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# MULTIFRACTION REDUCTION II: CONJECTURES FOR ARTIN-TITS GROUPS

PATRICK DEHORNOY

ABSTRACT. Multifraction reduction is a new approach to the Word Problem for Artin-Tits groups and, more generally, for the enveloping group  $\mathcal{U}(M)$  of a monoid  $M$  in which any two elements admit a greatest common divisor. This approach is based on a rewrite system  $\mathcal{R}(M)$  that extends free group reduction. In this paper, we show that for  $\mathcal{R}(M)$  to satisfy a weak form of convergence called semi-convergence is sufficient for solving the Word Problem for  $\mathcal{U}(M)$ , and we connect semi-convergence with other conditions involving  $\mathcal{R}(M)$ . We conjecture that all these properties are valid when  $M$  is an Artin-Tits monoid, and report about numerical experiments supporting this conjecture.

## 1. INTRODUCTION

Artin-Tits groups are those groups that admit a positive presentation  $\langle S \mid R \rangle$  where  $R$  contains at most one relation  $s\dots = t\dots$  for each pair of generators  $s, t$  in  $S$  and, when it does, the relation has the form  $stst\dots = tsts\dots$ , both sides of the same length [2, 3, 15, 4, 17]. This paper, which follows [7] but is self-contained, continues the investigation of multifraction reduction, a new approach to the Word Problem for Artin-Tits groups and, more generally, for the enveloping group  $\mathcal{U}(M)$  of a cancellative monoid  $M$  in which any two elements admit a left and a right greatest common divisor (“gcd-monoid”). This approach is based on a certain algebraic rewrite system  $\mathcal{R}(M)$ . It was proved in [7] that, if the monoid  $M$  satisfies an additional assumption called the 3-Ore condition, then the rewrite system  $\mathcal{R}(M)$  is convergent and every element of the enveloping group of  $M$  admits a unique representation by an irreducible multifraction, directly extending the classical result by Ø. Ore about representation by irreducible fractions.

In the current paper, we address the general case, when the 3-Ore condition is not assumed. In this case, the system  $\mathcal{R}(M)$  is not convergent, and there is no unique representation the elements of the group  $\mathcal{U}(M)$  by irreducible multifractions. However, we introduce a new, weaker condition called *semi-convergence*, and prove that most of the applications of convergence can be derived from semi-convergence, in particular a solution of the Word Problem for  $\mathcal{U}(M)$  whenever convenient finiteness conditions are satisfied. This makes the following conjecture crucial:

**Conjecture A.** *For every Artin-Tits monoid, the system  $\mathcal{R}(M)$  is semi-convergent.*

A proof of Conjecture A would imply the decidability of the Word Problem for every Artin-Tits group, a so far open question in the general case.

Here we discuss various conditions that imply semi-convergence and could lead to a proof of Conjecture A. We present a state-of-the-art description of the known results involving multifraction reduction, and report about computer experiments supporting Conjecture A and its variants.

The paper is organized as follows. We gather in Section 2 the needed prerequisites about multifractions, gcd-monoids, and the reduction system  $\mathcal{R}(M)$ . Semi-convergence and Conjecture A are introduced in Section 3, and their consequences are established. In Section 4, we analyze specific cases of reduction, mainly divisions and their extensions, tame reductions. This leads

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to a new property, stated as Conjecture **B**, which is stronger than Conjecture **A** but easier to experimentally check and maybe to establish. Then, we introduce in Section 5 cross-confluence, a new property of the system  $\mathcal{R}(M)$  that involves both  $\mathcal{R}(M)$  and its mirror-copy  $\tilde{\mathcal{R}}(M)$ . This leads to Conjecture **C** and its uniform version  $\mathbf{C}_{\text{unif}}$ , again stronger than Conjecture **A** but possibly more accessible. In Section 6, we analyze the case of small depth multifractions. We prove in particular that semi-convergence for multifractions of depth 2 is equivalent to  $M$  embedding into its enveloping group, and semi-convergence for multifractions of depth 4 is equivalent to a unique decomposition property for fractions in  $\mathcal{U}(M)$ . Finally, we gather in Section 7 reports about computer experiments and a few comments about further possible developments.

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## 2. MULTIFRACTION REDUCTION

In this introductory section, we recall the notions of a multifraction and of a gcd-monoid, as well as the definition of multifraction reduction [7].

**2.1. Multifractions.** If  $M$  is a monoid, we denote by  $\mathcal{U}(M)$  and  $\iota$  the enveloping group of  $M$  and the canonical morphism from  $M$  to  $\mathcal{U}(M)$ , characterized by the universal property that every morphism from  $M$  to a group factors through  $\iota$ . By definition, every element  $g$  of  $\mathcal{U}(M)$  can be expressed as

$$(2.1) \quad \iota(a_1)\iota(a_2)^{-1}\iota(a_3)\iota(a_4)^{-1}\cdots \quad \text{or} \quad \iota(a_1)^{-1}\iota(a_2)\iota(a_3)^{-1}\iota(a_4)\cdots,$$

with  $a_1, \dots, a_n$  in  $M$ . We shall investigate  $\mathcal{U}(M)$  using such expressions. In [7], without loss of generality, we only considered, expressions (2.1) where the first term  $\iota(a_1)$  is positive (possibly trivial, that is, equal to 1). Here, in particular in view of Sec. 5, we skip that condition, and allow for both signs in the first entry.

**Definition 2.2.** Let  $M$  be a monoid. Let  $\overline{M}$  be a disjoint copy of  $M$ ; call the elements of  $M$  (*resp.*  $\overline{M}$ ) positive (*resp.* negative). For  $n \geq 1$ , a *positive* (*resp.* *negative*)  $n$ -*multifraction* on  $M$  is a finite sequence  $(a_1, \dots, a_n)$  with entries in  $M \cup \overline{M}$ , alternating signs, and  $a_1$  in  $M$  (*resp.*  $\overline{M}$ ). The set of all multifractions (*resp.* all positive multifractions) completed with the empty sequence  $\emptyset$  is denoted by  $\mathcal{F}_{\pm}(M)$  (*resp.*  $\mathcal{F}(M)$ ). A multiplication is defined by

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_p) = \begin{cases} (a_1, \dots, a_{n-1}, a_n b_1, b_2, \dots, b_p) & \text{for } a_n \text{ and } b_1 \text{ in } M, \\ (a_1, \dots, a_{n-1}, b_1 a_n, b_2, \dots, b_p) & \text{for } a_n \text{ and } b_1 \text{ in } \overline{M}, \\ (a_1, \dots, a_n, b_1, \dots, b_p) & \text{otherwise,} \end{cases}$$

extended with  $\underline{a} \cdot \emptyset = \emptyset \cdot \underline{a} = \underline{a}$  for every  $\underline{a}$ .

We use  $\underline{a}, \underline{b}, \dots$  as generic symbols for multifractions, and  $a_i$  for the  $i$ th entry of  $\underline{a}$  counted from 1. For  $\underline{a}$  in  $\mathcal{F}_{\pm}(M)$ , the length of  $\underline{a}$  (number of entries) is called its *depth*, written  $\|\underline{a}\|$ . We identify an element  $a$  of  $M$  with the depth one positive multifraction  $(a)$ . Finite sequences will play the role of iterated fractions, and the following convention is then convenient:

**Notation 2.3.** For  $a_1, \dots, a_n$  in  $M$ , we put

$$(2.4) \quad a_1/\cdots/a_n := (a_1, \overline{a_2}, a_3, \overline{a_4}, \dots) \quad \text{and} \quad /a_1/\cdots/a_n := (\overline{a_1}, a_2, \overline{a_3}, a_4, \dots).$$

We say that  $i$  is *positive* (*resp.* *negative*) in  $\underline{a}$  if  $a_i$  (*resp.*  $\overline{a_i}$ ) occurs in the expansion of  $\underline{a}$ .

With this convention, we recover the notation of [7], where only positive multifractions are considered and  $\overline{M}$  remains hidden. Multifractions are adequately illustrated by associating with every element  $a$  of  $M$  an arrow labeled  $a$ , concatenating arrows to represent the product in  $M$ , and associating with every multifraction the path made of (the arrows of) the successive entries with alternating orientations. The rules for the multiplication of  $\underline{a}$  and  $\underline{b}$  can be read in the following diagrams:

$$\begin{aligned}
- n \text{ positive in } \underline{a}, 1 \text{ positive in } \underline{b}: & \quad \cdots \longleftarrow \frac{a_n}{\phantom{a_n}} \cdot \frac{b_1}{\phantom{b_1}} \longleftarrow \cdots = \cdots \longleftarrow \frac{a_n b_1}{\phantom{a_n b_1}} \longleftarrow \cdots \\
- n \text{ positive in } \underline{a}, 1 \text{ negative in } \underline{b}: & \quad \cdots \longleftarrow \frac{a_n}{\phantom{a_n}} \cdot \frac{b_1}{\phantom{b_1}} \longrightarrow \cdots = \cdots \longleftarrow \frac{a_n}{\phantom{a_n}} \frac{b_1}{\phantom{b_1}} \longrightarrow \cdots \\
- n \text{ negative in } \underline{a}, 1 \text{ positive in } \underline{b}: & \quad \cdots \longrightarrow \frac{a_n}{\phantom{a_n}} \cdot \frac{b_1}{\phantom{b_1}} \longleftarrow \cdots = \cdots \longrightarrow \frac{a_n}{\phantom{a_n}} \frac{b_1}{\phantom{b_1}} \longleftarrow \cdots \\
- n \text{ negative in } \underline{a}, 1 \text{ negative in } \underline{b}: & \quad \cdots \longrightarrow \frac{a_n}{\phantom{a_n}} \cdot \frac{b_1}{\phantom{b_1}} \longrightarrow \cdots = \cdots \longrightarrow \frac{b_1 a_n}{\phantom{b_1 a_n}} \longrightarrow \cdots
\end{aligned}$$

**Proposition 2.5.** (i) The set  $\mathcal{F}_{\pm}(M)$  equipped with  $\cdot$  and  $\emptyset$  is a monoid generated by  $M \cup \overline{M}$ , and  $\mathcal{F}(M)$  is the submonoid of  $\mathcal{F}_{\pm}(M)$  generated by  $M \cup \{1/a \mid a \in M\}$ . The family of depth one multifractions is a submonoid isomorphic to  $M$ .

(ii) Let  $\simeq$  be the congruence on  $\mathcal{F}_{\pm}(M)$  generated by  $(1, \emptyset)$  and the pairs  $(a/a, \emptyset)$  and  $(/a/a, \emptyset)$  with  $a$  in  $M$ . For  $\underline{a}$  in  $\mathcal{F}_{\pm}(M)$ , let  $\iota(\underline{a})$  be the  $\simeq$ -class of  $\underline{a}$ . Then the group  $\mathcal{U}(M)$  is (isomorphic to)  $\mathcal{F}_{\pm}(M)/\simeq$  and, for all  $a_1, \dots, a_n$  in  $M$ , we have

$$(2.6) \quad \iota(a_1/\cdots/a_n) = \iota(a_1)\iota(a_2)^{-1}\iota(a_3)\cdots, \quad \iota(/a_1/\cdots/a_n) = \iota(a_1)^{-1}\iota(a_2)\iota(a_3)^{-1}\cdots.$$

(iii) The restriction of  $\simeq$  to  $\mathcal{F}(M)$  is the congruence  $\simeq^+$  generated by  $(1, \emptyset)$  and the pairs  $(a/a, \emptyset)$  and  $(1/a/a, \emptyset)$  with  $a$  in  $M$ . The group  $\mathcal{U}(M)$  is also isomorphic to  $\mathcal{F}(M)/\simeq^+$ . The translation  $\underline{a} \mapsto 1 \cdot \underline{a}$  maps  $\mathcal{F}_{\pm}(M)$  onto  $\mathcal{F}(M)$  and, for all  $\underline{a}, \underline{b}$  in  $\mathcal{F}_{\pm}(M)$ , the relation  $\underline{a} \simeq \underline{b}$  is equivalent to  $1 \cdot \underline{a} \simeq^+ 1 \cdot \underline{b}$ .

*Proof.* The argument is similar to the proof of [7, Prop. 2.4], and we only point the differences due to using signed multifractions. For (i), associativity is checked directly, and the generating subsets for  $\mathcal{F}_{\pm}(M)$  and  $\mathcal{F}(M)$  follow from the equalities

$$(2.7) \quad a_1/\cdots/a_n = a_1 \cdot \overline{a_2} \cdot a_3 \cdot \overline{a_4} \cdot \cdots = a_1 \cdot 1/a_2 \cdot a_3 \cdot 1/a_4 \cdot \cdots :$$

both hold in  $\mathcal{F}_{\pm}(M)$ , and the second only involves positive multifractions.

For (ii) and (iii), for every  $a$  in  $\mathcal{F}_{\pm}(M)$ , the definition of  $\simeq$  implies  $\iota(\overline{a}) = \iota(a)^{-1}$  and, writing  $\iota^+(\underline{a})$  for the  $\simeq^+$ -class of  $\underline{a}$ , that of  $\simeq^+$  implies  $\iota^+(1/a) = \iota^+(a)^{-1}$ . Hence both  $\mathcal{F}_{\pm}(M)/\simeq$  and  $\mathcal{F}(M)/\simeq^+$  are groups generated by  $M$ . One easily checks that the latter groups satisfy the universal property defining  $\mathcal{U}(M)$ , and are therefore isomorphic to  $\mathcal{U}(M)$ . Then (2.6) directly follows from (2.7).

Next, for every  $\underline{a}$  in  $\mathcal{F}_{\pm}(M)$ , the product  $1 \cdot \underline{a}$  belongs to  $\mathcal{F}(M)$ . Then, for  $\underline{a}, \underline{b}$  in  $\mathcal{F}_{\pm}(M)$ , write  $\underline{a} \approx \underline{b}$  for  $1 \cdot \underline{a} \simeq^+ 1 \cdot \underline{b}$ . By considering all sign combinations and using relations like  $1/ab \simeq^+ 1/b \cdot 1/a$ , one checks that  $\approx$  is a congruence on  $\mathcal{F}_{\pm}(M)$ , and it contains the pairs  $(1, \emptyset)$ ,  $(a/a, \emptyset)$  and  $(/a/a, \emptyset)$  that generate  $\simeq$ , as one finds for instance  $1 \cdot /a/a = 1/a/a \simeq^+ \emptyset = 1 \cdot \emptyset$ . Hence  $\approx$  includes  $\simeq$ . In the other direction,  $\simeq$  contains pairs that generate  $\simeq^+$  and, being compatible with multiplication in  $\mathcal{F}_{\pm}(M)$ , it is in particular compatible with multiplication in  $\mathcal{F}(M)$ . So  $\simeq^+$  is included in  $\approx$ , and  $\underline{a} \simeq \underline{b}$  implies  $1 \cdot \underline{a} \simeq^+ 1 \cdot \underline{b}$  since 1 is invertible mod  $\simeq$ . Hence,  $\simeq$  is included in  $\approx$  and, finally,  $\simeq$  and  $\approx$  coincide, which means that, for all  $\underline{a}, \underline{b}$  in  $\mathcal{F}_{\pm}(M)$ , we have

$$(2.8) \quad \underline{a} \simeq \underline{b} \iff 1 \cdot \underline{a} \simeq^+ 1 \cdot \underline{b}.$$

For  $\underline{a}$  positive, we have  $1 \cdot \underline{a} = \underline{a}$ , so (2.8) implies in particular that  $\simeq^+$  is the restriction of  $\simeq$  to  $\mathcal{F}(M)$ , and the rest follows easily.  $\square$

Hereafter, we identify  $\mathcal{U}(M)$  with  $\mathcal{F}_{\pm}(M)/\simeq$  and  $\mathcal{F}(M)/\simeq^+$ . This representation is redundant in that, for every  $a$  in  $M$ , the inverse  $\iota(a)^{-1}$  of  $\iota(\underline{a})$  is represented both by the depth 1 negative multifraction  $/a$  and the depth 2 positive multifraction  $1/a$ .

We conclude this introduction with some terminology that will be used frequently:

**Definition 2.9.** A multifraction  $\underline{a}$  is called *unital* if  $\underline{a} \simeq \underline{1}$  holds, i.e., if  $\underline{a}$  represents 1 in the group  $\mathcal{U}(M)$ . It is called *trivial* if all entries are equal to 1 or  $\overline{1}$ . For  $n > 0$ , we write  $\underline{1}_n$  for  $1/\cdots/1$ ,  $n$  terms; for  $n < 0$ , we write  $\underline{1}_n$  for  $/1/1/\cdots/1$ ,  $|n|$  terms; in practice, we shall often omit the index and write  $\underline{1}$  for a trivial multifraction.

**2.2. Gcd-monoids.** The reduction process we investigate requires that the ground monoid is a gcd-monoid. We recall the basic definitions, referring to [7] (and [14]) for more details.

Let  $M$  be a monoid. For  $a, b$  in  $M$ , we say that  $a$  *left-divides*  $b$  or, equivalently, that  $b$  is a *right-multiple* of  $a$ , written  $a \leq b$ , if  $ax = b$  holds for some  $x$  in  $M$ . If  $M$  is cancellative monoid and 1 is the only invertible element in  $M$ , the left-divisibility relation is a partial order on  $M$ . In this case, when they exist, the greatest common  $\leq$ -lower bound of two elements  $a, b$  is called their *left-gcd*, denoted by  $a \wedge b$ , and their least common  $\leq$ -upper bound is called their *right-lcm*, denoted by  $a \vee b$ .

Symmetrically, we say that  $a$  *right-divides*  $b$  or, equivalently, that  $b$  is a *left-multiple* of  $a$ , written  $a \lesseqgtr b$ , if  $xa = b$  holds for some  $x$ . Under the same hypotheses,  $\lesseqgtr$  is a partial order on  $M$ , with the derived right-gcd and left-lcm written  $\tilde{\wedge}$  and  $\tilde{\vee}$ .

**Definition 2.10.** We say that  $M$  is a *gcd-monoid* if  $M$  is a cancellative monoid, 1 is the only invertible element in  $M$ , and any two elements of  $M$  admit a left-gcd and a right-gcd.

Typical examples of gcd-monoids are Artin-Tits monoids. Many more examples are known. In particular, every Garside or preGarside monoid [9, 14, 19] is a gcd-monoid.

Standard arguments [7, Lemma 2.15] show that a gcd-monoid admits *conditional right- and left-lcms*: it need not be true that any two elements admit a right-lcm, but any two elements that admit a common right-multiple admits a right-lcm, and similarly on the left.

The gcd and lcm operations of a gcd-monoid are connected in several ways with the product. We refer to [14, section II.2] for the (easy) proof of the rule for an iterated lcm:

**Lemma 2.11.** *If  $M$  is a gcd-monoid and  $a, b, c$  belong to  $M$ , then  $a \vee bc$  exists if, and only if,  $a \vee b$  and  $a' \vee c$  do, where  $a'$  is defined by  $a \vee b = ba'$ , and then we have*

$$(2.12) \quad a \vee bc = a \cdot b'c' = bc \cdot a''.$$

with  $a \vee b = ba' = ab'$  and  $a' \vee c = a'c' = ca''$ .

This implies in particular that  $a \leq bc$  holds if, and only if,  $a \vee b$  exists and  $a' \leq c$  holds, with  $a'$  defined by  $a \vee b = ba'$ .

**Lemma 2.13.** *If  $M$  is a gcd-monoid and  $a, b, a', b', c$  belong to  $M$  and satisfy  $ab' = ba'$ , then  $a \wedge b = a' \wedge c = 1$  implies  $a \wedge bc = 1$ .*

*Proof.* Assume  $x \leq a$  and  $x \leq bc$ . By Lemma 2.11,  $b \vee x$  must exist and, writing  $b \vee x = bx'$ , we must have  $x' \leq c$ . On the other hand,  $x \leq a$  implies  $x \leq ab' = ba'$ . So  $ba'$  is a common right-multiple of  $b$  and  $x$ , hence it is a right-multiple of their right-lcm  $bx'$ . As  $M$  is left-cancellative,  $bx' \leq ba'$  implies  $x' \leq a'$ . Hence  $x'$  left-divides both  $a'$  and  $c$ , and  $a' \wedge c = 1$  implies  $x' = 1$ , whence  $x \leq b$ . Then  $x$  left-divides both  $a$  and  $b$ , and  $a \wedge b = 1$  implies  $x = 1$ , hence  $a \wedge bc = 1$ .  $\square$

We shall also need the notion of a noetherian monoid. If  $M$  is a gcd-monoid, we use  $<$  for the proper version of left-divisibility:  $a < b$  holds if we have  $b = ax$  for some non-invertible  $x$ , i.e., for  $x \neq 1$ , and similarly for  $\tilde{<}$  vs.  $\lesseqgtr$ .

**Definition 2.14.** A monoid  $M$  is called *noetherian* if the relations  $<$  and  $\tilde{<}$  are well-founded, i.e., every nonempty subset of  $M$  has a  $<$ -minimal element and a  $\tilde{<}$ -minimal element.

Note that a monoid  $\langle S \mid R \rangle^+$  is noetherian whenever each relation in  $R$  is *homogeneous*, i.e., it has the form  $u = v$  with  $u, v$  of the same length: then,  $a < b$  implies that any  $S$ -word representing  $a$  is shorter than any  $S$ -word representing  $b$ , and an infinite  $<$ -descending sequence cannot exist. Artin-Tits monoids are typical examples.

**2.3. Reduction of multifractions.** Introduced in [7], our tool for investigating the congruence  $\simeq$  on multifractions is a family of partial depth-preserving transformations that, when defined, map a multifraction to a  $\simeq$ -equivalent multifraction. These transformations are written as an action on the right: when defined,  $\underline{a} \bullet R$  is the result of applying  $R$  to  $\underline{a}$ .

**Definition 2.15.** If  $M$  is a gcd-monoid and  $\underline{a}, \underline{b}$  lie in  $\mathcal{F}_\pm(M)$ , then, for  $i \geq 1$  and  $x$  in  $M$ , we declare that  $\underline{b} = \underline{a} \bullet R_{i,x}$  holds if we have  $\|\underline{b}\| = \|\underline{a}\|$ ,  $b_k = a_k$  for  $k \neq i-1, i, i+1$ , and there exists  $x'$  (necessarily unique) satisfying

$$\begin{array}{lll} \text{for } i = 1 \text{ positive in } \underline{a}: & b_i x = a_i, & b_{i+1} x = a_{i+1}, \\ \text{for } i = 1 \text{ negative in } \underline{a}: & x b_i = a_i, & x b_{i+1} = a_{i+1}, \\ \text{for } i \geq 2 \text{ positive in } \underline{a}: & b_{i-1} = x' a_{i-1}, \quad b_i x = x' a_i = x \tilde{\vee} a_i, & b_{i+1} x = a_{i+1}, \\ \text{for } i \geq 2 \text{ negative in } \underline{a}: & b_{i-1} = a_{i-1} x', \quad x b_i = a_i x' = x \vee a_i, & x b_{i+1} = a_{i+1}. \end{array}$$

We denote by  $\mathcal{R}(M)$  the family of all rules  $R_{i,x}$  with  $x \neq 1$ , write  $\underline{a} \Rightarrow \underline{b}$  if some rule of  $\mathcal{R}(M)$  maps  $\underline{a}$  to  $\underline{b}$ , and use  $\Rightarrow^*$  for the reflexive–transitive closure of  $\Rightarrow$ . A multifraction  $\underline{a}$  is called  $\mathcal{R}$ -reducible if at least one rule of  $\mathcal{R}(M)$  applies to  $\underline{a}$ , and  $\mathcal{R}$ -irreducible otherwise.

The reduction system  $\mathcal{R}(M)$  extends free reduction (deletion of factors  $x^{-1}x$  or  $xx^{-1}$ ): applying  $R_{i,x}$  to  $\underline{a}$  consists in removing  $x$  from  $a_{i+1}$  and push it through  $a_i$  using an lcm operation, see Fig. 1. A multifraction  $\underline{a}$  is eligible for  $R_{1,x}$  if, and only if,  $x$  divides both  $a_1$  and  $a_2$ , on the side coherent with their signs, and, eligible for  $R_{i,x}$  with  $i \geq 2$  if, and only if,  $x$  divides  $a_{i+1}$  and admits a common multiple with  $a_i$ , on the due side again.

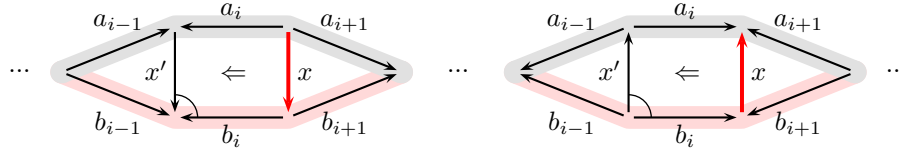


FIGURE 1. The reduction rule  $R_{i,x}$ : starting from  $\underline{a}$  (grey), we extract  $x$  from  $a_{i+1}$ , push it through  $a_i$  by taking the lcm of  $x$  and  $a_i$  (indicated by the small round arc), and incorporate the remainder  $x'$  in  $a_{i-1}$  to obtain  $\underline{b} = \underline{a} \bullet R_{i,x}$  (colored). The left hand side diagram corresponds to the case when  $i$  is negative in  $\underline{a}$ , i.e.,  $a_i$  is crossed negatively, the right hand one to the case when  $i$  is positive, i.e.,  $a_i$  is crossed positively, with opposite orientations of the arrows.

**Example 2.16.** Let  $M$  be the Artin-Tits monoid of type  $\tilde{A}_2$ , here written

$$\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}\mathbf{b}\mathbf{a} = \mathbf{b}\mathbf{a}\mathbf{b}, \mathbf{b}\mathbf{c}\mathbf{b} = \mathbf{c}\mathbf{b}\mathbf{c}, \mathbf{c}\mathbf{a}\mathbf{c} = \mathbf{a}\mathbf{c}\mathbf{a} \rangle^+,$$

and let  $\underline{a} := 1/\mathbf{c}/\mathbf{a}\mathbf{b}\mathbf{a}$ . Then  $\underline{a}$  is eligible for  $R_{2,\mathbf{a}}$  and  $R_{2,\mathbf{b}}$ , since  $\mathbf{a}$  and  $\mathbf{b}$  left-divide  $\mathbf{a}\mathbf{b}\mathbf{a}$  and admit a common right-multiple with  $\mathbf{c}$ . We find  $\underline{a} \bullet R_{2,\mathbf{a}} = \mathbf{a}\mathbf{c}/\mathbf{c}\mathbf{a}/\mathbf{b}\mathbf{a}$  and  $\underline{a} \bullet R_{2,\mathbf{b}} = \mathbf{b}\mathbf{c}/\mathbf{c}\mathbf{b}/\mathbf{a}\mathbf{b}$ . The latter are eligible for no reduction, hence they are  $\mathcal{R}$ -irreducible.

**Remark 2.17.** Most proofs involving reduction require separating cases according to the signs of the involved entries. To avoid this, we could merge  $\leq$  and  $\lesssim$  into a single relation  $\leq_i$  that would be, say,  $\leq$  for  $i$  even and  $\lesssim$  for  $i$  odd. But unifying formulas then requires introducing a new operation that is multiplication for  $i$  even and its opposite for  $i$  odd, which makes reading uneasy. For this reason, we do not adopt the convention.

We use  $R_{i,x}^+$  for the restriction of  $R_{i,x}$  to positive multifractions (those whose first entry lies in  $M$ ), and denote by  $\mathcal{R}^+(M)$  the family of all  $R_{i,x}^+$  with  $x \neq 1$ . It should be clear from the definition, and even more from the diagrams of Fig. 1, that  $R_{i,x}^+$  coincides with the rules  $R_{i,x}$  of [7, Def. 3.4]: if  $\underline{a}$  is a positive multifraction,  $a_i$  is positive for odd  $i$ , and negative for even  $i$ . The only new case is reduction at level 1 for a negative multifraction, defined to be the left-division by  $x$ .

We now state the basic properties of reduction needed below. They directly extend those established for positive multifractions in [7]. Verifying them in the general case is easy.

**Lemma 2.18.** Assume that  $M$  be a gcd-monoid.

- (i) The relation  $\Rightarrow^*$  is included in  $\simeq$ , i.e.,  $\underline{a} \Rightarrow^* \underline{b}$  implies  $\underline{a} \simeq \underline{b}$ .
- (ii) The relation  $\Rightarrow^*$  is compatible with the multiplication of  $\mathcal{F}_\pm(M)$ .
- (iii) For all  $\underline{a}, \underline{b}$  and  $p, q$ , the relation  $\underline{a} \Rightarrow^* \underline{b}$  is equivalent to  $\underline{1}_p \cdot \underline{a} \cdot \underline{1}_q \Rightarrow^* \underline{1}_p \cdot \underline{b} \cdot \underline{1}_q$ .

