Student understanding of the relation between tangent plane and the differential

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Action-Process-Object-Schema (APOS) is used to study students' understanding of the relationship between tangent planes and the differential. An initial conjecture, called a genetic decomposition, of mental constructions students may use in constructing their knowledge of planes, tangent planes, and the differential is proposed. It is tested with semi-structured interviews with 26 students. Results of the study suggest that students tend not to relate these ideas on their own and suggest ways to refine the initial genetic decomposition in order to help students to better understand these concepts.

Keywords: APOS, calculus, tangent plane, differential, function of two variables.

INTRODUCTION

Functions of several variables play a role of great importance in mathematics and the applied sciences. This study focuses on student understanding of the relation between two of the most basic ideas in the differential calculus of functions of two variables. Very little research has been published relating to this topic. An early work by Tall (1992) suggests using a geometric model to visualize differentials in three dimensions. Weber (2012) discussed the rate of change concept in the case of functions of two variables focusing on the use of covariational thinking to help students build a notion of rate of change in space. McGee and Russo (2015) used a model similar to that of Tall (op. cit.) in a study that applied semiotic representation theory to explore the effect of a semiotic chain in student understanding of partial and directional derivatives of functions of two variables. Martínez-Planell, Trigueros, and McGee (2015) applied APOS to study different components of the differential calculus of these functions: partial derivatives, planes, tangent planes, directional derivatives, and the differential. The present report expands their discussion of student understanding of the differential and its relation to the idea of tangent plane in accordance with Tall's model.

THEORETICAL BACKGROUND

Since APOS is a well-known theory we will only give a brief overview. For more information the reader may consult Arnon et al (2013).

In APOS, an Action is a transformation of a mathematical object that is perceived by the individual as external. It may be a step by step implementation of an explicitly

available set of rules or a rigid application of a memorized fact or algorithm. An individual is said to have an action conception of a given mathematical notion when he/she is limited to applying actions in problem solving activities involving the notion. As an individual repeats and reflects on an Action, it may be interiorized into a Process. A process is perceived as internal. An individual with a process conception of a mathematical idea may, without recurring to any external source, reflect on the steps of the process, omit steps, and anticipate the result without having to explicitly perform the process. A process may be coordinated with other processes, and it may also be reversed to the actions it came from as needed in a problem situation. As an individual needs to apply actions on a process he/she may come to see the process as a totality. When the individual is able to perform or imagine performing actions on a process it is said that the process has been encapsulated into an Object. An object may be de-encapsulated into the process it came from as needed in a problem situation. An individual with an object conception of a mathematical concept may recognize the applicability of the concept without any prompt in different problem situations, even in an unfamiliar context, as would be in a different discipline. A Schema for a particular mathematical idea is a coherent collection of actions, processes, objects, and other previously constructed schemas that are related to the mathematical idea. A schema is coherent in the sense that the different components of the schema are inter-related in the individual's mind and the individual can decide when a problem situation falls within the scope of the schema.

Even though one may think there is a linear progression from action, to process, to object, and then to having the different actions, processes, and objects organized in schemas, the progression is dialectical in nature, with partial developments, and passages back and forth between conceptions (Czarnocha, et al 1999). However, the theory is unequivocal in its recognition that a student's tendency to deal with problem situations in diverse mathematical tasks involving a particular mathematical concept is different depending on whether the student understands the concept as an action, a process, or an object.

