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# JUMPLETS

ROMAN ANDREEV<sup>†</sup>

ABSTRACT. In this note we describe a locally supported Riesz basis consisting of “jumplets” for the orthogonal complement of continuous splines on an interval in the space of discontinuous ones.

## 1. INTRODUCTION

Let  $T > 0$  and write  $J := [0, T]$ . For  $N \geq 1$  let a finite partition of  $J$  be given by

$$(1) \quad \mathcal{T} = \{0 =: t_0 < t_1 < \dots < t_N < t_{N+1} := T\}.$$

Let  $\mathbb{P}_d$  denote the space of real-valued polynomials on  $\mathbb{R}$  of degree at most  $d$ . We write

$$(2) \quad F := \{f \in L_2(J) : f|_{(t_{n-1}, t_n)} \in \mathbb{P}_{p_n}\} \quad \text{and} \quad E := F \cap H^1(J)$$

for the space  $F$  of piecewise polynomials on the interval  $[0, T]$  of given polynomial degrees  $p_n \geq 0$  and for its subset  $E \subset F$  of the continuous ones. Here and throughout,  $L_2$  and  $H^1$  refer to the usual Lebesgue and Sobolev spaces of real-valued functions on  $J$ .

In the next Section 2 we construct a locally supported Riesz basis for the  $L_2$  orthogonal complement of  $E$  in  $F$ . Its dimension is obviously  $N$ , which is the number of interior nodes of  $\mathcal{T}$ . We refer to  $E$  as the continuous part of  $F$ , and to its  $L_2$ -orthogonal complement in  $F$  as the discontinuous part of  $F$ . An application to adaptive approximation in  $L_2$  is discussed in Section 3.

## 2. CONSTRUCTION OF THE BASIS

If  $I \subset \mathbb{R}$  is an interval, we write  $(\cdot, \cdot)_I$  for the  $L_2(I)$  scalar product, and  $\|\cdot\|_I$  for the norm. The euclidean norm is denoted by  $|\cdot|$ . For integer  $d \geq 0$  let  $\ell_d \in \mathbb{P}_d$  be the Legendre polynomial on the unit interval  $[0, 1]$  characterized by  $\ell_d(1) = \sqrt{2d+1}$  and  $\int_0^1 \ell_d(t)q(t)dt = 0$  for any  $q \in \mathbb{P}_{d-1}$ . Then  $\{\ell_k : 0 \leq k \leq d\}$  is an orthonormal basis for  $\mathbb{P}_d \cap L_2([0, 1])$ .

For integer  $d \geq 0$  define the polynomial  $R_d := \sum_{k=0}^d \ell_k(1)\ell_k$ . Its key property is that  $(R_d, q)_{[0,1]} = q(1)$  for any polynomial  $q \in \mathbb{P}_d$ , as can be verified by expanding  $q$  into the Legendre basis. Further,  $\|R_d\|_{[0,1]} = 1 + d$ . Together, these observations imply  $R_d(1) = (R_d, R_d)_{[0,1]} = (1 + d)^2$ . Using  $\ell_k(0) = (-1)^k \ell_k(1)$  in the definition of  $R_d$  we also obtain  $R_d(0) = (-1)^d(1 + d)$ .

For each  $n = 1, \dots, N + 1$ , let  $J_n := (t_{n-1}, t_n)$  be the  $n$ -th subinterval in the partition  $\mathcal{T}$  from (1). Write  $\alpha_n := |t_n - t_{n-1}|^{-1}$  for the inverse of its length, and define the constants  $\beta_n := \alpha_n(1 + p_n)^2$ , where  $p_n$  is the polynomial degree from (2). Define the functions  $\psi_n^-$  and  $\psi_{n-1}^+$  in  $F$  having the values

$$(3) \quad \psi_n^-(t) := \alpha_n R_{p_n}(\alpha_n(t - t_{n-1})) \quad \text{and} \quad \psi_{n-1}^+(t) := \alpha_n R_{p_n}(\alpha_n(t_n - t))$$

if  $t \in J_n$ , and zero else. Note that  $\psi_{n-1}^+$  is the reflection of  $\psi_n^-$  about the vertical at the middle of the subinterval  $J_n$ . Since  $\beta_n$  is associated with both,  $\psi_{n-1}^+$  and  $\psi_n^-$ , we also write  $\beta_{n-1}^+ := \beta_n^- := \beta_n$ . These functions are designed towards the following observation.