*Proof.* (i) It directly follows from the definition (and from Fig. 1) that  $\underline{b} = \underline{a} \bullet R_{i,x}$  implies

$$\iota(a_{i-1})\iota(a_i)^{-1}\iota(a_{i+1}) = \iota(b_{i-1})\iota(b_i)^{-1}\iota(b_{i+1})$$

(resp.  $\iota(a_{i-1})^{-1}\iota(a_i)\iota(a_{i+1})^{-1} = \iota(b_{i-1})^{-1}\iota(b_i)\iota(b_{i+1})^{-1}$ ) for  $i$  negative (resp. positive) in  $\underline{a}$ .

(ii) Assume  $\underline{b} = \underline{a} \bullet R_{i,x}$ , and let  $\underline{c}$  be an  $r$ -multifraction. If the signs of  $r$  in  $\underline{c}$  and 1 in  $\underline{a}$  are different, then  $\underline{c} \cdot \underline{a}$  is the concatenation of  $\underline{c}$  and  $\underline{a}$ , similarly  $\underline{c} \cdot \underline{b}$  is the concatenation of  $\underline{c}$  and  $\underline{b}$ , and  $\underline{c} \cdot \underline{b} = (\underline{c} \cdot \underline{a}) \bullet R_{i+r,x}$  trivially holds.

If  $r$  is positive in  $\underline{c}$  and 1 is positive in  $\underline{a}$ , we have  $\underline{c} \cdot \underline{a} = c_1/\dots/c_{r-1}/c_r a_1/a_2/\dots$  and  $\underline{c} \cdot \underline{b} = c_1/\dots/c_{r-1}/c_r b_1/b_2/\dots$ , and we obtain  $\underline{c} \cdot \underline{b} = (\underline{c} \cdot \underline{a}) \bullet R_{i+r-1,x}$ : the point is that, if  $x$  both right-divides  $a_1$  and  $a_2$ , it a fortiori right-divides  $c_r a_1$  and  $a_2$ .

Finally, assume that  $r$  negative in  $\underline{c}$  and 1 is negative in  $\underline{a}$ . Then we find  $\underline{c} \cdot \underline{a} = c_1/\dots/a_1 c_r/a_2/\dots$  and  $\underline{c} \cdot \underline{b} = c_1/\dots/b_1 c_r/b_2/\dots$ : the argument is the same as above, mutatis mutandis: the assumption that  $\underline{a} \bullet R_{1,x}$  is defined means that  $x$  both left-divides  $a_1$  and  $a_2$ , which implies that it a fortiori left-divides  $a_1 c_r$  and  $a_2$ . Hence  $(\underline{c} \cdot \underline{a}) \bullet R_{i+r-1,x}$  is defined and we find  $\underline{c} \cdot \underline{b} = (\underline{c} \cdot \underline{a}) \bullet R_{i+r-1,x}$  again. This completes compatibility with left multiplication.

The compatibility on the right is similar: adding extra entries cannot destroy the eligibility for reduction. Let  $n = \|\underline{a}\|$ . Everything is trivial for  $i < n - 1$ , so we assume  $i = n - 1$ . If the signs of  $n$  in  $\underline{a}$  and of 1 in  $\underline{c}$  are different, the multiplication is a concatenation, and we obtain  $\underline{b} \cdot \underline{c} = (\underline{a} \cdot \underline{c}) \bullet R_{n-1,x}$  trivially. If  $n$  is positive in  $\underline{a}$  and 1 is positive in  $\underline{c}$ , the argument is the same as for  $R_{n-1,x}$ . Finally, assume that  $n$  is negative in  $\underline{a}$  and 1 is negative in  $\underline{c}$ . Then we find  $\underline{a} \cdot \underline{c} = a_1/\dots/a_{n-1}/c_1 a_n/c_2/\dots$  and  $\underline{b} \cdot \underline{c} = b_1/\dots/b_{n-1}/c_1 b_n/c_2/\dots$ . The assumption that  $\underline{a} \bullet R_{n-1,x}$  is defined means that  $x \tilde{\vee} a_{n-1}$  exists and  $x$  right-divides  $a_n$ , which implies that it a fortiori right-divides  $c_1 a_n$ . Hence  $(\underline{a} \cdot \underline{c}) \bullet R_{n-1,x}$  is defined, yielding  $\underline{b} \cdot \underline{c} = (\underline{a} \cdot \underline{c}) \bullet R_{n-1,x}$  again. Thus reduction is compatible with multiplication on the right.

(iii) That  $\underline{a} \Rightarrow \underline{b}$  implies  $\underline{1}_p \cdot \underline{a} \cdot \underline{1}_q \Rightarrow \underline{1}_p \cdot \underline{b} \cdot \underline{1}_q$  follows from (ii) directly. Conversely, assume  $\underline{1}_p \cdot \underline{b} \cdot \underline{1}_q = (\underline{1}_p \cdot \underline{a} \cdot \underline{1}_q) \bullet R_{i,x}$  with  $x \neq 1$ . Let  $n = \|\underline{a}\| = \|\underline{b}\|$ , and assume that the entries of  $\underline{a}$  occur in  $\underline{1}_p \cdot \underline{b} \cdot \underline{1}_q$  from  $r + 1$  to  $r + n$  (with  $r = p$  or  $r = p - 1$  according to the sign of  $p$  in  $\underline{1}_p$  and that of 1 in  $\underline{a}$ ). Then we necessarily have  $r + 1 \leq i < r + n$ . Indeed,  $i < r$  and  $i \geq r + n$  are impossible, since the  $(i + 1)$ st entry of  $\underline{1}_p \cdot \underline{a} \cdot \underline{1}_q$  is trivial. Moreover, in the case  $i = r + 1$ , the element  $x$  necessarily divides  $a_1$ , since, otherwise, the  $r$ th entry of  $\underline{1}_p \cdot \underline{b} \cdot \underline{1}_q$  could not be trivial. Hence,  $\underline{a} \bullet R_{i,x}$  is defined, and it must be equal to  $\underline{b}$ .  $\square$

Finally, we have a simple sufficient condition for termination.

**Lemma 2.19.** [7, Prop. 3.13] *If  $M$  is a noetherian gcd-monoid, then  $\mathcal{R}(M)$  is terminating: every sequence of reductions leads in finitely many steps to an  $\mathcal{R}$ -irreducible multifraction.*

We skip the proof, which is exactly the same in the signed case as in the positive case, and consists in observing that  $\underline{a} \Rightarrow \underline{b}$  forces  $\underline{b}$  to be strictly smaller than  $\underline{a}$  for some antilexicographical ordering on  $\mathcal{F}_{\pm}(M)$ .

**2.4. The convergent case.** The rewrite system  $\mathcal{R}(M)$  (as any rewrite system) is called *convergent* if every element, here every multifraction  $\underline{a}$ , admits a unique  $\mathcal{R}$ -irreducible reduct, usually denoted by  $\text{red}(\underline{a})$ . The main technical result of [7] is

**Proposition 2.20.** *If  $M$  is a noetherian gcd-monoid satisfying the 3-Ore condition:*

(2.21) *If three elements of  $M$  pairwise admit a common right- (resp. left-) multiple, then they admit a common right- (resp. left-) multiple,*

*then  $\mathcal{R}(M)$  is convergent.*

When a monoid  $M$  is eligible for Prop. 2.20, one easily deduces that two multifractions  $\underline{a}, \underline{b}$  represent the same element of  $\mathcal{U}(M)$  if, and only if,  $\text{red}(\underline{a}) \cdot \underline{1}_p = \text{red}(\underline{b}) \cdot \underline{1}_q$  holds for some  $p, q$  and, from there, that the monoid  $M$  embeds in its enveloping group  $\mathcal{U}(M)$  and every element of  $\mathcal{U}(M)$  is represented by a unique  $\widehat{\mathcal{R}}$ -irreducible multifraction, where  $\widehat{\mathcal{R}}(M)$  is obtained from  $\mathcal{R}(M)$  by adding a rule that removes trivial final entries. It is also proved in [7] that, under mild

additional finiteness assumptions on  $M$  (see Sec. 3 below), the relation  $\Rightarrow^*$  on  $M$  is decidable and, from there, so is the Word Problem for  $\mathcal{U}(M)$  when the 3-Ore condition is satisfied.

The above results are relevant for a number of gcd-monoids. We recall that an Artin-Tits monoid  $M = \langle S \mid R \rangle^+$  is said to be of *spherical type* if the Coxeter group obtained by adding to  $(S, R)$  the relation  $s^2 = 1$  for each  $s$  in  $S$  is finite [3]. And  $M$  is said to be of *type FC* if, for every subfamily  $S'$  of  $S$  such that, for all  $s, t$  in  $S'$ , there is a relation  $s\dots = t\dots$  in  $R$ , the submonoid of  $M$  generated by  $S'$  is spherical [1, 18].

**Proposition 2.22.** [7, Prop. 6.5] *An Artin-Tits monoid satisfies the 3-Ore condition if, and only if, it is of type FC.*

However, a number of Artin-Tits monoids fail to be of type FC and therefore are not eligible for Prop. 2.20, typically the monoid of type  $\tilde{A}_2$  considered in Example 2.16. So we are left with the question of either weakening the assumptions for Prop. 2.20, or using a conclusion weaker than convergence.

### 3. SEMI-CONVERGENCE

After showing in Subsection 3.1 that the 3-Ore assumption cannot be weakened when proving the convergence of  $\mathcal{R}(M)$ , we introduce in Subsection 3.2 a new property of  $\mathcal{R}(M)$  called *semi-convergence*, which, as the name suggests, is weaker than convergence. We conjecture (“Conjecture **A**”) that, for every Artin-Tits monoid, the system  $\mathcal{R}(M)$  is semi-convergent. We prove in Subsection 3.3 that most of the consequences known to follow from the convergence of  $\mathcal{R}(M)$  follow from its semi-convergence, in particular in terms of controlling the group  $\mathcal{U}(M)$  from inside the monoid  $M$  and solving its Word Problem for  $\mathcal{U}(M)$ . Finally, we describe in Subsection 3.4 several variants of semi-convergence.

**3.1. The strength of the 3-Ore condition.** A first attempt for improving Prop. 2.20 could be to establish the convergence of  $\mathcal{R}(M)$  from an assumption weaker than the 3-Ore condition. This approach fails, as the latter turns out to be not only sufficient, but also necessary. Hereafter we say that a monoid  $M$  satisfies the right (*resp.* left) 3-Ore condition when (2.21) is valid for right- (*resp.* left-) multiples. First, we recall

**Lemma 3.1.** [7, Lemma 2.12] *If  $M$  is a gcd-monoid, and  $a, b, c, d$  are elements of  $M$  satisfying  $ad = bc$ , then  $ad$  is the right-lcm of  $a$  and  $b$  if, and only if,  $c$  and  $d$  satisfy  $c \tilde{\wedge} d = 1$ .*

**Proposition 3.2.** *Let  $M$  be a gcd-monoid.*

- (i) *If  $\mathcal{R}^+(M)$  is convergent, then  $M$  satisfies the right 3-Ore condition.*
- (ii) *If  $\mathcal{R}(M)$  is convergent, then  $M$  satisfies the 3-Ore condition.*

*Proof.* (i) Assume that  $x, y, z$  belong to  $M$  and pairwise admit common right-multiples, hence right-lcms. Write

$$x \vee y = xy' = yx', \quad y \vee z = yz' = zy'', \quad x \vee z = xz'' = zx''.$$

Let  $\underline{a} := 1/x/y \vee z$ . As in Example 2.16, we find

$$\underline{b} = \underline{a} \cdot R_{2,y}^+ = y'/x'/z' \quad \text{and} \quad \underline{c} = \underline{a} \cdot R_{2,z}^+ = z''/x''/y''.$$

The assumption that  $\mathcal{R}^+(M)$  is convergent implies that  $\underline{b}$  and  $\underline{c}$  both reduce to  $\underline{d} := \text{red}(\underline{a})$ . By construction,  $\underline{d}$  is of depth 3, and  $d_3$  must be a common right-divisor of  $b_3$  and  $c_3$ , which are  $z'$  and  $y''$ . Now, Lemma 3.1 implies  $z' \tilde{\wedge} y'' = 1$ , whence  $d_3 = 1$ . Therefore, there exist  $d_1$  and  $d_2$  in  $M$  satisfying  $1/x/y \vee z \Rightarrow^* d_1/d_2/1$ . By Lemma 2.18, we deduce  $1/x/y \vee z \simeq d_1/d_2/1$ , hence, by (2.6),  $x^{-1}(y \vee z) = d_1 d_2^{-1}$  in  $\mathcal{U}(M)$ . This implies  $x d_1 = (y \vee z) d_2$  in  $\mathcal{U}(M)$ , hence in  $M$ , since the assumption that  $\mathcal{R}^+(M)$  is convergent implies that  $M$  embeds in  $\mathcal{U}(M)$ . It follows that  $x$  and  $y \vee z$  admit a common right-multiple, hence that  $x, y$ , and  $z$  admit a common right-multiple. Hence  $M$  satisfies the right 3-Ore condition.



(ii) The argument is symmetric, with negative multifractions. Assume that  $x, y, z$  belong to  $M$  and pairwise admit common left-multiples, hence left-lcms. Write

$$x \tilde{\vee} y = y'x = x'y, \quad y \tilde{\vee} z = z'y = y''z, \quad x \tilde{\vee} z = z''x = x''z.$$

Let  $\underline{a} := /1/x/y \vee z$ , in  $\mathcal{F}_{\pm}(M) \setminus \mathcal{F}(M)$ . Then we have

$$\underline{b} = \underline{a} \bullet R_{2,y} = /y'/x'/z' \quad \text{and} \quad \underline{c} = \underline{a} \bullet R_{2,z} = /z''/x''/y''.$$

The assumption that  $\mathcal{R}(M)$  is convergent implies that  $\underline{b}$  and  $\underline{c}$  admit a common  $\mathcal{R}$ -reduct, say  $\underline{d}$ . By construction,  $\underline{d}$  is of depth 3, and  $d_3$  must be a common left-divisor of  $b_3$  and  $c_3$ , which are  $z'$  and  $y''$ . By the symmetric counterpart of Lemma 3.1, we have  $z' \vee y'' = 1$ , whence  $d_3 = 1$ . Therefore, there exist  $d_1$  and  $d_2$  in  $M$  satisfying  $\underline{a} \xrightarrow{*} /d_1/d_2/1$ . By Lemma 2.18, we deduce  $/1/x/y \tilde{\vee} z \simeq /d_1/d_2/1$ , hence, by (2.6),  $x(y \tilde{\vee} z)^{-1} = d_1^{-1}d_2$  in  $\mathcal{U}(M)$ , whence  $d_1x = d_2(y \tilde{\vee} z)$  in  $\mathcal{U}(M)$ , hence in  $M$ . This shows that  $x$  and  $y \tilde{\vee} z$  admit a common left-multiple, hence that  $x, y$ , and  $z$  admit a common left-multiple. Hence  $M$  satisfies the left 3-Ore condition.  $\square$

Note that the argument for (i) cannot be used for (ii), because one should start with  $\underline{a} := /1/x/y \tilde{\vee} z$  and then we know nothing about the first entry(ies) of  $\text{red}(\underline{a})$ .

In principle, the right 3-Ore condition is slightly weaker than the full 3-Ore condition, and the convergence of  $\mathcal{R}^+(M)$  might be weaker than that of  $\mathcal{R}(M)$ . However, when  $M$  is an Artin-Tits monoid, all the above conditions are equivalent to  $M$  being of type FC and, therefore, none is weaker. So it seems hopeless to improve Prop. 2.20.

**3.2. Semi-convergence.** We are thus led to explore the other way, namely obtaining useful information about  $\mathcal{U}(M)$  from a property weaker than the convergence of  $\mathcal{R}(M)$ . This is the approach we develop in the rest of this paper.

When the system  $\mathcal{R}(M)$  is not convergent, a  $\simeq$ -class may contain several  $\mathcal{R}$ -irreducible multifractions, and there is no distinguished one: in Example 2.16, the automorphism that exchanges  $\mathbf{a}$  and  $\mathbf{b}$  exchanges the two  $\mathcal{R}$ -irreducible reducts of  $\underline{a}$ , making them indiscernible.

However, a direct consequence of convergence is

**Lemma 3.3.** *If  $M$  is a gcd-monoid and  $\mathcal{R}(M)$  is convergent, then, for every  $\underline{a}$  in  $\mathcal{F}_{\pm}(M)$ ,*

$$(3.4) \quad \text{If } \underline{a} \text{ is unital, then } \underline{a} \Rightarrow^* \underline{1} \text{ holds.}$$

Indeed, if  $\mathcal{R}(M)$  is convergent,  $\underline{a} \simeq \underline{1}$  implies (and, in fact, is equivalent to)  $\text{red}(\underline{a}) = \text{red}(\underline{1})$ , hence  $\text{red}(\underline{a}) = \underline{1}$ , since  $\underline{1}$  is  $\mathcal{R}$ -irreducible. Note that, by Lemma 2.18(i), the converse implication of (3.4) is always true:  $\underline{a} \Rightarrow^* \underline{1}$  implies that  $\underline{a}$  and  $\underline{1}$  represent the same element of  $\mathcal{U}(M)$ , hence that  $\underline{a}$  represents 1.

When  $\mathcal{R}(M)$  is not convergent, (3.4) still makes sense, and it is a priori a (much) weaker condition than convergence. This is the condition we shall investigate below:

**Definition 3.5.** If  $M$  is a gcd-monoid, we say that  $\mathcal{R}(M)$  (*resp.*  $\mathcal{R}^+(M)$ ) is *semi-convergent* if (3.4) holds for every  $\underline{a}$  in  $\mathcal{F}_{\pm}(M)$  (*resp.* in  $\mathcal{F}(M)$ ).

Thus Lemma 3.3 states that  $\mathcal{R}(M)$  is semi-convergent whenever it is convergent. By Prop. 2.22,  $\mathcal{R}(M)$  is semi-convergent for every Artin-Tits monoid  $M$  of type FC. But, as can be expected, semi-convergence is strictly weaker than convergence. We refer to [13] for the construction of monoids for which  $\mathcal{R}(M)$  is semi-convergent and not convergent.

The main conjecture we propose is:

**Conjecture A.** *For every Artin-Tits monoid  $M$ , the system  $\mathcal{R}^+(M)$  is semi-convergent.*

We shall report in Sec. 7 about experimental data supporting Conjecture A. For the moment, we just mention one example illustrating its predictions.

**Example 3.6.** Let  $M$  be the Artin-Tits monoid of type  $\tilde{A}_2$ . We saw in Example 2.16 that  $\mathcal{R}(M)$  is not convergent:  $\underline{a} = 1/c/aba$  admits the two distinct irreducible reducts  $ac/ca/ba$  and  $bc/cb/ab$ . When we multiply the former by the inverse of the latter (see Subsection 3.3

below), we obtain the 6-multifraction  $\underline{b} = \mathbf{ac/ca/ba/ab/cb/bc}$  which, by construction, is unital. Then Conjecture **A** predicts that  $\underline{b}$  must reduce to  $\underline{1}$ . This is indeed the case:

$$\underline{b} \bullet R_{3,ab} R_{4,cb} R_{5,bc} R_{1,ac} R_{2,cbc} R_{3,bc} R_{1,bc} = \underline{1}.$$

More generally, we can observe that, for every gcd-monoid  $M$ , if  $\underline{a}'$  and  $\underline{a}''$  are two reducts of an  $n$ -multifraction  $\underline{a}$ , then the  $2n$ -multifraction  $\underline{b}$  obtained by concatenating  $\underline{a}'$  and the inverse of  $\underline{a}''$  is never irreducible whenever  $\underline{a}''$  is nontrivial: by construction, we have  $b_n = a'_n$  and  $b_{n+1} = a''_n$ . An obvious induction shows that  $\underline{a} \Rightarrow^* \underline{a}'$  implies that  $a'_n$  divides  $a_n$  (on the left or on the right, according to the sign of  $n$  in  $\underline{a}$ ) and, similarly,  $a''_n$  divides  $a_n$ . Hence  $b_n$  and  $b_{n+1}$  admit a common multiple and, therefore,  $\underline{b}$  is eligible for some reduction  $R_{n,x}$  with  $x \neq 1$  whenever  $b_{n+1}$ , *i.e.*,  $a''_n$ , is not 1. Finally, if  $a''_n = 1$  holds, then  $\underline{b}$  is eligible for  $R_{2n-m, a''_m}$ , where  $m$  is the largest index such that  $a''_m$  is nontrivial.

**3.3. Applications of semi-convergence.** Most of the consequences of convergence already follow from semi-convergence—whence the interest of Conjecture **A**. We successively consider the possibility of controlling the congruence  $\simeq$ , the decidability of the Word Problem, and what is called Property H.

*Controlling  $\simeq$ .* In order to investigate  $\simeq$  without convergence, we introduce a new operation on multifractions to represent inverses.

**Notation 3.7.** We put  $\widetilde{\emptyset} = \emptyset$ , and, for every  $n$ -multifraction  $\underline{a}$ ,

$$(3.8) \quad \widetilde{\underline{a}} := \begin{cases} /a_n/\cdots/a_1 & \text{if } n \text{ is positive in } \underline{a}, \\ a_n/\cdots/a_1 & \text{if } n \text{ is negative in } \underline{a}. \end{cases}$$

**Lemma 3.9.** (i) For all multifractions  $\underline{a}, \underline{b}$ , we have  $\widetilde{\underline{a} \cdot \underline{b}} = \widetilde{\underline{b}} \cdot \widetilde{\underline{a}}$ .  
(ii) For every multifraction  $\underline{a}$ , we have  $\iota(\widetilde{\underline{a}}) = \iota(1 \cdot \widetilde{\underline{a}}) = \iota(\underline{a})^{-1}$ .

*Proof.* (i) Put  $\overline{a} = a$  for  $a$  in  $M$ . If  $\underline{a}$  written as a sequence in  $M \cup \overline{M}$  is  $(x_1, \dots, x_n)$ , then  $\widetilde{\underline{a}}$  is  $(\overline{x_n}, \dots, \overline{x_1})$ , and then  $\underline{a} \cdot \underline{b} = \widetilde{\underline{b}} \cdot \widetilde{\underline{a}}$  directly follows from the definition of the product.

Point (ii) comes from (2.6).  $\square$

Lemma 3.9(ii) says that, if  $\underline{a}$  represents an element  $g$  of  $\mathcal{U}(M)$ , then both  $\widetilde{\underline{a}}$  and  $1 \cdot \widetilde{\underline{a}}$  represent  $g^{-1}$ . By definition,  $\|\widetilde{\underline{a}}\| = \|\underline{a}\|$  always holds, and the operation  $\sim$  on  $\mathcal{F}_{\pm}(M)$  is involutive. However, this operation does not restrict to  $\mathcal{F}(M)$ : if  $\underline{a}$  is a positive multifraction, then  $\widetilde{\underline{a}}$  is positive if, and only if,  $\|\underline{a}\|$  is even. In order to represent inverses inside  $\mathcal{F}(M)$ , one can compose  $\sim$  with a left-translation by 1, thus representing the inverse of  $\underline{a}$  by  $1 \cdot \widetilde{\underline{a}}$ . The inconvenience is that involutivity is lost: for  $\|\underline{a}\|$  odd and  $\underline{b} = 1 \cdot \widetilde{\underline{a}}$ , we find  $1 \cdot \widetilde{\underline{b}} = 1 \cdot \underline{a} \neq \underline{a}$ .

**Lemma 3.10.** If  $M$  is a gcd-monoid and  $\mathcal{R}^+(M)$  is semi-convergent, then  $\underline{a} \simeq \underline{b}$  is equivalent to  $1 \cdot \underline{a} \cdot \widetilde{\underline{b}} \Rightarrow^* \underline{1}$  for all  $\underline{a}, \underline{b}$  in  $\mathcal{F}_{\pm}(M)$ .

*Proof.* As we have  $1 \simeq \emptyset$  and  $\underline{b} \cdot \widetilde{\underline{b}} \simeq \emptyset$  by Lemma 3.9,  $\underline{a} \simeq \underline{b}$  is equivalent to  $1 \cdot \underline{a} \cdot \widetilde{\underline{b}} \simeq \emptyset$ , hence, by (2.6), to  $1 \cdot \underline{a} \cdot \widetilde{\underline{b}}$  being unital, *i.e.*, to  $\iota(1 \cdot \underline{a} \cdot \widetilde{\underline{b}}) = 1$ . By construction,  $1 \cdot \underline{a} \cdot \widetilde{\underline{b}}$  lies in  $\mathcal{F}(M)$ . So, if  $\mathcal{R}^+(M)$  is semi-convergent,  $\iota(1 \cdot \underline{a} \cdot \widetilde{\underline{b}}) = 1$  is equivalent to  $1 \cdot \underline{a} \cdot \widetilde{\underline{b}} \Rightarrow^* \underline{1}$ .  $\square$

As a direct application, we obtain

**Proposition 3.11.** If  $M$  is a gcd-monoid and  $\mathcal{R}^+(M)$  is semi-convergent, then  $M$  embeds in its group  $\mathcal{U}(M)$ .

*Proof.* Assume  $a, b \in M$  and  $\iota(a) = \iota(b)$ , *i.e.*,  $a \simeq b$ . By Lemma 3.10, we must have  $a \cdot \widetilde{b} \Rightarrow^* \underline{1}$ , which is  $a/b \Rightarrow^* 1/1$ . By definition of reduction, this means that there exists  $x$  in  $M$  satisfying  $a/b \bullet R_{1,x}^+ = 1/1$ . This implies  $a = b (= x)$ .  $\square$

*The Word Problem for  $\mathcal{U}(M)$ .* If  $S$  is any set, we denote by  $S^*$  the free monoid of all  $S$ -words, using  $\varepsilon$  for the empty word. For representing group elements, we introduce signed  $S$ -words, defined to be words in  $S \cup \overline{S}$ , where  $\overline{S}$  is a disjoint copy of  $S$  consisting of one letter  $\overline{s}$  for each letter  $s$  of  $S$ . If  $w$  is a signed  $S$ -word, we denote by  $\overline{w}$  the signed word obtained from  $w$  by exchanging  $s$  and  $\overline{s}$  and reversing the order of letters. If  $M$  is a monoid,  $S$  is included in  $M$ , and  $w$  is an  $S$ -word, we denote by  $[w]^+$  the evaluation of  $w$  in  $M$ . We extend this notation to signed  $S$ -words by defining  $[w]^+$  to be the multifraction  $[w_1]^+ / \cdots / [w_n]^+$ , where  $(w_1, \dots, w_n)$  is the unique sequence of positive  $S$ -words such that  $w$  can be decomposed as  $w_1 \overline{w_2} w_3 \overline{w_4} \cdots$  with  $w_i \neq \varepsilon$  for  $1 < i \leq n$ .

**Lemma 3.12.** [7, Lemma 2.5] *For every monoid  $M$ , every generating family  $S$  of  $M$ , and every signed  $S$ -word  $w$ , the following are equivalent:*

- (i) *The word  $w$  represents 1 in  $\mathcal{U}(M)$ ;*
- (ii) *The multifraction  $[w]^+$  satisfies  $[w]^+ \simeq 1$  in  $\mathcal{F}_\pm(M)$ .*

Thus solving the Word Problem for the group  $\mathcal{U}(M)$  with respect to the generating set  $S$  amounts to deciding the relation  $[w]^+ \simeq 1$ , which takes place inside  $M$ .

A few more definitions are needed. First, a gcd-monoid  $M$  is called *strongly noetherian* if there exists a map  $\lambda : M \rightarrow \mathbb{N}$  satisfying, for all  $a, b$  in  $M$ ,

$$(3.13) \quad \lambda(ab) \geq \lambda(a) + \lambda(b), \quad \text{and} \quad \lambda(a) > 0 \text{ for } a \neq 1.$$

This condition is stronger than noetherianity, but it still follows from the existence of a presentation by homogeneous relations (same length on both sides): in this case, the word length induces a map  $\lambda$  as in (3.13). So every Artin-Tits monoid is strongly noetherian.

Next, we need the notion of a primitive element. Noetherianity implies the existence of *atoms*, namely elements that cannot be expressed as the product of two non-invertible elements. One shows [14, Coro.II.2.59] that, if  $M$  is a noetherian gcd-monoid, then a subfamily  $S$  of  $M$  generates  $M$  if, and only if, it contains all atoms of  $M$ .

**Definition 3.14.** [9] If  $M$  is a noetherian gcd-monoid, an element  $a$  of  $M$  is called *right-primitive* if it belongs to the smallest family  $X$  that contains the atoms of  $M$  and is such that, if  $a, b$  belong to  $X$  and  $a \vee b$  exists, then the element  $a'$  defined by  $a \vee b = ba'$  still belongs to  $X$ . *Left-primitive* elements are defined symmetrically. We say that  $a$  is *primitive* if it is right- or left-primitive.

