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# Rigged configurations of type $D_4^{(3)}$ and the filling map

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**Abstract.** We give a statistic preserving bijection from rigged configurations to a tensor product of Kirillov–Reshetikhin crystals  $\otimes_{i=1}^N B^{1,s_i}$  in type  $D_4^{(3)}$  by using virtualization into type  $D_4^{(1)}$ . We consider a special case of this bijection with  $B = B^{1,s}$ , and we obtain the so-called Kirillov–Reshetikhin tableaux model for the Kirillov–Reshetikhin crystal.

**Résumé.** Nous donnons une bijection préservant les statistiques entre les configurations grées et les produits tensoriels de cristaux de Kirillov–Reshetikhin  $\otimes_{i=1}^N B^{1,s_i}$  de type  $D_4^{(3)}$ , via une virtualisation en type  $D_4^{(1)}$ . Nous considérons un cas particulier de cette bijection pour  $B = B^{1,s}$  et obtenons ainsi les modèles de tableaux appelés Kirillov–Reshetikhin pour le cristal Kirillov–Reshetikhin.

**Keywords:** rigged configuration, Kirillov–Reshetikhin crystal, bijection

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## 1 Introduction

Rigged configurations were first introduced by Kerov, Kirillov, and Reshetikhin in [14, 15] as combinatorial objects that index solutions to the Bethe Ansatz for the Heisenberg spin chains. Rigged configurations were shown to be in bijection with semi-standard tableaux and classical highest weight elements of a tensor power of the vector representation in type  $A_n^{(1)}$ . This bijection was then extended to Littlewood–Richardson tableaux [16], to non-exceptional types [20], and to type  $E_6^{(1)}$  [19]. This bijection  $\Phi$  between rigged configurations and the tensor powers has been further expanded to include classically highest weight elements in a tensor product of certain Kirillov–Reshetikhin (KR) crystals [16, 21, 18, 27, 28].

Rigged configurations have been shown to display remarkable representation theoretic properties. A (classical) crystal structure was first given for simply-laced types [26], which was then extended to all finite types [27] and affine types [24]. While  $\Phi$  is defined recursively, making it difficult to work with, it preserves certain natural statistics (cocharge and energy), giving a bijective proof of the  $X = M$  conjecture of [5, 6]. Furthermore, the combinatorial  $R$ -matrix transforms into the identity map on rigged configurations under  $\Phi$ . Rigged configurations are also well-behaved under virtualization [21, 22, 27], a process of realizing a non-simply-laced type crystal inside of a simply-laced type, and embeddings  $B(\lambda) \hookrightarrow B(\mu)$  where  $\lambda \leq \mu$  component-wise, leading to a model for  $B(\infty)$  [24].

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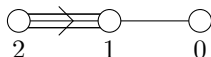


Fig. 2.1: Dynkin diagram of type  $D_4^{(3)}$ .

KR crystals in non-exceptional types were given a combinatorial model in [4] using Kashiwara–Nakashima tableaux [13]. The bijection  $\Phi$  has also lead to a new tableaux model for KR crystals, coined Kirillov–Reshetikhin (KR) tableaux, using filled rectangular tableaux [18, 25, 27]. The map between Kashiwara–Nakashima tableaux [13] and the KR tableaux is called the filling map.

The goal of this work is to extend  $\Phi$  to type  $D_4^{(3)}$  and describe the filling map. For this extended abstract, we will be focusing on the KR crystals  $B^{1,s}$  and the rigged configurations associated with tensor products of the form  $\bigotimes_{i=1}^N B^{1,s_i}$ . In particular, we show  $\Phi$  is a classical crystal isomorphism, and we describe the filling map for  $B^{1,s}$ . We do so by showing the filling map and bijection commute with the virtualization map, proving more special cases of many of the conjectures stated in [27].

This extended abstract is organized as follows. In Section 2, we give background on crystals, virtualization, and rigged configurations. In Section 3, we describe the bijection  $\Phi$ . In Section 4, we describe the filling map. In Section 5, we describe the virtualization map and our main results. In Section 6, we give possible extensions to  $B^{2,s}$  and some open questions. We conclude in Section 7 with some examples using Sage [29].

