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Equivariant Giambelli formula for the symplectic Grassmannians — Pfaffian Sum Formula

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Abstract. We prove an explicit closed formula, written as a sum of Pfaffians, which describes each equivariant Schubert class for the Grassmannian of isotropic subspaces in a symplectic vector space.

Résumé. On démontre une formule close explicite, écrite comme une somme de Pfaffiens, qui décrit toute classe de Schubert équivariante pour la Grassmannienne des sous-espaces isotropes dans un espace vectoriel symplectique.

Keywords: symplectic Grassmannian, Schubert calculus, Giambelli formula, Pfaffian, signed permutation, k -strict partition

1 Introduction

The classical Giambelli formula expresses a general Schubert class of the Grassmannian as the determinant of a matrix whose entries are the so-called *special Schubert classes*. A special Schubert class is defined by the locus of subspaces having excess intersection with a fixed linear subspace. These classes also coincide with the Chern classes of the universal quotient bundle over the Grassmannian. Various extensions of the formula have been obtained. The *Giambelli problem* consists in finding a “closed formula” for a Schubert class in terms of those special classes, and in the torus equivariant setting it is closely related to the theory of degeneracy loci of vector bundles (*cf.* [FP98], [Tam] and the references therein).

For the symplectic or orthogonal Grassmannians, there is a natural notion of special Schubert classes, which takes into account the isotropic conditions arising from the symplectic or orthogonal form. For the Grassmannian of maximal isotropic subspaces, the Giambelli formula, first found by Pragacz [Pra91], expresses a general Schubert class as a Pfaffian whose entries are appropriate quadratic polynomials in the special Schubert classes. Its natural equivariant version was obtained by Kazarian [Kaz00] in the context of degeneracy loci, and later it was also proved in [Ike07] and [IN09] in terms of equivariant cohomology by using more algebraic methods. Similarly to Pragacz’s formula, also Kazarian’s expression

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is given as a single Pfaffian. On the other hand, for the non-maximal isotropic Grassmannians, an answer to the (non-equivariant) Giambelli problem was given by Buch, Kresch, and Tamvakis [BKTb, BKTa]. Their formula expresses an arbitrary Schubert class as a polynomial in the special Schubert classes, it is defined by means of *Young's raising operators*. We can regard this polynomial expression as a certain “combinatorial interpolation” between the Jacobi-Trudi determinant and the Schur Pfaffian.

This paper is mainly concerned with the *equivariant* Giambelli problem for the non-maximal isotropic Grassmannians in the symplectic case. In [Wil10], Wilson employed the raising operators to define the *double theta polynomials*, and proved that these polynomials satisfy the equivariant Chevalley formula for the non-maximal symplectic Grassmannian. In [Wil10], it was further conjectured that the double theta polynomials would be equal to the double Schubert polynomials in [IMN11], a distinguished family of polynomials which are identified with the equivariant Schubert classes of the flag variety in the infinite rank setting.

The main result of this paper provides a formula expressing each double Schubert polynomial associated to the isotropic Grassmannians as a sum of Pfaffians whose entries are Wilson's double theta polynomials corresponding to the special Schubert classes. This immediately leads to a proof of Wilson's conjecture, because the raising operator formula can be rewritten as a Pfaffian sum by a formal computation. Our method to prove the main result is to use the *left divided difference operators* that are essential in the theory of (double) Schubert polynomials and exist only in its equivariant setup.

This article is a short version of the paper [IM15]. Detailed proofs are omitted. The only content which is not included in [IM15] is about a combinatorial description of the Bruhat order on a certain parabolic quotient of the type C Weyl group (Theorem 2.8).

2 Double Schubert polynomials of type C

We first recall the definition and fundamental properties of the double Schubert polynomials introduced in [IMN11]. Then we discuss the correspondences among the k -strict partitions, the characteristic indices and the parabolic quotients of the Weyl group.

2.1 Double Schubert polynomials

Weyl group for type C Let W_∞ be the infinite hyperoctahedral group which is defined by the generators $s_i, i = 0, 1, \dots$, and the relations

$$\begin{aligned} s_i^2 &= e \quad (i \geq 0), & s_i s_j &= s_j s_i \quad (|i - j| \geq 2), \\ s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0, & s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad (i \geq 1). \end{aligned} \tag{2.1}$$

We identify W_∞ with the group of all permutations w of $\mathbb{Z} \setminus \{0\}$ such that $w(i) \neq i$ for only finitely many $i \in \mathbb{Z} \setminus \{0\}$, and $\overline{w(i)} = w(\bar{i})$ for all i where $\bar{i} := -i$. The generators, often referred to as the *simple reflections*, are identified with the transpositions $s_0 = (1, \bar{1})$ and $s_i = (i + 1, i)(\bar{i}, \bar{i} + \bar{1})$ for $i \geq 1$. Let W_n be the subgroup of W_∞ generated by s_0, s_1, \dots, s_{n-1} . Or equivalently, it consists of elements $w \in W_\infty$ such that $w(i) = i$ for all $i > n$. The *one-line notation* of an element $w \in W_\infty$ is the sequence $w = (w(1)w(2)w(3)\dots)$. We often write the one-line notation of $w \in W_n$ by the finite sequence $(w(1)w(2)\dots w(n))$. The *length* of $w \in W_\infty$ is denoted by $\ell(w)$.