In APOS, research on student understanding of a particular mathematical concept starts by establishing a conjecture, called a genetic decomposition (GD), of specific mental constructions (in terms of the constructs of the theory) that students may do in order to come to understand the concept. The GD depends on the mathematics itself, the experience of the researcher teaching the concept, and any available data. A GD is not unique, different researchers may propose different genetic decompositions. What is important is that the GD needs to be supported by experimental data from students. What typically happens is that a preliminary genetic decomposition is proposed and the data obtained (usually with semi-structured interviews) shows that students make unexpected mental constructions and have difficulty with some of the mental constructions predicted in the GD. This leads to refining the genetic decomposition in order to reflect the constructions that students actually do and to

the development of activities to help students make the mental constructions with which they had difficulty. This ends a first cycle of research. The second research cycle would start with a classroom implementation of the newly developed student activities and would further refine the GD based on new interviews and classroom observations. These research cycles continue until they stabilize in a GD that serves to both, predict student behaviour and guide instruction. The present work uses APOS theory to study the level of cognitive development of students who completed a course using a traditional lecture/recitation model, as discussed in Arnon et al. (2013, p. 106). Thus, this is a report of a first cycle of APOS research.

GENETIC DECOMPOSITION

We now present a GD for plane and tangent plane. We also present a preliminary GD for the differential concept. This preliminary GD guided the development of the instruments for this study.

Plane

Given a non-vertical plane, the processes of slope of a line and fundamental plane (planes of the form $x=c$, $y=c$, $z=c$, for c constant) are coordinated into new processes of vertical change in the x and y directions, where it is recognized that vertical change in the x direction can be described as a function of the horizontal change in the x direction ($\Delta z_x = m_x \Delta x$), and similarly for vertical change in the y direction ($\Delta z_y = m_y \Delta y$). These processes are coordinated into a process of total vertical change on a plane in three-dimensional space so that total vertical change in any plane is given in terms of the sum of vertical changes in the directions of the coordinate axes: ($\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$ (see Figure 1). The need to perform actions which are treatments and conversions in and between representations (Duval, 2006) on the process of total vertical change promotes its encapsulation into the object conception of plane in three dimensions. In particular, the equation $z - z_0 = m_x(x - x_0) + m_y(y - y_0)$ can be seen as the vertical change on a plane with slopes m_x and m_y from an initial point (x_0, y_0, z_0) to a final generic point (x, y, z) and is also associated with the set of points (x, y, z) on a plane that contains the point (x_0, y_0, z_0) and has slopes m_x and m_y (point-slopes formula for a plane).

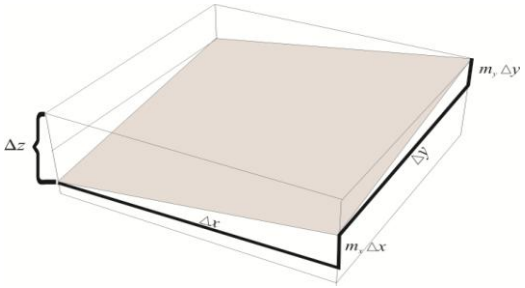

Figure 1: $\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$

Tangent Plane

The process of partial derivative is coordinated with that of plane into a new process where tangent planes to any surface at different points can be considered and computed. When there is a need to consider particular tangent planes and perform actions on them to describe the surface in terms of behaviour associated with its tangent plane(s), this process is encapsulated into an object conception of tangent plane.

The Differential (from the preliminary GD)

Treatment and conversion actions (Duval, 2006) are performed on the tangent plane process to recognize it as the differential.

To summarize, the above GD essentially proposes that students first do the mental construction of the process of total vertical change on a plane: $\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$. Then they coordinate this process with a process of partial derivative to obtain a process of tangent plane at $(x_0 y_0, f(x_0 y_0))$: $z - f(x_0 y_0) = f_x(x_0 y_0)\ (x - x_0) + f_y(x_0 y_0)\ (y - y_0)$. Students then do actions of notational change and of geometric interpretation on the process of tangent plane and interiorize these actions into a process of total differential: $df(\text{a,b}) = f_x(\text{a,b})\, dx + f_y(\text{a,b})\, dy$.