## 2 Background

### 2.1 Crystals

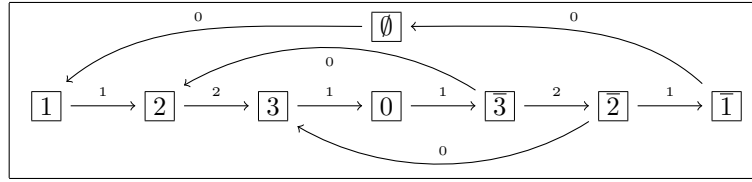
For this extended abstract, let  $\mathfrak{g}$  be the Kac–Moody algebra of type  $D_4^{(3)}$  with index set  $I = \{0, 1, 2\}$ , generalized Cartan matrix  $A = (A_{ij})_{i,j \in I}$ , weight lattice  $P$ , root lattice  $Q$ , fundamental weights  $\{\Lambda_i \mid i \in I\}$ , simple roots  $\{\alpha_i \mid i \in I\}$ , and simple coroots  $\{h_i \mid i \in I\}$ . There is a canonical pairing  $\langle \cdot, \cdot \rangle: P^\vee \times P \rightarrow \mathbb{Z}$  defined by  $\langle h_i, \alpha_j \rangle = A_{ij}$ , where  $P^\vee$  is the dual weight lattice. Let  $\mathfrak{g}_0$  denote the classical subalgebra of type  $G_2$  with index set  $I_0 = \{1, 2\}$ , weight lattice  $\bar{P}$ , root lattice  $\bar{Q}$ , fundamental weights  $\{\bar{\Lambda}_1, \bar{\Lambda}_2\}$ , and simple roots  $\{\bar{\alpha}_1, \bar{\alpha}_2\}$ .

An *abstract  $U_q(\mathfrak{g})$ -crystal* is a nonempty set  $\mathcal{B}$  together with a *weight function*  $\text{wt}: \mathcal{B} \rightarrow P$ , *crystal operators*  $e_a, f_a: \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$ , and maps  $\varepsilon_a, \varphi_a: \mathcal{B} \rightarrow \mathbb{Z} \sqcup \{-\infty\}$  for  $a \in I$ , subject to the conditions

1.  $\varphi_a(b) = \varepsilon_a(b) + \langle h_a, \text{wt}(b) \rangle$  for all  $a \in I$ ,
2. if  $e_a b \in \mathcal{B}$ , then  $\varepsilon_a(e_a b) = \varepsilon_a(b) - 1$ ,  $\varphi_a(e_a b) = \varphi_a(b) + 1$ , and  $\text{wt}(e_a b) = \text{wt}(b) + \alpha_a$ .
3. if  $f_a b \in \mathcal{B}$ , then  $\varepsilon_a(f_a b) = \varepsilon_a(b) + 1$ ,  $\varphi_a(f_a b) = \varphi_a(b) - 1$ , and  $\text{wt}(f_a b) = \text{wt}(b) - \alpha_a$ .
4.  $f_a b = b'$  if and only if  $b = e_a b'$  for  $b, b' \in \mathcal{B}$  and  $a \in I$ ,
5. if  $\varphi_a(b) = -\infty$  for  $b \in \mathcal{B}$ , then  $e_a b = f_a b = 0$ .

We define for all  $b \in \mathcal{B}$

$$\varepsilon_a(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid e_a^k b \neq 0\}, \quad \varphi_a(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid f_a^k b \neq 0\}. \quad (2.1)$$



**Fig. 2.2:** The KR crystal  $B^{1,1}$  of type  $D_4^{(3)}$  which is isomorphic to  $B(\bar{\Lambda}_1) \oplus B(0)$  as  $U_q(\mathfrak{g}_0)$ -crystals.

An abstract  $U_q(\mathfrak{g})$ -crystal with  $\varepsilon_a$  and  $\varphi_a$  defined as above is called a *regular* crystal.

Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be abstract  $U_q(\mathfrak{g})$ -crystals. The tensor product of crystals  $\mathcal{B}_2 \otimes \mathcal{B}_1$  is defined to be the Cartesian product  $\mathcal{B}_2 \times \mathcal{B}_1$  with the crystal structure

$$e_i(b_2 \otimes b_1) = \begin{cases} e_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1), \\ b_2 \otimes e_i b_1 & \text{if } \varepsilon_i(b_2) \leq \varphi_i(b_1), \end{cases} \quad \varepsilon_i(b_2 \otimes b_1) = \max(\varepsilon_i(b_2), \varepsilon_i(b_1) - \langle h_i, \text{wt}(b_2) \rangle)$$