The ring of Double Schubert Polynomials for type C Let $x = (x_1, x_2, \dots)$ be a sequence of indeterminates. Let $Q_r(x)$ ($r \geq 0$) be the functions defined by the generating function

$$\sum_{r=0}^{\infty} Q_r(x)u^r = \prod_{i=1}^{\infty} \frac{1 + x_i u}{1 - x_i u}. \tag{2.2}$$

Let Γ be the ring generated by $Q_r(x)$ ($r \geq 0$). Let \mathcal{R}_∞ be the polynomial ring $\Gamma[t, z]$ in the indeterminates $t = (t_1, t_2, \dots)$, and $z = (z_1, z_2, \dots)$ with the coefficients in Γ .

There is a distinguished $\mathbb{Z}[t]$ -basis \mathfrak{C}_w ($w \in W_\infty$) of \mathcal{R}_∞ called the *double Schubert polynomials of type C*. To characterize this basis, we first define two actions of W_∞ on \mathcal{R}_∞ (the *right* and *left* actions), and then introduce the *right* and *left divided difference operators*.

The right action on \mathcal{R}_∞ as a $\mathbb{Z}[t]$ -algebra For each $i \geq 1$, define the $\mathbb{Z}[t]$ -algebra automorphism s_i^z of \mathcal{R}_∞ by

$$s_i^z(z_i) = z_{i+1}, \quad s_i^z(z_{i+1}) = z_i, \quad s_i^z(z_j) = z_j \quad (j \neq i, i+1), \quad \text{and} \quad s_i^z(Q_r(x)) = Q_r(x).$$

Also define the $\mathbb{Z}[t]$ -algebra automorphism s_0^z of \mathcal{R}_∞ by

$$s_0^z(z_1) = -z_1, \quad s_0^z(z_i) = z_i \quad (i \geq 1), \quad \text{and} \quad s_0^z Q_r(x_1, x_2, \dots) = Q_r(z_1, x_1, x_2, \dots).$$

These satisfy the Coxeter relations (2.1), and therefore define an action of W_∞ on \mathcal{R}_∞ as a $\mathbb{Z}[t]$ -algebra.

The left action on \mathcal{R}_∞ as a $\mathbb{Z}[z]$ -algebra Consider the involution ω of the ring \mathcal{R}_∞ :

$$\omega(t_i) = -z_i, \quad \omega(z_i) = -t_i, \quad \omega(Q_r(x)) = Q_r(x).$$

For each $i \geq 0$, we define the $\mathbb{Z}[z]$ -algebra automorphism s_i^t of \mathcal{R}_∞ by $s_i^t := \omega s_i^z \omega$. In particular, we have

$$s_0^t Q_r(x_1, x_2, \dots) = Q_r(-t_1, x_1, x_2, \dots).$$

These satisfy the Coxeter relations (2.1), and therefore define an action of W_∞ on \mathcal{R}_∞ as a $\mathbb{Z}[z]$ -algebra.

Divided difference operators We define the *right* and *left divided difference operators* respectively by

$$\partial_i f = \frac{f - s_i^z f}{\omega(\alpha_i)} \quad \text{and} \quad \delta_i f = \frac{f - s_i^t f}{\alpha_i} \quad (i \geq 0, f \in \mathcal{R}_\infty),$$

where α_i ($i \geq 0$) are the *simple roots* defined by $\alpha_0 = 2t_1$ and $\alpha_i = t_{i+1} - t_i$ for all $i \geq 1$.

Theorem 2.1 (IMN11) *There exists a unique $\mathbb{Z}[t]$ -free basis $\{\mathfrak{C}_w(z, t; x) \mid w \in W_\infty\}$ of \mathcal{R}_∞ satisfying the equations*

$$\partial_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{ws_i} & \text{if } \ell(ws_i) < \ell(w) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \delta_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{s_i w} & \text{if } \ell(s_i w) < \ell(w) \\ 0 & \text{otherwise} \end{cases}, \tag{2.3}$$

for all $i \geq 0$, and such that \mathfrak{C}_w has no constant term except $\mathfrak{C}_e = 1$.

2.2 The k -Grassmannian elements and the invariant subring $\mathcal{R}_\infty^{(k)}$

Let $W_{(k)}$ be the subgroup of W_∞ generated by all s_i , $i \neq k$. Let $W_\infty^{(k)}$ be the set of minimum length coset representatives for $W_\infty/W_{(k)}$, and it is given by

$$W_\infty^{(k)} = \{w \in W_\infty \mid \ell(ws_i) > \ell(w) \text{ for all } i \neq k\}.$$

We denote $W_\infty^{(k)} \cap W_n$ by $W_n^{(k)}$. An element of $W_\infty^{(k)}$ is called k -Grassmannian and it is given by the following one-line notation:

$$w = (v_1 \cdots v_k \mid \overline{\zeta_1} \cdots \overline{\zeta_s} u_1 u_2 \cdots); \tag{2.4}$$

$$0 < v_1 < \cdots < v_k, \overline{\zeta_1} < \cdots < \overline{\zeta_s} < 0 < u_1 < u_2 < \cdots .$$

We insert a vertical line after v_k to indicate that w is regarded as a k -Grassmannian element. For example, $(13 \mid \overline{4}256 \cdots)$ is a 2-Grassmannian element in W_∞ .

Lemma 2.2 *Let $w \in W_\infty^{(k)}$. We have $\ell(s_0 w) < \ell(w)$ if and only if*

(L0) $w = (\cdots \mid \cdots \overline{1} \cdots)$, i.e., $\zeta_s = 1$.

