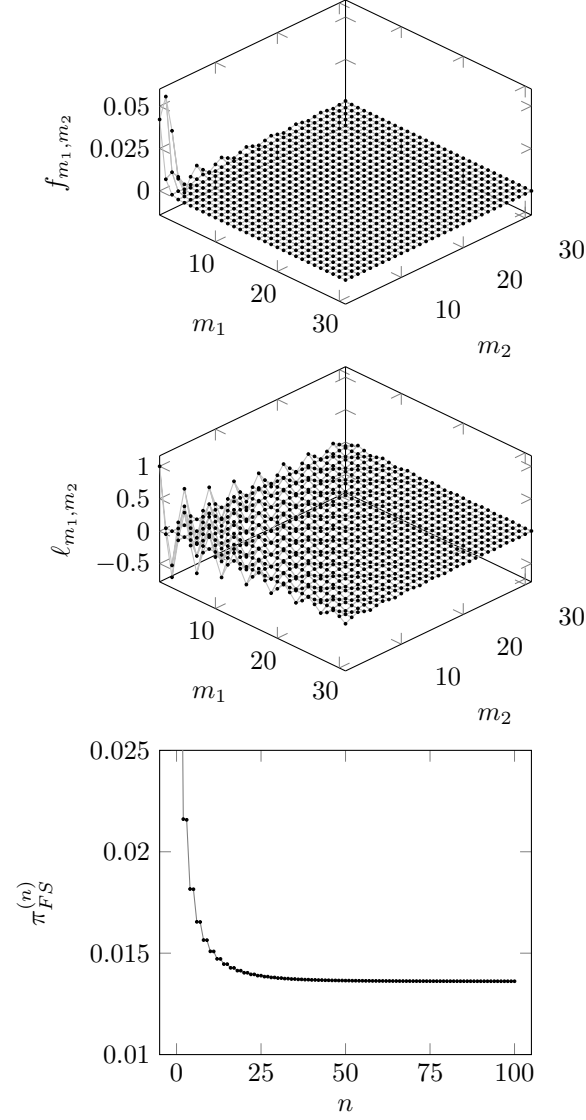


Figure 9: Forward start call option.

The Fourier coefficients (first row) and Hermite coefficients (second row) are displayed as a function of the double index  $(m_1, m_2)$ . The third row displays the price approximation as a function of the order  $n$  which utilizes all the coefficients with order  $m_1 + m_2 \leq n$ . The parameters values are  $t = 1/12$ ,  $T - t = 1/52$ ,  $x_0 = k = 0$ ,  $\kappa = 0.5$ ,  $v_0 = \theta = (0.25)^2$ ,  $\sigma = 0.25$ ,  $v_{min} = 10^{-4}$ ,  $v_{max} = 1$ , and  $\rho = -0.5$ .

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