METHOD

An instrument consisting of six questions was prepared to test student understanding of the different components of the GD. We are reporting mainly on one of these questions (problem 3a). However, when pertinent we will cite (although not always quote) student response to other questions (problems 1 and 2). The instrument was used in semi-structured interviews with 26 students who had just finished taking a multivariable calculus course. The 26 students were chosen from three sections that had different professors. Section T (9 students) used a traditional textbook (Stewart, 2006) and syllabus with all the homework problems chosen from the text. Section E1 was an experimental section (9 students) using the same textbook but with a set of activities designed to help students make the mental constructions in the preliminary GD. Section E3 was another experimental section (8 students) using the same textbook with activities for planes and tangent planes but not for the differential. In this section the differential was defined but was not discussed in class. They had the same set of textbook homework problems for the differential as section T. All three professors were experienced (over 20 years teaching), having taught the course many times, popular with students (as judged by student evaluations), and concerned with student learning. Each of the professors was asked to choose 9 students: 3 above average, 3 average, and 3 below average, providing as balanced a distribution as possible. One student did not show up. The interviews lasted from 40 to 60 minutes. At the same time they were interviewed, students produced written answers and the interviewer took notes of the hand gestures of the students. The interviews were

recorded, transcribed, individually analysed, and then discussed as a group and results were negotiated among the researchers. The questions of interest are reproduced below:

Problem 1. Students were given the plane below and were asked to find the slopes in the x and y directions (m_x and m_y), the total vertical change (Δz) for $\Delta x = 4$ and $\Delta y = 5$, and the equation of the plane. (Observe that $m_x = 3$, $m_y = 1$, and if $\Delta x = 4$ and $\Delta y = 5$ then $\Delta z = m_x \Delta x + m_y \Delta y = 3(4)+1(5)=17$. Also, the equation of the plane is $z-2=3(x-1)+1(y-2)$.)

Problem 2. Students were given the graph below and were asked for the sign (positive, negative, or zero) of $\partial f/\partial y\,(4.0,0.7)$ and of $D_{<-2,1>}f(4,0)$. (Observe that $\partial f/\partial y\,(4.0,0.7) < 0$ and $D_{<-2,1>}f(4,0) > 0$.)

Problem 3. The following plane is tangent to the graph of $z=f(x,y)$ at the point (1,2,0). (a) Find, if possible, the differential of f at the point (1,2), $df(1,2)$. (b) Find $D_{<1,1>}f(1,2)$. (Observe that since $m_x = 1$ and $m_y = 3$ then $df(1,2) = 1dx + 3dy$.)

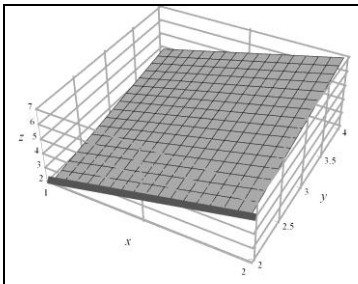

Figure for problem 1

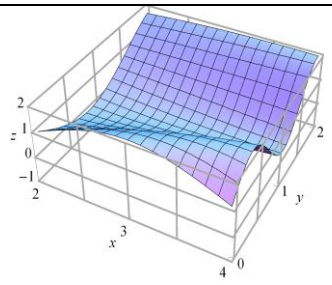

Figure for problem 2

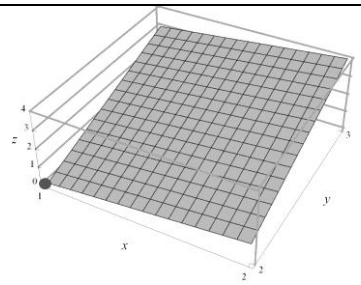

Figure for problem 3

RESULTS

None of the interviewed students clearly exhibited a process conception of the differential. Only one student (from section E1) could be considered to be in transition to a process conception of this concept, 6 showed an action conception (5 of them from section E1), and the other 19 students showed no knowledge or recollection of the concept.