Note that, in the above definition, nothing is required when  $a \vee b$  does not exist. The key technical result is as follows:

**Lemma 3.15.** [7, Prop 3.27] *If  $M$  is a strongly noetherian gcd-monoid with finitely many primitive elements and atom set  $S$ , then the relation  $[w]^+ \Rightarrow^* \underline{1}$  on  $(S \cup \overline{S})^*$  is decidable.*

This result is *not* trivial, because deciding whether a multifraction is eligible for some reduction requires to decide whether two elements of the ground monoid admit a common multiple, and this is the point, where the finiteness of the number of primitive elements occurs crucially, as it provides an a priori upper bound on the size of this possible common multiple. Then, we immediately deduce:

**Proposition 3.16.** *If  $M$  is a strongly noetherian gcd-monoid with finitely many primitive elements and  $\mathcal{R}^+(M)$  is semi-convergent, then the Word Problem for  $\mathcal{U}(M)$  is decidable.*

*Proof.* Let  $S$  be the atom set of  $M$ . By Lemma 3.15, the relation  $[w]^+ \Rightarrow^* \underline{1}$  on signed  $S$ -words is decidable. By (3.4) (and by Lemma 2.18(i)),  $[w]^+ \Rightarrow^* \underline{1}$  is equivalent to  $[w]^+ \simeq 1$ . Finally, by Lemma 3.12,  $[w]^+ \simeq 1$  is equivalent to  $w$  representing 1 in  $\mathcal{U}(M)$ . Hence the latter relation is decidable.  $\square$

Note that, because the multifraction  $[w]^+$  is always defined to be positive, we only need semi-convergence for  $\mathcal{R}^+(M)$  in the above argument.

In the particular case of Artin-Tits monoids, we deduce

**Corollary 3.17.** *If Conjecture A is true, then the Word Problem for every Artin-Tits group is decidable.*

*Proof.* Let  $M$  be an Artin-Tits monoid. We noted that  $M$  is a gcd-monoid [3], and that it is strongly noetherian. Next,  $M$  has finitely many primitive elements: this follows from (and, actually, is equivalent to) the result that every Artin-Tits monoid has a finite Garside family [11, 16]. Hence  $M$  is eligible for Prop. 3.16.  $\square$

Less ambitiously, the implication of Prop. 3.16 applies to each individual Artin-Tits monoid: for lack of a general proof of Conjecture A, we obtain a solution of the Word Problem for every Artin-Tits group for which  $\mathcal{R}^+(M)$  can be proved to be semi-convergent.

Let us conclude with algorithmic complexity. Lemma 3.15 says nothing about the complexity of reduction. We show now the existence of an upper bound for the number of reductions.

**Lemma 3.18.** *If  $M$  is a strongly noetherian gcd-monoid with finitely many primitive elements, then the number of reduction steps from an  $n$ -multifraction  $\underline{a}$  is at most  $F_n(\lambda(a_1), \dots, \lambda(a_n))$ , where  $\lambda$  satisfies (3.13),  $C$  is the maximum of  $\lambda(a) + 1$  for a primitive in  $M$ , and  $F_n$  is inductively defined by  $F_1(x) = x + 2$  and  $F_n(x_1, \dots, x_n) = (x_1 + 1)C^{F_{n-1}(x_2, \dots, x_n)}$ .*

*Proof.* An easy induction shows that the function  $F_n$  is increasing with respect to each variable and, for every  $1 \leq i < n$ , it satisfies the inequality

$$(3.19) \quad F_n(Cx_1, \dots, Cx_i, x_{i+1} - 1, x_{i+2}, \dots, x_n) < F_n(x_1, \dots, x_n).$$

For  $\underline{a}$  an  $n$ -multifraction, write  $T(\underline{a})$  for the maximal number of reduction steps from  $\underline{a}$  and  $\lambda(\underline{a})$  for  $(\lambda(a_1), \dots, \lambda(a_n))$ . We prove using induction on  $\Rightarrow$  the inequality  $T(\underline{a}) \leq F_n(\lambda(\underline{a}))$  for every  $n$ -multifraction  $\underline{a}$ . Assume  $\underline{a} \bullet R_{i,x} = \underline{b}$  with  $x$  an atom of  $M$  (what can assumed without loss of generality). We compare the sequences  $\lambda(\underline{a})$  and  $\lambda(\underline{b})$ . By definition,  $b_{i+1}$  is a proper divisor of  $a_{i+1}$ , which implies  $\lambda(b_{i+1}) < \lambda(a_{i+1})$ . Next,  $a_i$  is the product of at most  $\lambda(a_i)$  primitive elements of  $M$ , hence so is  $b_i$ , implying  $\lambda(b_i) \leq C\lambda(a_i)$ . Finally,  $a_{i-1}$  is the product of at most  $\lambda(a_{i-1})$  primitive elements of  $M$ , hence  $b_{i-1}$  is the product of at most  $\lambda(a_{i-1}) + 1$  primitive elements, implying  $\lambda(b_{i-1}) \leq C\lambda(a_{i-1})$ . Then the induction hypothesis implies  $T(\underline{b}) \leq F_n(\underline{b})$ , so, plugging the upper bounds for  $b_i$ , and using that  $F_n$  is increasing and (3.19), we find

$$T(\underline{b}) \leq F_n(C\lambda(a_1), \dots, C\lambda(a_i), \lambda(a_{i+1}) - 1, \lambda(a_{i+2}), \dots, \lambda(a_n)) < F_n(\lambda(a_1), \dots, \lambda(a_n)) = F_n(\lambda(\underline{a})),$$

and  $T(\underline{a}) \leq T(\underline{b}) + 1$  implies  $T(\underline{a}) \leq F_n(\underline{a})$ .  $\square$

The upper bound of Lemma 3.3 is not polynomial (very far from statistical data, which suggest a quadratic bound), but it is not very high either in the hierarchy of fast growing functions (it is “primitive recursive”). From there, one can easily deduce a similar upper bound (tower of exponentials) for the Word Problem for  $\mathcal{U}(M)$  when  $\mathcal{R}^+(M)$  is semi-convergent.

*Property H.* One says [10, 12, 20] that *Property H* is true for a presentation  $(S, R)$  of a monoid  $M$  if a signed  $S$ -word  $w$  represents 1 in  $\mathcal{U}(M)$  if, and only if, the empty word can be obtained from  $w$  using *special* transformations, namely positive and negative equivalence and left and right reversing. Positive equivalence means replacing a positive factor of  $w$  (no letter  $s^{-1}$ ) with an  $R$ -equivalent word, negative equivalence means replacing the inverse of a positive factor with the inverse of an  $R$ -equivalent word, whereas right-reversing consists in deleting some length two factor  $\bar{s}s$  or replacing some length two factor  $\bar{s}t$  with  $\bar{v}\bar{u}$  such that  $sv = tu$  is a relation of  $R$ , and left-reversing consists in deleting some length two factor  $s\bar{s}$  or replacing some length two factor  $s\bar{t}$  with  $\bar{u}\bar{v}$  such that  $vs = ut$  is a relation of  $R$ . Roughly speaking, what Property H says is that a word representing 1 can be transformed into the empty word without introducing new trivial factors  $s\bar{s}$  or  $\bar{s}s$ , a situation directly reminiscent of Dehn’s algorithm for hyperbolic groups, see [12, Section 1.2].

Say that a presentation  $(S, R)$  of a monoid  $M$  is a *right-lcm presentation* if  $R$  consists of one relation  $su = tv$  for each pair of generators  $s, t$  that admit a common right-multiple, with  $su$  and  $tv$  representing  $s \vee t$ . The standard presentation of an Artin-Tits monoid is a right-lcm presentation, and, symmetrically, a left-lcm presentation.

**Proposition 3.20.** *If  $M$  is a gcd-monoid and  $\mathcal{R}^+(M)$  is semi-convergent, Property H is true for every presentation of  $M$  that is an lcm presentation on both sides.*

The point is that applying a rule  $R_{i,x}$  to a multifraction  $[w]^+$  can be decomposed into a sequence of special transformations as defined above. The argument is the same as in the case when  $\mathcal{R}(M)$  is convergent [7, Prop. 5.19], and we do not repeat it.

Thus Conjecture A would imply the statement conjectured in [10]:

**Corollary 3.21.** *If Conjecture A is true, Property H is true for every Artin-Tits presentation.*

Let us mention the recent result [20] that Property H is true for the standard presentation of every sufficiently large Artin-Tits monoid, meaning lying in the closure of the family of Artin-Tits monoids of large type (no commutation relation) under amalgamated products over abelian standard parabolic subgroups.

**3.4. Alternative forms.** Here we mention several variants of semi-convergence.

**Proposition 3.22.** *If  $M$  is a noetherian gcd-monoid, then  $\mathcal{R}(M)$  (resp.  $\mathcal{R}^+(M)$ ) is semi-convergent if, and only if, for every  $\underline{a}$  in  $\mathcal{F}_\pm(M)$  (resp.  $\mathcal{F}(M)$ ),*

$$(3.23) \quad \text{If } \underline{a} \text{ is unital, then } \underline{a} \text{ is either trivial or reducible.}$$

*Proof.* Assume that  $\mathcal{R}(M)$  is semi-convergent, and let  $\underline{a}$  be a nontrivial unital multifraction in  $\mathcal{F}_\pm(M)$ . By definition,  $\underline{a} \Rightarrow^* \underline{1}$  holds. As  $\underline{a}$  is nontrivial, the reduction requires at least one step, so  $\underline{a}$  cannot be  $\mathcal{R}$ -irreducible. Hence, (3.23) is satisfied.

Conversely, assume (3.23). As  $M$  is noetherian, the rewrite system  $\mathcal{R}(M)$  is terminating, *i.e.*, the relation  $\Rightarrow$  admits no infinite descending sequence. Hence we can use induction on  $\Rightarrow$  to establish (3.4). So let  $\underline{a}$  be a unital multifraction in  $\mathcal{F}_\pm(M)$ . If  $\underline{a}$  is  $\Rightarrow$ -minimal, *i.e.*, if  $\underline{a}$  is  $\mathcal{R}$ -irreducible, then, by (3.23),  $\underline{a}$  must be trivial, *i.e.*, we have  $\underline{a} = \underline{1}$ , whence  $\underline{a} \Rightarrow^* \underline{1}$ . Otherwise,  $\underline{a}$  is  $\mathcal{R}$ -reducible, so there exist  $i, x$  such that  $\underline{b} = \underline{a} \bullet R_{i,x}$  is defined. By construction,  $\underline{b}$  is  $\simeq$ -equivalent to  $\underline{a}$ , hence it is unital. By the induction hypothesis, we have  $\underline{b} \Rightarrow^* \underline{1}$ . By transitivity of  $\Rightarrow^*$ , we deduce  $\underline{a} \Rightarrow^* \underline{1}$ . Hence  $\mathcal{R}(M)$  is semi-convergent.

The proof is similar for  $\mathcal{R}^+(M)$ . □

Condition (3.23) can be restricted to more special unital multifractions.

**Definition 3.24.** Call a multifraction  $\underline{a}$  *prime* if, for every  $i$  that is positive (resp. negative) in  $\underline{a}$ , the entries  $a_i$  and  $a_{i+1}$  admit no nontrivial common right- (resp. left-) divisor.

Since dividing adjacent entries by a common factor is a particular case of reduction, an  $\mathcal{R}$ -irreducible multifraction must be prime. The converse need not be true: for instance, the 6-multifraction  $\underline{b}$  of Example ?? is prime, and it is  $\mathcal{R}$ -reducible.

**Proposition 3.25.** *If  $M$  is a noetherian gcd-monoid, then  $\mathcal{R}(M)$  (resp.  $\mathcal{R}^+(M)$ ) is semi-convergent if, and only if, for every  $\underline{a}$  in  $\mathcal{F}_\pm(M)$  (resp.  $\mathcal{F}(M)$ ),*

$$(3.26) \quad \text{If } \underline{a} \text{ is unital and prime, then } \underline{a} \text{ is either trivial or reducible.}$$

*Proof.* By Prop. 3.22, the condition is necessary, since (3.26) is subsumed by (3.23). For the converse implication, assume (3.26). As for Prop. 3.22, we establish (3.4) using induction on  $\Rightarrow$ . Let  $\underline{a}$  be a unital multifraction in  $\mathcal{F}_\pm(M)$ . If  $\underline{a}$  is  $\mathcal{R}$ -irreducible, then it must be prime, for, otherwise, it is eligible for at least one division, which is a special case of reduction. Hence,  $\underline{a}$  must be  $\underline{1}$  by (3.26). Otherwise,  $\underline{a}$  is  $\mathcal{R}$ -reducible, there exist  $i, x$  such that  $\underline{b} = \underline{a} \bullet R_{i,x}$  is defined, the induction hypothesis implies  $\underline{b} \Rightarrow^* \underline{1}$ , hence  $\underline{a} \Rightarrow^* \underline{1}$ . Hence  $\mathcal{R}(M)$  is semi-convergent. The proof for  $\mathcal{R}^+(M)$  is similar. □

**Corollary 3.27.** *Conjecture A is true if, and only if, (3.26) holds for every Artin-Tits monoid  $M$  and every  $\underline{a}$  in  $\mathcal{F}(M)$ .*

We turn to another approach. Whenever the ground monoid  $M$  is noetherian, the rewrite systems  $\mathcal{R}(M)$  and  $\mathcal{R}^+(M)$  are terminating, hence they are convergent if, and only if, they are *confluent*, meaning that

$$(3.28) \quad \text{If we have } \underline{a} \Rightarrow^* \underline{b} \text{ and } \underline{a} \Rightarrow^* \underline{c}, \text{ there exists } \underline{d} \text{ satisfying } \underline{b} \Rightarrow^* \underline{d} \text{ and } \underline{c} \Rightarrow^* \underline{d}$$

(“diamond property”). We now observe that semi-convergence is equivalent to a weak form of confluence involving the unit multifractions  $\underline{1}$ .

**Proposition 3.29.** *If  $M$  is a gcd-monoid, then  $\mathcal{R}(M)$  (resp.  $\mathcal{R}^+(M)$ ) is semi-convergent if, and only if, for every  $\underline{a}$  in  $\mathcal{F}_\pm(M)$  (resp.  $\mathcal{F}(M)$ ),*

$$(3.30) \quad \text{The conjunction of } \underline{a} \Rightarrow^* \underline{b} \text{ and } \underline{a} \Rightarrow^* \underline{1} \text{ implies } \underline{b} \Rightarrow^* \underline{1}.$$

Relation (3.30) can be called  $\underline{1}$ -confluence for  $\underline{a}$ , since it corresponds to the special case  $\underline{c} = \underline{1}$  of (3.28): indeed, (3.28) with  $\underline{c} = \underline{1}$  claims the existence of  $\underline{d}$  satisfying  $\underline{b} \Rightarrow^* \underline{d}$  and  $\underline{1} \Rightarrow^* \underline{d}$ , and, as  $\underline{1}$  is  $\mathcal{R}^+$ -irreducible, we must have  $\underline{d} = \underline{1}$ , whence  $\underline{b} \Rightarrow^* \underline{1}$ , as asserted in (3.30). In order to establish Prop. 3.29, we need an auxiliary result, which connects  $\simeq$  with the symmetric closure of  $\Rightarrow^*$  and is a sort of converse for Lemma 2.18.

**Lemma 3.31.** *If  $M$  is a gcd-monoid and  $\underline{a}, \underline{b}$  belong to  $\mathcal{F}_\pm(M)$ , then  $\underline{a} \simeq \underline{b}$  holds if, and only if, there exist a finite sequence  $\underline{c}^0, \dots, \underline{c}^{2r}$  in  $\mathcal{F}_\pm(M)$  and  $p, q$  in  $\mathbb{Z}$  satisfying*

$$(3.32) \quad \underline{a} \cdot \underline{1}_p = \underline{c}^0 \Rightarrow^* \underline{c}^1 \ast \leftarrow \underline{c}^2 \Rightarrow^* \dots \ast \leftarrow \underline{c}^{2r} = \underline{b} \cdot \underline{1}_q.$$

*Proof.* For  $\underline{a}, \underline{b}$  in  $\mathcal{F}_\pm(M)$ , write  $\underline{a} \hat{\Rightarrow}^* \underline{b}$  if  $\underline{a} \cdot \underline{1}_p \Rightarrow^* \underline{b} \cdot \underline{1}_p$  holds for some  $p$ . By Lemma 2.18,  $\underline{a} \Rightarrow^* \underline{b}$  implies  $\underline{a} \cdot \underline{1}_r \Rightarrow^* \underline{b} \cdot \underline{1}_r$  and, therefore, the relation  $\hat{\Rightarrow}^*$  is transitive. It is also compatible with multiplication: on the left, this follows from Lemma 2.18 directly. On the right,  $\underline{a} \Rightarrow^* \underline{b} \cdot \underline{1}_p$  implies  $\underline{a} \cdot \underline{c} \Rightarrow^* \underline{b} \cdot \underline{1}_p \cdot \underline{c}$  for every  $\underline{c}$ , and we observe that  $\underline{1}_p \cdot \underline{c} \Rightarrow^* \underline{c} \cdot \underline{1}_p$  always holds. Hence, the symmetric closure  $\approx$  of  $\hat{\Rightarrow}^*$  is a congruence on  $\mathcal{F}_\pm(M)$ . As we have  $1 \Rightarrow^* \emptyset \cdot 1$ ,  $a/a \Rightarrow^* \emptyset \cdot \underline{1}_2$  and  $/a/a \Rightarrow^* \emptyset \cdot \underline{1}_{-2}$  for every  $a$  in  $M$ , the congruence  $\approx$  contains pairs that generate  $\simeq$ . Hence  $\underline{a} \simeq \underline{b}$  implies the existence of a zigzag in  $\hat{\Rightarrow}^*$  and its inverse connecting  $\underline{a}$  to  $\underline{b}$ . Taking the maximum of  $|r|$  for  $\underline{1}_r$  occurring in the zigzag, one obtains (3.32).  $\square$

*Proof of Prop. 3.29.* Assume that  $\mathcal{R}(M)$  is semi-convergent, and we have  $\underline{a} \Rightarrow^* \underline{b}$  and  $\underline{a} \Rightarrow^* \underline{1}$ . By Lemma 2.18, we have  $\underline{a} \simeq \underline{b}$  and  $\underline{a} \simeq \underline{1}$ , hence  $\underline{b} \simeq \underline{1}$ . As  $\mathcal{R}(M)$  is semi-convergent, this implies  $\underline{b} \Rightarrow^* \underline{1}$ . So (3.30) is satisfied, and  $\mathcal{R}(M)$  is  $\underline{1}$ -confluent.

Conversely, assume that  $\mathcal{R}(M)$  is  $\underline{1}$ -confluent. We first show using induction on  $k$  that, when we have a zigzag  $\underline{c}^0 \Rightarrow^* \underline{c}^1 \ast \leftarrow \underline{c}^2 \Rightarrow^* \underline{c}^3 \ast \leftarrow \underline{c}^4 \Rightarrow^* \dots$ , then  $\underline{c}^0 = \underline{1}$  implies  $\underline{c}^k \Rightarrow^* \underline{1}$  for every  $k$ . For  $k = 0$ , this is the assumption. For  $k$  even non-zero, we obtain  $\underline{c}^k \Rightarrow^* \underline{c}^{k-1} \Rightarrow^* \underline{1}$  using the induction hypothesis, whence  $\underline{c}^k \Rightarrow^* \underline{1}$  by transitivity of  $\Rightarrow^*$ . For  $k$  odd, we have  $\underline{c}^{k-1} \Rightarrow^* \underline{1}$  by the induction hypothesis and  $\underline{c}^{k-1} \Rightarrow^* \underline{c}^k$ , whence  $\underline{c}^k \Rightarrow^* \underline{1}$  by  $\underline{1}$ -confluence.

Now assume that  $\underline{a}$  is unital. Lemma 3.31 provides  $p, q, r$  and  $\underline{c}^0, \dots, \underline{c}^{2r}$  satisfying

$$\underline{1}_p = \underline{c}^0 \Rightarrow^* \underline{c}^1 \ast \leftarrow \underline{c}^2 \Rightarrow^* \dots \ast \leftarrow \underline{c}^{2r} = \underline{a} \cdot \underline{1}_q.$$

As shown above, we deduce  $\underline{a} \cdot \underline{1}_q \Rightarrow^* \underline{1}$ , whence  $\underline{a} \Rightarrow^* \underline{1}$  by Lemma 2.18. Hence  $\mathcal{R}(M)$  is semi-convergent.

Once again, the proof for  $\mathcal{F}(M)$  is the same.  $\square$

**Corollary 3.33.** *Conjecture **A** is true if, and only if, (3.30) holds for every Artin-Tits monoid  $M$  and every  $\underline{a}$  in  $\mathcal{F}(M)$ .*

Prop. 3.29 is important for testing Conjecture **A**, because it shows that, if  $\underline{a} \Rightarrow^* \underline{1}$  holds, then every sequence of reductions from  $\underline{a}$  inevitably leads to  $\underline{1}$ . In other words, any reduction strategy may be applied without loss of generality.

## 4. DIVISIONS AND TAME REDUCTIONS

When the system  $\mathcal{R}(M)$  is not convergent, it is not either confluent, and a multifraction may admit several reducts with no subsequent common reduct. However, by restricting to particular reductions, we can retrieve a (weak) form of confluence and let distinguished reducts appear. Investigated in Subsection 4.1, divisions provide a typical example. We show that, for every multifraction  $\underline{a}$ , there exists a well-defined maximal reduct  $\partial\underline{a}$  of  $\underline{a}$  accessible from  $\underline{a}$  by divisions. Then, in Subsection 4.2, we extend divisions into what we call tame reductions, which are those reductions that, in a sense, exclude no subsequent reduction opportunities. This leads us in Subsection 4.3 to Conjecture **B** about tame reductions from unital multifractions, which is stronger but more precise than Conjecture **A**.

**4.1. Divisions.** Divisions are the special cases of reduction when no remainder appears.

**Definition 4.1.** If  $M$  is a gcd-monoid and  $\underline{a}, \underline{b}$  belong to  $\mathcal{F}_{\pm}(M)$ , we declare that  $\underline{b} = \underline{a} \bullet D_{i,x}$  holds if we have  $\underline{b} = \underline{a} \bullet R_{i,x}$  and, in addition,  $x$  right- (*resp.* left-) divides  $a_i$  if  $i$  is positive (*resp.* negative) in  $\underline{a}$ . We use  $\mathcal{D}(M)$  for the family of all  $D_{i,x}$  with  $x \neq 1$ , write  $\underline{a} \Rightarrow_{\text{div}} \underline{b}$  if some rule of  $\mathcal{D}(M)$  maps  $\underline{a}$  to  $\underline{b}$ , and  $\Rightarrow_{\text{div}}^*$  for the reflexive–transitive closure of  $\Rightarrow_{\text{div}}$ .

So  $\underline{a} \bullet D_{i,x}$  is defined if, and only if,  $x$  divides  $a_i$  and  $a_{i+1}$  (on the due side), and applying  $D_{i,x}$  means dividing  $a_i$  and  $a_{i+1}$  by  $x$ . By definition,  $\underline{a}$  is  $\mathcal{D}$ -irreducible if, and only if, it is prime (Def. 3.24), *i.e.*, the gcds of adjacent entries (on the relevant side) are trivial.

Except in degenerated cases (*e.g.*, free monoids), the system  $\mathcal{D}(M)$  is not convergent: typically, for  $M = \langle \mathbf{a}, \mathbf{b} \mid \mathbf{aba} = \mathbf{bab} \rangle^+$  (Artin’s 3-strand braid monoid) and  $\underline{a} = \mathbf{a}/\mathbf{aba}/\mathbf{b}$ , we find  $\underline{a} \bullet D_{2,\mathbf{b}} = \mathbf{a}/\mathbf{ab}/1$  and  $\underline{a} \bullet D_{1,\mathbf{a}} = 1/\mathbf{ab}/\mathbf{b}$ , with no further division, and confluence can be restored only at the expense of applying some reduction  $R_{i,x}$ , here  $\underline{a} \bullet D_{2,\mathbf{b}} = \underline{a} \bullet D_{1,\mathbf{a}} R_{2,\mathbf{b}}$ . However, we shall see now that, for every multifraction  $\underline{a}$ , there exists a unique, well-defined maximal  $\mathcal{R}$ -reduct of  $\underline{a}$  that can be obtained using divisions.

It directly follows from the definition that, if  $M$  is a gcd-monoid and  $\underline{a}$  lies in  $\mathcal{F}_{\pm}(M)$ , then, for every  $i < \|\underline{a}\|$ , there exists a greatest division applying to  $\underline{a}$  at level  $i$ , namely  $D_{i,x}$  with  $x = a_i \tilde{\wedge} a_{i+1}$  (*resp.*  $a_i \wedge a_{i+1}$ ) if  $i$  is positive (*resp.* negative) in  $\underline{a}$ . Indeed, assuming for instance  $i$  positive in  $\underline{a}$ , the multifraction  $\underline{a} \bullet D_{i,x}$  is defined if, and only if,  $x$  right-divides both  $a_i$  and  $a_{i+1}$ , hence if, and only if,  $x$  right-divides the right-gcd  $a_i \tilde{\wedge} a_{i+1}$ .

**Notation 4.2.** if  $M$  is a gcd-monoid and  $\underline{a}$  is a multifraction on  $M$ , then, for  $i < \|\underline{a}\|$ , we write  $\underline{a} \bullet D_i^{\max}$  for  $\underline{a} \bullet D_{i,x}$  with  $x$  the gcd of  $a_i$  and  $a_{i+1}$  (on the due side).

**Lemma 4.3.** Say that a multifraction  $\underline{a}$  is  $j$ -prime if  $\underline{a} \bullet D_{j,y}$  is defined for no  $y \neq 1$ . If  $\underline{a}$  is  $j$ -prime, then so is  $\underline{a} \bullet D_{i,x}$  for all  $i, x$ .

*Proof.* Assume for instance  $j$  positive in  $\underline{a}$ , and let  $\underline{b} = \underline{a} \bullet D_{i,x}$ . We have either  $b_j = a_j$  (for  $i < j - 1$  and  $i > j$ ) or  $b_j \leq a_j$  (for  $i = j - 1$ ); similarly, we have either  $b_{j+1} = a_{j+1}$  (for  $i < j$  and  $i > j + 1$ ) or  $b_{j+1} \leq a_{j+1}$  (for  $i = j + 1$ ). Hence, in all cases, the assumption  $a_i \tilde{\wedge} a_{i+1} = 1$  implies  $b_i \tilde{\wedge} b_{i+1} = 1$ .  $\square$

Hence, if we start with  $\underline{a}$  and apply, in any order, maximal divisions  $D_i^{\max}$  in such a way that every level between 1 and  $\|\underline{a}\| - 1$  is visited at least one, we always finish with a prime multifraction. The latter may depend on the order of the divisions, but the following result shows that there exists a preferred choice.

**Proposition 4.4.** Let  $M$  be a gcd-monoid. For  $\underline{a}$  of depth  $n$  in  $\mathcal{F}_{\pm}(M)$ , put

$$(4.5) \quad \partial\underline{a} := \underline{a} \bullet D_{n-1}^{\max} D_{n-2}^{\max} \cdots D_1^{\max}.$$

Then,  $\partial\underline{a}$  is prime, and, for every  $\underline{b}$  in  $\mathcal{F}_{\pm}(M)$ ,

$$(4.6) \quad \underline{a} \Rightarrow_{\text{div}}^* \underline{b} \quad \text{implies} \quad \partial\underline{b} \Rightarrow^* \partial\underline{a}.$$

We begin with preparatory confluence results involving divisions and reductions.