Suppose $i \geq 1$. Then $\ell(s_i w) < \ell(w)$ if and only if one of the following holds:

(L1) $w = (\cdots \mid \cdots \overline{i+1} \cdots i \cdots)$, i.e., $\zeta_p = i + 1$ and $u_q = i$ for some p and q ;

(L2) $w = (\cdots i \cdots \mid \cdots \overline{i+1} \cdots)$, i.e., $\zeta_p = i + 1$ and $v_q = i$ for some p and q ;

(L3) $w = (\cdots i + 1 \cdots \mid \cdots i \cdots)$, i.e., $u_p = i$ and $v_q = i + 1$ for some p and q .

In particular, for any $i \in \{0, 1, 2, \dots\}$, $\ell(s_i w) = \ell(w) - 1$ implies $s_i w \in W_\infty^{(k)}$.

Let $\mathcal{R}_\infty^{(k)}$ be the $\mathbb{Z}[t]$ -subalgebra of \mathcal{R}_∞ fixed by the right action of $W_{(k)}$:

$$\mathcal{R}_\infty^{(k)} := \{f \in \mathcal{R}_\infty \mid s_i^z(f) = f \quad (\forall i \neq k)\}.$$

We can show that $\mathcal{R}_\infty^{(k)} = \bigoplus_{w \in W_\infty^{(k)}} \mathbb{Z}[t] \mathfrak{C}_w$. Furthermore, the double Schubert polynomials \mathfrak{C}_w for the k -Grassmannian elements $w \in W_\infty^{(k)}$ are characterized in terms of the left divided difference operators as follows. For $f \in \mathcal{R}_\infty$, let $f|_\emptyset$ denote the polynomial in $\mathbb{Z}[t]$ obtained from f by the substitutions $x_i \mapsto 0$ and $z_i \mapsto t_i$ for all i .

Proposition 2.3 *If a family $F_w, w \in W_\infty^{(k)}$ of elements of $\mathcal{R}_\infty^{(k)}$ satisfies the following conditions*

$$\delta_i F_w = \begin{cases} F_{s_i w} & \text{if } \ell(s_i w) < \ell(w); \\ 0 & \text{if } \ell(s_i w) > \ell(w); \end{cases} \tag{2.5}$$

$$F_w|_\emptyset = \delta_{w,e}, \tag{2.6}$$

then $F_w = \mathfrak{C}_w$ for all $w \in W_\infty^{(k)}$.

2.3 The k -strict partitions

A partition λ is a sequence of non-increasing non-negative integers $(\lambda_1, \lambda_2, \dots)$ where only finitely many λ_i 's are positive. It is called k -strict if no part greater than k is repeated. Let $\mathcal{P}_\infty^{(k)}$ be the set of all k -strict partitions. Let $\mathcal{P}_n^{(k)} \subset \mathcal{P}_\infty^{(k)}$ be the subset of all k -strict partitions whose Young diagrams fit in the $(n - k) \times (n + k)$ rectangle. We often denote λ by a finite sequence $(\lambda_1, \dots, \lambda_r)$, assuming $\lambda_i = 0$ for all $i > r$, and also denote the sum of the positive entries by $|\lambda|$.

Lemma 2.4 (BKTb) *There is a bijection $\mathcal{P}_\infty^{(k)} \cong W_\infty^{(k)}$ ($\lambda \leftrightarrow w_\lambda^{(k)}$) such that $|\lambda| = \ell(w_\lambda^{(k)})$.*

We describe the bijection below. Let $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}_\infty^{(k)}$. Let s be an index such that $\lambda_s > k$ and $\lambda_{s+1} \leq k$. Let $\zeta_i := \lambda_i - k$ for $i \in \{1, \dots, s\}$. Let $\mu = (\mu_1, \dots, \mu_k)$ be the conjugate of the partition $(\lambda_{s+1}, \lambda_{s+2}, \dots)$. For $i \in \{1, \dots, k\}$, define

$$v_i = s + \mu_{k+1-i} + i - \#\{p \mid \zeta_p > s + \mu_{k+1-i} + i\}.$$

The signed permutation w_λ is given by the one-line notation (2.4) where u_1, u_2, \dots are the increasing positive integers determined uniquely by v and ζ .

Conversely, let $w \in W_\infty^{(k)}$ with the one-line notation (2.4). Define a partition $\nu = (\nu_1, \nu_2, \dots)$ by $\nu_i = \#\{p \mid v_p > u_i\}$. Then a k -strict partition λ defined by setting $\lambda_i = \zeta_i + k$ if $1 \leq i \leq s$ and $\lambda_i = \nu_{i-s}$ if $s + 1 \leq i$.

Remark 2.5 *By the bijection in Lemma 2.4, the natural Bruhat order on $W_\infty^{(k)}$ does not correspond to the order on $\mathcal{P}_\infty^{(k)}$ defined by inclusions of Young diagrams. For example, consider 1-strict partitions $\lambda = (21)$ and $\mu = (4)$. The corresponding permutations are $w_\lambda^{(1)} = (3\bar{1}24) = s_2s_0s_1$ and $w_\mu^{(1)} = (1\bar{3}24) = s_2s_1s_0s_1$ respectively. We see that $w_\lambda^{(1)} \leq w_\mu^{(1)}$, while $\lambda \not\leq \mu$.*

2.4 Characteristic index and Bruhat order on $W_\infty^{(k)}$

The following indexing set for $W_\infty^{(k)}$ turns out to be useful for both the understanding of Bruhat order and the computation in the left divided difference operators.

