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A DDVV INEQUALITY FOR SUBMANIFOLDS OF WARPED PRODUCTS

JULIEN ROTH

ABSTRACT. We prove a DDVV inequality for submanifolds of warped products of the form $I \times_a \mathbb{M}^n(c)$ where I is an interval and $\mathbb{M}^n(c)$ a real space form of curvature c . As an application, we give a rigidity result for submanifolds of $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$.

RÉSUMÉ. Une inégalité de type DDVV pour les sous-variétés des produits tordus. Nous donnons une inégalité de type DDVV pour les sous-variétés des produits tordus de la forme $I \times_a \mathbb{M}^n(c)$ où I est un interval et $\mathbb{M}^n(c)$ un espace modèle réel de courbure constante c . Nous en déduisons un résultat de rigidité pour les sous-variétés de $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$.

Version française abrégée.

Soit (M^m, g) une variété riemannienne immergée isométriquement dans une variété riemannienne ambiante (N^{m+p}, \bar{g}) , de dimension $m + p$. Lorsque N est un espace modèle simplement connexe à courbure sectionnelle constante c , l'inégalité suivante est vérifiée :

$$(1) \quad \|H\|^2 \geq \rho + \rho^\perp - c,$$

où $\rho = \frac{2}{n(n-1)} \sum_{i < j} \langle R(e_i, e_j)e_j, e_i \rangle$ est la courbure scalaire normalisée de (M, g) et

$\rho^\perp = \frac{2}{n(n-1)} \sum_{i < j} \sum_{\alpha < \beta} \langle R^\perp(e_i, e_j)\xi_\alpha, \xi_\beta \rangle$ la courbure scalaire normale (également nor-

malisée), $\{e_1, \dots, e_n\}$ et $\{\xi_1, \dots, \xi_p\}$ étant respectivement des bases orthonormées locales de TM et $T^\perp M$. Cette inégalité est connue sous le nom de conjecture de DDVV car conjecturée par De Smets-Dillen-Verstraelen-Vrancken [2]. La conjecture a été démontrée récemment par Lu [6] et par Ge-Tang [4] indépendamment. Plus récemment, Chen et Cui [1] ont obtenu une inégalité comparable dans le cas des espaces produits $\mathbb{S}^n \times \mathbb{R}$ et $\mathbb{H}^n \times \mathbb{R}$. Le but de cette note est d'étendre le résultat de Chen-Cui pour les sous-variétés des produits tordus $I \times_a \mathbb{M}^n(c)$, c'est-à-dire $I \times \mathbb{M}^n(c)$ muni de la métrique $\tilde{g} = dt^2 + a(t)^2 g_{\mathbb{M}^n(c)}$, I étant un intervalle de \mathbb{R} et $a : I \rightarrow \mathbb{R}$ une fonction lisse ne s'annulant pas. Nous démontrons le résultat suivant.

Théorème 1. Soient $n > m \geq 2$ deux entiers. Soit M^m une sous-variété du produit tordu $I \times_a \mathbb{M}^n(c)$ avec courbure scalaire et courbure scalaire normale normalisées ρ et ρ^\perp et courbure moyenne H . L'inégalité suivante est vérifiée :

$$\|H\|^2 \geq \rho + \rho^\perp + \left(\frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left(1 - \frac{2}{n} \|T\|^2 \right) - \frac{2a''}{na} \|T\|^2.$$

Nous en déduisons un résultat pour les surfaces des produits tordus $\mathbb{R} \times_{e^{4\lambda t}} \mathbb{H}^n(c)$.

Corollaire 1. Soit M^m une sous-variété complète sans bord du produit tordu $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$ avec courbure scalaire et courbure scalaire normale normalisées ρ et ρ^\perp et courbure

moyenne H . Si $\|H\|^2 \leq \rho + \lambda^2$, alors

$$\|H\|^2 = \rho + \lambda^2, \quad \rho^\perp = 0, \quad n = 2 \quad \text{et} \quad \|T\| = 1.$$

En particulier, M est une surface du type $\mathbb{R} \times_{e^{\lambda t}} \gamma$, où γ est une courbe dans $\mathbb{H}^n(c)$.