**Lemma 2.1.**  $(\psi_n^\pm, f)_J = \lim_{\epsilon \rightarrow 0} f(t_n \pm |\epsilon|)$  for all  $f \in F$  and  $n = 1, \dots, N$ .

*Proof.* If  $f \in F$  then  $q : t \mapsto f(t_{n-1} + s/\alpha_n)$  is a polynomial of degree  $p_n$  on the unit interval  $[0, 1]$ . Since  $\psi_n^-$  is supported on  $J_n$ , we employ the definition of  $\psi_n^-$  and the properties of  $R_d$  to obtain  $(\psi_n^-, f)_J = (R_{p_n}, q)_{[0,1]} = q(1) = \lim_{\epsilon \rightarrow 0} f(t_n - |\epsilon|)$ . The proof for  $\psi_n^+$  is analogous.  $\square$

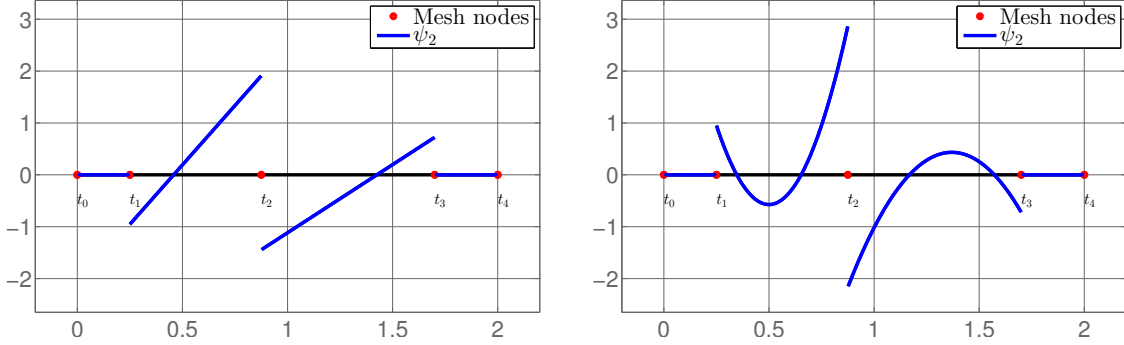


FIGURE 1. The basis function  $\psi_2$  from (4) for polynomial degrees  $p = 1$  (left) /  $p = 2$  (right).

The announced basis for the discontinuous part of  $F$  is now defined by

$$(4) \quad \psi_n := \frac{\psi_n^- - \psi_n^+}{\sqrt{\beta_n^- + \beta_n^+}}, \quad n = 1, \dots, N.$$

**Proposition 2.2.** *Assume that  $p_n \geq p_{\min} \geq 1$  in (2). Then,  $\Psi := \{\psi_n : n = 1, \dots, N\}$  is a Riesz basis for the  $L_2$ -orthogonal complement of  $E$  in  $F$ . More precisely, for all vectors  $c \in \mathbb{R}^N$ ,*

$$(5) \quad C_- |c|^2 \leq \|\Psi^T c\|_J^2 \leq C_+ |c|^2 \quad \text{with} \quad C_{\pm} = 1 \pm \frac{1}{1 + p_{\min}}.$$

*Proof.* Set  $\psi_n^\circ := \psi_n^- - \psi_n^+$ . Then  $(\psi_n^\circ, e)_J = \lim_{\epsilon \rightarrow 0} e(t_n - |\epsilon|) - \lim_{\epsilon \rightarrow 0} e(t_n + |\epsilon|) = 0$  for any  $e \in E$  by continuity of  $e$ . Hence, each  $\psi_n$  is  $L_2$ -orthogonal to  $E$ . By definition,  $\psi_n \in F$ . Therefore, once (5) has been established, the fact that the  $L_2$ -orthogonal complement of  $E$  in  $F$  is  $N$ -dimensional shows that  $\Psi$  is a basis for it. Now we compute

$$(6) \quad \gamma_{n,n-1}^\circ := (\psi_n^\circ, \psi_{n-1}^\circ)_J = -(\psi_n^-, \psi_{n-1}^+)_J = -\lim_{\epsilon \rightarrow 0} \psi_n^-(t_{n-1} + |\epsilon|) = \alpha_n (-1)^{1+p_n} (1 + p_n),$$

and similarly

$$(7) \quad \beta_n^\circ := (\psi_n^\circ, \psi_n^\circ)_J = \|\psi_n^-\|_J^2 + \|\psi_n^+\|_J^2 = \beta_n^- + \beta_n^+.$$