Definition 2.6 (Characteristic index) *For each $w \in W_\infty^{(k)}$ with the one-line notation (2.4), we define the associated characteristic index by*

$$\chi = (\chi_1, \chi_2, \dots) = (\zeta_1 - 1, \zeta_2 - 1, \dots, \zeta_s - 1, -u_1, -u_2, \dots).$$

In particular, if $w \in W_n^{(k)}$, then $\chi_i = -i - k$ for all $i > n - k$.

If χ and λ are the characteristic index and the k -strict partition associated to w respectively, then we have $\chi_j = \lambda_j - k - j + \#\{i \mid i < j, \lambda_i + \lambda_j > 2k + j - i\}$.

For a strictly decreasing sequence $\chi = (\chi_1, \chi_2, \dots)$ of integers, let

$$b_i^\chi = \begin{cases} \chi_i + 1 & \text{if } \chi_i \geq 0 \\ -\chi_i & \text{if } \chi_i < 0, \end{cases} \quad \text{for all } i = 1, 2, \dots$$

Definition 2.7 Let $\Xi_\infty^{(k)}$ be the set of all strictly decreasing sequences $\chi = (\chi_1, \chi_2, \dots)$ of integers such that b_i^χ 's are all distinct, $\{1, 2, \dots\} \setminus \{b_i^\chi, i = 1, 2, \dots\}$ is a finite set of cardinality k , and $\chi_i = -i - k$ for sufficiently large i . Let $\Xi_n^{(k)}$ be the subset of $\Xi_\infty^{(k)}$ consisting of χ such that $\chi_i = -i - k$ for all $i > n - k$. Define a partial order on $\Xi_\infty^{(k)}$ by

$$\chi \leq \chi' \quad \text{if and only if} \quad \chi_i \leq \chi'_i \quad \text{all } i = 1, 2, \dots$$

Theorem 2.8 The map $W_\infty^{(k)} \rightarrow \Xi_\infty^{(k)}$ associating the characteristic indices is an order preserving bijection. Furthermore this map also induces an order preserving bijection $W_n^{(k)} \cong \Xi_n^{(k)}$.

The proof of this theorem is postponed until Section 3.3.

Example 2.9 Let λ and μ be the partitions given in Remark 2.5. The corresponding characteristic indices are $\chi_\lambda = (0, -2, -4)$ and $\chi_\mu = (2, -2, -4)$ respectively. We find $\chi_\lambda \leq \chi_\mu$. This verifies Theorem 2.8 since we know $w_\lambda \leq w_\mu$ from Remark 2.5.

3 Equivariant Schubert classes of symplectic Grassmannians

We explain the fact that the double Schubert polynomials represent the equivariant Schubert classes.

3.1 Flag varieties of type C

Let $G = \text{Sp}_{2n}(\mathbb{C})$ be the complex symplectic group of rank n . Let B be a Borel subgroup of G , and T the maximal torus contained in B (see [IMN11] for the precise conventions). The flag variety \mathcal{Fl}_n is defined as the quotient space G/B . The Weyl group $N_G(T)/T$ is identified with W_n . For each $w \in W_n$, there is the point e_w in \mathcal{Fl}_n . Let B^- be the Borel subgroup such that $B \cap B^- = T$. The Schubert variety X_w is defined as the Zariski closure of B^- -orbit of the point $e_w \in \mathcal{Fl}_n$. The codimension of X_w is precisely the length $\ell(w)$ of w . Since X_w is a T -stable subvariety of \mathcal{Fl}_n it defines the fundamental class $[X_w]_T$ in the integral T -equivariant cohomology ring $H_T^*(\mathcal{Fl}_n)$. We call $[X_w]_T$ the equivariant Schubert class associated to $w \in W_n$. We regard $H_T^*(\mathcal{Fl}_n)$ as a $\mathbb{Z}[t]$ -algebra by identifying $H_T^*(pt) = \mathbb{Z}[t_1, \dots, t_n]$ and defining the action of t_i to be trivial for all $i > n$.

Theorem 3.1 ([IMN11]) There is a canonical $\mathbb{Z}[t]$ -algebra homomorphism $\pi_n : \mathcal{R}_\infty \rightarrow H_T^*(\mathcal{Fl}_n)$ such that π_n sends $\mathfrak{C}_w(z, t; x)$ to $[X_w]_T$ if $w \in W_n$ and to zero if $w \notin W_n$.

3.2 Symplectic Grassmannian and its Schubert varieties

We fix a non-negative integer k . For any positive integer $n \geq k$, let SG_n^k denote the Grassmannian of $(n - k)$ -dimensional isotropic subspaces in \mathbb{C}^{2n} equipped with a symplectic form. There is a maximal parabolic subgroup P_k of the symplectic group $G = \text{Sp}_{2n}(\mathbb{C})$ such that SG_n^k can be realized as the homogeneous space G/P_k .

Let $\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_1^*, \dots, \mathbf{e}_n^*$ be a standard symplectic basis of \mathbb{C}^{2n} , i.e., define the symplectic form by $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle \mathbf{e}_i^*, \mathbf{e}_j^* \rangle = 0, \langle \mathbf{e}_i, \mathbf{e}_j^* \rangle = \delta_{ij}$. For $1 \leq i \leq n$, define a complete flag

$$F^\bullet : F^n \subset \dots \subset F^1 \subset F^{\bar{1}} \subset \dots \subset F^{\bar{n}} \tag{3.1}$$

by $F^i = \langle \mathbf{e}_i, \dots, \mathbf{e}_n \rangle$ and $F^{\bar{i}} = \langle \mathbf{e}_i^*, \dots, \mathbf{e}_1^* \rangle + F^1$. For each $\lambda \in \mathcal{P}_n^{(k)}$, the corresponding Schubert variety Ω_λ with respect to F^\bullet can be defined as

$$\Omega_\lambda := \{V \in \text{SG}_n^k \mid \dim(V \cap \overline{F^{w_\lambda^{(k)}(k+j)}}) \geq j \quad (1 \leq j \leq n-k)\}. \quad (3.2)$$

