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# A DDVV INEQUALITY FOR SUBMANIFOLDS OF WARPED PRODUCTS

JULIEN ROTH

**ABSTRACT.** We prove a DDVV inequality for submanifolds of warped products of the form  $I \times_a \mathbb{M}^n(c)$  where  $I$  is an interval and  $\mathbb{M}^n(c)$  a real space form of curvature  $c$ . As an application, we give a rigidity result for submanifolds of  $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$ .

**RÉSUMÉ.** Une inégalité de type DDVV pour les sous-variétés des produits tordus. Nous donnons une inégalité de type DDVV pour les sous-variétés des produits tordus de la forme  $I \times_a \mathbb{M}^n(c)$  où  $I$  est un interval et  $\mathbb{M}^n(c)$  un espace modèle réel de courbure constante  $c$ . Nous en déduisons un résultat de rigidité pour les sous-variétés de  $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$ .

## Version française abrégée.

Soit  $(M^m, g)$  une variété riemannienne immergée isométriquement dans une variété riemannienne ambiante  $(N^{m+p}, \bar{g})$ , de dimension  $m + p$ . Lorsque  $N$  est un espace modèle simplement connexe à courbure sectionnelle constante  $c$ , l'inégalité suivante est vérifiée :

$$(1) \quad \|H\|^2 \geq \rho + \rho^\perp - c,$$

où  $\rho = \frac{2}{n(n-1)} \sum_{i < j} \langle R(e_i, e_j)e_j, e_i \rangle$  est la courbure scalaire normalisée de  $(M, g)$  et

$\rho^\perp = \frac{2}{n(n-1)} \sum_{i < j} \sum_{\alpha < \beta} \langle R^\perp(e_i, e_j)\xi_\alpha, \xi_\beta \rangle$  la courbure scalaire normale (également nor-

malisée),  $\{e_1, \dots, e_n\}$  et  $\{\xi_1, \dots, \xi_p\}$  étant respectivement des bases orthonormées locales de  $TM$  et  $T^\perp M$ . Cette inégalité est connue sous le nom de conjecture de DDVV car conjecturée par De Smets-Dillen-Verstraelen-Vrancken [2]. La conjecture a été démontrée récemment par Lu [6] et par Ge-Tang [4] indépendamment. Plus récemment, Chen et Cui [1] ont obtenu une inégalité comparable dans le cas des espaces produits  $\mathbb{S}^n \times \mathbb{R}$  et  $\mathbb{H}^n \times \mathbb{R}$ . Le but de cette note est d'étendre le résultat de Chen-Cui pour les sous-variétés des produits tordus  $I \times_a \mathbb{M}^n(c)$ , c'est-à-dire  $I \times \mathbb{M}^n(c)$  muni de la métrique  $\tilde{g} = dt^2 + a(t)^2 g_{\mathbb{M}^n(c)}$ ,  $I$  étant un intervalle de  $\mathbb{R}$  et  $a : I \rightarrow \mathbb{R}$  une fonction lisse ne s'annulant pas. Nous démontrons le résultat suivant.

**Théorème 1.** Soient  $n > m \geq 2$  deux entiers. Soit  $M^m$  une sous-variété du produit tordu  $I \times_a \mathbb{M}^n(c)$  avec courbure scalaire et courbure scalaire normale normalisées  $\rho$  et  $\rho^\perp$  et courbure moyenne  $H$ . L'inégalité suivante est vérifiée :

$$\|H\|^2 \geq \rho + \rho^\perp + \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left( 1 - \frac{2}{n} \|T\|^2 \right) - \frac{2a''}{na} \|T\|^2.$$

Nous en déduisons un résultat pour les surfaces des produits tordus  $\mathbb{R} \times_{e^{4\lambda t}} \mathbb{H}^n(c)$ .

**Corollaire 1.** Soit  $M^m$  une sous-variété complète sans bord du produit tordu  $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$  avec courbure scalaire et courbure scalaire normale normalisées  $\rho$  et  $\rho^\perp$  et courbure

moyenne  $H$ . Si  $\|H\|^2 \leq \rho + \lambda^2$ , alors

$$\|H\|^2 = \rho + \lambda^2, \quad \rho^\perp = 0, \quad n = 2 \quad \text{et} \quad \|T\| = 1.$$

En particulier,  $M$  est une surface du type  $\mathbb{R} \times_{e^{\lambda t}} \gamma$ , où  $\gamma$  est une courbe dans  $\mathbb{H}^n(c)$ .