$$f_i(b_2 \otimes b_1) = \begin{cases} f_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \\ b_2 \otimes f_i b_1 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \end{cases} \quad \varphi_i(b_2 \otimes b_1) = \max(\varphi_i(b_1), \varphi_i(b_2) + \langle h_i, \text{wt}(b_1) \rangle)$$

$$\text{wt}(b_2 \otimes b_1) = \text{wt}(b_2) + \text{wt}(b_1).$$

**Remark 2.1** Our tensor product convention is the opposite to that given in [12].

Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be two abstract  $U_q(\mathfrak{g})$ -crystals. A *crystal morphism*  $\psi: \mathcal{B}_1 \rightarrow \mathcal{B}_2$  is a map  $\mathcal{B}_1 \sqcup \{0\} \rightarrow \mathcal{B}_2 \sqcup \{0\}$  with  $\psi(0) = 0$  such that for  $b \in \mathcal{B}_1$

1. if  $\psi(b) \in \mathcal{B}_2$ , then  $\text{wt}(\psi(b)) = \text{wt}(b)$ ,  $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$ , and  $\varphi_i(\psi(b)) = \varphi_i(b)$ ;
2. we have  $\psi(e_i b) = e_i \psi(b)$  provided  $\psi(e_i b) \neq 0$  and  $e_i \psi(b) \neq 0$ ;
3. we have  $\psi(f_i b) = f_i \psi(b)$  provided  $\psi(f_i b) \neq 0$  and  $f_i \psi(b) \neq 0$ .

A crystal embedding or isomorphism is a crystal morphism such that the induced map  $\mathcal{B}_1 \sqcup \{0\} \rightarrow \mathcal{B}_2 \sqcup \{0\}$  is an embedding or bijection respectively. A crystal morphism is *strict* if it commutes with all crystal operators.

If an abstract  $U_q(\mathfrak{g})$ -crystal  $\mathcal{B}$  is isomorphic to the crystal basis of an integrable  $U_q(\mathfrak{g})$ -module, we simply say  $\mathcal{B}$  is a  $U_q(\mathfrak{g})$ -crystal. In particular, an irreducible highest weight  $U_q(\mathfrak{g}_0)$ -module with highest weight  $\lambda$  admits a crystal basis [11], which we denote by  $B(\lambda)$ . Moreover there is a unique element  $u_\lambda \in B(\lambda)$  such that  $\text{wt}(u_\lambda) = \lambda$  and  $e_a u_\lambda = 0$  for all  $a \in I_0$ . For each dominant integral weight  $\lambda = k_1 \bar{\Lambda}_1 + k_2 \bar{\Lambda}_2$ , we can associate a partition  $(k_1 + k_2, k_2)$ . We can realize  $B(\lambda)$  as semistandard tableaux of shape  $\lambda$  filled with entries in  $B(\bar{\Lambda}_1)$  whose crystal structure is given by embedding into  $B(\bar{\Lambda}_1)^{\otimes |\lambda|}$  using the reverse far-eastern reading word. The resulting tableaux were explicitly described by Kang and Misra [9].

## 2.2 Kirillov–Reshetikhin crystals

An important class of finite dimensional  $U'_q(\mathfrak{g})$ -representations are Kirillov–Reshetikhin (KR) modules  $W^{r,s}$  indexed by  $r \in I_0$  and  $s \in \mathbb{Z}_{>0}$ . KR modules are characterized by their Drinfeld polynomials [2, 3]

and correspond to the minimal affinization of  $B(s\bar{\Lambda}_r)$  [1]. The KR modules  $W^{1,s}$  admit a crystal basis called *Kirillov–Reshetikhin (KR) crystals* and denoted by  $B^{1,s}$ . As  $U_q(\mathfrak{g}_0)$ -crystals, we have  $B^{1,s} \cong \bigoplus_{k=1}^s B(k\bar{\Lambda}_1)$ , and  $B^{1,s}$  is a perfect crystal [10]. This means we can use a semi-infinite tensor product of  $B^{1,s}$  to realize highest weight  $U_q(\mathfrak{g})$ -crystals, see [7] for details.