In particular, the *special Schubert varieties* are the ones associated to one-line k -strict partitions:

$$\Omega_r = \{V \in \text{SG}_n^k \mid \dim(V \cap F^{r-k-1}) \geq 1\},$$

for $1 \leq r \leq n+k$. Their classes $[\Omega_r]_T$ are called the *special Schubert classes*. They are equal to the r -th equivariant Chern classes $c_r(\mathcal{Q})$ of the universal quotient bundle \mathcal{Q} over SG_n^k . For $1 \leq j \leq n-k$, we have

$$\dim \overline{F^{w_\lambda^{(k)}(k+j)}} = n - \chi_j. \quad (3.3)$$

This explains how naturally the characteristic index is defined.

There is the following commutative diagram :

$$\begin{array}{ccc} \mathcal{R}_\infty^{(k)} & \longrightarrow & \mathcal{R}_\infty \\ \downarrow \pi_n^{(k)} & & \downarrow \pi_n \\ H_T^*(\text{SG}_n^k) & \xrightarrow{p^*} & H_T^*(\mathcal{F}l_n) \end{array}$$

where the horizontal arrow p^* in the second row is the pullback of the natural projection $p : \mathcal{F}l_n \rightarrow \text{SG}_n^k$ and $\pi_n^{(k)}$ is obtained by restricting π_n to $\mathcal{R}_\infty^{(k)}$. If $\lambda \in \mathcal{P}_n^{(k)}$, we have $p^*[\Omega_\lambda]_T = [X_{w_\lambda^{(k)}}]_T$ and

$$\pi_n^{(k)}(\mathfrak{C}_{w_\lambda^{(k)}}(z, t; x)) = [\Omega_\lambda]_T.$$

In particular, the special Schubert class $[\Omega_r]_T$ of degree r is the image of $\mathfrak{C}_{w_r^{(k)}}(z, t; x)$ where $w_r^{(k)}$ is the element of $W_\infty^{(k)}$ corresponding to the partition with r boxes in one row.

3.3 Proof of Theorem 2.8

By the definition of the characteristic index, it is clear that the map is a bijection. Choose n such that $w, w' \in W_n^{(k)}$. Let χ and χ' be the characteristic indices of w and w' respectively. Let λ and λ' be the strict partitions corresponding to w and w' respectively. We reindex the flag F^\bullet defined in (3.1), and consider the complete flag $U^\bullet : 0 = U^n \subset \dots \subset U^{n-1} \dots \subset U^0 \subset U^{-1} \subset \dots \subset U^{-n} = \mathbb{C}^{2n}$ where for each $i \in \{1, 2, \dots, n\}$, $U^{-i} = F^{\bar{i}}$ and $U^i = F^{i-1}$. Note that $\dim U^i = n - i$ for all $-n \leq i \leq n$ by (3.3). Then the Schubert variety Ω_λ is given by

$$\Omega_\lambda = \{V \in \text{SG}_n^k \mid \dim(V \cap U^{\chi_i}) \geq i, i = 1, \dots, n-k\}.$$

It is well-known that $\Omega_\lambda \supset \Omega_{\lambda'}$ if and only if $w \leq w'$. Therefore $w \leq w'$ if and only if, for all $i = 1, \dots, n-k$, we have

$$U^{\chi'_i} \subset U^{\chi_i} \quad \text{i.e.} \quad \chi_i \leq \chi'_i.$$

4 Pfaffian sum formula for the equivariant Schubert classes

4.1 Double theta polynomials

Our goal is to give an explicit closed formula to describe $\mathfrak{C}_w(z, t; x)$ ($w \in W_\infty^{(k)}$) as a polynomial in terms of the double Schubert polynomials $\mathfrak{C}_{w_r}(z, t; x)$ corresponding to the special classes $[\Omega_r]_T$.

Definition 4.1 Define ${}_k\vartheta_r^{(l)}(x, z|t)$ for $l, r \geq 0$ by

$$\begin{aligned} \sum_{r=0}^{\infty} {}_k\vartheta_r^{(l)}(x, z|t) \cdot u^r &= \prod_{i=1}^{\infty} \frac{1+x_i u}{1-x_i u} \prod_{i=1}^k (1+z_i u) \prod_{i=1}^l (1-t_i u), \\ \sum_{r=0}^{\infty} {}_k\vartheta_r^{(-l)}(x, z|t) \cdot u^r &= \prod_{i=1}^{\infty} \frac{1+x_i u}{1-x_i u} \prod_{i=1}^k (1+z_i u) \prod_{i=1}^l \frac{1}{1+t_i u}. \end{aligned}$$

For $r < 0$, we set ${}_k\vartheta_r^{(l)}(x, z|t) = 0$. We omit k when it is made clear by the context. Under π_n , the indeterminates z_i correspond to the Chern roots of the tautological bundles and those theta polynomials ${}_k\vartheta_r^{(l)}(x, z|t)$ map to Chern classes of certain virtual bundles. In particular, Wilson in [Wil10, Prop. 6] proved

$$\mathfrak{C}_{w_r}(z, t; x) = {}_k\vartheta_r^{(r-k-1)}(x, z|t). \quad (4.1)$$

Although the above definition of the double theta polynomials appears slightly different from the one in Wilson's thesis [Wil10], one recovers Wilson's definition after applying appropriate changes of indices. See [IM15] for details. We can prove the following lemma from the definitions by direct computations.

