Note on rational 1–dimensional compact cycles.

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Note on rational 1−dimensional compact cycles. second version

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27/09/16

Abstract. We give a proof of the fact tha the subset of the rational curves form a closed analytic subset in the space of the 1−dimensional cycles of a complex space.

Key words. Rational curves - Cycles' space.

AMS Classification 2010. 32 C 25 - 32 G 10.

The aim of this short note is to prove the following result.

Theorem 0.0.1 *Let M be a complex space. Let R be the subset of the space $\mathcal{C}_1(M)$ consisting of compact rational $1-$dimensional cycles in M. Then R is a closed analytic subset in $\mathcal{C}_1(M)$.*

By a rational 1−dimensional cycle we mean that each irreducible component of such a cycle is a rational curve (may be singular).

Note that this result is classical in the projective context.

The proof of the theorem uses the following proposition.

Proposition 0.0.2 *Let $\pi : U \to V$ be a geometrically flat map between reduced complex spaces with one dimensional fibres. Assume that for a point $v_0 \in V$ the fibre $\pi^{-1}(v_0)$ has an irreducible component γ_0 which has genus ≥ 1. Then there exists an open neighbourhood V_1 of v_0 in V such that for any $v \in V_1$ the fibre $\pi^{-1}(v)$ has an irreducible component of genus ≥ 1.*

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In the previous statement, the genus of an irreducible compact curve is, by definition, the genus of its normalization.
Recall also that the geometric flatness assumption means that there exists a holomorphic map $\varphi : V \to \mathcal{C}_1(U)$ such that for each $v \in V$ we have $\pi^{-1}(v) = |\varphi(v)|$ and such that for v generic in V the $1-$cycle $\varphi(v)$ is reduced (that is to say that all multiplicities are equal to 1).
We begin by some general results in order to show that it is enough to prove the proposition and the theorem in the case when V is normal and when the fibres of π are connected.

Lemma 0.0.3 *Let $U \to V$ be a geometrically flat map between reduced complex spaces with holomorphic fibre map $\varphi : V \to \mathcal{C}_n(U)$. Let $\tau : \tilde{V} \to V$ be a proper modification of V and denote $\tilde{\pi} : \tilde{U} \to \tilde{V}$ the strict transform of π by τ. Then $\tilde{\pi}$ is geometrically flat and the fibre at a point $\tilde{v} \in \tilde{V}$ is the $n-$cycle $\varphi(\tau(\tilde{v})) \times \{\tilde{v}\}$ in $\tilde{U}$.*

PROOF. Let $S \subset V$ be the center of the modification τ. Then for $\tilde{v} \notin \tau^{-1}(S)$ the fibre of $\tilde{\pi}$ is $\varphi(\tau(\tilde{v})) \times \{\tilde{v}\}$ as a cycle in $\tilde{U} \subset U \times_V \tilde{V}$. As $\tilde{U}$ is a closed analytic subset in $U \times_V \tilde{V}$ and as the family of cycles $\tilde{v} \mapsto \varphi(\tau(\tilde{v})) \times \{\tilde{v}\}$ is an analytic family of compact cycles in $U \times_V \tilde{V}$ such that, for $\tilde{v}$ generic, they are contained in $\tilde{U}$, this is an analytic family of cycles in $\tilde{U}$ and this gives the holomorphic fibre map for $\tilde{\pi}$. ■

REMARKS.

1. If $n = 1$ and if R is the set of points in $\tilde{V}$ such that the fibre of $\tilde{\pi}$ is rational then $\tau(R)$ is the subset of V where the fibre of π is rational.

2. This lemma allows to assume that V is a normal complex space in the proofs of the proposition 0.0.2 and the theorem 0.0.1

Lemma 0.0.4 *Let $U \to V$ be a geometrically flat map between irreducible complex spaces and assume that V is normal. Let $\pi' : U \to W$ and $g : W \to V$ be a Stein factorization of π. Let $\nu : \tilde{W} \to W$ be the normalization of W and let $\tilde{\pi} : \tilde{U} \to \tilde{W}$ be the strict transform of π' by ν. Put $\tilde{g} : \tilde{W} \to V$ be the composition $\tilde{g} := \nu \circ g$. As $\tilde{\pi}$ is equi-dimensional proper and with a normal basis, it is geometrically flat with connected fibres. Also $\tilde{g}$ is proper, finite and surjective with a normal basis so it is geometrically flat with $0-$dimensional fibres.*
Let $\theta : \tilde{W} \to \mathcal{C}_n(\tilde{U})$ and $f : V \to \mathrm{Sym}^k(\tilde{W})$ the corresponding holomorphic fibre maps. Then the fibre map for the geometrically flat map $\tilde{\pi} \circ \tilde{g}$ is given by

$$f \circ \mathrm{Sym}^k(\theta) \circ Add : V \to \mathcal{C}_n(\tilde{U}),$$

where $Add : \mathrm{Sym}^k(\mathcal{C}_n(\tilde{U})) \to \mathcal{C}_n(\tilde{U})$ is the addition map of $n-$cycles in $\tilde{U}$.