Most students seemed to depend entirely on a symbolic representation, to the extent that seeing the "d" in the symbol $df(a,b)$ they concluded that the differential was some kind of slope or derivative. Indeed 19 of the 26 interviewed students showed this type of response. Tania is one such student. In problem 1 she showed no difficulty finding the slopes in the x and y directions of a given plane (m_x and m_y), finding the total vertical change (Δz) for $\Delta x = 4$ and $\Delta y = 5$, and writing the equation of the plane. Further, in problem 2 she correctly found the sign of the requested partial derivative by identifying it with the slope of a tangent line she drew on the given graph.

Tania: (after reading problem 3) What does it mean by the differential of f? Is that a slope?

Interviewer: The differential of *f* at a point. That was defined in class. What is the meaning of the differential of a function?

Tania: That was the slope at this point, isn't it?

Interviewer: No.

Tania: Ok, that would be, the vertical change.

Interviewer: OK

Tania: The point (1, 2) is this point, but to look for a vertical change I need two points, to be able to look for a *z*.

Note that Tania might be thinking of vertical change along the graph of the function (hence her need for two points). If this was the case, then she was not looking at the information about the function that may be obtained from the given tangent plane. This could indicate that she had not constructed the relation between the differential and the tangent plane. If she was thinking about vertical change along a plane, then she would seem not to have constructed a process of total vertical change on a plane, $\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$, suggesting that she succeeded in problem 1 by applying actions (using memorized formulas). In any case, she seemed not to need to perform any treatment actions (Duval, 2006) on the process of tangent plane to obtain a formula for the differential, nor was she doing any conversion action (Duval, 2006) to relate the analytical and graphic representations of the tangent plane in order to obtain the differential from the graph of the tangent plane, as conjectured in the GD. Considering that she might be showing difficulty thinking of *dx* and *dy* as independent variables the interviewer asked:

Interviewer: If I tell you that the differential has to do with the variables *dx* and *dy* as independent variables, do you remember what is the differential?

Tania: Let's see, *df* is equal to $m_x\, dx$ plus $m_y\, dy$.

Interviewer: Ok... and what is the differential then?

Tania: This would be *df* (she went on to correctly compute the total vertical change on the plane using $\Delta x = 1$ and $\Delta y = 1$ –rather than leaving *dx*, *dy* as independent variables- by calculating m_x and m_y).

Tania had shown that she could identify a partial derivative with the slope of a tangent line in problem 2. Further, in problem 3 she could compute m_x and m_y from the given tangent plane. Hence, it seems she was able to obtain $f_x(1,2)$ and $f_y(1,2)$ from the graph of the tangent plane. However, she needed the interviewer's comment to help her remember a formula and to link it with the given plane, and did not consider *dx* and *dy* as independent variables. This, and other similar cases, seemed to show the importance of recognizing the differential at a point as the total vertical change on the tangent plane as a function of the horizontal change (*dx*, *dy*) in the mental construction of the differential.

Ramon's performance on problem 1 suggested that he had a process conception of total vertical change on a plane $\Delta z = m_x\Delta x + m_y\Delta y$. He was able to explain this formula in his own words, showing that he could imagine the geometric interpretation of the different components of the formula. Further, when obtaining the equation of the given plane in problem 1, he made clear reference to the notion of total vertical change on a plane. However, when asked about the differential, he did not relate it to the total vertical change.

Ramón: I don't remember the formula for df.

Interviewer: And if I were to tell you that the differential of f gives the change in height along the tangent plane for horizontal changes of *dx* in the x direction and *dy* in the y direction?

Ramón: It is something like the formula for change, Δz… but I don't remember exactly how it was written… It was $df = m_x dx + m_y dy$… I don't know if it was something like this.

Interviewer: What would be m_x in this case?

Ramón: It would be the slope with respect to x and this is the slope with respect to y [referring to m_y].

Interviewer: Can you find them?

Ramón: Yes… in x it is equal to… it would be (1 – 0) divided by the horizontal change, which would be (2 – 1). The slope would be 1…

Interviewer: Now look for the slope in the y direction.