There is a statistic called *energy* defined on  $B = \bigotimes_{i=1}^N B^{1,s_i}$  [5]. First we define the *local energy function* on  $B^{1,s} \otimes B^{1,t}$  as follows. The combinatorial  $R$ -matrix is the unique  $U'_q(\mathfrak{g})$ -crystal isomorphism  $R: B^{1,s} \otimes B^{1,t} \rightarrow B^{1,t} \otimes B^{1,s}$  [10]. Let  $c' \otimes c = R(b \otimes b')$ .

$$H(e_i(b \otimes b')) = H(b \otimes b') + \begin{cases} -1 & i = 0 \text{ and } e_0(b \otimes b') = b \otimes e_0 b' \text{ and } e_0(c' \otimes c) = c' \otimes e_0 c, \\ 1 & i = 0 \text{ and } e_0(b \otimes b') = e_0 b \otimes b' \text{ and } e_0(c' \otimes c) = e_0 c' \otimes c, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

The local energy function is defined up to an additive constant [8], and so we normalize  $H$  by the condition  $H(1^s \otimes 1^t) = 0$  where  $1^k$  is row of length  $k$  filled with 1. Next we define  $D_{B^{1,s}}: B^{1,s} \rightarrow \mathbb{Z}$  by

$$D_{B^{1,s}}(b) = H(b \otimes b^\sharp) - H(1^s \otimes b^\sharp), \quad (2.3)$$

where  $b^\sharp$  is the unique element such that  $\varphi(b^\sharp) = s\Lambda_0$ . Then we define

$$D(b_N \otimes \cdots \otimes b_1) = \sum_{1 \leq i < j \leq N} H_i R_{i+1} R_{i+2} \cdots R_{j-1} + \sum_{j=1}^N D_{B^{1,s_j}} R_1 R_2 \cdots R_{j-1}, \quad (2.4)$$

where  $R_i$  and  $H_i$  are the combinatorial  $R$ -matrix and local energy function, respectively, acting on the  $i$ -th and  $(i + 1)$ -th factors and  $D_{B^{1,s_j}}$  acts on the rightmost factor. We say the *energy* of an element  $b \in B$  is  $D(b)$ .

### 2.3 Rigged configurations

Let  $\mathcal{H}_0 = I_0 \times \mathbb{Z}_{>0}$ . Consider a multiplicity array  $L = (L_i^{(a)} \in \mathbb{Z}_{\geq 0} \mid (a, i) \in \mathcal{H}_0)$  and a dominant integral weight  $\lambda$  of  $\mathfrak{g}_0$ . A  $(L; \lambda)$ -*configuration* is a sequence of partitions  $\nu = \{\nu^{(a)} \mid a \in I\}$  such that

$$\sum_{(a,i) \in \mathcal{H}_0} i m_i^{(a)} \bar{\alpha}_a = \sum_{(a,i) \in \mathcal{H}_0} i L_i^{(a)} \bar{\Lambda}_a - \lambda, \quad (2.5)$$

where  $m_i^{(a)}$  is the number of parts of length  $i$  in the partition  $\nu^{(a)}$ . We denote the set of  $(L, \lambda)$ -configurations by  $C(L, \lambda)$ . The *vacancy numbers* of  $\nu \in C(L; \lambda)$  are defined as

$$p_i^{(a)} = \sum_{j \geq 1} \min(i, j) L_j^{(a)} - \sum_{(b,j) \in \mathcal{H}_0} A_{ab} \min(i, j) m_j^{(b)}. \quad (2.6)$$

A *rigged configuration* of classical weight  $\lambda$  is a  $(L; \lambda)$ -configuration  $\nu$ , along with a sequence of multisets of integers  $J = \{J_i^{(a)} \mid (a, i) \in \mathcal{H}_0\}$  such that  $|J_i^{(a)}| = m_i^{(a)}$  (the size of  $J_i^{(a)}$ ) and  $\max J_i^{(a)} \leq p_i^{(a)}$ . (Often each  $J_i^{(a)}$  will be sorted in weakly decreasing order.) So to each row of length  $i$ , we have an integer  $x \in J_i^{(a)}$  and we call the pair  $(i, x)$  a *string*. The integers  $x \in J_i^{(a)}$  are called *label, rigging*, or

quantum number. The colabel of a string  $(i, x)$  is defined as  $p_i^{(a)} - x$ . A rigged configuration is *highest weight* if  $\min J_i^{(a)} \geq 0$  for all  $(a, i) \in \mathcal{H}_0$  and is *valid* if  $\max J_i^{(a)} \leq p_i^{(a)}$ . We say a string  $(i, a)$  is *singular* if  $p_i^{(a)} = x$  and is *quasi-singular* if  $p_i^{(a)} = x - 1$  and  $\max J_i^{(a)} \neq p_i^{(a)}$ .

