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QUASI-OPTIMALITY OF APPROXIMATE SOLUTIONS IN NORMED VECTOR SPACES

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Abstract. We discuss quasi-optimality of approximate solutions to operator equations in normed vector spaces, defined either by Petrov–Galerkin projection or by residual minimization. Examples demonstrate the sharpness of the estimates.

Let $X$ and $Y$ be real normed vector spaces. Let $B : X \rightarrow Y'$ be a linear operator. Fix $u \in X$ – the “unknown”. Let $X_h \times Y_h \subset X \times Y$ be nontrivial finite-dimensional subspaces. Abbreviate

\[
\gamma_h := \inf_{w \in X_h \setminus \{0\}} \frac{||Bw||_{Y_h}}{||w||_X} \quad \text{and} \quad \|B\| := \sup_{w \in \{u + X_h \setminus \{0\}} \frac{||Bw||_{Y_h}}{||w||_X}.
\]

Throughout, we assume the “discrete inf-sup condition”: $\gamma_h > 0$. We define $B_h : X_h \rightarrow Y_h'$ by $w \mapsto \langle Bu, w \rangle_{Y_h}$. In the first proposition we require $\dim X_h = \dim Y_h$. In the second we admit $\dim Y_h \geq \dim X_h$.

**Proposition 1.** Suppose $\dim X_h = \dim Y_h$. Then there exists a unique $u_h \in X_h$ such that

\[
\langle Bu_h, v \rangle = \langle Bu, v \rangle \quad \forall v \in Y_h.
\]

The mapping $u \mapsto u_h$ is linear with $||u_h||_X \leq \gamma_h^{-1}||Bu||_{Y_h}$ and satisfies the quasi-optimality estimate:

\[
||u - u_h||_X \leq (1 + \gamma_h^{-1}||B||) \inf_{w_h \in X_h} ||u - w_h||_X.
\]

**Proof.** The map $B_h$ is linear and injective by (1). It is bijective due to finite $\dim X_h = \dim Y_h = \dim Y'_h$. Thus a unique $u_h := B_h^{-1}(Bu)|_{Y_h}$ exists and $u \mapsto u_h$ is linear. By (1), $\gamma_h||u_h||_X \leq ||B_h u_h||_{Y_h'} = ||Bu||_{Y_h'}$. From $||u - u_h||_X \leq ||u - w_h||_X + ||w_h - u_h||_X$ and $\gamma_h||w_h - u_h||_X \leq ||B(u - w_h)||_{Y_h'} = ||Bu - Bu_h||_{Y_h'}$ we obtain (3).

**Proposition 2.** The set $U_h := \text{argmin}_{w_h \in X_h} ||Bu - Bw_h||_{Y_h'} \subset X_h$ of residual minimizers is nonempty, convex and bounded. Any $u_h \in U_h$ satisfies the quasi-optimality estimate

\[
||u - u_h||_X \leq (1 + 2\gamma_h^{-1}||B||) \inf_{w_h \in X_h} ||u - w_h||_X.
\]

**Proof.** The first statement is elementary: consider the metric projection of $(Bu)|_{Y_h} \in Y_h'$ onto $B_h X_h \subset Y_h'$. Quasi-optimality is obtained as above, except that $||B(u_h - w_h)||_{Y_h'} \leq ||Bu||_{Y_h} + ||Bu - Bu_h||_{Y_h'} \leq 2||Bu - Bu_h||_{Y_h'}$.

The set $U_h$ of minimizers is a singleton if the unit ball of $Y_h'$ is strictly convex. Since $Y_h$ is finite-dimensional, this is the case if and only if the norm of $Y_h$ is Gâteaux differentiable.

The constants in (3) and (4) are sharp: Take $X = Y = \mathbb{R}^2$ with the $||\cdot||_1$ norm. Then $||\cdot||_\infty$ is the norm of $Y'$. Take $u := (0, 1)$ and $B(w_1, w_2) := (w_1 + w_2, w_2)$. Set $X_h := \mathbb{R} \times \{0\}$ ($B$ is identity on $X_h$). Observe $||B|| = 1$.

- For (3) let $Y_h := \mathbb{R} \times \{0\}$. Then $||Bw_h||_{Y_h} = ||w_h||_X$ for all $w_h \in X_h$ gives $\gamma_h = 1$. Now, $u_h := (1, 0) \in X_h$ solves (2). In the quasi-optimality estimate we have $||u - u_h||_X = 2$ while $||u - w_h||_X = 1$ for $w_h = 0$.

- For (4) let $Y_h := Y$. Again, $\gamma_h = 1$. Since $Bu = (1, 1)$, the set of minimizers $U_h$ is the segment $[0, 2] \times \{0\}$. For $u_h := (2, 0) \in U_h$ we have $||u - u_h||_X = 3$ while $||u - w_h||_X = 1$ for $w_h = 0$. With a slight perturbation of the norms, say, we can achieve $U_h = \{u_h\}$ without essentially changing the distances.

If $X$ and $Y$ are Hilbert spaces and $B : X \rightarrow Y'$ is bounded by $||B||$ then in both propositions the mapping $P_h : X \rightarrow X$, $u \mapsto u_h$, is a well-defined bounded linear projection with $||P_h|| \leq \gamma_h^{-1}||B||$. The argument of


then improves the quasi-optimality estimate to $||u - u_h||_X \leq ||P_h|| \inf_{w_h \in X_h} ||u - w_h||_X$.

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