1. INTRODUCTION

Let (M^n, g) be a n -dimensional Riemannian manifold isometrically immersed into a $(n+p)$ -dimensional Riemannian manifold (N^{n+p}, \bar{g}) . When the ambient space is a real space form of constant sectional curvature c , we have the following pointwise inequality

$$(2) \quad \|H\|^2 \geq \rho + \rho^\perp - c,$$

where $\rho = \frac{2}{n(n-1)} \sum_{i < j} \langle R(e_i, e_j)e_j, e_i \rangle$ is the normalized scalar curvature of (M, g)

and $\rho^\perp = \frac{2}{n(n-1)} \sum_{i < j} \sum_{\alpha < \beta} \langle R^\perp(e_i, e_j)\xi_\alpha, \xi_\beta \rangle$ is the normalized normal curvature of the

immersion, where $\{e_1, \dots, e_n\}$ and $\{\xi_1, \dots, \xi_p\}$ are respectively orthonormal frames of TM and $T^\perp M$. This inequality, known as DDVV conjecture, was conjectured by De Smets-Dillen-Verstreelen-Vrancken in [2] and proved recently by Lu [6] and by Ge-Tang [4] independently. More recently, Chen and Cui generalized this inequality in the setting of product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.

In this note, we extend Chen-Cui result by proving a DDVV inequality for submanifolds of warped products $I \times_a \mathbb{M}^n(c)$ where $I \subset \mathbb{R}$ is an interval and $a : I \rightarrow \mathbb{R}$ is a nowhere vanishing smooth function. We denote by $\partial_t = \frac{\partial}{\partial t}$ the unit vector field tangent to the factor I . We prove the following result.

Theorem 1. *Let M^m be a submanifold of the warped product $I \times_a \mathbb{M}^n(c)$ with normalized scalar and normal scalar curvatures ρ and ρ^\perp and mean curvature H . Then, we have*

$$\|H\|^2 \geq \rho + \rho^\perp + \left(\frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left(1 - \frac{2}{n} \|T\|^2 \right) - \frac{2a''}{na} \|T\|^2,$$

where T is the part of ∂_t tangent to M .

Remark 1. *Note that, of course, we recover the DDVV inequality of [1] for product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ as well as for \mathbb{R}^{n+1} by taking $a = 1$, but we also recover the inequality for space forms. Indeed, \mathbb{S}^n and \mathbb{H}^n can be expressed in term of warped products. Namely, we have*

- (1) $\mathbb{S}^n = [0, 2\pi] \times_a \mathbb{S}^{n-1}$ with $a(t) = \sin(t)$. Hence the inequality of Theorem 1 becomes $\|H\|^2 \geq \rho + \rho^\perp - 1$.
- (2) $\mathbb{H}^n = [0, +\infty[\times_a \mathbb{S}^{n-1}$ with $a(t) = \sinh(t)$ or $\mathbb{H}^n = \mathbb{R} \times_a \mathbb{R}^{n-1}$ with $a(t) = e^{-t}$. For both cases, the inequality of Theorem 1 becomes $\|H\|^2 \geq \rho + \rho^\perp + 1$.

2. PRELIMINARIES

Let $\mathbb{M}^n(c)$ be the simply connected real space form of dimension n and constant curvature c . Let $I \subset \mathbb{R}$ an interval and $a : I \rightarrow \mathbb{R}$ be a nowhere vanishing smooth function. We consider the warped $\tilde{P}^{n+1} = I \times_a \mathbb{M}^n(c)$, that the product $I \times \mathbb{M}^n(c)$ endowed the metric $\tilde{g} = dt^2 + a(t)^2 g_{\mathbb{M}^n(c)}$. We denote by $\partial_t = \frac{\partial}{\partial t}$ the unit vector field tangent to the factor I . We recall (see [5] for instance) that the curvature tensor of $(\tilde{P}^{n+1}, \tilde{g})$ is given by

$$(3) \quad \begin{aligned} \tilde{R}(X, Y)Z &= \left(\frac{(a')^2}{a^2} - \frac{c}{a^2} \right) (\langle X, Z \rangle Y - \langle Y, Z \rangle X) \\ &+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) (\langle X, Z \rangle \langle Y, \partial_t \rangle \partial_t - \langle Y, Z \rangle \langle X, \partial_t \rangle \partial_t \\ &- \langle Y, \partial_t \rangle \langle Z, \partial_t \rangle X + \langle X, \partial_t \rangle \langle Z, \partial_t \rangle Y). \end{aligned}$$

Let (M^m, g) be a Riemannian manifold isometrically immersed into \tilde{P} . We denote by B its second fundamental form and A the shape operator defined for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$ by $\langle B(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$. Moreover, ∂_t can be written

$$\partial_t = T + \sum_{\alpha=1}^p f_\alpha \xi_\alpha,$$

where T is a vector field tangent to M , $\{\xi_1, \dots, \xi_p\}$ is a local orthonormal frame of $T^\perp M$ and f_1, \dots, f_p are smooth functions over M . We will simply denote A_{ξ_α} by A_α .