**Example 2.2** Rigged configurations will be depicted with vacancy numbers on the left and labels on the right. For example,

$$5 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline 1 & & & \\ \hline \end{array} 3 \quad -2 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} -2$$

is a rigged configuration of weight  $2\bar{\Lambda}_1 + \bar{\Lambda}_2$  with  $L$  is given by  $L_1^{(1)} = L_2^{(1)} = L_1^{(2)} = 1$  with all other  $L_i^{(a)} = 0$ . See Section 7 on how to construct this example in Sage.

Denote by  $\text{RC}^*(L; \lambda)$  the set of valid highest weight rigged configurations. Rigged configurations have an abstract  $U_q(\mathfrak{g}_0)$ -crystal structure [27]. To obtain the weight, we first note that we can compute the classical weight by

$$\overline{\text{wt}}(\nu, J) = \sum_{(a,i) \in \mathcal{H}_0} i(L_i^{(a)}\bar{\Lambda}_a - m_i^{(a)}\bar{\alpha}_a). \tag{2.7}$$

We can extend this to  $\text{wt}: \text{RC}(L; \lambda) \rightarrow P$  by  $\text{wt}(\nu, J) = k_0\Lambda_0 + \overline{\text{wt}}(\nu, J)$ , where  $k_0$  is such that  $\langle \text{wt}(\nu, J), c \rangle = 0$  with  $c$  the canonical central element of  $\mathfrak{g}$  (i.e., we make  $\text{wt}(\nu, J)$  be level 0). Explicitly, if  $\overline{\text{wt}}(\nu, J) = c_1\bar{\Lambda}_1 + c_2\bar{\Lambda}_2$ , then we have  $k_0 = -2c_1 - 3c_2$ . Next we recall the crystal operators.

**Definition 2.3** Let  $\mathfrak{g}_0$  be a Lie algebra of finite type and  $L$  a multiplicity array. Let  $(\nu, J)$  be a valid rigged configuration. Fix  $a \in I_0$  and let  $x$  be the smallest label of  $(\nu, J)^{(a)}$ , the strings associated to  $\nu^{(a)}$ .

1. If  $x \geq 0$ , then set  $e_a(\nu, J) = 0$ . Otherwise, let  $\ell$  be the minimal length of all strings in  $(\nu, J)^{(a)}$  which have label  $x$ . The rigged configuration  $e_a(\nu, J)$  is obtained by replacing the string  $(\ell, x)$  with the string  $(\ell - 1, x + 1)$  and changing all other labels so that all colabels remain fixed.
2. If  $x > 0$ , then add the string  $(1, -1)$  to  $(\nu, J)^{(a)}$ . Otherwise, let  $\ell$  be the maximal length of all strings in  $(\nu, J)^{(a)}$  which have label  $x$  and replace the string  $(\ell, x)$  by the string  $(\ell + 1, x - 1)$ . In both cases, change all other labels so that all colabels remain fixed. If the result is a valid rigged configuration, then it is  $f_a(\nu, J)$ . Otherwise  $f_a(\nu, J) = 0$ .

**Remark 2.4** The condition for highest weight rigged configurations matches with the usual crystal theoretic definition; i.e., that  $e_a(\nu, J) = 0$  for all  $(\nu, J) \in \text{RC}^*(L; \lambda)$ .

**Example 2.5** Let  $(\nu, J)$  be the rigged configuration from Example 2.2. Then

$$e_1(\nu, J) = 0, \quad e_2(\nu, J) = 2 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} 0 \quad -1 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} -1, \\ f_1(\nu, J) = 3 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} 1 \quad -1 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} -1, \quad f_2(\nu, J) = 0.$$

Let  $\text{RC}(L; \lambda)$  denote the set generated from  $\text{RC}^*(L; \lambda)$  by the crystal operators. Let  $\text{RC}(L)$  be the closure under the crystal operators of the set  $\text{RC}^*(L) = \bigsqcup_{\lambda \in P^+} \text{RC}^*(L; \lambda)$ .