## 1. INTRODUCTION

Let  $(M^n, g)$  be a  $n$ -dimensional Riemannian manifold isometrically immersed into a  $(n+p)$ -dimensional Riemannian manifold  $(N^{n+p}, \bar{g})$ . When the ambient space is a real space form of constant sectional curvature  $c$ , we have the following pointwise inequality

$$(2) \quad \|H\|^2 \geq \rho + \rho^\perp - c,$$

where  $\rho = \frac{2}{n(n-1)} \sum_{i < j} \langle R(e_i, e_j)e_j, e_i \rangle$  is the normalized scalar curvature of  $(M, g)$

and  $\rho^\perp = \frac{2}{n(n-1)} \sum_{i < j} \sum_{\alpha < \beta} \langle R^\perp(e_i, e_j)\xi_\alpha, \xi_\beta \rangle$  is the normalized normal curvature of the

immersion, where  $\{e_1, \dots, e_n\}$  and  $\{\xi_1, \dots, \xi_p\}$  are respectively orthonormal frames of  $TM$  and  $T^\perp M$ . This inequality, known as DDVV conjecture, was conjectured by De Smets-Dillen-Verstrealen-Vrancken in [2] and proved recently by Lu [6] and by Ge-Tang [4] independently. More recently, Chen and Cui generalized this inequality in the setting of product spaces  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$ .

In this note, we extend Chen-Cui result by proving a DDVV inequality for submanifolds of warped products  $I \times_a \mathbb{M}^n(c)$  where  $I \subset \mathbb{R}$  is an interval and  $a : I \rightarrow \mathbb{R}$  is a nowhere vanishing smooth function. We denote by  $\partial_t = \frac{\partial}{\partial t}$  the unit vector field tangent to the factor  $I$ . We prove the following result.

**Theorem 1.** *Let  $M^m$  be a submanifold of the warped product  $I \times_a \mathbb{M}^n(c)$  with normalized scalar and normal scalar curvatures  $\rho$  and  $\rho^\perp$  and mean curvature  $H$ . Then, we have*

$$\|H\|^2 \geq \rho + \rho^\perp + \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left( 1 - \frac{2}{n} \|T\|^2 \right) - \frac{2a''}{na} \|T\|^2,$$

where  $T$  is the part of  $\partial_t$  tangent to  $M$ .

**Remark 1.** *Note that, of course, we recover the DDVV inequality of [1] for product spaces  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$  as well as for  $\mathbb{R}^{n+1}$  by taking  $a = 1$ , but we also recover the inequality for space forms. Indeed,  $\mathbb{S}^n$  and  $\mathbb{H}^n$  can be expressed in term of warped products. Namely, we have*

- (1)  $\mathbb{S}^n = [0, 2\pi] \times_a \mathbb{S}^{n-1}$  with  $a(t) = \sin(t)$ . Hence the inequality of Theorem 1 becomes  $\|H\|^2 \geq \rho + \rho^\perp - 1$ .
- (2)  $\mathbb{H}^n = [0, +\infty[ \times_a \mathbb{S}^{n-1}$  with  $a(t) = \sinh(t)$  or  $\mathbb{H}^n = \mathbb{R} \times_a \mathbb{R}^{n-1}$  with  $a(t) = e^{-t}$ . For both cases, the inequality of Theorem 1 becomes  $\|H\|^2 \geq \rho + \rho^\perp + 1$ .

## 2. PRELIMINARIES

Let  $\mathbb{M}^n(c)$  be the simply connected real space form of dimension  $n$  and constant curvature  $c$ . Let  $I \subset \mathbb{R}$  an interval and  $a : I \rightarrow \mathbb{R}$  be a nowhere vanishing smooth function. We consider the warped  $\tilde{P}^{n+1} = I \times_a \mathbb{M}^n(c)$ , that the product  $I \times \mathbb{M}^n(c)$  endowed the metric  $\tilde{g} = dt^2 + a(t)^2 g_{\mathbb{M}^n(c)}$ . We denote by  $\partial_t = \frac{\partial}{\partial t}$  the unit vector field tangent to the factor  $I$ . We recall (see [5] for instance) that the curvature tensor of  $(\tilde{P}^{n+1}, \tilde{g})$  is given by

$$(3) \quad \begin{aligned} \tilde{R}(X, Y)Z &= \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) (\langle X, Z \rangle Y - \langle Y, Z \rangle X) \\ &+ \left( \frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) (\langle X, Z \rangle \langle Y, \partial_t \rangle \partial_t - \langle Y, Z \rangle \langle X, \partial_t \rangle \partial_t \\ &- \langle Y, \partial_t \rangle \langle Z, \partial_t \rangle X + \langle X, \partial_t \rangle \langle Z, \partial_t \rangle Y). \end{aligned}$$

Let  $(M^m, g)$  be a Riemannian manifold isometrically immersed into  $\tilde{P}$ . We denote by  $B$  its second fundamental form and  $A$  the shape operator defined for any  $X, Y \in \Gamma(TM)$  and  $\xi \in \Gamma(T^\perp M)$  by  $\langle B(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$ . Moreover,  $\partial_t$  can be written

$$\partial_t = T + \sum_{\alpha=1}^p f_\alpha \xi_\alpha,$$

where  $T$  is a vector field tangent to  $M$ ,  $\{\xi_1, \dots, \xi_p\}$  is a local orthonormal frame of  $T^\perp M$  and  $f_1, \dots, f_p$  are smooth functions over  $M$ . We will simply denote  $A_{\xi_\alpha}$  by  $A_\alpha$ .