Set  $\delta_n := \gamma_{n,n-1}^\circ / \sqrt{\beta_n^\circ \beta_{n-1}^\circ}$ . Then the Gramian of  $\Psi$  is the tridiagonal symmetric  $N \times N$  matrix

$$(8) \quad M_\Psi := \begin{pmatrix} 1 & \delta_2 & & & \\ \delta_2 & 1 & \delta_3 & & \\ & \delta_3 & 1 & \ddots & \\ & & \ddots & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Note  $c^T M_\Psi c = |c|^2 + \sum_{n=2}^N 2\delta_n c_n c_{n-1}$  for any  $c \in \mathbb{R}^N$ , so we wish to compare the magnitude of the second term to  $|c|^2$ . Observe

$$(9) \quad |2\delta_n c_n c_{n-1}| = |\gamma_{n,n-1}^\circ| \times 2 \frac{|c_n|}{\sqrt{\beta_n^\circ}} \frac{|c_{n-1}|}{\sqrt{\beta_{n-1}^\circ}} \leq \frac{\beta_n}{1 + p_{\min}} \times \left( \frac{c_n^2}{\beta_n^\circ} + \frac{c_{n-1}^2}{\beta_{n-1}^\circ} \right),$$

so that  $\sum_{n=2}^N |2\delta_n c_n c_{n-1}| \leq \frac{1}{1+p_{\min}} |c|^2$ . Therefore,  $C_- |c|^2 \leq c^T M_\Psi c \leq C_+ |c|^2$ , with  $C_{\pm} = 1 \pm \frac{1}{1+p_{\min}}$ .  $\square$

On a uniform mesh  $\mathcal{T}$  and with uniform polynomial degree  $p$  one obtains  $\delta_n = -\frac{1}{2} \frac{(-1)^p}{1+p}$  for each  $n$  in the Gramian (8). It is a tridiagonal symmetric Toeplitz matrix with eigenvalues  $\{1 + \frac{1}{1+p} \cos \frac{k\pi}{N+1} : k = 1, \dots, N\}$ . Letting  $N \rightarrow \infty$  shows that the constants in (5) cannot be improved in general.

The fact that the Gramian (8) is well-conditioned (5), together with Lemma 2.1, allows for fast computation of the discontinuous part of a function  $f \in F$ . Let  $\Delta \in \mathbb{R}^N$  denote the vector of jumps of  $f$ , whose components are  $\Delta_n = \lim_{\epsilon \rightarrow 0} \{f(t_n - |\epsilon|) - f(t_n + |\epsilon|)\}$  for  $n = 1, \dots, N$ . Then Lemma

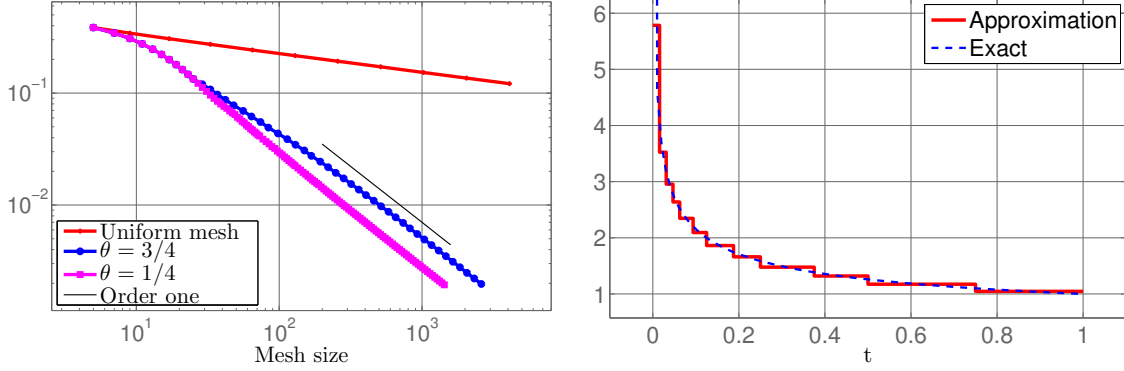


FIGURE 2. Left:  $L_2$  error achieved by the adaptive approximation algorithm for  $g : t \mapsto t^{-1/3}$  on  $J = (0, 1)$  by piecewise constant functions, and the error of the  $L_2$  best approximation of  $g$  on a uniform mesh, as a function of the mesh size. Right:  $g$  and its adaptive approximation  $g_4$  for  $\theta = 3/4$ .