**Theorem 2.6 ([27])** *Let  $\mathfrak{g}_0$  be a Lie algebra of finite type. For  $(\nu, J) \in \text{RC}^*(L; \lambda)$ , let  $X_{(\nu, J)}$  be the closure of  $(\nu, J)$  under  $e_a, f_a$  for  $a \in I_0$ . Then  $X_{(\nu, J)} \cong B(\lambda)$  as  $U_q(\mathfrak{g}_0)$ -crystals.*

There is a statistic called *cocharge* on rigged configurations given by

$$\text{cc}(\nu, J) = \frac{1}{2} \sum_{a, b \in I_0} \sum_{i, j \in \mathbb{Z}_{>0}} (\alpha_a | \alpha_b) \min(i, j) m_i^{(a)} m_j^{(b)} + \sum_{(a, i) \in \mathcal{H}_0} \sum_{x \in J_i^{(a)}} x. \tag{2.8}$$

Moreover cocharge is invariant under  $e_a$  and  $f_a$  for  $a \in I_0$  [27].

### 2.4 Virtual crystals

Let  $\widehat{\mathfrak{g}}$  be the Kac–Moody algebra with index set  $\widehat{I}$  of type  $D_4^{(1)}$  and  $\widehat{\mathfrak{g}}_0$  be of type  $D_4$ . We consider the diagram folding  $\phi: \widehat{I} \searrow I$  defined by  $\phi(0) = 0, \phi(2) = 1$ , and  $\phi(1) = \phi(3) = \phi(4) = 2$ . The folding  $\phi$  restricts to a diagram folding of type  $\widehat{\mathfrak{g}}_0 \searrow \mathfrak{g}_0$ , and by abuse of notation, we also denote this folding by  $\phi$ .

**Remark 2.7** *To simplify our notation, for any object  $X$  or  $\overline{X}$  of  $\mathfrak{g}_0$ , we denote the corresponding object of  $\widehat{\mathfrak{g}}_0$  by  $\widehat{X}$ .*

Furthermore, the folding  $\phi$  induces an embedding of weight lattices  $\Psi: \overline{P} \longrightarrow \widehat{P}$  given by

$$\overline{\lambda}_a \mapsto \sum_{b \in \phi^{-1}(a)} \widehat{\Lambda}_b, \quad \overline{\alpha}_a \mapsto \sum_{b \in \phi^{-1}(a)} \widehat{\alpha}_b. \tag{2.9}$$

This gives an embedding of crystals as sets  $v: B(\lambda) \longrightarrow B(\Psi(\lambda))$ , and let  $V(\lambda)$  denote the image of  $v$ . We can define a crystal structure on  $V$  which is induced from the crystal  $B(\Psi(\lambda))$  by

$$\begin{aligned} e^v &:= \prod_{b \in \phi^{-1}(a)} \widehat{e}_b, & f^v &:= \prod_{b \in \phi^{-1}(a)} \widehat{f}_b, \\ \varepsilon_a^v &:= \widehat{\varepsilon}_x, & \varphi_a^v &:= \widehat{\varphi}_x, \\ \text{wt} &:= \Psi^{-1} \circ \widehat{\text{wt}}, \end{aligned} \tag{2.10}$$

where we fix some  $x \in \phi^{-1}(a)$ . We say the pair  $(V(\lambda), B(\Psi(\lambda)))$  is a *virtual crystal* and the isomorphism  $v$  is the *virtualization map*.

**Proposition 2.8 ([27])** *Let  $\mathfrak{g}_0$  be of finite type. Then we have  $B(\lambda) \cong V(\lambda)$  as  $U_q(\mathfrak{g}_0)$ -crystals.*

In particular, we can define a virtualization map on rigged configurations by

$$\widehat{\nu}^{(b)} = \nu^{(a)}, \tag{2.11a}$$

$$\widehat{J}_i^{(b)} = J_i^{(a)} \tag{2.11b}$$

for all  $b \in \phi^{-1}(a)$  [27].

## 3 The bijection $\Phi$

Consider a tensor product of KR crystals  $B = \bigotimes_{i=1}^N B^{r_i, s_i}$ . We write  $\text{RC}(B)$  for  $\text{RC}(L)$  with  $L_i^{(a)}$  equal to the number of factors  $B^{a, i}$  occurring in  $B$ . In this section, we describe the map  $\Phi: \text{RC}(B) \longrightarrow B$ .

### 3.1 The basic algorithm $\delta$

We begin by describing the basic step  $\delta: \text{RC}(B^{1,1} \otimes B^*) \rightarrow \text{RC}(B^*)$ , where  $B^*$  is some tensor product of KR crystals. Each step  $\delta$  returns some element  $b \in B^{1,1}$ , which we use to create  $B$ . We note that this is the special case of the algorithm given in [17] for type  $D_4^{(3)}$ .