From the expression of the curvature tensor of \tilde{P} , we get immediately the Gauss, Codazzi and Ricci equations for a submanifold of \tilde{P} . Namely, if we denote by R and R^\perp the curvature tensor of (M, g) and the normal curvature respectively, we have the following

Proposition 2.1. *The Gauss, Codazzi and Ricci equations of the immersion of M into \tilde{P} are respectively*

$$\begin{aligned} \langle R(X, Y)Z, W \rangle &= \langle B(Y, Z), B(X, W) \rangle - \langle B(Y, W), B(X, Z) \rangle \\ &+ \left(\frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left(\langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle \right) \\ &+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) \left(\langle X, Z \rangle \langle Y, T \rangle \langle W, T \rangle - \langle Y, Z \rangle \langle X, T \rangle \langle W, T \rangle \right. \\ &\left. - \langle Y, T \rangle \langle Z, T \rangle \langle X, W \rangle + \langle X, T \rangle \langle Z, T \rangle \langle Y, W \rangle \right), \end{aligned}$$

$$\langle (\tilde{\nabla}_X B)(Y, Z), \xi_\alpha \rangle = \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) f_\alpha \left(\langle Y, T \rangle \langle X, Z \rangle - \langle X, T \rangle \langle Y, Z \rangle \right),$$

$$\langle R^\perp(X, Y)\nu, \xi \rangle = \langle [A_\nu, A_\xi]X, Y \rangle.$$

The proof is straightforward from the expression of \tilde{R} .

Finally, we recall that the DDVV conjecture was reduced to the following algebraic result (see [3]) proved by Lu.

Theorem ([6]). *Let $n, p \geq 2$ be two integers and M_1, M_2, \dots, M_p be some $n \times n$ real symmetric and trace-free matrices. Then, we have*

$$\sum_{\alpha, \beta=1}^p \|[M_\alpha, M_\beta]\|^2 \leq \left(\sum_{\alpha=1}^p \|M_\alpha\|^2 \right)^2.$$

Now, we are able to prove Theorem 1.

3. PROOF OF THEOREM 1

First, from the definition of ρ and using the Gauss equation, we have

$$\begin{aligned}
\rho &= \frac{2}{n(n-1)} \sum_{i < j} \langle R(e_i, e_j) e_j, e_i \rangle \\
&= \frac{1}{n(n-1)} \sum_{i \neq j} \langle R(e_i, e_j) e_j, e_i \rangle \\
&= \frac{1}{n(n-1)} \sum_{i \neq j} \left(\langle B(e_j, e_j), B(e_i, e_i) \rangle - \|B(e_i, e_j)\|^2 + \left(\frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \right. \\
&\quad \left. + \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) (\langle T, e_i \rangle^2 + \langle T, e_j \rangle^2) \right) \\
&= \left(\frac{(a')^2}{a^2} - \frac{c}{a^2} \right) + \frac{1}{n(n-1)} \left(n^2 \|H\|^2 - \|B\|^2 + 2(n-1) \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) \|T\|^2 \right)
\end{aligned}$$

Now, we set $\tau = B - Hg$ the traceless part of the second fundamental form. Clearly, we have $\|\tau\|^2 = \|B\|^2 - n\|H\|^2$. Hence we get

$$(4) \quad \rho = \left(\frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left(1 - \frac{2}{n} \|T\|^2 \right) - \frac{2a''}{na} \|T\|^2 + \|H\|^2 - \frac{1}{n(n-1)} \|\tau\|^2.$$

Moreover, for any $\alpha \in \{1, \dots, p\}$, we set $S_\alpha : TM \rightarrow TM$ the operator defined by $\langle S_\alpha X, Y \rangle = \langle \tau(X, Y), \xi_\alpha \rangle$. Obviously, we have $S_\alpha = A_\alpha - \langle H, \xi_\alpha \rangle \text{Id}$ and $[A_\alpha, A_\beta] = [S_\alpha, S_\beta]$. From the Ricci Equation, given in Proposition 2.1, we have

$$\rho^\perp = \frac{1}{n(n-1)} \sqrt{\sum_{\alpha, \beta=1}^p \|[A_\alpha, A_\beta]\|^2} = \frac{1}{n(n-1)} \sqrt{\sum_{\alpha, \beta=1}^p \|[S_\alpha, S_\beta]\|^2}.$$