**Lemma 4.7.** *Assume that both  $\underline{a} \bullet R_{i+1,x}$  and  $\underline{a} \bullet D_{i,y}$  are defined. Then we have  $\underline{a} \bullet R_{i+1,x} D_{i,z} = \underline{a} \bullet D_{i,y} R_{i+1,x}$ , where  $z$  is determined by the equalities  $a_i = ay$ ,  $x \vee a = av$ , and  $vz = v \vee y$  (resp.  $a_i = ya$ ,  $x \tilde{\vee} a = va$ , and  $zv = v \tilde{\vee} y$ ) for  $i$  positive (resp. negative) in  $\underline{a}$ . Moreover, if  $y$  is maximal for  $\underline{a}$ , meaning that  $y$  is the gcd of  $a_i$  and  $a_{i+1}$  on the due side, then  $z$  is maximal for  $\underline{a} \bullet R_{i+1,x}$ .*

*Proof.* (Fig. 2) Assume that  $i$  is positive in  $\underline{a}$ , so  $a_{i+1}$  is negative in  $\underline{a}$ . Put  $\underline{b} := \underline{a} \bullet R_{i+1,x}$  and  $\underline{c} := \underline{a} \bullet D_{i,y}$ . By definition, there exists  $x'$  satisfying

$$\begin{aligned} b_{i-1} &= a_{i-1}, & b_i &= a_i x', & x b_{i+1} &= a_{i+1} x' = x \vee a_{i+1}, & x b_{i+2} &= a_{i+2}, \\ c_{i-1} &= a_{i-1}, & c_i y &= a_i, & c_i y &= a_i, & c_{i+2} &= a_{i+2}. \end{aligned}$$

As  $a_{i+1}$  is  $c_{i+1}y$ , Lemma 2.11 implies the existence of  $u$ ,  $v$ , and  $z$  satisfying

$$(4.8) \quad b_{i+1} = uz \quad \text{with} \quad c_{i+1}v = xu = x \vee c_{i+1} \quad \text{and} \quad yx' = vz = y \vee v.$$

By construction, we have  $b_i = a_i x' = c_i y x' = c_i v z$ , which shows that  $z$  right-divides both  $b_i$  and  $b_{i+1}$ . Hence  $\underline{d} := \underline{b} \bullet D_{i,z}$  is defined, and we have

$$(4.9) \quad d_{i-1} = a_{i-1}, \quad d_i = c_v, \quad d_{i+1} = u, \quad d_{i+1} = b_{i+2}.$$

On the other hand, by assumption,  $x$  left-divides  $c_{i+2}$ , which is  $a_{i+2}$ , and  $x$  and  $c_{i+1}$  admit a common right-multiple, namely their right-lcm  $xu$ . Hence,  $\underline{c} \bullet R_{i,x}$  is defined, and comparing with (4.9) directly yields the expected equality  $\underline{c} \bullet R_{i,x} = \underline{b} \bullet D_{i,z}$ .

It remains to prove that, if  $y$  is maximal for  $\underline{a}$ , then  $z$  is maximal for  $\underline{b}$ . So assume  $y = a_i \tilde{\wedge} a_{i+1}$ . We deduce  $c_i \tilde{\wedge} c_{i+1} = 1$ . On the other hand, by Lemma 3.1, the assumption  $c_{i+1}v = c_{i+1} \vee x$  implies  $u \tilde{\wedge} v = 1$ . Then (the symmetric counterpart of) Lemma 2.13 implies  $c_i v \tilde{\wedge} u = 1$ , whence  $b_i \tilde{\wedge} b_{i+1} = z$ .

Of course, a symmetric proof applies when  $i$  is negative in  $\underline{a}$ . □

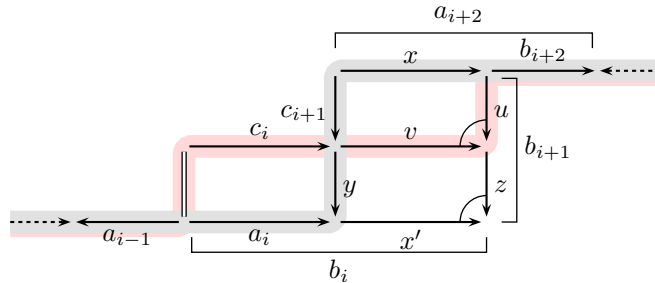


FIGURE 2. Proof of Lemma 4.7: if  $\underline{a}$  is eligible both for  $R_{i+1,x}$  and  $D_{i,y}$ , we can start with either and converge to the colored path.

**Lemma 4.10.** *Assume  $\underline{b} = \underline{a} \bullet R_{i,x}$ , and  $j \neq i, i+1, i-2$  (resp.  $j = i-2$ ). Put  $\underline{a}' := \underline{a} \bullet D_j^{\max}$  and  $\underline{b}' := \underline{b} \bullet D_j^{\max}$ . Then we have  $\underline{b}' = \underline{a}' \bullet R_{i,x}$  (resp.  $\underline{b}' = \underline{a}' \bullet R_{i,x} D_{i-2}^{\max}$ ).*

*Proof.* For  $j \leq i-3$  or  $j \geq i+2$ , the reduction  $R_{i,x}$  does not change the  $j$ th and  $(j+1)$ st entries, so the greatest division at level  $j$  remains the same, and commutation is straightforward. For  $j = i-1$ , Lemma 4.7 gives  $\underline{b}' = \underline{a}' \bullet R_{i,x}$ .

Assume  $j = i-2$ . Then the reduction  $R_{i,x}$  does not change the  $j$ th entry, but it possibly increases the  $(j+1)$ st entry. So, if  $D_{j,y}$  is the maximal  $j$ -division for  $\underline{a}$ , then  $D_{j,y}$  applies to  $\underline{b}$ , but it need not be the maximal  $j$ -division for  $\underline{b}$ . Expanding the definitions, we obtain the commutation relation  $\underline{a} \bullet R_{i,x} D_{j,y} = \underline{a} \bullet D_{j,y} R_{i,x}$ , meaning  $\underline{b} \bullet D_{j,y} = \underline{a}' \bullet R_{i,x}$ , together with  $\underline{b}' = (\underline{b} \bullet D_{j,y}) D_j^{\max}$ , whence  $\underline{b}' = \underline{a}' \bullet R_{i,x} D_{i-2}^{\max}$ , as expected. □

**Lemma 4.11.** *If  $\underline{a}$  and  $\underline{b}$  are  $j$ -prime for  $j \geq i$ , then  $b = \underline{a} \bullet R_{i+1,x}$  implies  $\partial \underline{a} \Rightarrow^* \partial \underline{b}$ .*



*Proof.* By assumption, we have  $\partial \underline{a} = \underline{a} \cdot D_{i-1}^{\max} \cdots D_1^{\max}$  and  $\partial \underline{b} = \underline{b} \cdot D_{i-1}^{\max} \cdots D_1^{\max}$ . Put  $\underline{a}^i := \underline{a}$ ,  $\underline{b}^i := \underline{b}$  and, inductively,  $\underline{a}^j = \underline{a}^{j+1} \cdot D_j^{\max}$ ,  $\underline{b}^j = \underline{b}^{j+1} \cdot D_j^{\max}$  for  $j$  decreasing from  $i-1$  to  $1$ , yielding  $\partial \underline{a} = \underline{a}^1$  and  $\partial \underline{b} = \underline{b}^1$ . We prove using induction on  $j$  decreasing from  $i$  to  $1$  that, for every  $y$ , there exist  $x_0, x_1, \dots, x_k$  with  $i-2k > j$  satisfying  $\underline{b}^j = \underline{a}^j \cdot R_{i+1, x_0} R_{i-1, x_1} \cdots R_{x-2k+1, x_k}$ . By assumption, the property is true for  $j = i$ , with  $k = 0$  and  $x_0 = x$ . Assume  $i > j \geq 1$ . By induction hypothesis, we have  $\underline{b}^{j+1} = \underline{a}^{j+1} \cdot R_{i+1, x_0} R_{i-1, x_1} \cdots R_{i-2k+1, x_k}$  for some  $x_0, \dots, x_k$  with  $i-2k > j+1$ . By repeated applications of Lemma 4.10, we deduce that each reduction  $R_{i-2\ell+1, x_\ell}$  commutes with  $D_j^{\max}$ , except the last one in the case  $i-2k = j+2$ , in which case Lemma 4.10 prescribes to add one more reduction (a division)  $R_{i-2k-1, x_{k+1}}$ . In this way, we obtain either  $\underline{b}^j = \underline{a}^j \cdot R_{i+1, x_0} R_{i-1, x_1} \cdots R_{i-2k+1, x_k}$ , or  $\underline{b}^j = \underline{a}^j \cdot R_{i+1, x_0} R_{i-1, x_1} \cdots R_{i-2k+1, x_k} R_{i-2k-1, x_{k+1}}$ , and the induction continues.  $\square$

We can now complete the argument.

*Proof of Prop. 4.4.* It follows from the definition that  $\underline{a} \cdot D_{n-1}^{\max}$  is  $(n-1)$ -prime, then that  $\underline{a} \cdot D_{n-1}^{\max} D_{n-2}^{\max}$  is  $(n-1)$ - and  $(n-2)$ -prime, etc., hence that  $\partial \underline{a}$  is  $i$ -prime for  $1 \leq i < n$ , hence it is prime.

We now establish (4.6), *i.e.*, prove that  $\underline{a} \Rightarrow_{\text{div}}^* \underline{b}$  implies  $\partial \underline{b} \Rightarrow^* \partial \underline{a}$ . For an induction, it is sufficient to prove that  $\underline{a} \Rightarrow_{\text{div}} \underline{b}$  implies  $\partial \underline{b} \Rightarrow^* \partial \underline{a}$ . So, we assume  $\underline{b} = \underline{a} \cdot D_{i,z}$ , and aim at proving  $\partial \underline{b} \Rightarrow_{\text{div}}^* \partial \underline{a}$ . We compare the successive divisions occurring in the determination of  $\partial \underline{a}$  and  $\partial \underline{b}$ . Assuming  $\|\underline{a}\| = n$ , put  $\underline{a}^n := \underline{a}$ ,  $\underline{b}^n := \underline{b}$ , and let  $\underline{a}^i$  (*resp.*  $\underline{b}^i$ ) be obtained from  $\underline{a}^{i+1}$  (*resp.*  $\underline{b}^{i+1}$ ) by applying  $D_i^{\max}$ , so that we finally have  $\partial \underline{a} = \underline{a}^1$  and  $\partial \underline{b} = \underline{b}^1$ . We assume that  $i$  is positive in  $\underline{a}$ .

Consider first  $j \geq i+2$ . Applying Lemma 4.10, we inductively obtain  $\underline{b}^j = \underline{a}^j \cdot D_{i,z}$ , implying the commutativity of the  $n-i-2$  left hand squares in the diagram of Fig. 3.

Now let  $x = \underline{a}^{i+1} \wedge \underline{a}^{i+2}$ . By definition, we have  $\underline{a}^{i+1} = \underline{a}^{i+2} \cdot D_{i+1,x}$ . By Lemma 4.7, there exists  $\underline{c}$  and  $z'$  satisfying  $\underline{c} = \underline{a}^{i+2} \cdot D_{i+1,x} D_{i,z'} = \underline{a}^{i+2} \cdot D_{i,z} R_{i+1,x}$ , which reads  $\underline{c} = \underline{a}^{i+1} D_{i,z'} = \underline{b}^{i+2} \cdot R_{i+1,x}$ .

Next,  $\underline{c}$  is obtained from  $\underline{a}^{i+1}$  by some  $i$ -division, whereas  $\underline{a}^i$  is obtained from  $\underline{a}^{i+1}$  by the maximal  $i$ -division, hence  $\underline{a}^i$  must be obtained from  $\underline{c}$  by some further  $i$ -division, namely the maximal  $i$ -division for  $\underline{c}$ . So,  $\underline{a}^i = \underline{c} \cdot D_i^{\max}$  holds.

On the other hand,  $\underline{b}^{i+1}$  is obtained from  $\underline{b}^{i+2}$  by the maximal  $(i+1)$ -division, namely  $D_{i+1,y}$  with  $y = b_{i+1} \wedge b_{i+2}$ . As we have  $x = a_{i+2} \wedge a_{i+2}$  and  $b_{i+2} = a_{i+2}$ , the relation  $b_{i+1} \leq a_{i+1}$  implies  $y \leq x$ , say  $x = yx'$ . It follows that reducing  $x$  at level  $i+1$  in  $\underline{b}^{i+2}$  amounts to first dividing by  $y$  and then reducing  $x'$ , *i.e.*, we have  $\underline{b} \cdot R_{i+1,x} = \underline{b} \cdot D_{i+1,y} R_{i+1,x'} = \underline{b}^{i+1} \cdot R_{i+1,x'}$ .

Now, two reductions apply to  $\underline{b}^{i+1}$ , namely  $R_{i+1,x'}$ , which leads to  $\underline{c}$ , and  $D_i^{\max}$ , which is, say,  $D_{i+1,y}$  and leads to  $\underline{b}^i$ . Applying Lemma 4.7 again, we obtain the existence of  $y'$  satisfying  $\underline{b}^{i+1} \cdot R_{i+1,x'} D_{i,y'} = \underline{b}^{i+1} \cdot D_{i,y} R_{i+1,x'}$ , which boils down to  $\underline{c} \cdot D_{i,y'} = \underline{b}^i \cdot R_{i+1,x'}$ . Moreover, as  $D_{i,y}$  is the maximal  $i$ -division applying to  $\underline{b}^{i+1}$ , Lemma 4.7 implies that  $D_{i,y'}$  is the maximal  $i$ -division applying to  $\underline{c}$ . We obtained above  $\underline{a}^i = \underline{c} \cdot D_i^{\max}$ , so we deduce  $\underline{a}^i = \underline{b}^i \cdot R_{i+1,x'}$ .

From there, we are in position for applying Lemma 4.11 (with  $\underline{a}$  and  $\underline{b}$  interchanged): we have  $\underline{a}^i = \underline{b}^i \cdot R_{i+1,x'}$ , and, by construction,  $\underline{a}^i$  and  $\underline{b}^i$  are  $j$ -prime for every  $j \geq i$ . Then Lemma 4.11 ensures  $\underline{b}^1 \Rightarrow^* \underline{a}^1$ , which is  $\partial \underline{b} \Rightarrow^* \partial \underline{a}$ .  $\square$

**Example 4.12.** In the Artin-Tits monoid of type  $\tilde{A}_2$ , let  $\underline{a} = \text{ab/aba/aca}$ . One finds  $\partial \underline{a} = \text{ab/ba/ca}$ , an  $\mathcal{R}$ -irreducible multifraction. If we denote by  $\text{Irr}(\underline{a})$  the family of all  $\mathcal{R}$ -irreducible reducts of  $\underline{a}$ , this example shows that  $\text{Irr}(\partial \underline{a})$ , which, by construction, is included in  $\text{Irr}(\underline{a})$ , since  $\underline{a} \Rightarrow^* \partial \underline{a}$  holds, may be a strict subset of the latter: we have  $\underline{a} \cdot D_{1,\text{ba}} R_{2,c} = \text{cb/bc/ac}$ , whence  $\text{Irr}(\underline{a}) = \{\partial \underline{a}, \text{cb/bc/ac}\}$ , but  $\text{Irr}(\partial \underline{a}) = \{\partial \underline{a}\}$ . On the other hand, this example also shows that  $\underline{a} \Rightarrow_{\text{div}} \underline{b}$  does not imply  $\text{Irr}(\partial \underline{b}) \subseteq \text{Irr}(\partial \underline{a})$ : for  $\underline{b} = \underline{a} \cdot D_{1,\text{ba}} = 1/\text{b/aca}$ , we have  $\text{Irr}(\partial \underline{b}) = \text{Irr}(\underline{b}) = \{\text{cb/bc/ac}, \text{ab/ba/ca}\} \not\subseteq \text{Irr}(\partial \underline{a}) = \{\text{ab/ba/ca}\}$ . This is the most we can expect, since  $\underline{a} \Rightarrow_{\text{div}} \underline{b}$  implies  $\underline{b} \Rightarrow^* \partial \underline{a}$ , and, therefore,  $\text{Irr}(\partial \underline{a}) \subseteq \text{Irr}(\partial \underline{b})$ .



**Lemma 4.17.** *If  $M$  is a noetherian gcd-monoid and  $\underline{a}$  is a multifraction on  $M$ , an  $i$ -reducer  $x$  for  $\underline{a}$  is tame if, and only if,  $x$  divides every maximal  $i$ -reducer for  $\underline{a}$ .*

*Proof.* Assume that  $x$  is a tame  $i$ -reducer for  $\underline{a}$ , and  $y$  is a maximal  $i$ -reducer for  $\underline{a}$ . By Lemma 4.16,  $a_i$ ,  $x$ , and  $y$  admit a common multiple, hence an lcm, and, therefore, the lcm of  $x$  and  $y$  is again an  $i$ -reducer for  $\underline{a}$ . The assumption that  $y$  is maximal implies that this lcm is  $y$ , i.e., that  $x$  divides  $y$  (on the due side).

Conversely, assume that  $x$  is an  $i$ -reducer for  $\underline{a}$  that divides every maximal  $i$ -reducer. Let  $y$  be an arbitrary  $i$ -reducer for  $\underline{a}$ . As  $M$  is noetherian,  $y$  divides at least one maximal  $i$ -reducer, say  $z$ . By assumption,  $x$  divides  $z$ , hence so does the lcm of  $x$  and  $y$ . Since  $z$  is an  $i$ -reducer for  $\underline{a}$ , so is its divisor the lcm of  $x$  and  $y$  and, therefore,  $\underline{a} \cdot R_{i,x}$  and  $\underline{a} \cdot R_{i,y}$  admit a common reduct. Hence  $x$  is tame for  $\underline{a}$ .  $\square$

**Proposition 4.18.** *If  $M$  is a noetherian gcd-monoid and  $\underline{a}$  is a multifraction on  $M$ , there exists for every  $i < \|\underline{a}\|$  a unique greatest tame  $i$ -reducer for  $\underline{a}$ , namely the gcd of all maximal  $i$ -reducers for  $\underline{a}$ . The latter is a multiple of the gcd of  $a_i$  and  $a_{i+1}$ .*

*Proof.* Let  $z$  be the gcd (on the relevant side) of all maximal  $i$ -reducers for  $\underline{a}$ . Then  $z$ , and every divisor  $x$  of  $z$ , divides every maximal  $i$ -reducer for  $\underline{a}$ , hence, by Lemma 4.17, it is a tame  $i$ -reducer for  $\underline{a}$ .

Conversely, if  $x$  is a tame  $i$ -reducer for  $\underline{a}$ , then, by Lemma 4.17,  $x$  divides every maximal  $i$ -reducer for  $\underline{a}$ , hence it divides their gcd  $z$ . Hence  $z$  is the greatest tame  $i$ -reducer for  $\underline{a}$ .

Finally, assume that  $x$  divides  $a_i$  and  $a_{i+1}$ . Then  $\underline{a} \cdot D_{i,x}$ , hence a fortiori  $\underline{a} \cdot R_{i,x}$  is defined, so  $x$  is an  $i$ -reducer for  $\underline{a}$ . Let  $y$  be any  $i$ -reducer for  $\underline{a}$ . Then the lcm of  $a_i$  and  $y$  is a common multiple of  $a_i$ ,  $x$ , and  $y$ . Hence, by Lemma 4.16,  $x$  is a tame  $i$ -reducer for  $\underline{a}$ . This applies in particular when  $x$  is the gcd of  $a_i$  and  $a_{i+1}$ .  $\square$

The (unique) element  $x$  whose existence is stated in Prop. 4.18 will be called the *greatest tame  $i$ -reducer* for  $\underline{a}$ . It only depends on the entries  $a_i$  and  $a_{i+1}$ , and on the sign of  $i$  in  $\underline{a}$ . The following example shows that it may be strictly larger than the gcd of  $a_i$  and  $a_{i+1}$ .

**Example 4.19.** In the Artin-Tits monoid of type  $\tilde{A}_2$ , let  $\underline{a} := 1/a/cabab$ . There exist two maximal 2-reducers for  $\underline{a}$ , namely  $caa$  and  $cab$ , both wild. The greatest tame 2-reducer is their left-gcd  $ca$ , itself a proper multiple of the left-gcd of  $a$  and  $cabab$ , which is 1.

If  $M$  is a noetherian gcd-monoid, starting from an arbitrary multifraction  $\underline{a}$  and repeatedly performing (maximal) tame reductions leads in finitely many steps to a  $\simeq$ -equivalent multifraction that is *tame-irreducible*, meaning eligible for no tame reduction. By Prop. 4.18, a division is always tame, so a tame-irreducible multifraction is prime. Adapting the proof of Prop. 3.25 yields:

**Proposition 4.20.** *If  $M$  is a noetherian gcd-monoid, then  $\mathcal{R}(M)$  (resp.  $\mathcal{R}^+(M)$ ) is semi-convergent if, and only if, for every  $\underline{a}$  in  $\mathcal{F}_{\pm}(M)$  (resp.  $\mathcal{F}(M)$ ),*

$$(4.21) \quad \text{If } \underline{a} \text{ is unital and tame-irreducible, then } \underline{a} \text{ is either trivial or reducible.}$$

**Corollary 4.22.** *Conjecture A is true if, and only if, (4.21) holds for every Artin-Tits monoid  $M$  and every  $\underline{a}$  in  $\mathcal{F}(M)$ .*

The above results suggest to investigate tame-irreducible multifractions more closely. By definition, only wild reductions may apply to a tame-irreducible multifraction. A possible approach in order to establish (4.21) could be to study  $\underline{1}$ -confluence for tame-irreducible multifractions, typically looking for the various irreducible reducts arising from different wild reductions. One could suspect that the latter are disjoint, but this is not true:

**Example 4.23.** Always in the Artin-Tits monoid of type  $\tilde{A}_2$ , let  $\underline{a}$  be the 6-multifraction  $1/c/aba/bc/a/bcb$ . Then  $\underline{a}$  admits no tame reducer, but it admits four wild reducers, namely  $a$  and  $b$  at level 2, and  $b$  and  $c$  at level 5. The associated 5-reducts of  $\underline{a}$  admit no common reduct, but the 2-reducts do, since one finds

$$(\underline{a} \cdot R_{2,a}) \cdot R_{5,c}R_{3,b}R_{1,ac}R_{2,b}R_{3,c} = bc/accb/ca/ab/ca/cb = (\underline{a} \cdot R_{2,b}) \cdot R_{5,c}R_{3,cb}.$$

We conjecture that such a confluence is impossible with the (wild) reductions occurring at the highest possible level.

Tame reductions do not share all properties of divisions. Whereas all div-reducts of  $\underline{a}$  admit  $\partial\underline{a}$  as a common reduct, some tame reducts of  $\underline{a}$  may admit no common reduct:

**Example 4.24.** We observed in Example 4.28 that  $\mathbf{a}$  and  $\mathbf{b}$  are wild 2-reducers for  $1/c/aba$ . In the same context, let  $\underline{a} = 1/c/1/1/aba$ . Because  $a_4$  is trivial, every divisor of  $a_5$  provides a tame 4-reducer for  $\underline{a}$ . Hence  $\mathbf{a}$  and  $\mathbf{b}$  are tame 4-reducers for  $\underline{a}$ , and, from there,  $ac/ca/ba/1/1$  and  $bc/cb/ab/1/1$ , which are both irreducible, can be reached from  $\underline{a}$  using tame reduction steps, respectively  $R_{3,a}R_{1,a}R_{3,ba}$  and  $R_{3,b}R_{1,b}R_{3,ab}$ .

**4.3. The operator  $\text{red}_t$ .** For every multifraction  $\underline{a}$  and for every  $i$ , there exists a unique greatest division applying to  $\underline{a}$  at level  $i$ . By Prop. 4.18, there exists similarly a unique greatest tame reduction applying to  $\underline{a}$  at level  $i$ . On the shape of Notation 4.2, we put

**Notation 4.25.** If  $M$  is a noetherian gcd-monoid and  $\underline{a}$  is a multifraction on  $M$ , then, for  $i < \|\underline{a}\|$ , we write  $\underline{a} \cdot R_i^{\max}$  for  $\underline{a} \cdot R_{i,x}$ , where  $x$  is the greatest tame  $i$ -reducer for  $\underline{a}$ . For  $\alpha = (i_1, \dots, i_\ell)$  a sequence in  $\{1, \dots, \|\underline{a}\| - 1\}$ , we write  $\underline{a} \cdot R_\alpha^{\max}$  for  $\underline{a} \cdot R_{i_1}^{\max} \dots R_{i_\ell}^{\max}$ .

**Example 4.26.** When  $M$  satisfies the 3-Ore condition, every reducer is tame, and  $\underline{a} \cdot R_i^{\max}$  coincides with the maximal  $i$ -reduct of  $\underline{a}$ , as used in [7, Sec. 6]. Otherwise, wild reducers may exist and  $\underline{a} \cdot R_i^{\max}$  need not be a maximal  $i$ -reduct of  $\underline{a}$ : in the Artin-Tits monoid of type  $\tilde{A}_2$ , for  $\underline{a} = 1/c/aba$ , we have  $\underline{a} \cdot R_2^{\max} = \underline{a}$ , since there is no nontrivial tame 2-reducer.

It is shown in [7, Sec. 6] that, if  $M$  is a noetherian gcd-monoid satisfying the 3-Ore condition, then there exists, for each depth  $n$ , a universal recipe  $U(n)$  for going from a depth  $n$  multifraction  $\underline{a}$  on  $M$  to the unique  $\mathcal{R}$ -irreducible reduct  $\text{red}(\underline{a})$  of  $\underline{a}$ . Copying the definition of  $U(n)$ , we introduce:

**Definition 4.27.** If  $M$  is a noetherian gcd-monoid, then, for every depth  $n$  multifraction  $\underline{a}$  on  $M$ , we put  $\text{red}_t(\underline{a}) := \underline{a} \cdot R_{U(n)}^{\max}$ , with  $U(n)$  empty for  $n = 0, 1$  and inductively defined by  $U(n) = (1, 2, \dots, n-1) \frown U(n-2)$  for  $n \geq 2$ .

Thus,  $\text{red}_t(\underline{a})$  is obtained from  $\underline{a}$  by iteratively performing the maximal tame reduction at the successive levels prescribed by the universal rule  $U(n)$ . By [7, Prop. 5.13], if  $M$  is a noetherian gcd-monoid satisfying the 3-Ore condition, then, for every  $\underline{a}$  in  $\mathcal{F}_\pm(M)$ , we have  $\text{red}(\underline{a}) = \text{red}_t(\underline{a})$ . In this case,  $\underline{b} \Rightarrow^* \text{red}_t(\underline{a})$  holds for any reduct of  $\underline{a}$ , and  $\text{red}_t(\underline{a})$  is always  $\mathcal{R}$ -irreducible. It is easy to see that these properties do not extend to the general case:

**Example 4.28.** In the Artin-Tits monoid of type  $\tilde{A}_2$ , let  $\underline{a} := 1/c/aba$ . Then  $U(3) = (1, 2)$  leads to  $\text{red}_t(\underline{a}) = \underline{a} \cdot R_1^{\max} R_2^{\max} = \underline{a}$ , since 1 and  $c$  have no nontrivial common divisor, and there is no tame 2-reducer for  $\underline{a}$ . On the other hand, both  $\mathbf{a}$  and  $\mathbf{b}$  are 2-reducers for  $\underline{a}$ , and neither  $\underline{a} \cdot R_{2,a} \Rightarrow^* \text{red}_t(\underline{a})$  nor  $\underline{a} \cdot R_{2,b} \Rightarrow^* \text{red}_t(\underline{a})$  holds.

Next, let  $\underline{b} := ac/aca/aba$ . We find  $\text{red}_t(\underline{b}) = \underline{b} \cdot D_{1,ac} = 1/c/aba$ , to be compared with  $\partial\underline{b} = \underline{b} \cdot D_{2,a} = ac/ca/ba$ . Then  $\partial\underline{b} \Rightarrow^* \text{red}_t(\underline{b})$  fails, so  $\underline{b} \Rightarrow_{\text{div}}^* \underline{b}'$  does not imply  $\underline{b}' \Rightarrow^* \text{red}_t(\underline{b})$ .

Finally, let  $\underline{c} := 1/c/aba/cb$ . We find  $\text{red}_t(\underline{c}) = \underline{c} \cdot D_{3,b} = 1/c/ba/c$ , which is not irreducible, nor even tame-irreducible: we have  $\text{red}_t^2(\underline{c}) = \text{red}_t(\underline{c}) \cdot R_{2,b} R_{3,c} = bc/cb/a/c$ .

These negative features however say nothing about tame reductions for a unital multifraction. We propose:

**Conjecture B.** *If  $M$  is an Artin-Tits monoid, then  $\text{red}_t(\underline{a}) = \underline{1}$  holds for every unital multifraction  $\underline{a}$  in  $\mathcal{F}(M)$ .*

By definition,  $\underline{a} \Rightarrow^* \text{red}_t(\underline{a})$  holds, so  $\text{red}_t(\underline{a}) = \underline{1}$  is a strengthening of  $\underline{a} \Rightarrow^* \underline{1}$  in which we assert not only that  $\underline{a}$  reduces to  $\underline{1}$  but also that it goes to  $\underline{1}$  in some prescribed way. Thus:

**Fact 4.29.** *Conjecture B implies Conjecture A.*

Although Conjecture B is more demanding than Conjecture A, it might be easier to establish (or to contradict), as it predicts a definite equality rather than an existential statement. By the results of [7], Conjecture B is true for every Artin-Tits monoid of FC type.

## 5. CROSS-CONFLUENCE

We introduce a new property of reduction that involves both the rewrite system  $\mathcal{R}(M)$  and a symmetric counterpart  $\tilde{\mathcal{R}}(M)$  of the latter that we introduce in Subsection 5.1. Some results about the termination of the joint system  $\mathcal{R}(M) \cup \tilde{\mathcal{R}}(M)$  are established in Subsection 5.2. Then we introduce cross-confluence, a new notion of confluence that combines  $\mathcal{R}(M)$  and  $\tilde{\mathcal{R}}(M)$ , in Subsection 5.3. This leads us to Conjecture **C**, another approach for establishing Conjecture **A**, and to its uniform version Conjecture **C<sub>unif</sub>**, which is discussed in Subsection 5.4.