Lemma 4.2 (a) For all $i \geq 0$, we have $\delta_i \vartheta_r^{(l)} = 0$ if $l \neq \pm i$ and $\delta_i \vartheta_r^{(l)} = \vartheta_{r-1}^{(l-1)}$ if $l = \pm i$.

(b) For $i \neq 0$, we have $\delta_i (\vartheta_r^{(i)} \cdot \vartheta_s^{(-i)}) = \vartheta_{r-1}^{(i-1)} \cdot \vartheta_s^{(-i-1)} + \vartheta_r^{(i-1)} \cdot \vartheta_{s-1}^{(-i-1)}$.

4.2 Multi Schur-Pfaffian

For each $i \geq 1$, let $c^{(i)}$ be an infinite sequence of indeterminates $c_r^{(i)}, r \in \mathbb{Z}$. Consider the polynomial ring $\mathbb{Z}[c] := \mathbb{Z}[c_r^{(i)}, i \geq 1, r \in \mathbb{Z}]$. The multi Schur-Pfaffian

$$\text{Pf}[c_{r_1}^{(1)} \dots c_{r_m}^{(m)}] \in \mathbb{Z}[c]$$

is defined as follows:

- for $m = 1$ we set $\text{Pf}[c_r^{(1)}] = c_r^{(1)}$.
- for $m = 2$, we set $\text{Pf}[c_{r_1}^{(1)} c_{r_2}^{(2)}] = c_{r_1}^{(1)} c_{r_2}^{(2)} + 2 \sum_{s=1}^{r_2} (-1)^s c_{r_1+s}^{(1)} c_{r_2-s}^{(2)}$.
- for any odd $m \geq 3$, we set $\text{Pf}[c_{r_1}^{(1)} \dots c_{r_m}^{(m)}] = \sum_{s=1}^m (-1)^{s-1} c_{r_s}^{(s)} \cdot \text{Pf}[c_{r_1}^{(1)} \dots \widehat{c_{r_s}^{(s)}} \dots c_{r_m}^{(m)}]$.
- for any even $m \geq 4$, we set $\text{Pf}[c_{r_1}^{(1)} \dots c_{r_m}^{(m)}] = \sum_{s=2}^m (-1)^s \text{Pf}[c_{r_1}^{(1)} c_{r_s}^{(s)}] \cdot \text{Pf}[c_{r_2}^{(2)} \dots \widehat{c_{r_s}^{(s)}} \dots c_{r_m}^{(m)}]$.

The following properties follow from the above definition of Pfaffian by the induction on m .

Proposition 4.3

- (1) If $\text{Pf}[c_r^{(l)} c_r^{(l)}] = 0$, then we have $\text{Pf}[c_{r_1}^{(l_1)} \dots c_r^{(l)} c_r^{(l)} \dots c_{r_m}^{(l_m)}] = 0$.
- (2) If $\text{Pf}[c_r^{(l)} c_s^{(l)}] + \text{Pf}[c_s^{(l)} c_r^{(l)}] = 0$, then we have

$$\text{Pf}[c_{r_1}^{(l_1)} \dots c_r^{(l)} c_s^{(l)} \dots c_{r_m}^{(l_m)}] + \text{Pf}[c_{r_1}^{(l_1)} \dots c_s^{(l)} c_r^{(l)} \dots c_{r_m}^{(l_m)}] = 0.$$

4.3 Pfaffian sum formula (Main Theorem)

Definition 4.4 For all $(r_1, \dots, r_m), (l_1, \dots, l_m) \in \mathbb{Z}^m$, let

$$\text{Pf} \left[\vartheta_{r_1}^{(l_1)} \vartheta_{r_2}^{(l_2)} \dots \vartheta_{r_m}^{(l_m)} \right] := \text{Pf} \left[c_{r_1}^{(1)} c_{r_2}^{(2)} \dots c_{r_m}^{(m)} \right] \Big|_{c=\vartheta^{(l)}}$$

where $|_{c=\vartheta^{(l)}}$ means that we substitute $\vartheta_s^{(l_i)}$ for $c_s^{(i)}$ for all $i \in \{1, \dots, m\}$ and $s \in \mathbb{Z}$.

Let $\Delta = \{(i, j) \mid 1 \leq i < j \leq n - k\}$. Define a subset $D(\lambda)$ of Δ by $D(\lambda) := \{(i, j) \in \Delta \mid \chi_i + \chi_j < 0\}$. The main result of this paper is the following.

Theorem 4.5 (Main Theorem) Let λ be a k -strict partition in $\mathcal{P}_n^{(k)}$. We have

$$\mathfrak{C}_{w_\lambda^{(k)}} = \sum_{I \subset D(\lambda)} \text{Pf} \left[\vartheta_{\lambda_1 + a_I^1}^{(\chi_1)} \dots \vartheta_{\lambda_{n-k} + a_I^{n-k}}^{(\chi_{n-k})} \right], \tag{4.2}$$

where I runs over all subsets of $D(\lambda)$ and $a_s^I = \#\{j \mid (s, j) \in I\} - \#\{i \mid (i, s) \in I\}$.

