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# Note on rational 1–dimensional compact cycles.

Daniel Barlet\*

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**ABSTRACT.** We give a proof of the fact that the subset of the rational curves form a closed analytic subset in the space of the 1–dimensional cycles of a complex space.

**KEY WORDS.** Rational curves - Cycles' space.

**AMS CLASSIFICATION 2010.** 32 C 25 - 32 G 10.

The aim of this short note is to prove the following result.

**Theorem 0.0.1** *Let  $M$  be a complex space. Let  $R$  be the subset of the space  $\mathcal{C}_1(M)$  consisting of compact rational 1–dimensional cycles in  $M$ . Then  $R$  is a closed analytic subset in  $\mathcal{C}_1(M)$ .*

By a rational 1–dimensional cycle we mean that each irreducible component of such a cycle is a rational curve (may be singular).

Note that this result is classical in the projective context.

**PROOF.** Consider first an irreducible component  $V$  of  $\mathcal{C}_1(M)$  such that the generic cycle in  $V$  is reduced. Let  $U_0 \subset V \times M$  be the graph of the tautological family parametrized by  $V$  (this is a reduced cycle-graph from our assumption on  $V$ ) given by

$$U_0 := \{(v, x) \in V \times M / x \in |v|\}.$$

Let  $\pi_0 : U_0 \rightarrow V$  the projection. Now fix a relatively compact open set  $W$  in  $V$  and let  $U := \pi_0^{-1}(W)$  and  $\pi : U \rightarrow W$  the restriction of  $\pi_0$  to  $U$ . The inclusion map of  $W$  in  $\mathcal{C}_1(U)$  will be denoted by  $\varphi$ .

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Now, thanks to the proposition IV 7.1.2 in [B-M 1], the number of irreducible components (counting multiplicities) of the cycle  $\varphi(v)$  for  $v \in W$  is bounded by some integer  $k$  depending only on  $W$ : As for each integer  $k$  the subset

$$F_k := \{v \in V / v \text{ has at least } k \text{ irreducible components, counting multiplicities}\}$$

is a closed analytic subset in  $V$ , the fact that  $V = \cup_{k \geq 1} F_k$  implies the existence, for the compact set  $\bar{W}$ , of an integer  $k_W$  such that  $\bar{W} \subset F_{k_W}$ . Fix now  $k := k_W$ .

Consider now in  $U \times \mathbb{P}_1$  the subset  $\Gamma$  of compact 1-dimensional cycles  $\gamma$  which are sum of  $l \in [1, k]$  irreducible curves (distinct or not) each having degree 1 on  $\mathbb{P}_1$  and such that the direct image of  $\gamma$  in  $U$  is a cycle in  $\varphi(W)$ .

CLAIM. The subset  $\Gamma \subset \mathcal{C}_1(U \times \mathbb{P}_1)$  defined by the two conditions above is a closed analytic subset in  $\mathcal{C}_1(U \times \mathbb{P}_1)$ .

PROOF OF THE CLAIM. Let  $p : U \times \mathbb{P}_1 \rightarrow \mathbb{P}_1$  and  $p' : U \times \mathbb{P}_1 \rightarrow U$  be the projections and  $\pi' : U \times \mathbb{P}_1 \rightarrow W$  the composition of the projection  $p'$  with  $\pi$ . Then we have two holomorphic maps given by the direct images of compact 1-cycles in  $U \times \mathbb{P}_1$  (see [B-M 1] IV th. 3.5.4)

$$p_* : \mathcal{C}_1(U \times \mathbb{P}_1) \rightarrow \mathbb{N}[\mathbb{P}_1] \simeq \mathcal{C}_1(\mathbb{P}_1)$$

and

$$p'_* : \mathcal{C}_1(U \times \mathbb{P}_1) \rightarrow \mathcal{C}_1(U).$$

Now, as the map  $\varphi$  is proper (see [B-M 1] th. III 6.2.1 and th. IV 2.7.20) and injective<sup>1</sup>, the subset  $\varphi(W)$  is closed and analytic in  $\mathcal{C}_1(U)$  and so the subset defined by  $T := (p'_*)^{-1}(\varphi(W))$  in  $\mathcal{C}_1(U \times \mathbb{P}_1)$  is also analytic and closed. The subset  $S_1 := p_*^{-1}(1[\mathbb{P}_1])$  is also closed and analytic in  $\mathcal{C}_1(U \times \mathbb{P}_1)$ . For each  $l \in [1, k]$  the image  $S_l$  of the addition map  $S_1^l \rightarrow \mathcal{C}_1(U \times \mathbb{P}_1)$  is also analytic and closed because the addition map of cycles is holomorphic and proper (and finite). Then the union of  $S_l$  for  $l \in [1, k]$  is a closed analytic subset. Then the intersection  $\Gamma := T \cap (\cup_{l=1}^k S_l)$  is analytic and closed in  $\mathcal{C}_1(U \times \mathbb{P}_1)$ ; so the claim is proved.