**5.1. Right reduction.** The rewrite rules  $R_{i,x}$  consist in pushing the factor  $x$  to the left, and will henceforth be called *left* reductions. We can also consider symmetric counterparts, naturally called *right* reductions, where elements are pushed to the right.

**Definition 5.1.** If  $M$  is a gcd-monoid and  $\underline{a}, \underline{b}$  lie in  $\mathcal{F}_\pm(M)$ , then, for  $i \geq 1$  and  $x$  in  $M$ , we declare that  $\underline{b} = \underline{a} \bullet \tilde{R}_{i,x}$  holds if we have  $\|\underline{b}\| = \|\underline{a}\|$ ,  $b_k = a_k$  for  $k \neq i-1, i, i+1$ , and there exists  $x'$  (necessarily unique) satisfying

$$\begin{array}{ll} \text{for } i < \|\underline{a}\| \text{ positive in } \underline{a}: & xb_{i-1} = a_{i-1}, \quad xb_i = a_i x' = x \vee a_i, \quad b_{i+1} = a_{i+1} x', \\ \text{for } i < \|\underline{a}\| \text{ negative in } \underline{a}: & b_{i-1} x = a_{i-1}, \quad b_i x = x' a_i = x \tilde{\vee} a_i, \quad b_{i+1} = x' a_{i+1}, \\ \text{for } i = \|\underline{a}\| \text{ positive in } \underline{a}: & xb_{i-1} = a_{i-1}, \quad xb_i = a_i, \\ \text{for } i = \|\underline{a}\| \text{ negative in } \underline{a}: & b_{i-1} x = a_{i-1}, \quad b_i x = a_i. \end{array}$$

We denote by  $\tilde{\mathcal{R}}(M)$  the family of all  $\tilde{R}_{i,x}$  with  $x \neq 1$ , write  $\underline{a} \tilde{\Rightarrow} \underline{b}$  if some rule of  $\tilde{\mathcal{R}}(M)$  maps  $\underline{a}$  to  $\underline{b}$ , and use  $\tilde{\Rightarrow}^*$  for the reflexive–transitive closure of  $\tilde{\Rightarrow}$ .

The action of  $\tilde{R}_{i,x}$  is symmetric of that of  $R_{i,x}$ : one extracts  $x$  from  $a_{i-1}$ , lets it cross  $a_i$  using an lcm, and incorporates the resulting remainder in  $a_{i+1}$ , thus carrying  $x$  from level  $i-1$  to level  $i+1$ , see Fig. 4. As in the case of  $R_{1,x}$ , the action of  $\tilde{R}_{n,x}$  for  $n = \|\underline{a}\|$  is adapted to avoid creating a  $(n+1)$ st entry.

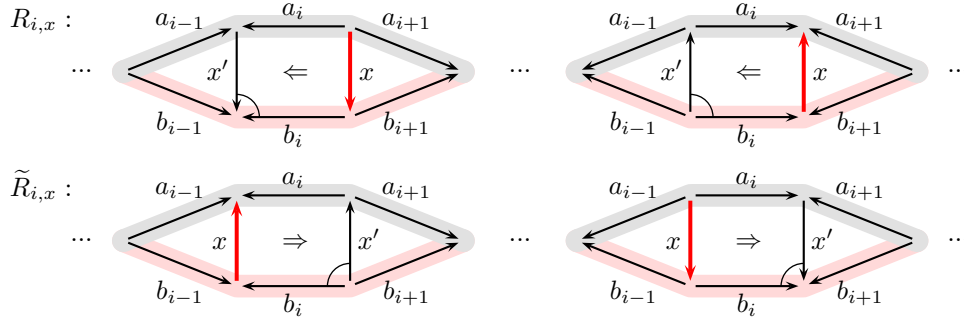


FIGURE 4. Comparing the left reduction  $R_{i,x}$  and the right reduction  $\tilde{R}_{i,x}$ : in the first case (top), one pushes the factor  $x$  from  $a_{i+1}$  to  $a_{i-1}$  through  $a_i$ , in the second case (bottom), one pushes  $x$  from  $a_{i-1}$  to  $a_{i+1}$  through  $a_i$ .

**Remark 5.2.** Right reduction is *not* an inverse of left reduction: when the reduced factor  $x$  crosses (in one direction or the other) the entry  $a_i$ , using the lcm operation cancels common factors. Typically, if  $x$  divides  $a_i$ , then both  $R_{i,x}$  and  $\tilde{R}_{i,x}$  amount to dividing by  $x$  and, therefore, their actions coincide. See Remark 5.30 for more on this.

The properties of right reduction are, of course, similar to those of left reduction. In fact, there exists a simple duality between left and right reduction.

**Lemma 5.3.** *If  $M$  is a gcd-monoid, then, for all  $\underline{a}, \underline{b}$  in  $\mathcal{F}_\pm(M)$ ,*

$$(5.4) \quad \underline{a} \Rightarrow \underline{b} \text{ is equivalent to } \underline{\tilde{a}} \tilde{\Rightarrow} \underline{\tilde{b}}.$$

*Proof.* Comparing the definitions shows that, if  $\underline{a}$  and  $\underline{b}$  have depth  $n$ , then  $\underline{b} = \underline{a} \cdot R_{i,x}$  is equivalent to  $\underline{\tilde{b}} = \underline{\tilde{a}} \cdot \tilde{R}_{n+1-i,x}$ , whence (5.4).  $\square$

**Lemma 5.5.** *Assume that  $M$  is a gcd-monoid.*

- (i) *The relation  $\tilde{\Rightarrow}^*$  is included in  $\simeq$ , i.e.,  $\underline{a} \tilde{\Rightarrow}^* \underline{b}$  implies  $\underline{a} \simeq \underline{b}$ .*
- (ii) *The relation  $\tilde{\Rightarrow}^*$  is compatible with the multiplication of  $\mathcal{F}_{\pm}(M)$ .*
- (iii) *For all  $\underline{a}, \underline{b}$  and  $p, q$ , the relation  $\underline{a} \tilde{\Rightarrow}^* \underline{b}$  is equivalent to  $\underline{1}_p \cdot \underline{a} \cdot \underline{1}_q \tilde{\Rightarrow}^* \underline{1}_p \cdot \underline{b} \cdot \underline{1}_q$ .*
- (iv) *If  $\underline{a}, \underline{b}$  belong to  $\mathcal{F}(M)$ , then  $\underline{a} \Rightarrow \underline{b}$  is equivalent to  $1 \cdot \underline{\tilde{a}} \Rightarrow 1 \cdot \underline{\tilde{b}}$ , and  $\underline{a} \tilde{\Rightarrow} \underline{b}$  is equivalent to  $1 \cdot \underline{\tilde{a}} \tilde{\Rightarrow} 1 \cdot \underline{\tilde{b}}$ .*

*Proof.* (i) By (5.4),  $\underline{a} \tilde{\Rightarrow}^* \underline{b}$  implies  $\underline{\tilde{a}} \Rightarrow^* \underline{\tilde{b}}$ , whence  $\iota(\underline{\tilde{a}}) = \iota(\underline{\tilde{b}})$  by Lemma 2.18(i), hence  $\iota(\underline{a}) = \iota(\underline{b})$  by Lemma 3.9, i.e.,  $\underline{a} \simeq \underline{b}$ .

(ii) For all  $\underline{c}, \underline{d}$ , the relation  $\underline{a} \tilde{\Rightarrow}^* \underline{b}$  implies  $\underline{\tilde{a}} \Rightarrow^* \underline{\tilde{b}}$ , whence  $\underline{\tilde{d}} \cdot \underline{\tilde{a}} \cdot \underline{\tilde{c}} \Rightarrow^* \underline{\tilde{d}} \cdot \underline{\tilde{b}} \cdot \underline{\tilde{c}}$  by Lemma 2.18(ii), which is  $(\underline{c} \cdot \underline{a} \cdot \underline{d}) \tilde{\Rightarrow}^* (\underline{c} \cdot \underline{b} \cdot \underline{d})$  by Lemma 3.9, and finally  $\underline{c} \cdot \underline{a} \cdot \underline{d} \Rightarrow^* \underline{c} \cdot \underline{b} \cdot \underline{d}$  by (5.4) again.

(iii) Using duality as above, the result follows now from Lemma 2.18(iii).

(iv) By (5.4),  $\underline{a} \Rightarrow \underline{b}$  is equivalent to  $\underline{\tilde{a}} \tilde{\Rightarrow} \underline{\tilde{b}}$ , hence, by (iii), it is also equivalent to  $1 \cdot \underline{a} \Rightarrow 1 \cdot \underline{b}$ . Similarly, by (5.4) again,  $\underline{a} \tilde{\Rightarrow} \underline{b}$  is equivalent to  $\underline{\tilde{a}} \tilde{\Rightarrow} \underline{\tilde{b}}$ , hence, by Lemma 2.18, it is also equivalent to  $1 \cdot \underline{\tilde{a}} \tilde{\Rightarrow} 1 \cdot \underline{\tilde{b}}$ . (Because the translation  $\underline{a} \mapsto 1 \cdot \underline{a}$  is not surjective, the previous equivalences do not coincide.)  $\square$

By (5.4), an infinite sequence of  $\tilde{\mathcal{R}}$ -reduction steps would provide an infinite sequence of  $\mathcal{R}$ -reduction steps, and vice versa, so  $\tilde{\mathcal{R}}(M)$  is terminating if, and only if,  $\mathcal{R}(M)$  is. Comparing irreducible elements is straightforward:

**Lemma 5.6.** *If  $M$  is a gcd-monoid, a multifraction  $\underline{a}$  is  $\mathcal{R}$ -irreducible if, and only if,  $\underline{\tilde{a}}$  is  $\tilde{\mathcal{R}}$ -irreducible. For  $\underline{a}$  positive with  $\|\underline{a}\|$  even,  $\underline{a}$  is  $\mathcal{R}^+$ -irreducible if, and only if,  $\underline{\tilde{a}}$  is  $\mathcal{R}^+$ -irreducible.*

*Proof.* Assume  $\underline{\tilde{a}} \tilde{\Rightarrow} \underline{\tilde{b}}$ . By (5.4), we deduce  $\underline{a} = \underline{\tilde{a}} \Rightarrow \underline{\tilde{b}}$ , hence  $\underline{a}$  is not  $\mathcal{R}$ -irreducible. So  $\underline{a}$  being  $\mathcal{R}$ -irreducible implies that  $\underline{\tilde{a}}$  is  $\tilde{\mathcal{R}}$ -irreducible. Conversely, assume  $\underline{\tilde{a}} \Rightarrow \underline{\tilde{b}}$ . By (5.4), we deduce  $\underline{a} = \underline{\tilde{a}} \tilde{\Rightarrow} \underline{\tilde{b}}$ , hence  $\underline{a}$  is not  $\tilde{\mathcal{R}}$ -irreducible. So  $\underline{a}$  being  $\tilde{\mathcal{R}}$ -irreducible implies that  $\underline{\tilde{a}}$  is  $\mathcal{R}$ -irreducible.

Because all  $\mathcal{R}$ -reducts of a positive multifraction are positive, a positive multifraction is  $\mathcal{R}^+$ -irreducible if, and only if, it is  $\mathcal{R}$ -irreducible. If  $\underline{a}$  is positive and  $\|\underline{a}\|$  is even,  $\underline{\tilde{a}}$  is positive, and then the  $\mathcal{R}^+$ -irreducibility of  $\underline{\tilde{a}}$  makes sense, and the rest is clear.  $\square$

**Proposition 5.7.** *For every gcd-monoid  $M$ , the following are equivalent:*

- (i) *The system  $\mathcal{R}(M)$  is convergent (resp. semi-convergent);*
- (ii) *The system  $\tilde{\mathcal{R}}(M)$  is convergent (resp. semi-convergent);*
- (iii) *The systems  $\mathcal{R}^+(M)$  and  $\tilde{\mathcal{R}}^+(M)$  are convergent (resp. semi-convergent).*

*Proof.* We begin with convergence. Assume that  $\mathcal{R}(M)$  is convergent. Let  $\underline{a}$  belong to  $\mathcal{F}_{\pm}(M)$ . Let  $\underline{\tilde{b}} = \text{red}(\underline{\tilde{a}})$ . By definition, we have  $\underline{\tilde{a}} \Rightarrow^* \underline{\tilde{b}}$ . Then, by Lemmas 5.5 and 5.6, we have  $\underline{a} \tilde{\Rightarrow}^* \underline{\tilde{b}}$  and  $\underline{\tilde{b}}$  is  $\tilde{\mathcal{R}}$ -irreducible. Assume that  $\underline{c}$  is  $\tilde{\mathcal{R}}$ -irreducible and  $\underline{a} \tilde{\Rightarrow}^* \underline{c}$  holds. By Lemmas 5.5 and 5.6 again, we deduce  $\underline{\tilde{a}} \Rightarrow^* \underline{\tilde{c}}$  and  $\underline{\tilde{c}}$  is  $\tilde{\mathcal{R}}$ -irreducible. The assumption that  $\mathcal{R}(M)$  is convergent implies  $\underline{\tilde{c}} = \underline{\tilde{b}}$ , whence  $\underline{c} = \underline{\tilde{b}}$ . Hence  $\underline{\tilde{b}}$  is the only  $\tilde{\mathcal{R}}$ -irreducible  $\tilde{\mathcal{R}}$ -reduct of  $\underline{a}$ . Therefore,  $\tilde{\mathcal{R}}(M)$  is convergent, and (i) implies (ii). The converse implication is proved in the same way, so (i) is equivalent to (ii).

Since all  $\mathcal{R}$ -reducts and  $\tilde{\mathcal{R}}$ -reducts of a positive multifraction are positive, it is clear that (i) implies that  $\mathcal{R}^+(M)$  is convergent, and (ii) implies that  $\tilde{\mathcal{R}}^+(M)$  is convergent. Conversely, assume that both  $\mathcal{R}^+(M)$  and  $\tilde{\mathcal{R}}^+(M)$  are convergent. Let  $\underline{a}$  be an arbitrary multifraction on  $M$ . Assume first that  $M$  is positive. Let  $\underline{\tilde{b}}$  be the unique  $\mathcal{R}^+$ -irreducible reduct of  $\underline{a}$ . As all  $\mathcal{R}$ -reducts of  $\underline{a}$  are positive,  $\underline{\tilde{b}}$  is also the unique  $\tilde{\mathcal{R}}$ -irreducible reduct of  $\underline{a}$ . Assume now that  $\underline{a}$  is negative. Then  $\underline{\tilde{a}}$  is positive. Let  $\underline{\tilde{b}}$  be the unique  $\tilde{\mathcal{R}}^+$ -irreducible reduct of  $\underline{\tilde{a}}$ . As all  $\mathcal{R}$ -reducts

of a positive multifraction are positive,  $\underline{b}$  is  $\widetilde{\mathcal{R}}$ -irreducible. By Lemmas 5.5 and 5.6,  $\widetilde{\underline{b}}$  is  $\mathcal{R}$ -irreducible, and  $\underline{a} \Rightarrow^* \underline{b}$  holds. Now assume that  $\underline{c}$  is  $\mathcal{R}$ -irreducible and  $\underline{a} \Rightarrow^* \underline{c}$  holds. Then  $\widetilde{\underline{c}}$  is  $\mathcal{R}$ -irreducible and  $\widetilde{\underline{a}} \Rightarrow^* \widetilde{\underline{c}}$  holds. As  $\widetilde{\underline{a}}$  is positive and  $\widetilde{\mathcal{R}}^+(M)$  is convergent, we deduce  $\widetilde{\underline{c}} = \widetilde{\underline{b}}$ , whence  $\underline{c} = \underline{b}$ . Hence  $\mathcal{R}(M)$  is convergent, and (iii) implies (i). This completes the argument for convergence.

Assume now that  $\mathcal{R}(M)$  is semi-convergent. Let  $\underline{a}$  be a unital multifraction. By Lemma 2.18,  $\widetilde{\underline{a}}$  is unital as well, hence we must have  $\widetilde{\underline{a}} \Rightarrow^* \underline{1}$ , which, by (5.4), implies  $\underline{a} \Rightarrow^* \underline{1} = \underline{1}$ . Hence  $\widetilde{\mathcal{R}}(M)$  is semi-convergent. The converse implication is similar, so (i) and (ii) are equivalent in this case as well. On the other hand, (i) and (ii) clearly imply (iii). Finally, assume that both  $\mathcal{R}^+(M)$  and  $\widetilde{\mathcal{R}}^+(M)$  are semi-convergent. Let  $\underline{a}$  be a unital multifraction. If  $\underline{a}$  is positive, the assumption that  $\mathcal{R}^+(M)$  is semi-convergent implies  $\underline{a} \Rightarrow^* \underline{1}$ . If  $\underline{a}$  is negative, then  $\widetilde{\underline{a}}$  is positive, and the assumption that  $\widetilde{\mathcal{R}}^+(M)$  is semi-convergent implies  $\widetilde{\underline{a}} \Rightarrow^* \underline{1}$ , whence  $\underline{a} \Rightarrow^* \underline{1} = \underline{1}$  by (5.4). Hence  $\mathcal{R}(M)$  is semi-convergent. So (iii) implies (i), which completes the argument for semi-convergence.  $\square$

In the convergent case, the above proof implies, with obvious notation,  $\widetilde{\text{red}}(\underline{a}) = (\text{red}(\widetilde{\underline{a}}))^\sim$ .

**Remark 5.8.** It is shown in [7, Sec. 3] that, when left reduction is considered, cutting final trivial entries essentially does not change reduction. This is not true for right reduction: deleting trivial final entries can change the reducts (as deleting trivial initial entries does in the case of left reduction).

**5.2. Mixed termination.** If  $M$  is a noetherian gcd-monoid, then, by Lemma 2.19, the rewrite system  $\mathcal{R}(M)$  and  $\widetilde{\mathcal{R}}(M)$  are terminating, meaning that there is no infinite sequence of  $\mathcal{R}$ - or  $\widetilde{\mathcal{R}}$ -reductions. Let us first observe that a stronger finiteness result holds:

**Proposition 5.9.** *If  $M$  is a finitely generated noetherian gcd-monoid, then every multifraction  $\underline{a}$  on  $M$  admits only finitely many left reducts, and finitely many right reducts.*

*Proof.* As  $M$  is noetherian and contains no nontrivial invertible element, a subfamily of  $M$  is generating if, and only if, it contains all atoms. Hence, the assumption that  $M$  is finitely generated implies that the atom set  $\mathcal{A}(M)$  of  $M$  is finite. Let  $\underline{a}$  belong to  $\mathcal{F}_\pm(M)$ . We construct a tree  $T_{\underline{a}}$ , whose nodes are pairs  $(\underline{b}, s)$ , where  $\underline{b}$  is a left reduct of  $\underline{a}$  and  $s$  is a finite sequence in  $\mathcal{A}(M) \times \mathbb{N}$ : the root of  $T_{\underline{a}}$  is  $(\underline{a}, \varepsilon)$ , and the sons of  $(\underline{b}, s)$  are all pairs  $(\underline{b} \bullet R_{i,x}, s \frown (i, x))$  such that  $\underline{b} \bullet R_{i,x}$  is defined. As  $\mathcal{A}(M)$  is finite, for every multifraction  $\underline{b}$ , the number of pairs  $(i, x)$  with  $x$  in  $\mathcal{A}(M)$  and  $\underline{b} \bullet R_{i,x}$  defined is finite. Hence each node in  $T_{\underline{a}}$  has finitely many immediate successors. On the other hand, the assumption that  $M$  is noetherian implies that  $\mathcal{R}(M)$  is terminating and, therefore, the tree  $T_{\underline{a}}$  has no infinite branch. Hence, by König's lemma,  $T_{\underline{a}}$  is finite. As every left reduct of  $\underline{a}$  appears (maybe more than once) in  $T_{\underline{a}}$ , the number of such reducts is finite.

The argument for right reducts is symmetric.  $\square$

When left- and right reduction are mixed, the situation is more complicated, as the well-orders witnessing for the termination of left and right reduction are not the same. It is easy to see that Prop. 5.9 does not extend in general:

**Example 5.10.** In the Artin-Tits of type  $\widetilde{A}_2$ , let  $\underline{a} = 1/a/bc/1$ . Then we can compute  $\underline{a} \bullet R_{2,b} \widetilde{R}_{3,a} = ba/b/ca/ac$ , whence, repeating three times,

$$\underline{a} \bullet R_{2,b} \widetilde{R}_{3,a} R_{2,c} \widetilde{R}_{3,b} R_{2,a} \widetilde{R}_{3,c} = bacbac/a/bc/acbacb = bacbac \cdot \underline{a} \cdot acbacb.$$

Hence the multifractions  $\underline{a} \bullet (R_{2,b} \widetilde{R}_{3,a} R_{2,c} \widetilde{R}_{3,b} R_{2,a} \widetilde{R}_{3,c})^p$  make for  $p \geq 0$  an infinite non-terminating (and non-periodic) sequence with respect to  $\mathcal{R}(M) \cup \widetilde{\mathcal{R}}(M)$ .

By contrast, let us mention without detailed proof a finiteness result valid when the ground monoid  $M$  is a Garside monoid [9, 14], *i.e.*, a strongly noetherian gcd-monoid possessing in addition an element  $\Delta$  (“Garside element”) whose left- and right-divisors coincide, generate  $M$ , and are finite in number.

**Proposition 5.11.** *If  $M$  is a Garside monoid, then, for every multifraction  $\underline{a}$  on  $M$ , the family of all  $(\mathcal{R} \cup \widetilde{\mathcal{R}})$ -reducts of  $\underline{a}$  is finite.*

*Proof (sketch).* Let  $\Delta$  be a Garside element in  $M$  and let  $\underline{a}$  be a multifraction on  $M$ . Then there exists a positive integer  $d$  such that the path associated with  $\underline{a}$  can be drawn in the finite fragment of the Cayley graph of  $M$  made of the divisors of  $\Delta^d$ : this is the notion of a path “drawn in  $\text{Div}(\Delta^d)$ ” as considered in [8]. Then the family of all paths drawn in  $\text{Div}(\Delta^d)$  is closed under the special transformations alluded to in Subsec. 3.3, which implies that all multifractions that can be derived from  $\underline{a}$  using  $\Rightarrow$  and  $\widetilde{\Rightarrow}$  are drawn in the same finite fragment  $\text{Div}(\Delta^d)$  of the Cayley graph. As we consider multifractions with a fixed depth, only finitely many of them can be drawn in a finite fragment of a Cayley graph.  $\square$

The argument extends to every Artin-Tits monoid  $M$  of type FC, replacing the finite family of divisors of the Garside element  $\Delta$  with the union of the finitely many finite families of divisors of the Garside elements  $\Delta_I$ , where  $I$  is a family of atoms of  $M$  that generates a spherical type submonoid of  $M$ .

**5.3. Cross-confluence.** We now introduce a new notion of confluence that combines left- and right reduction.

**Definition 5.12.** If  $M$  is a gcd-monoid, we say that  $\mathcal{R}(M)$  (*resp.*  $\mathcal{R}^+(M)$ ) is *cross-confluent* if, for all  $\underline{a}, \underline{b}, \underline{c}$  in  $\mathcal{F}_{\pm}(M)$  (*resp.* in  $\mathcal{F}(M)$ ),

$$(5.13) \quad \text{If we have } \underline{a} \widetilde{\Rightarrow}^* \underline{b} \text{ and } \underline{a} \widetilde{\Rightarrow}^* \underline{c}, \text{ there exists } \underline{d} \text{ satisfying } \underline{b} \Rightarrow^* \underline{d} \text{ and } \underline{c} \Rightarrow^* \underline{d}.$$

Dually, we say that  $\widetilde{\mathcal{R}}(M)$  (*resp.*  $\widetilde{\mathcal{R}}^+(M)$ ) is *cross-confluent* if, for all  $\underline{a}, \underline{b}, \underline{c}$  in  $\mathcal{F}_{\pm}(M)$  (*resp.* in  $\mathcal{F}(M)$ ),

$$(5.14) \quad \text{If we have } \underline{a} \Rightarrow^* \underline{b} \text{ and } \underline{a} \Rightarrow^* \underline{c}, \text{ there exists } \underline{d} \text{ satisfying } \underline{b} \widetilde{\Rightarrow}^* \underline{d} \text{ and } \underline{c} \widetilde{\Rightarrow}^* \underline{d}.$$

In the case of  $\mathcal{R}(M)$ , owing to the equivalence (5.4), cross-confluence can also be stated as a property involving  $\mathcal{R}(M)$  exclusively, namely

$$(5.15) \quad \text{If we have } \widetilde{\underline{a}} \Rightarrow^* \widetilde{\underline{b}} \text{ and } \widetilde{\underline{a}} \Rightarrow^* \widetilde{\underline{c}}, \text{ there exists } \underline{d} \text{ satisfying } \underline{b} \Rightarrow^* \underline{d} \text{ and } \underline{c} \Rightarrow^* \underline{d},$$

and similarly for  $\widetilde{\mathcal{R}}(M)$ . In the case of  $\mathcal{R}^+(M)$ , we also have an alternative statement involving  $\Rightarrow^*$  exclusively, but it takes the less symmetric form

$$(5.16) \quad \text{If we have } 1 \cdot \widetilde{\underline{a}} \Rightarrow^* 1 \cdot \widetilde{\underline{b}} \text{ and } 1 \cdot \widetilde{\underline{a}} \Rightarrow^* 1 \cdot \widetilde{\underline{c}}, \\ \text{there exists } \underline{d} \text{ satisfying } \underline{b} \Rightarrow^* \underline{d} \text{ and } \underline{c} \Rightarrow^* \underline{d},$$

**Proposition 5.17.** *For every gcd-monoid  $M$ , the following are equivalent:*

- (i) *The system  $\mathcal{R}(M)$  is cross-confluent;*
- (ii) *The system  $\widetilde{\mathcal{R}}(M)$  is cross-confluent;*
- (iii) *The systems  $\mathcal{R}^+(M)$  and  $\widetilde{\mathcal{R}}^+(M)$  are cross-confluent.*

*Proof.* Assume that  $\mathcal{R}(M)$  is cross-confluent. Applying (5.13) to  $\widetilde{\underline{a}}, \widetilde{\underline{b}}$ , and  $\widetilde{\underline{c}}$  and using (5.4), we deduce (5.14), so  $\widetilde{\mathcal{R}}(M)$  is also cross-confluent. Doing the same from (5.14) returns to (5.13). So (i) and (ii) are equivalent.