Example 4.6 Let $k = 1, n = 5$. Let $\lambda = (5, 3, 2, 1)$ be a k -strict partition. Then $w_\lambda^{(1)} = 5|4\bar{2}\bar{1}3$ and $D(\lambda) = \{(2, 4), (3, 4)\}$. We have

$$\mathfrak{C}_{w_\lambda^{(1)}} = \text{Pf}[\vartheta_5^{(3)} \vartheta_3^{(1)} \vartheta_2^{(0)} \vartheta_1^{(-3)}] + \text{Pf}[\vartheta_5^{(3)} \vartheta_4^{(1)} \vartheta_2^{(0)} \vartheta_0^{(-3)}] + \text{Pf}[\vartheta_5^{(3)} \vartheta_3^{(1)} \vartheta_3^{(0)} \vartheta_0^{(-3)}].$$

Once we read the formula in terms of raising operators, the following corollary is immediate.

Corollary 4.7 If $D(\lambda) = \Delta$, in particular, if λ is contained in the $(n - k) \times k$ rectangle, then $\mathfrak{C}_{w_\lambda^{(k)}}$ is a single determinant $\det(\vartheta_{\lambda_i + j - i}^{(\chi_i)})_{1 \leq i, j \leq n - k}$ ([Tam, §1]). If $D(\lambda) = \emptyset$, in particular, if λ is a strict partition containing the $(n - k) \times k$ rectangle, then $\mathfrak{C}_{w_\lambda^{(k)}}$ is a single Pfaffian $\text{Pf}[\vartheta_{\lambda_1}^{(\chi_1)} \dots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}]$.

In the case when λ is contained in the $(n - k) \times k$ rectangle, the result was proved by Wilson [Wil10].

4.4 Related results

Anderson and Fulton [AF] defined a notion of vexillary signed permutation in type B,C, and D. They showed the double Schubert polynomials associated to vexillary signed permutations are given by explicit Pfaffian formulas. Naruse [Nar] also independently proved a formula that expresses the corresponding double Schubert polynomials as a specialization of the factorial Q - and P -functions. Since our formula also expresses some Schubert classes as single Pfaffians, there is an overlap between our results and the results of [AF]. However, not all k -Grassmannian permutations are vexillary and there are non-vexillary k -Grassmannian permutations whose corresponding classes are written as single Pfaffians, e.g. $13|5\bar{4}2$ is not vexillary but $\mathfrak{C}_{13|5\bar{4}2}$ is a single Pfaffian as above.

5 Proof of Main Theorem

5.1 Key results on multi Schur-Pfaffian with entries $\vartheta_r^{(l)}$

The proof of our main theorem is essentially based on the following two propositions. We can show them by using the inductive definition of the multi Schur-Pfaffian together with Lemma 4.2.

Proposition 5.1 *Suppose $l \geq 0$ and $r > k + l$. Then $\text{Pf}[\vartheta_{r_1}^{(l_1)} \cdots \vartheta_r^{(l)} \vartheta_r^{(l)} \cdots \vartheta_{r_m}^{(l_m)}] = 0$.*

Proposition 5.2 (a) *Let $i \geq 0$. If $l_p \neq \pm i$ for all $p \in \{1, \dots, m\}$, then $\delta_i \text{Pf}[\vartheta_{r_1}^{(l_1)} \cdots \vartheta_{r_m}^{(l_m)}] = 0$.*

(b) *Let $i \geq 0$. Suppose that $l_p \in \{\pm i\}$ for some $p \in \{1, \dots, m\}$ and that $l_q \notin \{\pm i\}$ for all $q \neq p$.*

$$\delta_i \text{Pf}[\vartheta_{r_1}^{(l_1)} \cdots \vartheta_{r_p}^{(l_p)} \cdots \vartheta_{r_m}^{(l_m)}] = \text{Pf}[\vartheta_{r_1}^{(l_1)} \cdots \vartheta_{r_p-1}^{(l_p-1)} \cdots \vartheta_{r_m}^{(l_m)}].$$

(c) *Let $i > 0$. Suppose that $l_p = i$ and $l_q = -i$ for some $p < q$ and that $l_s \neq \pm i$ for all $s \notin \{p, q\}$. Then we have*

$$\begin{aligned} & \delta_i \text{Pf}[\vartheta_{r_1}^{(l_1)} \cdots \vartheta_{r_p}^{(i)} \cdots \vartheta_{r_q}^{(-i)} \cdots \vartheta_{r_m}^{(l_m)}] \\ = & \text{Pf}[\vartheta_{r_1}^{(l_1)} \cdots \vartheta_{r_p-1}^{(i-1)} \cdots \vartheta_{r_q}^{(-i-1)} \cdots \vartheta_{r_m}^{(l_m)}] + \text{Pf}[\vartheta_{r_1}^{(l_1)} \cdots \vartheta_{r_p}^{(i-1)} \cdots \vartheta_{r_q-1}^{(-i-1)} \cdots \vartheta_{r_m}^{(l_m)}]. \end{aligned}$$

5.2 Key lemma

For $w \in W_n^{(k)}$, we denote the corresponding function of the right hand side of (4.2) by $\Theta_w^{(n,k)}$. The group $W_n^{(k)}$ has a unique longest element denoted by $w_{max}^{(n,k)}$, which is known to be of order two. Then we can define an involution $W_n^{(k)} \rightarrow W_n^{(k)}$ ($w \mapsto w^\vee$) by $w^\vee = w w_{max}^{(n,k)}$. For $w \in W_\infty$, let $\delta_w := \delta_{s_{i_1}} \cdots \delta_{s_{i_\ell}}$ where $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word expression.