2.1 and definition (4) of  $\psi_n$  imply  $b_n := (\psi_n, f)_J = \Delta_n / \sqrt{\beta_n^- + \beta_n^+}$ . The coefficients  $c \in \mathbb{R}^N$  of the discontinuous part of  $f$  with respect to the basis (4) satisfy the linear system  $M_\Psi c = b$ , which can be quickly solved approximatively by the conjugate gradient method. This may be cheaper than projecting onto  $E$  and then taking the difference because the dimension of  $E$  is significantly larger than  $N$  when the polynomial degrees  $p_n$  in (2) are large.

### 3. APPLICATION: ADAPTIVE APPROXIMATION

As an application of the above construction we describe an algorithm for the adaptive approximation of a given function  $g \in L_2(J)$ . Suppose  $\mathcal{T}_i$ ,  $i = 0, 1, 2, \dots$ , is a sequence of meshes as in the introduction, which are nested,  $\mathcal{T}_i \subset \mathcal{T}_{i+1}$ . Let  $F_i \subset Y$  be the space of piecewise affine splines on with respect to  $\mathcal{T}_i$ , and set  $E_i := F_i \cap H^1(J)$ , which is then the space of continuous piecewise affine splines. Note that  $F_i = E_i' + E_i$ , where the prime denotes the derivative. Let  $E_i^\perp$  denote the  $L_2$ -orthogonal complement of  $E_i$  in  $F_i$ , and let  $Q_i^\perp : L_2(J) \rightarrow E_i^\perp$  be the surjective  $L_2$ -orthogonal projection.

Suppose  $g_i \in E_i'$  is an approximation of  $g$ . Then we consider  $g_i^\perp := Q_i^\perp g_i \in E_i^\perp$ , and use the coefficients of  $g_i^\perp$  with respect to the Riesz basis (4) for  $E_i^\perp$  as error indicators for the marking of subintervals to be adaptively refined. The adaptive algorithm is as follows. Let  $\mathcal{T}_0$  be given. Fix a threshold parameter  $\theta \in (0, 1]$ . For each  $i = 0, 1, 2, \dots$ , do: **1)** Compute the  $L_2$ -orthogonal projection  $g_i$  of  $g$  onto  $E_i'$ , or an approximation thereof. **2)** Compute the projection  $g_i^\perp := Q_i^\perp g_i$ . Set  $N_i = \#(\mathcal{T}_i \cap (0, T))$  for the number of interior nodes in  $\mathcal{T}_i$ , and  $[N_i] := \{1, \dots, N_i\}$ . Let  $c \in \mathbb{R}^{N_i}$  be the vector of the coefficients of  $g_i^\perp \in E_i^\perp$  with respect to the Riesz basis (4) for  $E_i^\perp$ . Select  $M_i \subset [N_i]$  of minimal size such that  $\sum_{n \in M_i} c_n^2 \geq \theta^2 |c|^2$ . **3)** If  $\mathcal{T}_i = \{0 = t_0 < t_1 < \dots < t_{N_i} < t_{N_i+1} = T\}$ , let the new mesh  $\mathcal{T}_{i+1}$  contain  $\mathcal{T}_i$ , and the new nodes  $\frac{1}{2}(t_{n-1} + t_n)$  and  $\frac{1}{2}(t_n + t_{n+1})$  for all  $n \in M_i$ .

The Riesz basis property guarantees that the part of the indicator  $g_i^\perp$  corresponding to the marked subset  $M_i$  carries a fraction of its total  $L_2(J)$  norm that is comparable to  $\theta$ .

As an example we take  $g : t \mapsto t^{-1/3}$  on  $J := [0, T]$  with  $T := 1$ . We set  $\mathcal{T}_0 := \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$  for the initial mesh. Instead of the  $L_2$ -orthogonal projection we use the 4-point Gauss–Legendre quadrature rule on each subinterval to obtain  $g_i$ . The error  $\|g - g_i\|_J$  for the above adaptive is shown in Figure 2. Due to the lack of smoothness of the given function  $g$ , it is approximated by piecewise constant functions on a uniform mesh with a rate of  $\approx 1/6$  with respect to the mesh size  $\#\mathcal{T}_i$ , while adaptivity recovers the asymptotic rate of one.

This note was mainly written while at RICAM, Linz (AT), 2014.