Now let  $R(W) := \{v \in W / \varphi(v) \text{ is rational}\}$ ; recall that, by definition, a 1-cycle  $\gamma$  is rational if and only if each irreducible component of  $\gamma$  is rational.

We want to show that  $\varphi^{-1}(p'_*(\Gamma))$  coincides with  $R(W)$ .

First remark that the subset  $\varphi^{-1}(p'_*(\Gamma))$  is analytic and closed because on the closed analytic subset  $\Gamma$  the direct image  $(p')_*$  is proper on  $\varphi(W)$  :

As  $\varphi$  is proper, for any compact set  $L$  in  $\varphi(W)$  there exists a compact set  $K$  in  $W$  such that  $L \subset \varphi(K)$ . Fix a compact set  $K$  in  $W$ . Then the volume (for any continuous hermitian metric on  $U$ ) of the cycles  $\varphi(v)$ ,  $v \in K$ , is uniformly bounded (see [B.78] or [B-M 1] th.III 6.2.1). So the cycles in  $U \times \mathbb{P}_1$  with bounded degree

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<sup>1</sup>In fact a proper embedding ; but we dont need this point.

on  $\mathbb{P}_1$  with direct image equal to some  $\varphi(v), v \in K$ , has also a bounded volume. And such a cycle is geometrically contained in  $\pi^{-1}(K) \times \mathbb{P}_1$  which is compact, so they are in a compact set in  $\mathcal{C}_1(U \times \mathbb{P}_1)$  and the properness over  $\varphi(W)$  of  $p'_*$  is proved.

Take  $\gamma \in R(W)$ ; it has at most  $k$  irreducible components. Each irreducible component of  $\gamma$  is the direct image by  $p'$  of the graph of a holomorphic map  $\mathbb{P}_1 \rightarrow U$ . Then the sum of these graphs is a cycle in  $\Gamma$ .

Conversely, each element in  $\Gamma$  is the direct image by  $p'$  of the sum of  $l \in [1, k]$  graphs of holomorphic maps  $\mathbb{P}_1 \rightarrow U$  : an irreducible 1-cycle in  $U \times \mathbb{P}_1$  which has degree 1 on  $\mathbb{P}_1$  is necessarily such a graph ! So we have proved that  $R(W)$  is a closed analytic subset in  $W$ .

As  $V$  is locally compact, the subset  $R(V)$  of points in  $V$  such that the corresponding cycle is rational is a closed analytic subset in  $V$ .

To obtain the general case (so the case of an irreducible component of  $\mathcal{C}_1(M)$  has a generic cycle which is not reduced) it is enough to use the following easy lemma.

**Lemma 0.0.2** *Let  $S$  be a normal complex space and  $f : S \rightarrow \mathcal{C}_n(M)$  be a holomorphic map from  $S$  to the space of compact  $n$ -cycles of a complex space  $M$ . There exist integers  $m_1, \dots, m_h$  and holomorphic maps  $|f_j| : S \rightarrow \mathcal{C}_n(M)$  such that  $f = \sum_{j=1}^h m_j \cdot |f_j|$  and such that for each  $j \in [1, h]$  the generic cycle in the analytic family defined by the map  $|f_j|$  is reduced.*

This lemma is an easy consequence of the theorem IV 3.4.1 of [B-M 1].

To conclude the proof it is enough to remark that in the situation of the lemma above with  $n = 1$ , the cycle  $f(s)$  is rational if and only if each cycle  $|f_j|(s)$  is rational, to apply the result already obtain to the analytic families of 1-cycles in  $M$  with generic reduced member parametrized by the normalization  $\tilde{V}$  of the irreducible component  $V$ , and to remark that  $R(V)$  is equal to  $\nu_*(R(\tilde{V}))$  where  $\nu : \tilde{V} \rightarrow V$  is the normalization map. ■

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