Ramón: I would do it in the same way… it would be (4 – 1)/ (3 – 2) … which is 3. This is the slope with respect to y. The Δx would be the dx, but at the point (1, 2), that is, at x = 1, y = 2, around here… so it is this point here… I couldn't calculate it.

Interviewer: And if I were to tell you that *dx* and *dy* are independent variables? That is, that stays like that as a function.

Ramón: I don't have the change in x which is a very small number, no… I couldn't look for it there… the product of $m_x dx$ gives a change, vertical, but the dx alone only tells me it is a very small horizontal change in this figure…

This suggests that students need to explore the relationship between the differential and the notion of total vertical change on a plane and that the GD should be revised to make this explicit. It may also be observed that Ramón resists thinking of dx and dy as independent variables.

Some students, like Karla, seemed to lose sight of the function once the tangent plane is given. Karla was able to quickly find and justify the sign of the partial and directional derivatives from the graph of the function given in problem 2.

Interviewer: Will that be positive, negative, or zero? (Referring to the slope of a tangent line to the surface on problem 2 that Karla correctly drew at the given base point and in the requested y direction.)

Karla: Negative… because as y increases, z decreases.

Interviewer: And how about the directional derivative?

Karla: I would say that positive

Interviewer: Why?

Karla: Because… I take this (referring to the direction vector) as Δx and Δy… and as we move this way (pointing in the right direction) z is increasing.

However, later when working problem 3:

Interviewer: Could you tell me from the drawing of that plane there what is the meaning of directional derivative?

Karla: With this? [Pointing to the given plane in problem 3.]

Interviewer: So you drew a line segment on the plane where y=2 [see Figure 2].

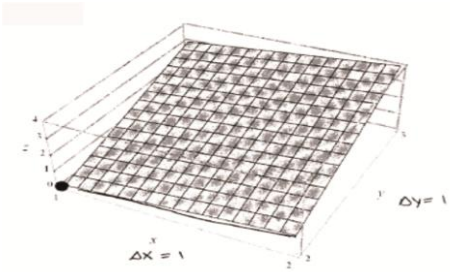

Figure 2: Karla's drawing on problem 3

When working with problem 3, Karla seemed unable to relate the tangent plane to the function when it was not present in the graph, or to say anything about the partial derivatives, showing evidence of not considering the tangent plane as a local approximation of the function.

All results obtained from students´ interviews were similar. They suggest making explicit the construction of dx and dy as independent variables when considering the differential of the function and they also suggest that the coordination between processes on the function (like those for partial derivatives, directional derivatives, and vertical change) and the same processes on the tangent plane, is important in the mental construction of the differential. This also suggests the need to make constructions related to treatments and conversions (Duval, 2006) on the tangent plane in order to construct a process conception of the differential.

Results of this study suggest a refinement of the preliminary GD in order to make it a better model of students' constructions. It is also important to incorporate this refinement in activities designed for instruction in order to help students make the mental constructions necessary for a process conception of the differential.

DISCUSSION AND CONCLUSIONS

The notion of the differential appears to be very difficult for students in this study. All the interviewed students had already finished a course on two-variable functions, but still showed they could not even remember what the differential is in this context. Students' performance during the interview clearly evidenced they had not constructed the necessary processes involved in relating the notion of vertical change on a plane, Δz, and that of the differential, $df(a,b)$. Students' difficulties seem to be due to the fact that, in students' minds, the notion of differential remains isolated from that of the tangent plane. Further, the construction of the relation between processes on the graph of a function and those performed on a corresponding tangent plane seems absent in students' constructions. It also seems students need to explicitly construct a process involved in the recognition that the differential at a point $df(a,b)$ is a function of two independent variables, (dx, dy) representing horizontal change, which is associated to the vertical change (on the tangent plane) of the original function.