Since, the operators S_α are symmetric and trace-free, we can apply the theorem of Lu at any point of M to get

$$\sum_{\alpha, \beta=1}^p \|[S_\alpha, S_\beta]\|^2 \leq \left(\sum_{\alpha=1}^p \|S_\alpha\|^2 \right)^2.$$

Thus,

$$\rho^\perp \leq \frac{1}{n(n-1)} \sum_{\alpha=1}^n \|S_\alpha\|^2 = \frac{1}{n(n-1)} \|\tau\|^2.$$

Reporting this in (4), we obtain

$$\|H\|^2 \geq \rho + \rho^\perp + \left(\frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left(1 - \frac{2}{n} \|T\|^2 \right) - \frac{2a''}{na} \|T\|^2,$$

which concludes the proof. \square

4. AN APPLICATION TO SUBMANIFOLDS OF $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$

We finish this note by the following application of Theorem 1 to submanifolds of warped product of the type $\mathbb{R} \times_a \mathbb{H}^n(c)$ where a is the real function defined by $a(t) = e^{\lambda t}$ with λ a real constant.

Corollary 1. *Let M^m be a submanifold of the warped product $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$ with normalized scalar and normal scalar curvatures ρ and ρ^\perp and mean curvature H . Then, we have*

$$\|H\|^2 \geq \rho + \rho^\perp + \lambda^2 - ce^{-2\lambda t} \left(1 - \frac{2}{n} \|T\|^2 \right).$$

Proof: This comes directly from Theorem 1, with the fact that $\frac{(a')^2}{a^2} - \frac{c}{a^2} = \lambda^2 - ce^{-2\lambda t}$ and $\frac{a''}{a^2} = \lambda^2$. Hence the term $\left(\frac{(a')^2}{a^2} - \frac{c}{a^2}\right) \left(1 - \frac{2}{n}\|T\|^2\right) - \frac{2a''}{na}\|T\|^2$ becomes $\lambda^2 - ce^{-2\lambda t} \left(1 - \frac{2}{n}\|T\|^2\right)$. \square

Comparing $\|H\|^2$ with ρ is a natural question which leads to rigidity results. Indeed, by the Gauss formula, we know that, for hypersurfaces of space forms, ρ is up to a constant (which is the sectional curvature k of the ambient space form) the second mean curvature H_2 , that is the second elementary symmetric polynomial in the principal curvatures. Moreover, it is a classical fact that $H^2 \geq H_2$ with equality at umbilical points. Hence, assuming $H^2 \leq \rho - k$ implies that M is a hypersphere. In this spirit, and using the above DDVV inequality, we give the following rigidity result.

Corollary 2. *Let M^m be a complete submanifold without boundary of the warped product $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$ with normalized scalar and normal scalar curvatures ρ and ρ^\perp and mean curvature H . If $\|H\|^2 \leq \rho + \lambda^2$, then*

$$\|H\|^2 = \rho + \lambda^2, \quad \rho^\perp = 0, \quad n = 2 \quad \text{and} \quad \|T\| = 1.$$

Hence, M is a surface of the type $\mathbb{R} \times_{e^{\lambda t}} \gamma$, where γ is a curve in $\mathbb{H}^n(c)$.

Proof: First note that since $n \geq 2$ and $\|T\|^2 \leq 1$ and $c < 0$, we have $ce^{-2\lambda t} \left(1 - \frac{2}{n}\|T\|^2\right) \leq 0$. Note also that, by definition, $\rho^\perp \geq 0$. Hence, from Corollary 1, $\|H\|^2 \leq \rho + \lambda^2$ is possible if and only if $\|H\|^2 = \rho + \lambda^2$, $\rho^\perp = 0$, $n = 2$ and $\|T\| = 1$. Since $n = 2$, then M is a surface and the fact that $\|T\| = 1$ implies that $T = \partial_t$ and so M is of the type $I \times_{e^{\lambda t}} \gamma$, where γ is a curve in $\mathbb{H}^n(c)$. Since we assume that M is complete and without boundary, $I = \mathbb{R}$. This concludes the proof. \square

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