Lemma 5.3 *Let $w \in W_n^{(k)}$ and $w_{max}^{(n,k)}$ the longest element in $W_n^{(k)}$. We have $\Theta_w^{(n,k)} = \delta_{w^\vee} \Theta_{w_{max}^{(n,k)}}$.*

Proof: Let $\lambda \in \mathcal{P}_n^{(k)}$ be the k -strict partition corresponding to $w \in W_n^{(k)}$. We proceed by the induction on $\ell(w^\vee)$. The claim obviously holds if $\ell(w^\vee) = 0$. Suppose $\ell(w^\vee) \geq 1$. Then there is $i \in \{0, 1, \dots, n-1\}$ such that $w = s_i w_{\lambda'} \in W_n^{(k)}$ for some $\lambda' \in \mathcal{P}_n^{(k)}$ satisfying $\ell(\lambda') = \ell(\lambda) + 1$. By the induction hypothesis we have

$$\delta_{w_{\lambda'}^\vee} \Theta_{w_{max}^{(n,k)}}^{(n,k)} = \sum_{I \subset D(\lambda')} \text{Pf} \left[\vartheta_{\lambda'_1 + a_1^I}^{(\chi'_1)} \cdots \vartheta_{\lambda'_{n-k} + a_{n-k}^I}^{(\chi'_{n-k})} \right], \quad (5.1)$$

where $\chi' = (\chi'_1, \dots, \chi'_{n-k})$ is the characteristic index of λ' . Lemma 2.2 gives the 4 possible cases of the characteristic index χ' in terms of χ . In each case, we apply δ_i to the right hand side of (5.1) by using Proposition 5.2 and obtain $\Theta_w^{(n,k)}$. Since $\delta_{w^\vee} = \delta_i \delta_{w_{\lambda'}^\vee}$, the desired formula holds. \square

5.3 Proof of Main Theorem

First note that the polynomial defined by $\Theta_w^{(n,k)}$ is independent of n , i.e., for all $m \geq n$, and $w \in W_n^{(k)} \subset W_m^{(k)}$, we have $\Theta_w^{(m,k)} = \Theta_w^{(n,k)}$. We can see this fact from the definition of the multi-Schur Pfaffian. Thus, for each $w \in W_\infty^{(k)}$, we can define Θ_w to be $\Theta_w^{(n,k)}$ by choosing an n so such that $w \in W_n^{(k)}$. Proof of the main theorem is completed if we prove the following.

Proposition 5.4 *Let $w \in W_\infty^{(k)}$. We have $\Theta_w = \mathfrak{C}_w$.*

Proof: By Proposition 2.3, it suffices to show that the family of functions Θ_w ($w \in W_\infty^{(k)}$) satisfies (2.5) and (2.6).

By observing that $\ell(s_i w) = \ell(w) - 1$ implies $\delta_i \delta_{w^\vee} = \delta_{(s_i w)^\vee}$, the first half of the condition (2.5) follows from Lemma 5.3. For the second half of the condition (2.5), first we see that $\delta_j \Theta_{w_{max}^{(n,k)}} = 0$ for all $j \neq k$ from Proposition 5.1 and Proposition 5.2 (a). Now the claim follows from

- If $\ell(s_i w) = \ell(w) + 1$ and $s_i w \in W_\infty^{(k)}$, then $\delta_i \delta_{w^\vee} = 0$.
- If $\ell(s_i w) = \ell(w) + 1$ and $s_i w \notin W_\infty^{(k)}$, then there exists $j \neq k$ such that $\delta_i \delta_{w^\vee} = \delta_{w^\vee} \delta_j$.

It remains to show that the condition (2.6) holds. It is obvious that $\Theta_e|_e = 1$. For $w \neq 0$, we expand Θ_w as a polynomial in terms of the ϑ -functions $\vartheta_r^{(l)}$ by using the definition of Pfaffian. Note that each monomial contains $\vartheta_{\lambda_1 + a_1^+ + j}^{(\chi_1)}$ as its first factor. We finish the proof by showing the substitution makes this factor vanish by using the following easy observation: $\vartheta_m^{(l)}|_\emptyset = 0$ if $\max\{l + k, 0\} < m$. □

5.4 An alternative proof of Main Theorem

Let $w_{n,0}$ be the longest element in W_n . In [IMN11, Theorem 1.2], the authors proved that $\mathfrak{C}_{w_{n,0}}$ can be obtained from the (factorial) Q -function by a certain specialization of variables. In terms of Pfaffian and the theta polynomials ${}_k \vartheta_r^l := {}_k \vartheta_r^l(x, z|t)$, we have

$$\mathfrak{C}_{w_{n,0}}(x, z|t) = \text{Pf} \left[{}_{n-1} \vartheta_{2n-1}^{(n-1)} \quad {}_{n-2} \vartheta_{2n-3}^{(n-2)} \cdots {}_1 \vartheta_3^{(1)} \quad {}_0 \vartheta_1^{(0)} \right].$$

Proposition 5.2 and Proposition 5.1 are valid even when the values of k of ϑ 's are distinct in the Pfaffian. It turns out that the following extra relation of ${}_k \vartheta_r^l$ allows us to derive the Pfaffian sum formula of $\mathfrak{C}_{w_{max}^{(n,k)}}$ by consecutively applying divided difference operators to $\mathfrak{C}_{w_{n,0}}$: if $l \geq 0$ and $k > 0$,

$${}_k \vartheta_r^{(l)} = {}_{k-1} \vartheta_r^{(l+1)} + (t_{l+1} + z_k) \cdot {}_{k-1} \vartheta_{r-1}^{(l)}.$$

Thus Lemma 5.3 implies the main theorem by Theorem 2.1. See [IM15, Section 8]) for the detail.

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