Next, all  $\mathcal{R}$ - and  $\widetilde{\mathcal{R}}$ -reducts of a positive multifraction are positive, so, if  $\mathcal{R}(M)$  is cross-confluent, its restriction to  $\mathcal{F}(M)$  is also cross-confluent. Similarly, if  $\widetilde{\mathcal{R}}(M)$  is cross-confluent, its restriction to  $\mathcal{F}(M)$  is cross-confluent. Hence (i) implies (iii). Finally, assume that  $\mathcal{R}^+(M)$  and  $\widetilde{\mathcal{R}}^+(M)$  are cross-confluent, and  $\underline{a}, \underline{b}, \underline{c}$  belong to  $\mathcal{F}_{\pm}(M)$  and satisfy  $\underline{a} \widetilde{\Rightarrow}^* \underline{b}$  and  $\underline{a} \widetilde{\Rightarrow}^* \underline{c}$ . If  $\underline{a}$ , hence  $\underline{b}$  and  $\underline{c}$  as well, are positive, the assumption that  $\mathcal{R}^+(M)$  is cross-confluent implies the existence of  $\underline{d}$  satisfying (5.13). Assume now that  $\underline{a}$ , hence  $\underline{b}$  and  $\underline{c}$ , are negative. Then  $\widetilde{\underline{a}}, \widetilde{\underline{b}}, \widetilde{\underline{c}}$  are positive, and we have  $\widetilde{\underline{a}} \Rightarrow^* \widetilde{\underline{b}}$  and  $\widetilde{\underline{a}} \Rightarrow^* \widetilde{\underline{c}}$ . The assumption that  $\widetilde{\mathcal{R}}^+(M)$  is cross-confluent implies the existence of  $\underline{d}$  satisfying  $\widetilde{\underline{b}} \widetilde{\Rightarrow}^* \underline{d}$  and  $\widetilde{\underline{c}} \widetilde{\Rightarrow}^* \underline{d}$ . By (5.4), we deduce  $\underline{b} \Rightarrow^* \underline{d}$  and  $\underline{c} \Rightarrow^* \underline{d}$ . So (5.13) holds,  $\mathcal{R}(M)$  is cross-confluent, and (iii) implies (i).  $\square$

Cross-confluence is intermediate between convergence and semi-convergence:



**Proposition 5.18.** *Assume that  $M$  is a noetherian gcd-monoid.*

- (i) *If  $\mathcal{R}(M)$  is convergent, then  $\mathcal{R}(M)$  is cross-confluent.*
- (ii) *If  $\mathcal{R}(M)$  is cross-confluent, then  $\mathcal{R}(M)$  is semi-convergent.*
- (iii) *Mutatis mutandis, the same implications hold for  $\mathcal{R}^+(M)$ .*

We begin with an auxiliary result:

**Lemma 5.19.** *Assume that  $M$  is a gcd-monoid.*

- (i) *If  $\mathcal{R}(M)$  is cross-confluent, then, for every  $\underline{a}$  in  $\mathcal{F}_\pm(M)$ , the relations  $\underline{a} \Rightarrow^* \underline{1}$  and  $\underline{a} \cong^* \underline{1}$  are equivalent.*
- (ii) *If  $\mathcal{R}^+(M)$  is cross-confluent, then, for every  $\underline{a}$  in  $\mathcal{F}(M)$ , the relation  $\underline{a} \cong^* \underline{1}$  implies  $\underline{a} \Rightarrow^* \underline{1}$ .*

*Proof.* (i) Assume  $\underline{a} \in \mathcal{F}_\pm(M)$  and  $\underline{a} \Rightarrow^* \underline{1}$ . By Prop. 5.17, the cross-confluence of  $\mathcal{R}(M)$  implies that of  $\widetilde{\mathcal{R}}(M)$ . By definition, we also have  $\underline{a} \Rightarrow^* \underline{a}$ . Then (5.14) implies the existence of  $\underline{d}$  satisfying  $\underline{1} \cong^* \underline{d}$  and  $\underline{a} \cong^* \underline{d}$ . By definition,  $\underline{1}$  is  $\widetilde{\mathcal{R}}$ -irreducible, so  $\underline{1} \cong^* \underline{d}$  implies  $\underline{d} = \underline{1}$ , whence  $\underline{a} \cong^* \underline{1}$ . Conversely, assume  $\underline{a} \cong^* \underline{1}$ . We have  $\underline{a} \cong^* \underline{a}$ , and (5.13) implies the existence of  $\underline{d}$  satisfying  $\underline{a} \Rightarrow^* \underline{d}$  and  $\underline{1} \Rightarrow^* \underline{d}$ , whence  $\underline{d} = \underline{1}$ , and  $\underline{a} \Rightarrow^* \underline{1}$ .

(ii) When  $\underline{a}$  lies in  $\mathcal{F}(M)$ , the latter argument remains valid, and it shows again that  $\underline{a} \cong^* \underline{1}$  implies  $\underline{a} \Rightarrow^* \underline{1}$ . (By contrast, the former argument need not extend, as there is a priori no reason why the cross-confluence of  $\mathcal{R}^+(M)$  should imply that of  $\widetilde{\mathcal{R}}^+(M)$ .)  $\square$

*Proof of Prop. 5.18.* (i) Let  $\underline{a}$  belong to  $\mathcal{F}_\pm(M)$ , and assume  $\underline{a} \cong^* \underline{b}$  and  $\underline{a} \cong^* \underline{c}$ . By Lemma 5.5, we have  $\underline{a} \simeq \underline{b} \simeq \underline{c}$ , whence  $\underline{b} \Rightarrow^* \underline{d}$  and  $\underline{c} \Rightarrow^* \underline{d}$ , where  $\underline{d}$  is the (unique)  $\mathcal{R}$ -irreducible reduct of  $\underline{a}$ . Hence (5.13) is satisfied.

(ii) Assume that  $\mathcal{R}(M)$  is cross-confluent, and we have  $\underline{a} \Rightarrow^* \underline{1}$  and  $\underline{a} \Rightarrow^* \underline{b}$  for some  $\underline{a}$  in  $\mathcal{F}_\pm(M)$ . By (5.14), which is valid since, by Prop. 5.17, the cross-confluence of  $\mathcal{R}(M)$  implies that of  $\widetilde{\mathcal{R}}(M)$ , there exists  $\underline{d}$  satisfying  $\underline{b} \cong^* \underline{d}$  and  $\underline{1} \cong^* \underline{d}$ . Since  $\underline{1}$  is  $\widetilde{\mathcal{R}}$ -irreducible, we must have  $\underline{d} = \underline{1}$ , whence  $\underline{b} \cong^* \underline{1}$ . By Lemma 5.19, we deduce  $\underline{b} \Rightarrow^* \underline{1}$ . Hence  $\mathcal{R}(M)$  is  $\underline{1}$ -confluent and, therefore, by Prop. 3.29, it is semi-convergent.

(iii) For (i), the argument is the same in the case  $\underline{a} \in \mathcal{F}(M)$ . For (ii), assume that  $\mathcal{R}^+(M)$  is cross-confluent, and we have  $\underline{a} \Rightarrow^* \underline{1}$  and  $\underline{a} \Rightarrow^* \underline{b}$  for some  $\underline{a}$  in  $\mathcal{F}(M)$ . By Lemma 5.5(iv), we have  $1 \cdot \widetilde{\underline{a}} \cong^* 1 \cdot \underline{1} = \underline{1}$  and  $1 \cdot \underline{a} \cong^* 1 \cdot \underline{b}$ , and  $1 \cdot \widetilde{\underline{a}}$  is positive. As  $\mathcal{R}^+(M)$  is cross-confluent, we deduce the existence of  $\underline{d}$  satisfying  $\underline{1} \Rightarrow^* \underline{d}$  and  $1 \cdot \widetilde{\underline{b}} \Rightarrow^* \underline{d}$ . As  $\underline{1}$  is  $\mathcal{R}$ -irreducible, we have  $\underline{d} = \underline{1}$ , whence  $1 \cdot \widetilde{\underline{b}} \Rightarrow^* \underline{1}$ , and, by Lemma 5.5(iv) again,  $\underline{b} \cong^* \underline{1}$ . By Lemma 5.19(ii), we deduce  $\underline{b} \Rightarrow^* \underline{1}$ . Hence  $\mathcal{R}^+(M)$  is  $\underline{1}$ -confluent and, by Prop. 3.29, it is semi-convergent.  $\square$

We propose the following conjecture, which, by Prop. 5.18, implies Conjecture **A**:

**Conjecture C.** *If  $M$  is an Artin-Tits monoid, then  $\mathcal{R}^+(M)$  is cross-confluent.*

Conjecture **C** is different from Conjectures **A** and **B** in that it predicts something for every multifraction, and not only for unital ones. So, in a sense, it is more structural. We now observe that some elementary cases of cross-confluence are always valid.

**Lemma 5.20.** *Assume that  $M$  is a gcd-monoid and  $\underline{a}$  is a multifraction on  $M$  such that  $\underline{a} \bullet R_{i,x}$  is defined. If  $i$  is positive (resp. negative) in  $\underline{a}$ , let  $x'$  and  $\widehat{x}$  be defined by  $x'a_i = a_i \widetilde{\vee} x$  and  $\widehat{x} = a_i \widetilde{\wedge} x$  (resp.  $a_i x' = a_i \vee x$  and  $\widehat{x} = a_i \wedge x$ ). Then, we have*

$$(5.21) \quad \underline{a} \bullet R_{i,x} \widetilde{R}_{i,x'} = \underline{a} \bullet D_{i,\widehat{x}}.$$

*Proof.* (Fig. 5) Let  $\underline{a}' = \underline{a} \bullet R_{i,x}$ . Assuming  $i$  negative in  $\underline{a}$ , we have

$$(5.22) \quad a'_{i-1} = a_{i-1} x', \quad x a'_i = a_i x' = a_i \vee x, \quad x a'_{i+1} = a_{i+1}.$$

By construction,  $x'$  right-divides  $a'_{i-1}$ , and  $x'$  and  $a'_i$  admit a common left-multiple, namely  $a_i x'$ . Hence  $\underline{a}'' = \underline{a}' \bullet \widetilde{R}_{i,x'}$  is defined, and it is determined by

$$(5.23) \quad a''_{i-1} x' = a'_{i-1}, \quad a''_i x' = x'' a'_i = a'_i \widetilde{\vee} x', \quad a''_{i+1} = x'' a'_{i+1}.$$

By definition, the left-lcm  $x''a'_i$  of  $x'$  and  $a'_i$  left-divides their common left-multiple  $xa'_i$ , which implies the existence of  $\hat{x}$  satisfying  $x = \hat{x}x''$ , and, from there,  $a_ix' = xa'_i = \hat{x}x''a'_i = \hat{x}a''_ix'$ , whence  $a_i = \hat{x}a''_i$ . Merging (5.22) and (5.23), we deduce  $a''_{i-1} = a_{i-1}$ ,  $\hat{x}a''_i = a_i$ , and  $\hat{x}a''_{i+1} = a_{i+1}$ , which shows that  $\underline{a}''$  is obtained from  $\underline{a}$  by left-dividing the  $i$ th and  $(i+1)$ st entries by  $\hat{x}$ , *i.e.*, by applying the division  $D_{i,\hat{x}}$ .

The argument is symmetric when  $i$  is positive in  $\underline{a}$ .  $\square$

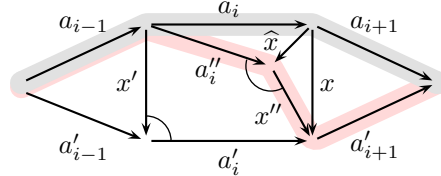


FIGURE 5. Composing the left reduction  $R_{i,x}$  and the inverse right reduction  $\tilde{R}_{i,x'}$  amounts to performing the division  $D_{i,\hat{x}}$  where  $\hat{x}$  is the gcd of  $a_i$  and  $x$ .

When  $\underline{a} \cdot \tilde{R}_{i,x}$  is defined, a symmetric argument gives

$$(5.24) \quad \underline{a} \cdot \tilde{R}_{i,x} R_{i,x'} = \underline{a} \cdot D_{i-1,\hat{x}},$$

where  $x'$  and  $\hat{x}$  are now defined by  $a_ix' = a_i \vee x$  and  $\hat{x} = a_i \wedge x$  (*resp.*  $x'a_i = a_i \tilde{\vee} x$  and  $\hat{x} = a_i \tilde{\wedge} x$ ) if  $i$  is positive (*resp.* negative) in  $\underline{a}$ . The index of the division is shifted ( $i-1$  instead of  $i$ ) relatively to (5.21), because, as a set of pairs,  $D_{i,x}$  is included in  $\tilde{R}_{i+1,x}$ .

We deduce that cross-confluence for one reduction is always valid:

**Proposition 5.25.** *If  $M$  is a gcd-monoid, then, for all  $\underline{a}, \underline{b}, \underline{c}$  in  $\mathcal{F}_{\pm}(M)$ ,*

$$(5.26) \quad \text{If we have } \underline{a} \overset{\sim}{\Rightarrow} \underline{b} \text{ and } \underline{a} \overset{\sim}{\Rightarrow} \underline{c}, \text{ there exists } \underline{d} \text{ satisfying } \underline{b} \overset{*}{\Rightarrow} \underline{d} \text{ and } \underline{c} \overset{*}{\Rightarrow} \underline{d}.$$

$$(5.27) \quad \text{If we have } \underline{a} \Rightarrow \underline{b} \text{ and } \underline{a} \Rightarrow \underline{c}, \text{ there exists } \underline{d} \text{ satisfying } \underline{b} \overset{*}{\Rightarrow} \underline{d} \text{ and } \underline{c} \overset{*}{\Rightarrow} \underline{d}.$$

*Proof.* Assume  $\underline{b} = \underline{a} \cdot \tilde{R}_{i,x}$  and  $\underline{c} = \underline{a} \cdot \tilde{R}_{j,y}$ . By Lemma 5.20, we have

$$\underline{b} \cdot R_{i,x'} = \underline{a} \cdot D_{i,\hat{x}} \quad \text{and} \quad \underline{c}' \cdot R_{j,y'} = \underline{a} \cdot D_{j,\hat{y}}$$

for some  $x', \hat{x}$  and  $y', \hat{y}$ . By Prop. 4.4,  $\partial \underline{a}$  is a common reduct of all multifractions obtained from  $\underline{a}$  by a division, hence in particular of  $\underline{a} \cdot D_{i,x'}$  and  $\underline{a} \cdot D_{j,y'}$ . Thus  $\underline{b} \overset{*}{\Rightarrow} \partial \underline{a}$  and  $\underline{c} \overset{*}{\Rightarrow} \partial \underline{a}$  hold, hence (5.26) is satisfied with  $\underline{d} = \partial \underline{a}$ .

The proof of (5.27) is symmetric, using (5.24) instead of (5.21), and  $\underline{d} = \partial \underline{a}$  again.  $\square$

In view of extending Prop. 5.25, one can wonder whether, if  $\underline{b}$  is obtained by applying  $k$  right reductions to  $\underline{a}$ , then applying  $k$  well chosen left reductions to  $\underline{b}$  can provide a multifraction  $\underline{c}$  obtained by (possibly  $k$ ) divisions from  $\underline{a}$ . This fails in general.

**Example 5.28.** In the Artin-Tits monoid of type  $\tilde{A}_2$ , consider  $\underline{a} = 1/a/ca/cb/b$  and  $\underline{b} = \underline{a} \cdot \tilde{R}_{3,a} \tilde{R}_{5,b} = 1/1/ca/cb/1$ , (which is  $\tilde{\mathcal{R}}$ -irreducible). The only way to left reduce  $\underline{b}$  is to start with  $R_{1,ca}$ , leading to  $\underline{c} = ca/1/cb/1$ , which is not reachable from  $\underline{a}$  by two, or any number, of divisions.

In the above example, we have  $\underline{c} = \underline{a} \cdot R_{4,b} R_{2,cac}$ , and therefore there is no contradiction with the weaker conclusion that  $\underline{c}$  is obtained both from  $\underline{a}$  and  $\underline{b}$  by applying two left reductions. The following example shows that the latter property is not true in general.

**Example 5.29.** In the Artin-Tits monoid of type  $\tilde{A}_2$ , consider  $\underline{a} = ca/cb/bc/ba$  and  $\underline{b} = \underline{a} \cdot \tilde{R}_{3,bc} \tilde{R}_{2,ca} \tilde{R}_{4,a} = 1/1/ac/ab$ . Then (as predicted by Conjecture C),  $\underline{a}$  and  $\underline{b}$  admit common left reducts, but the latter are  $\underline{c}$  and  $\underline{c} \cdot D_{2,c}$ , with  $\underline{c} = ac/cab/c/1 = \underline{a} \cdot R_{2,b} R_{3,a} D_{2,a} R_{3,b} D_{1,aa} D_{2,b}$ , not reachable from  $\underline{a}$  using less than six left reductions. What is surprising here is that  $\underline{b}' = \underline{a} \cdot \tilde{R}_{3,bc} \tilde{R}_{2,ca} = 1/1/aca/aba$  left reduces to  $\underline{a}$ , whereas  $\underline{b} = \underline{b}' \cdot D_{3,a}$  only left reduces to  $\underline{c}$ , very far from  $\underline{a}$ : one single division may change left reducts completely.

**Remark 5.30.** That cross-confluence is difficult to prove is paradoxical, as right reduction is close to being an inverse of left reduction. Indeed, provided the ground monoid  $M$  is noetherian, every reduction is a product of atomic reductions, *i.e.*, reductions of the type  $R_{i,x}$  or  $\tilde{R}_{i,x}$  with  $x$  an atom of  $M$ . Now, if  $x$  is an atom, the gcd of  $x$  and  $a_i$  is either  $a_i$ , meaning that  $x$  divides  $a_i$ , or 1. In the former case,  $\underline{a} \bullet R_{i,x}$  is  $\underline{a} \bullet D_{i,x}$ , whereas, in the latter, Lemma 5.20 implies  $\underline{a} = (\underline{a} \bullet R_{i,x}) \bullet \tilde{R}_{i,x'}$ , *i.e.*, left reducing  $x$  in  $\underline{a}$  is the inverse of right reducing  $x'$ . Thus, using  $\Rightarrow_{\text{at}}$  for atomic left reduction,  $\Rightarrow^*$  is generated by  $\Rightarrow_{\text{at}} \cup \Rightarrow_{\text{div}}$ , and  $\Leftarrow^*$  by  $\Leftarrow_{\text{at}} \cup \Rightarrow_{\text{div}}$ . Cross-confluence between  $\Rightarrow_{\text{at}}$  and  $\Leftarrow_{\text{at}}$  is trivial, whereas, according to Lemma 4.7 and the results of [7], confluence between  $\Rightarrow_{\text{at}}$  and  $\Rightarrow_{\text{div}}$  is always true. Thus, one might hope that mixing both types will not change the situation significantly. This is not true: Example 5.33 below shows that composing one reduction and one division may be sufficient to quit tame reductions.

**5.4. Uniform cross-confluence.** Prop. 5.25 shows not only that, for all elementary right reducts  $\underline{b}, \underline{c}$  of  $\underline{a}$ , there exists  $\underline{d}$  witnessing for cross-confluence, but even that there exists  $\underline{d}$  only depending on  $\underline{a}$ , namely  $\partial \underline{a}$ , that witnesses for all elementary right reducts of  $\underline{a}$  simultaneously. This suggests to consider a strong version of cross-confluence:

**Definition 5.31.** If  $M$  is a gcd-monoid, we say that  $\mathcal{R}(M)$  (*resp.*  $\mathcal{R}^+(M)$ ) is *uniformly cross-confluent* if there exists a map  $\nabla$  from  $\mathcal{F}_{\pm}(M)$  (*resp.*  $\mathcal{F}(M)$ ) to itself such that, for every  $\underline{a}$  in  $\mathcal{F}_{\pm}(M)$  (*resp.* in  $\mathcal{F}(M)$ ), the relation  $\underline{b} \Rightarrow^* \nabla \underline{a}$  holds for every right reduct  $\underline{b}$  of  $\underline{a}$ .

Defining  $\nabla \underline{a} = \text{red}(\underline{a})$  in the convergent case, we can strengthen Prop. 5.18 into:

**Proposition 5.32.** *If  $M$  is a noetherian gcd-monoid and  $\mathcal{R}(M)$  is convergent, then  $\mathcal{R}(M)$  is uniformly cross-confluent.*

We propose:

**Conjecture  $\mathbf{C}_{\text{unif}}$ .** *If  $M$  is an Artin-Tits monoid, then  $\mathcal{R}^+(M)$  is uniformly cross-confluent.*

Conjecture  $\mathbf{C}_{\text{unif}}$ , which implies Conjecture  $\mathbf{C}$  and, therefore, Conjecture  $\mathbf{A}$ , is more demanding, but one might hope that an explicit definition of  $\nabla$  could reduce a possible proof to a series of verifications. However, finding a close formula for  $\nabla$  seems difficult: natural candidates could be a least common ancestor of all  $\mathcal{R}$ -irreducible left reducts of  $\underline{a}$ , or (in view of Prop. 5.32)  $\text{red}_t(\underline{a})$ . These choices do not work: when  $\underline{a}$  has more than one  $\mathcal{R}$ -irreducible left reduct, it may be that the only possible choice for  $\nabla(\underline{a})$  is one of the latter.

**Example 5.33.** In the Artin-Tits monoid of type  $\tilde{A}_2$ , consider  $\underline{a} := \text{a/bac/bb/aca}$  and  $\underline{b} = \underline{a} \bullet \tilde{R}_{2,\text{a}} D_{3,\text{c}} = 1/\text{cbac}/\text{ccb}/\text{ca}$ . Then  $\underline{a}$  has two  $\mathcal{R}$ -irreducible left reducts, namely  $\underline{a}' = 1/\text{bac}/\text{cb}/\text{ca}$  and  $\underline{a}'' = \text{a/baac}/\text{ab}/\text{ac}$ , and the only common left reduct of  $\underline{a}$  and  $\underline{b}$  is  $\underline{a}'$ . In particular,  $\underline{b}$  does not left reduce to  $\text{red}_t(\underline{a})$ , which is  $\text{a/ac}/\text{b}/\text{aca}$ .

## 6. FINITE APPROXIMATIONS

The semi-convergence of  $\mathcal{R}(M)$  and Conjectures  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$ , involve multifractions of arbitrary depth. Further results appear in the particular case of small depth multifractions. The cases of depth 2 and, more interestingly, of depth 4 are addressed in Subsections 6.1 and 6.2, where connections with the embeddability in the group and the uniqueness of fractional decompositions, respectively, are established. An application of the latter to partial orderings of the group is established in Subsection 6.3.

**6.1. The  $n$ -semi-convergence property.** The rewrite system  $\mathcal{R}(M)$  has been called semi-convergent if (3.4) holds for every multifraction on  $M$ , *i.e.*, if  $\underline{a}$  being unital implies  $\underline{a} \Rightarrow^* \underline{1}$ .

**Definition 6.1.** If  $M$  is a gcd-monoid, we say that  $\mathcal{R}(M)$  (*resp.*  $\mathcal{R}^+(M)$ ) is  *$n$ -semi-convergent* if (3.4) holds for every  $n$ -multifraction  $\underline{a}$  in  $\mathcal{F}_{\pm}(M)$  (*resp.*  $\mathcal{F}(M)$ ).

Accordingly, we shall use Conjecture  $\mathbf{A}_n$  for the restriction of Conjecture  $\mathbf{A}$  to depth  $n$  multifractions, and similarly for  $\mathbf{B}$  and  $\mathbf{C}$ . Some easy connections exist. Of course, if  $\mathcal{R}(M)$  is  $n$ -semi-convergent, then so is its subsystem  $\mathcal{R}^+(M)$ .

**Lemma 6.2.** *If  $M$  is a gcd-monoid and  $\mathcal{R}(M)$  (resp.  $\mathcal{R}^+(M)$ ) is  $n$ -semi-convergent, it is  $p$ -semi-convergent for  $p < n$ .*

*Proof.* Let  $\underline{a}$  be a nontrivial unital  $p$ -multifraction, with  $p < n$ . There exists  $r$  (equal to  $n - p$  or  $n - p + 1$ ) such that  $\underline{a} \cdot \underline{1}_r$  has width  $n$ , and it is also nontrivial and unital. As  $\mathcal{R}(M)$  is  $n$ -semi-convergent, we have  $\underline{a} \cdot \underline{1}_r \Rightarrow^* \underline{1}$ , which implies  $\underline{a} \Rightarrow^* \underline{1}$  by Lemma 2.18(iii). So  $\mathcal{R}(M)$  is  $p$ -semi-convergent.  $\square$

On the other hand, by repeating the proof of Prop. 3.22, we obtain

**Lemma 6.3.** *If  $M$  is a noetherian gcd-monoid, then  $\mathcal{R}(M)$  (resp.  $\mathcal{R}^+(M)$ ) is  $n$ -semi-convergent if, and only if, (3.23) holds for every  $n$ -multifraction  $\underline{a}$  in  $\mathcal{F}_\pm(M)$  (resp.  $\mathcal{F}(M)$ ), i.e., if  $\underline{a}$  is unital, then it is either trivial or reducible.*

We now address the cases of small depth. The case of depth one is essentially trivial, in that it follows from a sufficiently strong form of noetherinity and does not really involve the algebraic properties of the monoid:

**Proposition 6.4.** *Assume that  $M$  is a gcd-monoid that admits a length function, namely a map  $\lambda : M \rightarrow \mathbb{N}$  satisfying, for all  $a, b$  in  $M$ ,*

$$(6.5) \quad \lambda(ab) = \lambda(a) + \lambda(b), \quad \text{and} \quad \lambda(a) > 0 \text{ for } a \neq 1.$$

*Then the system  $\mathcal{R}(M)$  is 1-semi-convergent.*

Condition (6.5) is strong noetherianity (3.13) with  $\geq$  replaced by  $=$ . It holds in every Artin-Tits monoid and, more generally, in every monoid with a homogeneous presentation.

*Proof.* Extend the map  $\lambda$  to  $\mathcal{F}_\pm(M)$  by  $\lambda(\underline{a}) := \sum_{i \text{ positive in } \underline{a}} \lambda(a_i) - \sum_{i \text{ negative in } \underline{a}} \lambda(a_i)$ . Then  $\lambda$  is a homomorphism from the monoid  $\mathcal{F}_\pm(M)$  to  $(\mathbb{N}, +)$ , and, for every  $a$  in  $M$ , we have  $\lambda(a/a) = \lambda(/a/a) = \lambda(1) = \lambda(\emptyset) = 0$ . By Prop. 2.5, the latter pairs generate  $\simeq$  as a congruence, hence  $\lambda$  is invariant under  $\simeq$ . Then  $a \neq 1$  implies  $\lambda(a) \neq \lambda(1)$ , whence  $a \not\approx 1$ . Hence the only unital 1-multifraction is 1, and  $\mathcal{R}(M)$  is 1-semi-convergent.  $\square$

The case of depths 2 and 3 turn out to be directly connected with the embeddability of the considered monoid in its enveloping group.

**Proposition 6.6.** *If  $M$  is a gcd-monoid, the following are equivalent:*

- (i) *The system  $\mathcal{R}^+(M)$  is 2-semi-convergent.*
- (ii) *The system  $\mathcal{R}(M)$  is 2-semi-convergent.*
- (iii) *The system  $\mathcal{R}(M)$  is 3-semi-convergent.*
- (iv) *The monoid  $M$  embeds in  $\mathcal{U}(M)$ .*

*Proof.* Assume that  $\mathcal{R}^+(M)$  is 2-semi-convergent. Let  $a, b$  two elements of  $M$  satisfying  $\iota(a) = \iota(b)$ . By (2.6), we have  $a/b \simeq 1$ . The assumption that  $\mathcal{R}^+(M)$  is 2-semi-convergent implies  $a/b \Rightarrow^* \underline{1}$ . By definition, this means that there exists  $x$  in  $M$  satisfying  $a = x = b$ , whence  $a = b$ . So  $M$  embeds in  $\mathcal{U}(M)$ . Hence (i) implies (iv).

Clearly (ii) implies (i), and (iii) implies (ii) by Lemma 6.2.

Finally, assume that  $M$  embeds in  $\mathcal{U}(M)$ , and let  $\underline{a}$  be a positive nontrivial unital 3-multifraction. By assumption, we have  $\underline{a} \simeq \underline{1}$ , whence  $\iota(a_1)\iota(a_2)^{-1}\iota(a_3) = 1$  in  $\mathcal{U}(M)$  by (2.6), and, therefore,  $\iota(a_2) = \iota(a_3)\iota(a_1) = \iota(a_3a_1)$  in  $\mathcal{U}(M)$ . As  $M$  embeds in  $\mathcal{U}(M)$ , this implies  $a_2 = a_3a_1$  in  $M$ . Then  $\underline{a}$  has the form  $a_1/a_3a_1/a_3$ , implying  $\underline{a} \bullet D_{1,a_1}D_{2,a_3} = 1/1/1$ . Thus  $\mathcal{R}^+(M)$  is 3-semi-convergent. The argument is the same for a negative 3-multifraction  $\underline{a}$ , finding now  $\iota(a_1)^{-1}\iota(a_2)\iota(a_3)^{-1} = 1$ , whence  $a_2 = a_1a_3$ , and  $\underline{a} \bullet D_{1,a_1}D_{2,a_3} = /1/1/1$ . Hence  $\mathcal{R}(M)$  is 3-semi-convergent. So (iv) implies (iii).  $\square$

**Corollary 6.7.** *For every Artin-Tits monoid  $M$ , the system  $\mathcal{R}(M)$  is  $n$ -semi-convergent for  $n \leq 3$ .*

*Proof.* By Lemma 6.2,  $\mathcal{R}(M)$  is 1-semi-convergent. Next, it is known [21] that  $M$  embeds into  $\mathcal{U}(M)$ . Hence, by Prop. 6.6,  $\mathcal{R}(M)$  is 2- and 3-semi-convergent.  $\square$

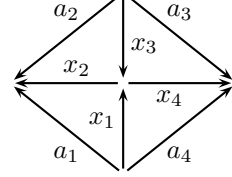
In other words, Conjecture  $\mathbf{A}_n$  is true for  $n \leq 3$ .

**6.2. Multifractions of depth 4.** We now address 4-semi-convergence, which turns out to give rise to interesting phenomena. We begin with preliminary results about unital multifractions that are in some sense the simplest ones.

**Definition 6.8.** If  $M$  is a monoid and  $\underline{a}$  is an  $n$ -multifraction on  $M$ , with  $n$  even, we say that  $(x_1, \dots, x_n)$  is a *central cross* for  $\underline{a}$  if we have

$$a_i = \begin{cases} x_i x_{i+1} & \text{for } i \text{ positive in } \underline{a}, \\ x_{i+1} x_i & \text{for } i \text{ negative in } \underline{a}, \end{cases}$$

with the convention  $x_{n+1} = x_1$ .