Students' results show that they conceive the differential as an empty symbolic interpretation as 'some kind of derivative' or, in the best case, a procedure where small values are substituted for dx and dy. The above mentioned constructions need to be taken into account in order to help students give meaning to the differential of a two variable function. As the preliminary GD did not describe all of the students' constructions that were shown to be needed in this study, a refined GD taking explicitly into account those constructions that this study showed to be missing was designed as part of the contribution of this study. This new model needs, of course, to be used in the design of instructional activities and tested with students. The refined GD follows.

REFINED GENETIC DECOMPOSITION FOR THE DIFFERENTIAL OF TWO-VARIABLE FUNCTIONS

Perform the treatment actions of graphically comparing the tangent plane to a two variable function at a given point in order to form a construct of the tangent plane as a local approximation of the function. Do actions to express the point-slopes equation for the tangent plane at a given point (a, b, f (a,b)) as the differential $df(a,b)=f_x(a,b)dx+f_y(a,b)dy$ together with treatment and conversion actions that relate processes of partial derivatives and vertical change on a function with the same processes on the tangent plane. These actions are interiorized into processes of total vertical change, $\Delta z=m_x\Delta x+m_y\Delta y$ and the differential. These processes are coordinated into a new process that enables students to relate them. Perform actions needed to evaluate the differential at a fixed point (a,b) for different values of dx and dy. Interiorize these actions into a process that recognizes that given a function f and a point (a,b), the differential $df(a,b)$ is the total vertical change on the tangent plane expressed as a function of the horizontal changes dx and dy (see Figure 3).

Reflection on the action of computing the differential at different points allows interiorization of the differential into a process where the functional dependence of the differential on the starting point (*a*,*b*) is recognized. This process is coordinated with that of function so that the consideration of the differential as a two-variable function is made possible. When actions need to be applied, for example, to find specific properties of the differential, it may be encapsulated into an object.

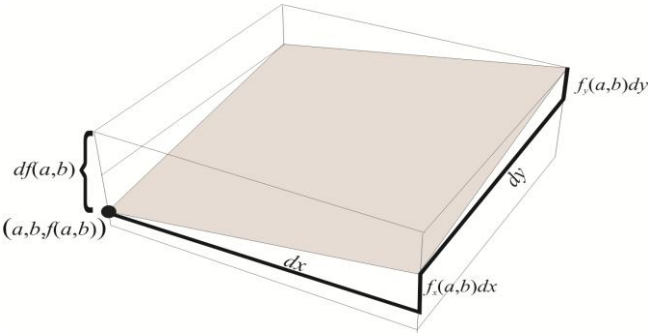

Figure 3: The differential

REFERENCES

Arnon, I., Cotrill, J., Dubinsky, E., Octaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2013). APOS Theory: A framework for research and curriculum development in mathematics education. Springer Verlag: New York.

Czarnocha, B., Dubinsky, E., Prabhu, V., & Vidakovic, D. (1999). One theoretical perspective in undergraduate mathematics education research. In O. Zaslavsky (Ed.), Proceedings of the 23rd Conference of PME (Vol. 1, pp. 95–110). Haifa, Israel: PME.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. Educational Studies in Mathematics, 61(1), 103-131.

Martínez-Planell, R., Trigueros, M. & McGee, D., (2015a). On students' understanding of the differential calculus of functions of two variables. *Journal of Mathematical Behavior* (38), pp. 57-86.

McGee, D. & Moore-Russo, D. (2015). Impact of explicit presentation of slopes in three dimensions on students' understanding of derivatives in multivariable calculus. To appear in *International Journal of Science and Mathematics Education*.

Stewart, J. (2006). *Calculus: Early Transcendentals, 6E*. United States: Thompson Brooks/Cole

Tall, David. (1992) Visualizing differentials in two and three dimensions. Teaching Mathematics and Its Applications, Vol.11 (No.1). pp. 1-7.

Weber E. D. (2012). Students' Ways of Thinking about Two-Variable Functions and Rate of Change in Space. Ph.D. Dissertation. Arizona State University.