From the expression of the curvature tensor of  $\tilde{P}$ , we get immediately the Gauss, Codazzi and Ricci equations for a submanifold of  $\tilde{P}$ . Namely, if we denote by  $R$  and  $R^\perp$  the curvature tensor of  $(M, g)$  and the normal curvature respectively, we have the following

**Proposition 2.1.** *The Gauss, Codazzi and Ricci equations of the immersion of  $M$  into  $\tilde{P}$  are respectively*

$$\begin{aligned} \langle R(X, Y)Z, W \rangle &= \langle B(Y, Z), B(X, W) \rangle - \langle B(Y, W), B(X, Z) \rangle \\ &+ \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) (\langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle) \\ &+ \left( \frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) (\langle X, Z \rangle \langle Y, T \rangle \langle W, T \rangle - \langle Y, Z \rangle \langle X, T \rangle \langle W, T \rangle \\ &- \langle Y, T \rangle \langle Z, T \rangle \langle X, W \rangle + \langle X, T \rangle \langle Z, T \rangle \langle Y, W \rangle), \end{aligned}$$

$$\langle (\tilde{\nabla}_X B)(Y, Z), \xi_\alpha \rangle = \left( \frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) f_\alpha (\langle Y, T \rangle \langle X, Z \rangle - \langle X, T \rangle \langle Y, Z \rangle),$$

$$\langle R^\perp(X, Y)\nu, \xi \rangle = \langle [A_\nu, A_\xi]X, Y \rangle.$$

The proof is straightforward from the expression of  $\tilde{R}$ .

Finally, we recall that the DDVV conjecture was reduced to the following algebraic result (see [3]) proved by Lu.

**Theorem ([6]).** *Let  $n, p \geq 2$  be two integers and  $M_1, M_2, \dots, M_p$  be some  $n \times n$  real symmetric and trace-free matrices. Then, we have*

$$\sum_{\alpha, \beta=1}^p \|[M_\alpha, M_\beta]\|^2 \leq \left( \sum_{\alpha=1}^p \|M_\alpha\|^2 \right)^2.$$

Now, we are able to prove Theorem 1.

## 3. PROOF OF THEOREM 1

First, from the definition of  $\rho$  and using the Gauss equation, we have

$$\begin{aligned}
\rho &= \frac{2}{n(n-1)} \sum_{i < j} \langle R(e_i, e_j)e_j, e_i \rangle \\
&= \frac{1}{n(n-1)} \sum_{i \neq j} \langle R(e_i, e_j)e_j, e_i \rangle \\
&= \frac{1}{n(n-1)} \sum_{i \neq j} \left( \langle B(e_j, e_j), B(e_i, e_i) \rangle - \|B(e_i, e_j)\|^2 + \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \right. \\
&\quad \left. + \left( \frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) (\langle T, e_i \rangle^2 + \langle T, e_j \rangle^2) \right) \\
&= \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) + \frac{1}{n(n-1)} \left( n^2 \|H\|^2 - \|B\|^2 + 2(n-1) \left( \frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) \|T\|^2 \right)
\end{aligned}$$

Now, we set  $\tau = B - Hg$  the traceless part of the second fundamental form. Clearly, we have  $\|\tau\|^2 = \|B\|^2 - n\|H\|^2$ . Hence we get

$$(4) \quad \rho = \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left( 1 - \frac{2}{n} \|T\|^2 \right) - \frac{2a''}{na} \|T\|^2 + \|H\|^2 - \frac{1}{n(n-1)} \|\tau\|^2.$$

Moreover, for any  $\alpha \in \{1, \dots, p\}$ , we set  $S_\alpha : TM \rightarrow TM$  the operator defined by  $\langle S_\alpha X, Y \rangle = \langle \tau(X, Y), \xi_\alpha \rangle$ . Obviously, we have  $S_\alpha = A_\alpha - \langle H, \xi_\alpha \rangle \text{Id}$  and  $[A_\alpha, A_\beta] = [S_\alpha, S_\beta]$ . From the Ricci Equation, given in Proposition 2.1, we have

$$\rho^\perp = \frac{1}{n(n-1)} \sqrt{\sum_{\alpha, \beta=1}^p \|[A_\alpha, A_\beta]\|^2} = \frac{1}{n(n-1)} \sqrt{\sum_{\alpha, \beta=1}^p \|[S_\alpha, S_\beta]\|^2}.$$