The diagram of Def. 6.8 shows that a multifraction that admits a central cross is unital: if  $(x_1, \dots, x_n)$  is a central cross for a positive multifraction  $\underline{a}$ , we find

$$\iota(\underline{a}) = \iota(x_1 x_2) \iota(x_3 x_2)^{-1} \iota(x_3 x_4) \cdots (x_1 x_n)^{-1} = 1,$$

and similarly when  $\underline{a}$  is negative. It follows from the definition that a sequence  $(x_1, \dots, x_n)$  is a central cross for a positive multifraction  $\underline{a}$  if, and only if,  $(x_2, \dots, x_n, x_1)$  is a central cross for  $/a_2/\cdots/a_n/a_1$ . So, we immediately obtain

**Lemma 6.9.** *For every monoid  $M$  and every even  $n$ , a positive  $n$ -multifraction  $\underline{a}$  admits a central cross if, and only if, the negative multifraction  $/a_2/\cdots/a_n/a_1$  does.*

Multifractions with a central cross always behave nicely in terms of reduction:

**Lemma 6.10.** *If  $M$  is a gcd-monoid and  $\underline{a}$  is a multifraction on  $M$  that admits a central cross, then  $\text{red}_t(\underline{a}) = \underline{1}$  holds.*

*Proof.* We prove the result using induction on  $n \geq 2$  even, and assuming  $\underline{a}$  positive. Assume that  $(x_1, \dots, x_n)$  is a central cross for  $\underline{a}$ . For  $n = 2$ , the assumption boils down to  $a_1 = a_2 = x_1 x_2$ , directly implying  $\underline{a} \bullet R_1^{\max} = 1/1$ . Assume  $n \geq 4$  and, say,  $\underline{a}$  positive. Let  $x := x_1 \tilde{\wedge} x_3$ , with  $x_1 = x'_1 x$  and  $x_3 = x'_3 x$ . Then we have  $a_1 \tilde{\wedge} a_2 = x x_2$ , whence  $\underline{a} \bullet R_1^{\max} = x'_1/x'_3/a_3/\cdots/a_n$ . As  $a_3 = x_3 x_4$  expands into  $a_3 = x'_3 x x_4$ , this can be rewritten as  $\underline{a} \bullet R_1^{\max} = x'_1/x'_3/x'_3 x x_4/\cdots/a_n$ . We deduce

$$(6.11) \quad \underline{a} \bullet R_1^{\max} R_2^{\max} = x'_1 x x_4 / 1/1 / a_4 / \cdots / a_n = x_1 x_4 / 1/1 / a_4 / \cdots / a_n$$

with  $a_4 = x_1 x_4$  for  $n = 4$ , and  $a_4 = x_5 x_4$  for  $n \geq 6$ . In every case, the subsequent action of  $R_3^{\max} \cdots R_{n-1}^{\max}$  is to push  $a_4$ , then  $a_5$ , etc. until  $a_n$ , through  $1/1$ , leading to

$$(6.12) \quad \underline{a} \bullet R_1^{\max} \cdots R_{n-1}^{\max} = x_1 x_4 / a_4 / \cdots / a_n / 1/1.$$

For  $n = 4$ , (6.12) reads  $\underline{a} \bullet R_1^{\max} R_2^{\max} R_3^{\max} = x_1 x_4 / x_1 x_4 / 1/1$ , and a further application of  $R_1^{\max}$  yields  $\underline{1}$ . For  $n \geq 6$ , the assumption that  $(x_1, \dots, x_n)$  is a central cross for  $\underline{a}$  implies that  $(x_1, x_4, \dots, x_n)$  is a central cross for the  $(n-2)$ -multifraction  $x_1 x_4 / a_4 / \cdots / a_n$ . The induction hypothesis for the latter gives  $\text{red}_t(x_1 x_4 / a_4 / \cdots / a_n) = \underline{1}$ , which expands into

$$(6.13) \quad x_1 x_4 / a_4 / \cdots / a_n \bullet R_{U(n-2)}^{\max} = \underline{1}_{n-2}.$$

By Lemma 2.18(iii), (6.13) implies  $x_1 x_4 / a_4 / \cdots / a_n / 1/1 \bullet R_{U(n-2)}^{\max} = \underline{1}_n$ . Merging with (6.12), we deduce  $\underline{a} \bullet R_{1 \cdots n-1}^{\max} R_{U(n-2)}^{\max} = \underline{1}_n$ , which is  $\underline{a} \bullet R_{U(n)}^{\max} = \underline{1}_n$ , i.e.,  $\text{red}_t(\underline{a}) = \underline{1}$ .

The argument is similar when  $\underline{a}$  is negative. □

We now concentrate on 4-multifractions. A sort of transitivity of central crosses holds.

**Lemma 6.14.** *Assume that  $M$  is a gcd-monoid,  $\underline{a}, \underline{b}$  are 4-multifractions admitting a central cross, and  $c_1 a_4 = c_4 b_1$  and  $c_2 a_3 = c_3 b_2$  holds. Then  $c_1 a_1 / c_2 a_2 / c_3 b_3 / c_4 b_4$  admits a central cross. In particular,  $a_1 / a_2 / b_3 / b_4$  admits a central cross for  $a_4 = b_1$  and  $a_3 = b_2$ .*

*Proof.* (Fig. 6) Let  $(x_1, \dots, x_4)$  and  $(y_1, \dots, y_4)$  be central crosses for  $\underline{a}$  and  $\underline{b}$ , respectively. By assumption, we have  $c_2x_3x_4 = c_2a_3 = c_3b_2 = c_3y_3y_2$ , so  $x_4$  and  $y_2$  admit a common left-multiple, say  $x_4 \tilde{\vee} y_2 = xx_4 = yy_2$ , and there exists  $z_3$  satisfying  $c_2a_3 = z_3xx_4 = z_3yy_2 = c_3b_2$ . Then  $a_3 = x_3x_4$  and  $b_2 = y_3y_2$  respectively imply

$$(6.15) \quad c_2x_3 = z_3x \quad \text{and} \quad c_3y_3 = z_3y.$$

Arguing similarly from  $c_1x_1x_4 = c_1a_4 = c_4b_1 = c_4y_1y_2$ , we deduce the existence of  $z_1$  satisfying  $c_1a_4 = z_1xx_4 = z_1yy_2 = c_4b_1$ , leading to

$$(6.16) \quad c_1x_1 = z_1x \quad \text{and} \quad c_4y_1 = z_1y.$$

Then  $(z_1, xx_2, z_3, yy_4)$  is a central cross for  $c_1a_1/c_2a_2/c_3b_3/c_4b_4$ .  $\square$

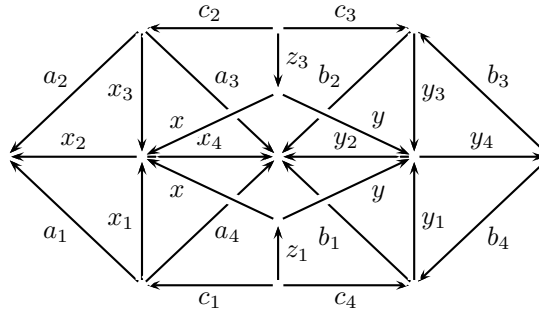


FIGURE 6. Transitivity of the existence of a central cross.

We deduce that reduction preserves the existence of a central cross in both directions.

**Lemma 6.17.** *Assume that  $M$  is a gcd-monoid and  $\underline{a}, \underline{b}$  are 4-multifractions on  $M$  satisfying  $\underline{a} \Rightarrow^* \underline{b}$ . Then  $\underline{a}$  admits a central cross if, and only if,  $\underline{b}$  does.*

*Proof.* It is enough to prove the result for  $\underline{a} \Rightarrow \underline{b}$ , say  $\underline{b} = \underline{a} \bullet R_{i,x}$ . Assume that  $\underline{a}$  is positive and  $(x_1, \dots, x_4)$  is a central cross for  $\underline{a}$ . Our aim is to construct a central cross for  $\underline{b}$  from that for  $\underline{a}$ . Consider the case  $i = 2$ , see Fig. 7. Let  $a_2x' = x \vee a_2$ . By definition, we have

$$(6.18) \quad b_1 = a_1x', \quad xb_2 = a_2x', \quad xb_3 = a_3, \quad b_4 = a_4.$$

By assumption,  $x$  and  $x_3$  admit a common right-multiple, namely  $a_3$ , hence they admit a right-lcm, say  $x \vee x_3 = xy_3 = x_3y$ . We have  $xb_2 = a_2x' = x_3(x_2x')$ , hence the right-lcm of  $x$  and  $x_3$  left-divides  $xb_2$ , i.e., we have  $xy_3 \leq xb_2$ , whence  $y_3 \leq b_2$ , say  $b_2 = y_3y_2$ .

Next, we have  $x_3x_2x' = a_2x' = xb_2 = xy_3y_2 = x_3yy_2$ , whence  $x_2x' = yy_2$  by left-cancelling  $x_3$ . Put  $y_1 = x_1y$ . We find  $b_1 = a_1x' = x_1x_2x' = x_1yy_2 = y_1y_2$ .

On the other hand, by assumption, as both  $x$  and  $x_3$  left-divide  $a_3$ , their right-lcm  $x_3y$  left-divide  $a_3$ , which is  $x_3x_4$ . So we have  $x_3y \leq x_3x_4$ , whence  $y \leq x_4$ , say  $x_4 = yy_4$ . Then we find  $xb_3 = a_3 = x_3x_4 = x_3yy_4 = xy_3y_4$ , whence  $b_3 = y_3y_4$ .

Finally, we have  $b_4 = a_4 = x_1x_4 = x_1yy_4 = y_1y_4$ . So  $(y_1, \dots, y_4)$  is a central cross for  $\underline{b}$ . The argument for  $i = 3$  is similar, mutatis mutandis, and so is the one for  $i = 1$ : in this case, the counterpart of  $x'$  is trivial, which changes nothing. Finally, the case when  $\underline{a}$  is negative is treated symmetrically. So  $\underline{b}$  admits a central cross whenever  $\underline{a}$  does.

For the other direction, assume again that  $\underline{a}$  and  $\underline{b}$  are positive, and that  $(y_1, \dots, y_4)$  is a central cross for  $\underline{b}$ . Assume again  $i = 2$ , and (6.18). The equality  $xb_2 = a_2x$  implies that  $(a_1, 1, a_2, x')$  is a central cross for  $a_1/a_2/xb_2/b_1$ . On the other hand,  $(y_1, y_2, xy_3, y_4)$  is a central cross for  $b_1/xb_2/a_3/a_4$ . So both  $a_1/a_2/xb_2/b_1$  and  $b_1/xb_2/a_3/a_4$  admit a central cross. By Lemma 6.14, this implies that  $a_1/a_2/a_3/a_4$  admits a central cross. So  $\underline{a}$  admits a central cross whenever  $\underline{b}$  does.  $\square$

The above proof does not use the assumption that  $xb_2$  is the lcm of  $x$  and  $a_2$ , but only the fact that the equalities of (6.18) hold for some  $x, x'$ , which is the case, in particular, for

$\underline{b} = \underline{a} \bullet \widetilde{R}_{i,x'}$ . So it also shows that every right reduct of a 4-multifraction with a central cross admits a central cross, and conversely.

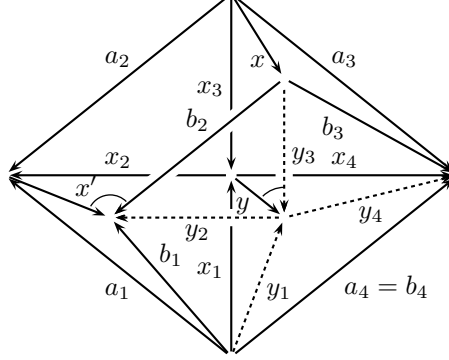


FIGURE 7. Construction of a central cross for  $\underline{a} \bullet R_{i,x}$  starting from one for  $\underline{a}$ .

We deduce a complete description of the 4-multifractions that reduce to  $\underline{1}$ :

**Proposition 6.19.** *If  $M$  is a gcd-monoid, then, for every 4-multifraction  $\underline{a}$  on  $M$ , the following are equivalent:*

- (i) *The relation  $\underline{a} \Rightarrow^* \underline{1}$  holds.*
- (ii) *The relation  $\text{red}_t(\underline{a}) = \underline{1}$  holds.*
- (iii) *The multifraction  $\underline{a}$  admits a central cross.*

*Proof.* The 4-multifractions  $1/1/1/1$  and  $/1/1/1/1$  both admit the central cross  $(1, 1, 1, 1)$ . Hence, by Lemma 6.17, every 4-multifraction satisfying  $\underline{a} \Rightarrow^* 1/1/1/1$  or  $\underline{a} \Rightarrow^* /1/1/1/1$  admits a central cross as well. Hence (i) implies (iii).

Next, Lemma 6.10 says that  $\text{red}_t(\underline{a}) = \underline{1}$  holds for every multifraction  $\underline{a}$  that admits a central cross, so (iii) implies (ii).

Finally, (ii) implies (i) by definition.  $\square$

We return to the study of  $n$ -semi-convergence for  $\mathcal{R}(M)$ , here for  $n = 4, 5$ .

**Proposition 6.20.** *If  $M$  is a gcd-monoid, the following are equivalent:*

- (i) *The system  $\mathcal{R}^+(M)$  is 4-semi-convergent.*
- (ii) *The system  $\mathcal{R}(M)$  is 4-semi-convergent.*
- (iii) *The system  $\mathcal{R}(M)$  is 5-semi-convergent.*
- (iv) *Every unital 4-multifraction in  $\mathcal{F}(M)$  admits a central cross.*

*Proof.* Assume that  $\mathcal{R}^+(M)$  is 4-semi-convergent. Let  $\underline{a}$  be a unital positive 4-multifraction. As  $\mathcal{R}^+(M)$  is 4-semi-convergent, we have  $\underline{a} \Rightarrow^* \underline{1}$ . By Prop. 6.19, we deduce that  $\underline{a}$  admits a central cross. Hence (i) implies (iv).

By definition, (ii) implies (i), and, by Lemma 6.2, (iii) implies (ii).

Finally, assume (iv), and let  $\underline{a}$  be a unital 5-multifraction on  $M$ . Assume first that  $\underline{a}$  is positive. Then the positive 4-multifraction  $a_5 a_1 / a_2 / a_3 / a_4$  is unital as well, hence, by assumption, it admits a central cross  $(x_1, \dots, x_4)$ , which expands into  $\underline{a} = a_1 / x_3 x_2 / x_3 x_4 / x_1 x_4 / a_5$  with  $a_5 a_1 = x_1 x_2$ . Then we obtain

$$\begin{aligned} \underline{a} &\Rightarrow^* a_1 / x_2 / 1 / x_1 / a_5 && \text{via } D_{2,x_3} D_{3,x_4} \\ &\Rightarrow^* a_1 / x_1 x_2 / a_5 / 1 / 1 = a_1 / a_5 a_1 / a_5 / 1 / 1 && \text{via } R_{3,x_1} R_{4,a_5} \\ &\Rightarrow^* 1 / 1 / 1 / 1 / 1 && \text{via } D_{1,a_1} D_{2,a_5}. \end{aligned}$$

If  $\underline{a}$  is negative, then  $a_2 / a_3 / a_4 / a_1 a_5$  is unital and positive, hence admits a central cross by assumption, leading to  $\underline{a} = /a_1 / x_1 x_2 / x_3 x_2 / x_3 x_4 / a_5$  with  $a_1 a_5 = x_1 x_4$ , and to  $\underline{a} \Rightarrow^* \underline{1}$ , this time via  $D_{2,x_2} D_{3,x_3} R_{3,x_4} R_{4,a_5} D_{1,a_1} D_{2,a_5}$ . Thus, every unital 5-multifraction on  $M$  reduces to  $\underline{1}$ , and  $\mathcal{R}(M)$  is 5-semi-convergent. So (iv) implies (iii).  $\square$

We now establish alternative forms for the point (iv) in Prop.6.20, connected with the uniqueness of the expression by irreducible fractions.

**Proposition 6.21.** *For every gcd-monoid  $M$ , the following are equivalent:*

- (i) *Every unital 4-multifraction in  $\mathcal{F}(M)$  admits a central cross.*
- (ii) *For all  $a, b, c, d$  in  $M$  satisfying  $\iota(a/b) = \iota(c/d)$ , there exist  $x, y$  in  $M$  satisfying*

$$(6.22) \quad a = x(a \tilde{\wedge} b), \quad b = y(a \tilde{\wedge} b), \quad c = x(c \tilde{\wedge} d), \quad d = y(c \tilde{\wedge} d).$$

- (iii) *All  $a, b, c, d$  in  $M$  satisfying  $\iota(a/b) = \iota(c/d)$  and  $a \tilde{\wedge} b = 1$  satisfy  $a \leq c$  and  $b \leq d$ .*
- (iv) *All  $a, b, c, d$  in  $M$  satisfying  $\iota(a/b) = \iota(c/d)$  and  $a \tilde{\wedge} b = c \tilde{\wedge} d = 1$  satisfy  $a = c$  and  $b = d$ .*
- (v) *All  $a, b, c$  in  $M$  satisfying  $a \tilde{\wedge} b = b \wedge c = 1$  and  $\iota(a/b/c) \in \iota(M)$  satisfy  $b = 1$ .*

Before establishing Prop.6.21, we begin with two characterizations of 4-multifractions with a central cross:

**Lemma 6.23.** *If  $M$  is a gcd-monoid, a positive 4-multifraction  $\underline{a}$  on  $M$  admits a central cross if, and only if, there exist  $x, y$  in  $M$  satisfying*

$$(6.24) \quad a_1 = x(a_1 \tilde{\wedge} a_2), \quad a_2 = y(a_1 \tilde{\wedge} a_2), \quad a_3 = y(a_3 \tilde{\wedge} a_4), \quad a_4 = x(a_3 \tilde{\wedge} a_4),$$

*if, and only if, there exist  $x', y'$  in  $M$  satisfying*

$$(6.25) \quad a_1 = (a_1 \tilde{\wedge} a_4)x', \quad a_2 = (a_2 \tilde{\wedge} a_3)x', \quad a_3 = (a_2 \tilde{\wedge} a_3)y', \quad a_4 = (a_1 \tilde{\wedge} a_4)y'.$$

*Proof.* Assume that  $\underline{a}$  is positive and  $(x_1, \dots, x_4)$  is a central cross for  $\underline{a}$ . Write  $x_1 = x(x_1 \tilde{\wedge} x_3)$  and  $x_3 = y(x_1 \tilde{\wedge} x_3)$ . Then we have  $x \tilde{\wedge} y = 1$ , and  $a_1 = x(x_1 \tilde{\wedge} x_3)x_2$ ,  $a_2 = y(x_1 \tilde{\wedge} x_3)x_2$ , whence  $a_1 \tilde{\wedge} a_2 = (x_1 \tilde{\wedge} x_3)x_2$ , and  $a_1 = x(a_1 \tilde{\wedge} a_2)$ ,  $a_2 = y(a_1 \tilde{\wedge} a_2)$ . On the other hand, we also find  $a_3 = x_3x_4 = y(x_1 \tilde{\wedge} x_3)x_4$ ,  $a_4 = x_1x_4 = x(x_1 \tilde{\wedge} x_3)x_4$ , whence  $a_3 \tilde{\wedge} a_4 = (x_1 \tilde{\wedge} x_3)x_4$ , and  $a_3 = y(a_3 \tilde{\wedge} a_4)$ ,  $a_4 = x(a_3 \tilde{\wedge} a_4)$ . So we found  $x, y$  satisfying (6.24).

The proof for (6.25) is similar, writing  $x_2 = (x_2 \wedge x_4)x'$  and  $x_4 = (x_2 \wedge x_4)y'$  and deducing  $a_1 \wedge a_4 = x_1(x_2 \wedge x_4)$  and  $a_2 \wedge a_3 = x_3(x_2 \wedge x_4)$ .

In the other direction, if (6.24) is satisfied,  $(x, a_1 \tilde{\wedge} a_2, y, a_3 \tilde{\wedge} a_4)$  is a central cross for  $\underline{a}$ , whereas, if (6.25) is satisfied, so is  $(a_1 \wedge a_4, x', a_2 \wedge a_3, y)$ .  $\square$

*Proof of Prop. 6.21.* If  $\iota(a/b) = \iota(c/d)$  holds, the 4-multifraction  $a/b/d/c$  is unital, hence it admits a central cross if (i) is true. In this case, Lemma 6.23 gives (6.22). So (i) implies (ii).

Next, applying (ii) in the case  $a \tilde{\wedge} b = 1$  gives  $a = x$  and  $b = y$ , whence  $a \leq c$  and  $b \leq d$ . If, in addition, we have  $c \tilde{\wedge} d = 1$ , we similarly obtain  $c \leq a$  and  $d \leq b$ , whence  $a = c$  and  $b = d$ . So (ii) implies (iii) and (iv). On the other hand, (iv) implies (iii). Indeed, assume  $\iota(a/b) = \iota(c/d)$  with  $a \tilde{\wedge} b = 1$ . Let  $e = c \tilde{\wedge} d$ , with  $c = c'e$  and  $d = d'e$ . Then we have  $c' \tilde{\wedge} d' = 1$  and  $\iota(a/b) = \iota(c'/d')$ . Then (iv) implies  $a = c' \leq c$  and  $b = d' \leq d$ .

Now, let  $a, b, c$  in  $M$  satisfy  $a \tilde{\wedge} b = b \wedge c = 1$  and  $\iota(a/b/c) = \iota(d)$  for some  $d$  in  $M$ . Then  $\iota(a/b) = \iota(d/c)$  holds so, if (ii) holds, we have  $a \leq d$  and  $b \leq c$ . Then the assumption  $b \wedge c = 1$  implies  $b = 1$ . So (iii) implies (v).

Finally, assume that  $\underline{a}$  is a unital 4-multifraction in  $\mathcal{F}(M)$ . Write  $a_2 = xa'_2$  and  $a_3 = xa'_3$  with  $x = a_2 \wedge a_3$ , so that  $a'_2 \wedge a'_3 = 1$  holds. Next, write  $a_1 = a'_1y$  and  $a'_2 = a''_2$  with  $y = a_1 \tilde{\wedge} a'_2$ , so that  $a'_1 \tilde{\wedge} a''_2 = 1$  holds. Then we have  $\iota(a'_1/a''_2/a'_3) = \iota(a_1/a'_2/a'_3) = \iota(a_1/a_2/a_3)$ , and the assumption  $\iota(\underline{a}) = 1$  implies  $\iota(a_1/a_2/a_3) = \iota(a_4) \in \iota(M)$ . Hence  $\iota(a'_1/a''_2/a'_3)$  lies in  $\iota(M)$  with  $a'_1 \tilde{\wedge} a''_2 = 1$  and  $a'_2 \wedge a'_3 = 1$ , whence a fortiori  $a''_2 \wedge a'_3 = 1$ . If (v) is true, we deduce  $a''_2 = 1$ . This means that  $(a'_1, y, x, a'_3)$  is a central cross for  $\underline{a}$ . So (v) implies (i).  $\square$

Putting things together, we obtain:

**Corollary 6.26.** *Conjectures  $\mathbf{A}_4$  and  $\mathbf{B}_4$  are equivalent, and they are equivalent to the property that, for every Artin-Tits monoid  $M$ , every element of  $\mathcal{U}(M)$  of the form  $ab^{-1}$  with  $a, b$  in  $M$  admits only one such expression with  $a \tilde{\wedge} b = 1$ .*

*Proof.* Let  $M$  be an Artin-Tits monoid and  $\underline{a}$  be a unital 4-multifraction on  $M$ . By Prop.6.19,  $\underline{a} \Rightarrow^* \underline{1}$  implies  $\text{red}_t(\underline{a}) = \underline{1}$ , so Conjecture  $\mathbf{A}_4$  implies Conjecture  $\mathbf{B}_4$ . On the other hand, by



Prop. 6.19 again, the property “ $\underline{a}$  unital implies  $\underline{a} \Rightarrow^* \underline{1}$ ” is equivalent to “ $\underline{a}$  unital implies  $\underline{a}$  admits a central cross” and, by Prop. 6.21, the latter is equivalent to the uniqueness of fractional decompositions.  $\square$

**6.3. An application to partial orderings on  $\mathcal{U}(M)$ .** If  $M$  is a Garside monoid, it is known [14, Sec. II.3.2] that the left- and right-divisibility relations of  $M$  can be extended into well-defined partial orders on the enveloping group  $\mathcal{U}(M)$  by declaring  $g \leq h$  (resp.  $g \lesssim h$ ) for  $g^{-1}h \in M$  (resp.  $gh^{-1} \in M$ ). The construction extends to every gcd-monoid, and we show that lattice properties are preserved whenever  $\mathcal{R}(M)$  is 4-semi-convergent.

**Proposition 6.27.** (i) *If  $M$  is a gcd-monoid that embeds into  $\mathcal{U}(M)$ , then declaring  $g \leq h$  for  $g^{-1}h \in \iota(M)$  provides a partial order on  $\mathcal{U}(M)$  that extends left-divisibility on  $M$ .*

(ii) *If  $\mathcal{R}(M)$  is 4-semi-convergent, any two elements of  $\mathcal{U}(M)$  that admit a common  $\leq$ -lower bound (resp. a common  $\leq$ -upper bound) admit a greatest one (resp. a lowest one).*

*Proof.* (i) We identify  $M$  with its image under  $\iota$ . As  $M$  is a semigroup in  $\mathcal{U}(M)$ , the relation  $\leq$  is transitive on  $\mathcal{U}(M)$ , and it is antisymmetric, as 1 is the only invertible element of  $M$ . Hence  $\leq$  is a partial order on  $\mathcal{U}(M)$ . By definition, it extends left-divisibility on  $M$ .

(ii) Assume that  $g$  and  $h$  admit a common  $\leq$ -lower bound  $f$  (see Fig. 8 left). This means that we have  $g = fx$  and  $h = fy$  for some  $x, y$  in  $M$ . Let  $f_0 := f(x \wedge y)$ . Write  $x = (x \wedge y)x_0$ ,  $y = (x \wedge y)y_0$ . Then we have  $g = f_0x_0$  and  $h = f_0y_0$ , whence  $f_0 \leq g$  and  $f_0 \leq h$ . Now assume that  $f_1$  is any common  $\leq$ -lower bound of  $g$  and  $h$ , say  $g = f_1x_1$  and  $h = f_1y_1$ . In the group  $\mathcal{U}(M)$ , we have  $x_0x_1^{-1} = y_0y_1^{-1}$ . Hence, the 4-multifraction  $x_0/x_1/y_1/y_0$  is unital, so, by assumption, it admits a central cross. Then Lemma 6.23 provides  $x', y'$  satisfying

$$x_0 = (x_0 \wedge y_0)x', \quad x_1 = (x_1 \wedge y_1)x', \quad y_1 = (x_1 \wedge y_1)y', \quad y_0 = (x_0 \wedge y_0)y'.$$

By definition of  $f_0$ , we have  $x_0 \wedge y_0 = 1$ , whence  $x' = x_0$  and  $y' = y_0$ , and, from there,  $f_1(x_1 \wedge y_1) = f_0$ , hence  $f_1 \leq f_0$ , in  $\mathcal{U}(M)$ . So  $f_0$  is a greatest  $\leq$ -lower bound for  $g$  and  $h$ .

The argument for lowest  $\leq$ -upper bound is symmetric. Assume that  $g$  and  $h$  admit a common  $\leq$ -upper bound  $f$  (see Fig. 8 right). Write  $f = gx = hy$ , then  $x = x_0(x \tilde{\wedge} y)$  and  $y = y_0(x \tilde{\wedge} y)$ . Put  $f_0 := gx_0$ . We have  $f_0(x \tilde{\wedge} y) = f = hy = hy_0(x \tilde{\wedge} y)$ , whence  $hy_0 = f_0$ . So  $f_0$  is a common  $\leq$ -upper bound of  $g$  and  $h$ . Now assume that  $f_1$  is any common  $\leq$ -upper bound of  $g$  and  $h$ , say  $f_1 = g_1x_1 = hy_1$ . In  $\mathcal{U}(M)$ , we have  $x_0y_0^{-1} = x_1y_1^{-1}$ , i.e., the 4-multifraction  $x_0/y_0/y_1/x_1$  is unital, hence it admits a central cross. Then Lemma 6.23 provides  $x, y$  satisfying

$$x_0 = x(x_0 \tilde{\wedge} y_0), \quad y_0 = y(x_0 \tilde{\wedge} y_0), \quad x_1 = x(x_1 \tilde{\wedge} y_1), \quad y_1 = y(x_1 \tilde{\wedge} y_1).$$

The definition of  $f_0$  gives  $x_0 \tilde{\wedge} y_0 = 1$ , whence  $x = x_0$  and  $y = y_0$ , leading in  $\mathcal{U}(M)$  to  $f_0(x_1 \tilde{\wedge} y_1) = f_1$ , hence  $f_0 \leq f_1$ . So  $f_0$  is a lowest  $\leq$ -upper bound for  $g$  and  $h$ .  $\square$

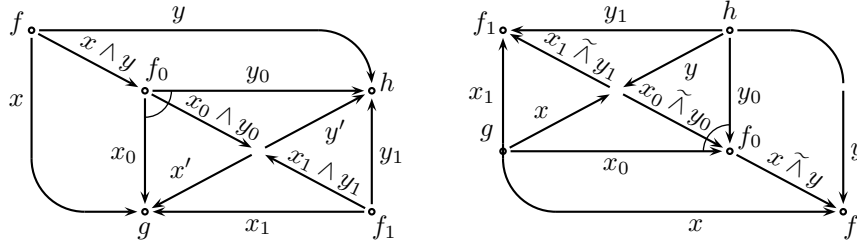


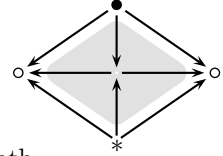
FIGURE 8. Conditional lattice property for the poset  $(\mathcal{U}(M), \leq)$ : greatest lower bound on the left, lowest upper bound on the right.