PROOF. Remember that, by definition of a Stein reduction, the fibre of g of a point $v \in V$ is the set of connected components of $\pi^{-1}(v)$ and that the fibre of π' at a point $w \in g^{-1}(v)$ is the connected component of $\pi^{-1}(v)$ given by w. So π' and also $\tilde{\pi}$ have connected fibres. As g is proper (and finite) and as W is irreducible the image by g of non normal points in W is a closed analytic subset in V with no interior point. So it is clear that the holomorphic map $f \circ \mathrm{Sym}^k(\theta) \circ Add$ is a fibre map for $\tilde{\pi} \circ \tilde{g}$ at the generic points of V. This is enough to conclude. ■

Corollary 0.0.5 *In the situation of the previous lemma with $n = 1$, let $\tilde{R}$ be the subset of $\tilde{W}$ of points such the fibre of $\tilde{\pi}$ is rational. Then the subset of point in V such that the fibre of π is rational is equal to $R := \{v \in V \ / \ f(v) \in \mathrm{Sym}^k(\tilde{R})\}$. Then, if $\tilde{R}$ is closed (resp. analytic), so is R.*

PROOF. This corollary is clear because a compact curve is rational if and only if each of its connected component is rational. ■

REMARK. With the previous results, it is enough to prove the proposition 0.0.2 and the theorem 0.0.1 with the following extra assumptions : V is normal and all fibres of π are connected.

PROOF OF THE PROPOSITION. We shall use the following result (see [B.80]): Let C be a reduced compact curve in a complex space M, and let L be a holomorphic line bundle on C. Then there exists an open neighbourhood M' of C in M and a holomorphic line bundle $\mathcal{L}$ on M' inducing L on C. Moreover, if L is topologically trivial on C we may choose $\mathcal{L}$ topologically trivial on M'.
Note that the last point is not stated in *loc. cit* but is a trivial consequence of the proof given there.

Consider now $C_0 = |\varphi(v_0)| = \pi^{-1}(v_0)$ and let $\nu : \tilde{C}_0 \to C_0$ be the normalization of C_0. Define the coherent sheaf $\mathcal{F} := \nu_*(\mathcal{O}_{\tilde{C}_0})$ on C_0. We have an exact sequence of coherent sheaves

$$0 \to \mathcal{O}_{C_0} \to \mathcal{F} \to Q \to 0$$

where Q has support in a finite set. Then $H^1(C_0, Q)$ vanishes and we have a surjective map

$$H^1(C_0, \mathcal{O}_{C_0}) \to H^1(C_0, \mathcal{F}) \to 0.$$

Using the surjectivity above, we can find a topologically trivial line bundle L on C_0 which is not holomorphically trivial on each non rational component of C_0. Thanks to the result quoted above and to the properness of π we can find an open neighbourhood V_0 of v_0 in V an a line bundle $\mathcal{L}$ on $U_0 := \pi^{-1}(V_0)$ which is topologically trivial on U_0 and induces L on C_0. Now let Z be the subset of the space $\mathcal{C}_1(\mathcal{L})$ of

connected[1] compact $1-$cycles in $\mathcal{L}$ such that their direct image on U_0 is contained in $\varphi(V_0) \subset \mathcal{C}_1(U_0)$. This subset Z is a closed analytic subset in $\mathcal{C}_1(\mathcal{L})$ because $\varphi(V_0)$ is a closed analytic subset in $\mathcal{C}_1(U_0)$ as $\varphi : V_0 \to \mathcal{C}_1(U_0)$ is a proper holomorphic map and as the direct image by the projection $p : \mathcal{L} \to U_0$ is holomorphic.
Remark that for each $v \in V_0$ the set Z contains the cycle $\varphi(v)$ of $\mathcal{L}$ which is the zero section of the restriction of $\mathcal{L}$ to $|\varphi(v)|$ with suitable multiplicities, in order that its direct image on U_0 is equal to $\varphi(v)$. This defines a closed holomorphic embedding of $\varphi(V_0)$ in $Z \subset \mathcal{C}_1(\mathcal{L})$.
We shall show now that the direct image map $f_* : Z \to \varphi(V_0)$ for compact $1-$cycles induced by the projection $\mathcal{C}_1(\mathcal{L}) \to \mathcal{C}_1(U_0)$ has positive dimensional fiber at $\varphi(v) \in Z$ when $\varphi(v)$ is rational.
Assume that for some $v \in V_0$ the $1-$cycle $C := \varphi(v)$ is rational. Then the restriction of the line bundle $\mathcal{L}$ on C is holomorphically trivial and any compact $1-$dimensional cycle in $\mathcal{L}_{|C}$ can be move (by vertical translation). So any point in Z in the fibre of f_* over $C = \varphi(v)$ is not isolated. This proves our assertion.
Now C_0 has at least one irreducible component, say γ, which is not rational; so the corresponding point of $\varphi(v_0) \in Z$ is isolated in its fibre for f_*. Indeed, the zero section is the only reduced compact $1-$dimensional cycle in $L_{|\gamma}$ as $L_{|\gamma}$ is topologically trivial but not holomorphically trivial by construction. So any connected compact $1-$cycle near-by $\varphi(v_0)$ in $Z \cap f_*^{-1}(C_0)$ must have support in the zero section of L on each non rational component of C_0. As $\varphi(v_0)$ is connected this implies that on a rational component of $\varphi(v_0)$ which meets an irrational component, the corresponding component of a near-by cycle to the cycle $\varphi(v_0)$ in $f_*^{-1}(C_0)$ has to vanish at some point (the intersection with some non rational component). Then the corresponding component of such a cycle is the zero section over this rational component (we have only constant sections on rational components). As we assume C_0 connected and as the cycles in Z are connected, we conclude that $\varphi(v_0) \in Z$ is an isolated point in its fibre of f_*.
Now the subset T of points $t \in Z$ such that the dimension at t of the fibre of f_* is at least equal to 1 is a closed analytic set in Z. The intersection of T with the closed embedding of $\varphi(V_0)$ is Z defines a closed analytic subset in $\varphi(V_0)$ and then also of V_0 which contains the subset of rational fibers of π in V_0. As we have shown that v_0 is not in this closed analytic subset, we obtain an open neighourhood V_1 of v_0 in V_0 such that for any $v \in V_1$ the cycle $\varphi(v)$ is not rational. ■