Since, the operators  $S_\alpha$  are symmetric and trace-free, we can apply the theorem of Lu at any point of  $M$  to get

$$\sum_{\alpha, \beta=1}^p \|[S_\alpha, S_\beta]\|^2 \leq \left( \sum_{\alpha=1}^p \|S_\alpha\|^2 \right)^2.$$

Thus,

$$\rho^\perp \leq \frac{1}{n(n-1)} \sum_{\alpha=1}^n \|S_\alpha\|^2 = \frac{1}{n(n-1)} \|\tau\|^2.$$

Reporting this in (4), we obtain

$$\|H\|^2 \geq \rho + \rho^\perp + \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left( 1 - \frac{2}{n} \|T\|^2 \right) - \frac{2a''}{na} \|T\|^2,$$

which concludes the proof.  $\square$

4. AN APPLICATION TO SUBMANIFOLDS OF  $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$ 

We finish this note by the following application of Theorem 1 to submanifolds of warped product of the type  $\mathbb{R} \times_a \mathbb{H}^n(c)$  where  $a$  is the real function defined by  $a(t) = e^{\lambda t}$  with  $\lambda$  a real constant.

**Corollary 1.** *Let  $M^m$  be a submanifold of the warped product  $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$  with normalized scalar and normal scalar curvatures  $\rho$  and  $\rho^\perp$  and mean curvature  $H$ . Then, we have*

$$\|H\|^2 \geq \rho + \rho^\perp + \lambda^2 - ce^{-2\lambda t} \left( 1 - \frac{2}{n} \|T\|^2 \right).$$

**Proof:** This comes directly from Theorem 1, with the fact that  $\frac{(a')^2}{a^2} - \frac{c}{a^2} = \lambda^2 - ce^{-2\lambda t}$  and  $\frac{a''}{a^2} = \lambda^2$ . Hence the term  $\left(\frac{(a')^2}{a^2} - \frac{c}{a^2}\right) \left(1 - \frac{2}{n}\|T\|^2\right) - \frac{2a''}{na}\|T\|^2$  becomes  $\lambda^2 - ce^{-2\lambda t} \left(1 - \frac{2}{n}\|T\|^2\right)$ .  $\square$

Comparing  $\|H\|^2$  with  $\rho$  is a natural question which leads to rigidity results. Indeed, by the Gauss formula, we know that, for hypersurfaces of space forms,  $\rho$  is up to a constant (which is the sectional curvature  $k$  of the ambient space form) the second mean curvature  $H_2$ , that is the second elementary symmetric polynomial in the principal curvatures. Moreover, it is a classical fact that  $H^2 \geq H_2$  with equality at umbilical points. Hence, assuming  $H^2 \leq \rho - k$  implies that  $M$  is a hypersphere. In this spirit, and using the above DDVV inequality, we give the following rigidity result.

**Corollary 2.** *Let  $M^m$  be a complete submanifold without boundary of the warped product  $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$  with normalized scalar and normal scalar curvatures  $\rho$  and  $\rho^\perp$  and mean curvature  $H$ . If  $\|H\|^2 \leq \rho + \lambda^2$ , then*

$$\|H\|^2 = \rho + \lambda^2, \quad \rho^\perp = 0, \quad n = 2 \quad \text{and} \quad \|T\| = 1.$$

Hence,  $M$  is a surface of the type  $\mathbb{R} \times_{e^{\lambda t}} \gamma$ , where  $\gamma$  is a curve in  $\mathbb{H}^n(c)$ .

**Proof:** First note that since  $n \geq 2$  and  $\|T\|^2 \leq 1$  and  $c < 0$ , we have  $ce^{-2\lambda t} \left(1 - \frac{2}{n}\|T\|^2\right) \leq 0$ . Note also that, by definition,  $\rho^\perp \geq 0$ . Hence, from Corollary 1,  $\|H\|^2 \leq \rho + \lambda^2$  is possible if and only if  $\|H\|^2 = \rho + \lambda^2$ ,  $\rho^\perp = 0$ ,  $n = 2$  and  $\|T\| = 1$ . Since  $n = 2$ , then  $M$  is a surface and the fact that  $\|T\| = 1$  implies that  $T = \partial_t$  and so  $M$  is of the type  $I \times_{e^{\lambda t}} \gamma$ , where  $\gamma$  is a curve in  $\mathbb{H}^n(c)$ . Since we assume that  $M$  is complete and without boundary,  $I = \mathbb{R}$ . This concludes the proof.  $\square$

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