Of course, we have a symmetric extension for the right-divisibility relation.

**Question 6.28.** Is the assumption that  $M$  embeds in  $\mathcal{U}(M)$  sufficient to ensure that conditional greatest lower bounds and lowest upper bounds for  $\leq$  exist in  $\mathcal{U}(M)$ ?

**6.4. Multifractions of depth 6 and beyond.** The simple results established in the case of 4-multifractions do not extend to depth 6 and beyond. In particular, unital 6-multifractions need not admit a central cross: for instance, in the Artin-Tits monoid of type  $\tilde{A}_2$ , let  $\underline{a} := aba/aca/cac/bcb/bcb/bab$  and  $\underline{b} := ab/ac/ca/cb/bc/ba$ . Then we have  $\underline{a} \Rightarrow \underline{b} \Rightarrow^* \underline{1}$ , and  $\underline{a}$  admits a central cross, but  $\underline{b}$  does not, as it is prime. So the counterparts of Lemma 6.17 and Prop. 6.19 fail.

What Prop. 6.19 says is that, if we define  $(\Gamma_4, *)$  to be the pointed graph on the right, then, for every 4-multifraction  $\underline{a}$  that reduces to  $\underline{1}$ , there exists a labelling of the edges of  $\Gamma_4$  by elements of  $M$  such that the outer labels from  $*$  are  $a_1, \dots, a_4$  and the labels in each triangle induce equalities in  $M$ .



We shall see now that Conjecture **B** predicts similar results for every depth.

**Definition 6.29.** For  $n \geq 6$  even, let  $(\Gamma_n, *)$  be the graph obtained by appending  $n - 2$  adjacent copies of  $\Gamma_4$  around  $(\Gamma_{n-2}, *)$  starting from  $*$ , with alternating orientations, and connecting the last copy of  $(\Gamma_4, *)$  with the first one, see Fig. 9. We say that an  $n$ -multifraction on a monoid  $M$  admits a  $\Gamma_n$ -tiling if there exists a labeling of the edges of  $\Gamma_n$  by elements of  $M$  such that the outer labels from  $*$  are  $a_1, \dots, a_n$  and the labels in each triangle induce equalities in  $M$ .

Then, Prop. 6.19 says that every 4-multifraction reducing to  $\underline{1}$  admits a  $\Gamma_4$ -tiling.

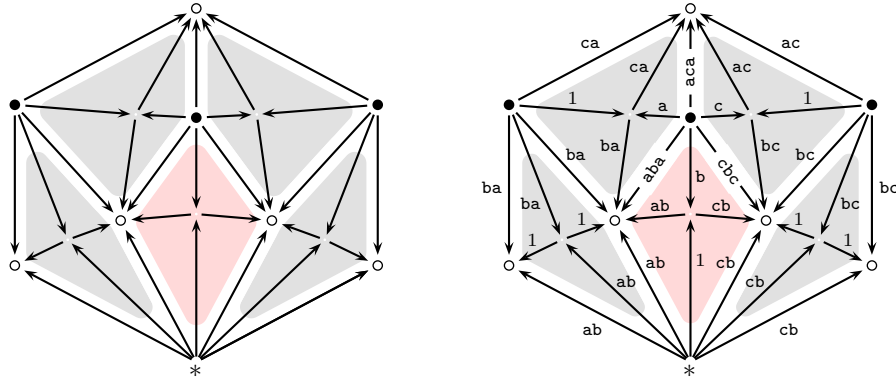
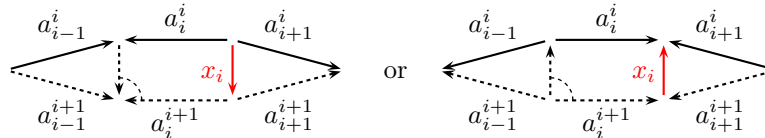


FIGURE 9. The graph  $\Gamma_6$ : on the left, the naked graph, with four juxtaposed copies of  $\Gamma_4$  around one (colored) copy of  $\Gamma_4$ ; it has fourteen vertices, namely four springs (one inner), five wells (two inner), and five 4-prongs; on the right, the  $\Gamma_6$ -tiling for the 6-multifraction  $ab/ba/ca/ac/bc/ab$  in the Artin-Tits monoid of type  $\tilde{A}_2$  resulting from the reduction in Ex. 3.6. Conjecture **B** predicts that every 6-multifraction representing  $\underline{1}$  in an Artin-Tits monoid admits such a  $\Gamma_6$ -tiling.

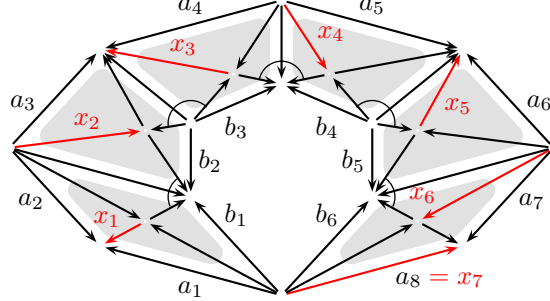
**Proposition 6.30.** *Let  $M$  be an Artin-Tits monoid. If Conjecture **B** is true, then every unital  $n$ -multifraction on  $M$  (with  $n$  even) admits a  $\Gamma_n$ -tiling, see Fig. 9.*

*Proof.* Let  $\underline{a}$  be a unital  $n$ -multifraction on  $M$ . Conjecture **B** predicts the equality  $\underline{a} \bullet R_{U(n)}^{\max} = \underline{1}$ . We shall see that the latter (and, more generally, any equality of the form  $\underline{a} \bullet R_{U(n), \underline{x}} = \underline{1}$ , with  $\underline{x}$  maximal or not) implies that  $\underline{a}$  admits a  $\Gamma_n$ -tiling. In view on an induction, assume  $\underline{a} \bullet R_{1, x_1} \cdots R_{n-1, x_{n-1}} = \underline{b} \bullet \underline{1}_2$ , for some (necessarily unital)  $(n - 2)$ -multifraction  $\underline{b}$ . Let  $\underline{a}^0 := \underline{a}$  and, inductively,  $\underline{a}^i := \underline{a}^{i-1} \bullet R_{i, x_i}$ . We start from a loop of edges with alternating orientations labeled  $\underline{a}^0$ . Then we inductively complete the graph using  $n - 1$  steps of the type



according to the sign of  $i$  in  $\underline{a}$ . Because the final two entries of  $\underline{a}^{n-1}$  are trivial, the last two steps take a simpler form:  $a_n^{n-1} = 1$  is equivalent to  $a_n = x_{n-1}$ , whereas  $a_{n-1}^{n-1} = 1$  is equivalent to  $x_{n-2}^{-1} a_{n-1} \lesssim a_n$ .

In this way, we obtain an  $M$ -labeling of an annular graph made of  $n - 2$  copies of  $\Gamma_4$ , whose outer boundary is labeled  $\underline{a}$ , and whose inner boundary is labeled  $\underline{b}$ , see the diagram on the right (here in the case going from 8 to 6). Then we repeat the process with  $\underline{b}$ , etc., until a 4-multifraction is reached, and we conclude using Prop. 6.19. This construction exactly corresponds to the inductive definition of  $\Gamma_n$ .  $\square$



It is well-known that, if a multifraction represents 1 in a group, then it can be realised as the boundary of a van Kampen diagram, which can be viewed as a finite fragment of the Cayley graph. But what is remarkable here is the existence of one single universal shape, with prescribed springs, wells, and 4-prongs, that works for every unital  $n$ -multifraction at the same time.

Finally, one may wonder whether some counterparts of Lemma 6.10 and 6.17 might hold with  $\Gamma_n$  replacing  $\Gamma_4$ : maybe they do, but the natural argument for proving them requires that the ground monoid satisfies the 3-Ore condition, in which case it is known that every unital  $n$ -multifraction admits an  $\Gamma_n$ -tiling, thus making the results trivial.

## 7. MISCELLANEA

The main three properties addressed in this paper are Conjectures **A**, **B**, and **C** (together with the uniform version **C**<sub>unif</sub> of the latter), which involve arbitrary Artin-Tits monoids, and are known to be true for those of FC type. Testing these statements with a computer is easy, and we report about experiments that, altogether, support the conjectures and provide some experimental evidence. The involved program is available at [6], and the experiments are easy to repeat and confirm.

We begin with a few remarks about implementation options (Subsection 7.1), then report about the obtained data (Subsection 7.2). Finally, we conclude with a few hints about further properties of reduction, including several counter-examples (Subsection 7.3).

### 7.1. Implementation options.

*Choice of the monoid.* We are interested in Artin-Tits monoids  $M$  such that  $\mathcal{R}(M)$  is not convergent, hence not of FC type. It is natural to look for monoids with a maximal convergence defect, meaning that the proportion of multifractions with more than one irreducible reduct is maximal. As can be expected, the ratio is maximal for the Artin-Tits monoid of type  $\tilde{A}_2$ , and, more generally, those with all relations of length 3 exactly: as divergent reducts may arise only with counter-examples to the 3-Ore property, it is natural that this happens more frequently when all atoms give rise to such counter-examples. Another advantage of  $\tilde{A}_2$  and, more generally,  $K_{n,3}$ , the Artin-Tits monoid whose Dynkin diagram is the complete graph with  $n$  vertices and all edges labeled 3, is the existence of an explicit description of primitive elements (namely 1, atoms, and products of two distinct atoms) providing a better efficiency (and 100% correctness with no termination problem) for the implementation of the monoid operations (equality, lcms, gcfs, etc). Therefore, we mostly concentrated on  $\tilde{A}_2$ , considered as the critical type (but any other choice is possible with [6]).

*Generation of random multifractions.* Exhaustively enumerating multifractions up to a given length (sum of the lengths of the entries) is difficult, as, even in the case of 3 atoms, that are more than  $2.6 \times 10^9$  multifractions of length up to 12. Therefore it is more realistic to use samples of random multifractions. Generating random elements of the monoid and, from there,

random multifractions, is easy via random words in the atom alphabet (with a bias due to the relations).

Generating random unital multifractions is more delicate. As the density of unital multifractions is negligible, generating random multifractions and selecting those that are unital is not a good option (in addition, it requires a prior solution to the Word Problem, which exists for  $\tilde{\mathbb{A}}_2$  but not in general). Two methods have been used. The first one (“brownian motion”) is to follow the definition of  $\simeq$ , thus starting with an empty word and randomly adding or deleting pairs  $s/s$  and applying the Artin-Tits relations. Inserting right- and left-reversing steps (the special transformations of Property H, see Subsec. 3.3) improves the efficiency.

The second method (“lcm-expansions”) consists in starting from a multifraction that admits a random central cross, hence is unital of a very special type, and deriving new, more generic, unital multifractions as follows:

**Definition 7.1.** (Fig. 10) If  $M$  is a gcd-monoid and  $\underline{a}$  is a unital  $n$ -multifraction on  $M$ , with  $n$  even, we say that  $\underline{b}$  is an *lcm-expansion* of  $\underline{a}$  if  $\|\underline{b}\| = \|\underline{a}\|$  holds and, for each  $i$ , there exist decompositions  $a_i = a'_i a''_i$ ,  $b_i = b'_i b''_i$  satisfying  $a'_i b''_{i-1} = a'_{i+1} b''_i = a'_i \vee a'_{i+1}$  for  $i$  negative in  $\underline{a}$ , and  $b'_{i-1} a''_i = b'_i a''_{i+1} = a''_i \tilde{\vee} a''_{i+1}$  for  $i$  positive in  $\underline{a}$ , with indices modulo  $n$ , *i.e.*,  $n + 1$  means 1.

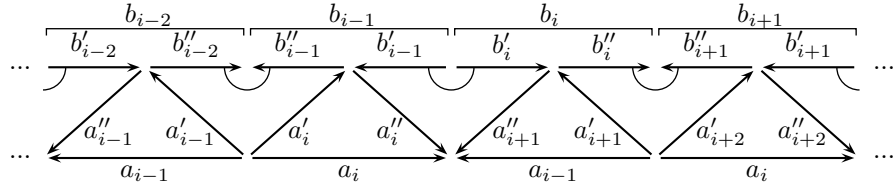


FIGURE 10. Lcm-expansion, here for  $i$  positive in  $\underline{a}$ ; the shift of the indices for  $\underline{b}$  ensures that  $\underline{b}$  has the same sign as  $\underline{a}$ . Note the symmetry: starting with right-divisors leads to the same notion.

For  $\underline{b}$  an lcm-expansion of  $\underline{a}$ , one reads on Fig. 10 the equality  $\iota(\underline{b}) = \iota(a'_1 b''_n)^{-1} \iota(\underline{a}) \iota(a'_1 b''_n)$ , hence lcm-expansion preserves unitality. Constructing lcm-expansions of  $\underline{a}$  is easy: with the notation of Def. 7.1, once a left-divisor  $a'_i$  of  $a_i$  is chosen for each  $i$ , all the remaining elements  $a''_i, b'_i, b''_i$  are determined, and then so is  $\underline{b}$  (but the choice leads to an lcm-expansion, only if the lcms exist). The advantage is that the depth is controlled (which is more difficult with brownian motion), the inconvenience is that there is no guarantee that generic multifractions are obtained (but the expansion procedure can be iterated).

*Maximal vs. atomic reduction steps.* Implementing reduction is straightforward, once the lattice operations of the monoid are available. As a composition of  $i$ -reductions is again an  $i$ -reduction, one might think of restricting to maximal reduction steps, *i.e.*, considering reducts  $\underline{a} \cdot R_{i,x}$  where  $x$  is a maximal  $i$ -reducer for  $\underline{a}$ . This is not a good option, as some reducts may be missed:

**Example 7.2.** In the Artin-Tits monoid of type  $\tilde{\mathbb{A}}_2$ , let  $\underline{a} = \mathbf{ab/ba/ca/bcbc}$ . The only maximal reductions from  $\underline{a}$  are  $R_{3,cc}$  followed by  $D_{1,ab}$ , leading to the unique irreducible  $1/\mathbf{ab/ca/cb}$ . However, one also finds  $\underline{a} \cdot R_{3,c} D_{1,ab} R_{2,c} R_{3,b} = \mathbf{cb/abbc/ba/bc}$ , a second irreducible reduct of  $\underline{a}$  not reachable by maximal steps.

This however does not contradict the following (surprising?) result:

**Proposition 7.3.** *If  $M$  is a strongly noetherian gcd-monoid, and  $\underline{b}, \underline{c}$  are irreducible reducts of some multifraction, then there exists a finite sequence of maximal reductions and inverses of maximal reductions connecting  $\underline{b}$  to  $\underline{c}$ .*

*Proof (sketch).* Write  $\underline{b} \bowtie \underline{c}$  when there exists a sequence as in the statement. Using induction on the ordinal  $\lambda(\underline{a})$ , where  $\lambda$  satisfies (3.13), one proves the following general criterion: If  $\bowtie$  is an equivalence relation on  $\mathcal{F}_{\pm}(M)$  such that, for every multifraction  $\underline{a}$  that is  $i$ -prime and  $j$ -irreducible for  $j \neq i$  and every 1-reduct  $\underline{a}'$  of  $\underline{a}$ , there exists a irreducible reduct  $\underline{b}$  of  $\underline{a}'$  satisfying  $\underline{a} \bowtie \underline{b}$ , then, for every  $\underline{a}$  in  $\mathcal{F}_{\pm}(M)$ , any two irreducible reducts of  $\underline{a}$  are  $\bowtie$ -equivalent.  $\square$

## 7.2. Experimental data.

*Conjectures A and B.* Testing the two of them is essentially the same thing: one generates a random unital multifraction  $\underline{a}$ , and one checks  $\underline{a} \Rightarrow^* \underline{1}$  in the former case,  $\text{red}_t(\underline{a}) = \underline{1}$  in the latter. Because semi-convergence implies 1-confluence, one can fix any reduction strategy for checking  $\underline{a} \Rightarrow^* \underline{1}$ , for instance looking at each step for the smallest level  $i$  and the first atom  $x$  such that  $R_{i,x}$  applies. Although computing  $\text{red}_t$  is slightly slower, as, at each step, all  $i$ -reducers have to be determined in order to take their gcds, both computations are fast. Precise numbers are not really significant here; in type  $\tilde{A}_2$  or  $K_{4,3}$ , the typical order of magnitude is  $5 \times 10^4$  (*resp.*  $1.5 \times 10^4$ ) random unital multifractions of length 20 and depth 4 (*resp.* length 40 and depth 6) per hour of computation. In other types (*e.g.*,  $\tilde{A}_3$ , or  $\tilde{C}_2$ ), efficiency is diminished by a factor 10 approximately.

No counter-example to Conjecture **A** or **B** was ever found. As the density of visited (unital) multifractions becomes negligible when the length grows, the significance of such data is questionable. However, it may be noticed that, for the many properties considered in this paper and discarded by counter-examples, the length of the latter (all found by random search) is never more than 12 or so: this does not say anything about a possible counter-example to Conjecture **A** or **B** but, at the least, this shows that the considered lengths are not ridiculous.

*Conjecture A<sub>4</sub>.* For the special case of Conjecture **A<sub>4</sub>** (and of Conjecture **B<sub>4</sub>**, which, by Coro. 6.26, is equivalent), an exhaustive search makes sense, by systematically considering all possibilities for the central cross (up to a certain length).

**Fact 7.4.** *For the Artin-Tits monoid of type  $\tilde{A}_2$ , Conjecture **A<sub>4</sub>** is true for all lcm-expansions of all multifractions that admits a central cross with entries of length at most 2.*

By contrast, the procedure applied to the monoid  $M_{C,4}$  of [13, Prop. 6.9] duly finds a counter-example, namely an irreducible lcm-expansion of a 4-multifraction with a central cross (with rays of length 1). What seems to discard a similar counter-example in an Artin-Tits monoid is the fact that Artin-Tits relations preserve the atoms occurring in an element (the “support”), but, even for this weak form of Conjecture **A<sub>4</sub>**, we have no proof so far.

*Conjectures C and C<sub>unif</sub>.* Testing these conjectures is easier in that it involves arbitrary, not necessarily unital multifractions, but it is more difficult in that it requires to determine all right-reducts of a multifraction  $\underline{a}$  and then, for each pair of them, to determine all their left-reducts. The complexity of constructing the tree  $T_{\underline{a}}$  (as used in the proof of Prop. 5.9) and its right counterpart  $\tilde{T}_{\underline{a}}$  increases quickly with the depth. Beyond depth 4, the total number of reducts often becomes large (usually a few ones, but possibly several thousands), resulting in a huge computation time. Typically, in the current version of the program, one can test Conjecture **C<sub>unif</sub>** for about  $5 \times 10^3$  (*resp.*  $4 \times 10^3$ ) random multifractions of length 20 (*resp.* length 30) and depth 3 per hour of computation. Going to depth 4 diminishes the speed by a factor 50.

To overcome these bounds, we also tested (without size limitation) the following instance of Conjecture **C**: starting with a multifraction  $\underline{a}$ , we determine the (not necessarily distinct) right-reducts  $\underline{b}_1, \dots, \underline{b}_4$  and left-reducts  $c_1, \dots, c_4$  of  $\underline{a}$  obtained using the four natural strategies (levels from bottom or top, atoms in lexicographical or antilexicographical order), and check the property  $\exists k \forall j (\underline{b}_j \Rightarrow^* \underline{c}_k)$ , thus checking Conjecture **C** for  $\{\underline{b}_1, \dots, \underline{b}_4\}$ . The cost is then comparable as the one for Conjecture **B**, with about  $5 \times 10^4$  (*resp.*  $5 \times 10^3$ ) random tries for length 20 (*resp.* 30) per hour of computation. The stronger conclusion  $\forall k \forall j (\underline{b}_j \Rightarrow^* \underline{c}_k)$  is almost always valid but, on the shape of Ex. 5.33, exceptions occur.

**7.3. Further questions.** We point to a few natural questions involving reduction. Most of them remain open, or gave rise to counter-examples.

*Normal forms for reductions.* Distinguished expressions for sequences of reductions could be obtained by identifying skew commutation relations, typically of the form  $R_{i,x}R_{j,y} \rightleftharpoons R_{j,y'}R_{i,x'}$ , meaning that, if  $\underline{a} \bullet R_{i,x}R_{j,y}$  is defined, then so is  $\underline{a} \bullet R_{j,y'}R_{i,x'}$  and the results are equal. Typically, one could try to push divisions to one side, so that all remaining steps are invertible. This approach does not work well, as exceptions always appear. In the same vein, in view of a possible induction and building on the universal scheme  $U(n)$  that works in the 3-Ore case, one could conjecture that every reduction sequence is equivalent to one where the highest level occurs only once, or that, if an  $n$ -multifraction  $\underline{a}$  is  $i$ -irreducible for  $i < n - 1$ , then reducing  $\underline{a}$  can be done by a sequence of the form  $R_{n-1,x_1}R_{n-3,x_3}R_{n-5,x_5} \cdots$ . This need not be the case.

**Example 7.5.** In the Artin-Tits of type  $\tilde{A}_2$ , let  $\underline{a} = 1/\mathbf{ba}/\mathbf{cb}/\mathbf{ca}/\mathbf{ab}$ . Then  $\underline{a}$  is  $i$ -irreducible for  $i \leq 3$ , but the only sequence from  $\underline{a}$  to an irreducible reduct is  $R_{4,\mathbf{a}}R_{2,\mathbf{bc}}R_{3,\mathbf{a}}R_{4,\mathbf{b}}$ , discarding the above two conjectures. Of course, Conjecture **B** is not contradicted, because  $\underline{a}$  is not unital.

In the same direction, one can study local confluence between left reductions  $R_{i,x}$  and right reductions  $\bar{R}_{j,y}$ , with the hope of obtaining normal forms useful for cross-confluence. In almost all cases, there exists indeed local confluence solutions. However, the case  $j = i + 2$  remains problematic in general. Moreover, using local confluence for an induction is unclear, because there is no common well-founded relation underlying both left and right reduction and, except in type FC, a multifraction may admit infinitely many left-right reducts.

*Homomorphisms.* As reduction is constructed using multiplication and lcm operations, it is preserved by morphisms that preserves these operations, namely lcm-morphisms [5]: if  $\phi$  is an lcm-morphism from a gcd-monoid  $M$  to a gcd-monoid  $M'$ , then  $\underline{a} \Rightarrow^* \underline{b}$  implies  $\phi(\underline{a}) \Rightarrow^* \phi(\underline{b})$  for all  $\underline{a}, \underline{b}$  in  $\mathcal{F}_{\pm}(M)$ . This however is not easy to use for, say, Conjecture **A**, because the implications go in the wrong direction. If we study the semi-convergence of  $\mathcal{R}(M)$ , mapping  $M$  to a gcd-monoid  $M'$  that satisfies the 3-Ore condition does not help: if  $\underline{a}$  is unital in  $\mathcal{F}_{\pm}(M)$ , then  $\phi(\underline{a})$  is unital in  $\mathcal{F}_{\pm}(M')$ , so  $\phi(\underline{a}) \Rightarrow^* \underline{1}$  holds, but deducing  $\underline{a} \Rightarrow^* \underline{1}$  is problematic. In the other direction, if  $M'$  satisfies the 3-Ore condition and  $\phi'$  is an lcm-morphism from  $M'$  to  $M$ , then  $\text{Im}\phi'$  is included in some part of  $M$  where the 3-Ore condition is satisfied, and knowing that  $\mathcal{R}(M')$  is (semi)-convergent will not help for multifractions on  $M$  outside  $\text{Im}\phi'$ .

Morphisms (lcm-morphisms or not) might be useful for establishing particular properties, like Conjecture **A**<sub>4</sub>, *i.e.*, the fact that every unital 4-multifraction admits a central cross. If  $M$  is the Artin-Tits monoid of type  $\tilde{A}_2$ , one may think of using the classical embedding from  $M$  to the Artin-Tits group of type  $B_3$ , namely  $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{abab} = \mathbf{baba}, \mathbf{bcb} = \mathbf{cbc}, \mathbf{ac} = \mathbf{ca} \rangle$ , that maps  $\mathbf{a}$  to  $\mathbf{a}^{-1}\mathbf{b}^{-1}\mathbf{cba}$ , and  $\mathbf{b}$  and  $\mathbf{c}$  to themselves, but the image is not included in the monoid. A probably better choice is to map  $M$  to the Artin-Tits monoid  $M'$  of type  $D_4$ , namely  $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \mid \mathbf{ada} = \mathbf{dad}, \mathbf{bdb} = \mathbf{bdb}, \mathbf{cdc} = \mathbf{cdc} \rangle^+$ , by  $\phi(s) = \mathbf{d}s$  for  $s = \mathbf{a}, \mathbf{b}, \mathbf{c}$ . If  $\underline{a}$  is a unital 4-multifraction on  $M$ , then  $\phi(\underline{a})$  admits a central cross in  $M'$ : to deduce that  $\underline{a}$  admits a central cross in  $M$ , it suffices to show that at least one of the central crosses for  $\phi(\underline{a})$  lies in  $\text{Im}\phi$ . The problem is that  $\text{Im}\phi$  is not closed under right-divisor in  $M'$ : for instance,  $\phi(\mathbf{a}) \leq \phi(\mathbf{bac})$  holds in  $M'$ , but  $\mathbf{a} \leq \mathbf{bac}$  fails in  $M$ .

By the way, the following natural question seems to be open:

**Question 7.6** (F. Wehrung). Does the above morphism  $\phi$  induce an embedding from the Artin-Tits group of type  $\tilde{A}_2$  into the Artin-Tits group of type  $D_4$ ?

*Reduction graphs.* Almost nothing is known about the structural properties of the graph formed by the reducts of a multifraction, for instance their possible lattice properties: if we have  $\underline{a} \Rightarrow^* \underline{b}$  and  $\underline{a} \Rightarrow^* \underline{c}$  and if  $\underline{b}$  and  $\underline{c}$  admit a common reduct, does there exist a common reduct  $\underline{d}$  of  $\underline{b}$  and  $\underline{c}$  such that every common reduct of  $\underline{b}$  and  $\underline{c}$  is a reduct of  $\underline{d}$ ? This is frequently true, but not always:

**Example 7.7.** In the Artin-Tits monoid of type  $\tilde{A}_2$ , consider  $\underline{a} = 1/\mathbf{a}/\mathbf{bcb}/\mathbf{bcb}/\mathbf{a}$  (which is unital). One finds  $\underline{b} = \underline{a} \bullet R_{2,\mathbf{b}} = \mathbf{ba}/\mathbf{ab}/\mathbf{cb}/\mathbf{bcb}/\mathbf{a}$  and  $\underline{c} = \underline{a} \bullet R_{2,\mathbf{c}} = \mathbf{ca}/\mathbf{ac}/\mathbf{bc}/\mathbf{bcb}/\mathbf{a}$ . Then  $\underline{b}$

and  $\underline{c}$  admit the two symmetric maximal common reducts, namely

$$\begin{aligned}\underline{d}' &= \underline{b} \bullet D_{3,b} = \underline{c} \bullet R_{3,b} D_{1,ca} R_{2,b} = \mathbf{ba/ab/c/bc/a}, \\ \underline{d}'' &= \underline{b} \bullet R_{3,c} D_{1,ba} R_{2,c} = \underline{c} \bullet D_{3,c} = \mathbf{ca/ac/b/cb/a},\end{aligned}$$

and there is no common reduct  $\underline{d}$  of  $\underline{b}$  and  $\underline{c}$  of which  $\underline{d}'$  and  $\underline{d}''$  are reducts.

A possible conclusion in view of the long list of counter-examples described in this paper could be that there is no hope for many further general properties of reduction, implying that a possible proof of semi-convergence has to involve the specific properties of the ground monoid in a deep way. In particular, a proof of Conjecture **A**, **B**, or **C** should require developing new specific tools for Artin-Tits groups. The results of [13] may suggest approaches.

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