Corollary 0.0.6 *Let $M := U \times \mathbb{P}_1$ be a reduced complex space and let $p : M \to \mathbb{P}_1$ be the projection. Let $p_* : \mathcal{C}_1(M) \to \mathcal{C}_1(\mathbb{P}_1) \simeq \mathbb{N}.[\mathbb{P}_1]$ the direct image for compact $1-$cycles. Define $X := p_*^{-1}(1.[\mathbb{P}_1])$. As p_* is holomorphic, this is a closed analytic subset in $\mathcal{C}_1(M)$. Let S_0 be the subset of $\mathcal{C}_1(M)$ of irreducible cycles which are in X. This is a Zariski open subset in X (see [B-M] prop. IV 7.1.2). Then the closure S_1 of S_0 in X contains only rational cycles.*

[1] see th. IV 7.2.1 for the analyticity of this condition on compact cycles.

PROOF. Let $\pi : U \to V := S_1$ be the projection of the graph of the tautological family of $1-$cycles parametrized by S_1. As the generic cycle in this family is irreducible, by definition of S_0, all fibres of π are connected. Also the generic fibres are rational because for $s \in S_0$ it is reduced and isomorphic to $\mathbb{P}_1$. If there is a non rational cycle in S_1, then there exists, thanks to the previous proposition, a non empty open set of non rational cycles in this family. But S_0 is open and dense in S_1 this gives a contradiction. ■

PROOF OF THE THEOREM. Let V be an irreducible component of $\mathcal{C}_1(M)$. As we may normalize V thanks to the lemma 0.0.3, we can assume that the generic cycle in V is reduced. Let W be a relatively compact open set in V. Then there exists, thanks to the proposition IV 7.1.2 in [B-M 1], an integer $k \geq 1$ such that for any $v \in W$ the cycle v has at most k irreducible components.
Denote $\pi : U \to W$ the projection of the graph of the tautological family of compact curves in M parametrized by W.
Define the subset $S_l \subset \mathcal{C}_1(U \times \mathbb{P}_1)$ as the image by the addition map of cycles of $(S_1)^l$ in $\mathcal{C}_1(U \times \mathbb{P}_1)$. As the addition map is proper and finite, S_l is a closed analytic subset of $\mathcal{C}_1(U \times \mathbb{P}_1)$ for each integer $l \geq 1$.
Let $q : U \times \mathbb{P}_1 \to U$ the projection. We shall prove that the subset of rational cycles in the family $(v)_{v \in W}$ is exactly given by the subset

$$R := \big(\cup_{l=1}^{k} \ q_*(S_l) \big) \cap W$$

which is a closed analytic subset in W because the map q is proper and the subset $S_l \subset \mathcal{C}_1(U \times \mathbb{P}_1)$ is a closed analytic subset.
First remark that each cycle in R is rational as the direct image of a rational cycle is rational and we proved that each cycle in S_1 is rational in corollary 0.0.6.
Conversely, let $v \in W$ such that v is rational. Then there exists an integer $l \in [1, k]$ and l holomorphic generically injective[2] maps (distinct or not) $f_1, \ldots, f_l$ from $\mathbb{P}_1$ to U such that the sum of there images is v and with graphs $G_1, \ldots, G_l$ which are points in S_0. Then $v = \sum_{i=1}^{l} q_*(G_i)$ and so v is in R.
To conclude the proof it is enough to say that a subset which is closed and analytic on any open relatively compact subset in V is a closed analytic subset in V. ■

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[2] the normalization maps.