Set  $\ell_0 = 1$ . Do the following process for  $a = 1$ . Find the minimal integer  $i \geq \ell_{a-1}$  such that  $\nu^{(a)}$  has a singular string of length  $i$ . If no such  $i$  exists, then set  $b = a$  and  $\ell_a = \infty$  and terminate. Otherwise set  $\ell_a = i$  and repeat the above process for  $a = 2$ .

Suppose the process has not terminated. We remove the selected (singular) string of length  $\ell_1$  from consideration. If there are no singular or quasi-singular strings in  $\nu^{(a)}$  larger than  $\ell_2$  or if  $\ell_2 = \ell_1$  and there is only one string of length  $\ell_1$  in  $\nu^{(1)}$ , then set  $b = 3$  and terminate. Otherwise find the smallest  $i \geq \ell_2$  that satisfies one of the following three mutually exclusive conditions:

- (S)  $J^{(1,i)}$  is singular and  $i > 1$ ;
- (P)  $J^{(1,i)}$  is singular and  $i = 1$ ;
- (Q)  $J^{(1,i)}$  is quasi-singular.

If (P) holds, set  $b = \emptyset$ , and  $\ell_3 = i$  and terminate. If (S) holds, set  $\ell_3 = i - 1$ ,  $\bar{\ell}_3 = i$ , say case (S) holds for  $a = n$ , and continue. If (Q) holds, find the minimal  $j > i$  such that (S) holds. If no such  $j$  exists, set  $b = 0$  and terminate. Else set  $\bar{\ell}_3 = j$  and say case (Q, S) holds and continue.

Suppose the process has not terminated, and let  $a = 2$ . If  $\ell_a = \bar{\ell}_{a+1}$ , then set  $\bar{\ell}_a = \ell_a$ , afterwards reset  $\ell_a = \bar{\ell}_a - 1$ , and say case (S2) holds for  $a$ . Otherwise find the minimal index  $i \geq \bar{\ell}_{a+1}$  such that  $\nu^{(a)}$  has a singular string of length  $i$ . If no such  $i$  exists, set  $b = \bar{a} + \bar{1}$  and terminate. Otherwise set  $\bar{\ell}_a = i$  and repeat this for  $a = 1$  (there must exist at least two singular strings if  $\bar{\ell}_3 = \bar{\ell}_1$  and case (S2) does not hold). If the process has not terminated, set  $b = \bar{1}$ .

Set all undefined  $\ell_a$  and  $\bar{\ell}_a$  for  $a = 1, 2, 3$  to  $\infty$ .

### 3.2 Change in the rigged configuration

The rigged configurations change under  $\delta$  as follows. We first remove a box from  $\bar{\ell}_a$  in  $\nu^{(a)}$  for  $a = 1, 2$ , and if case (S2) holds for  $a$ , we remove another box from that particular row, otherwise we remove a box from  $\ell_a$ . If case (S) holds, then remove two boxes from  $\bar{\ell}_3$  and make the resulting string singular. If case (Q) holds, remove a box from  $\ell_3$  and make the resulting string singular. If case (Q, S) holds, then we remove both boxes corresponding to  $\ell_3$  and  $\bar{\ell}_3$ , but we make the smaller one (i.e. the row corresponding to  $\ell_3$ ) singular and the larger one quasi-singular. Also make all the changed strings in  $\nu^{(2)}$  singular.

**Remark 3.1** We can determine the inverse algorithm by roughly doing the opposite of the above; in particular, selecting largest (quasi)singular strings at most as long as before.

**Example 3.2** Using the rigged configuration  $(\nu, J)$  from Example 2.2 and  $B = B^{1,1} \otimes B^{1,2} \otimes B^{2,1}$ . Applying the map  $\delta$ , we get  $b = 3$  and

$$\delta(\nu, J) = 3 \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} 3 \quad -2 \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} -2$$



### 3.3 Extending to arbitrary rectangles

We now extend  $\Phi$  to  $B = \otimes_{i=1}^N B^{1,s_i}$  by defining the map

$$\text{ls}: \text{RC}(B^{1,s} \otimes B^*) \longrightarrow \text{RC}(B^{1,1} \otimes B^{1,s-1} \otimes B^*),$$

which is known as *left-split*. On the rigged configurations, the map  $\text{ls}$  is the identity (but perhaps increases the vacancy numbers) and a strict crystal embedding. Thus iterating  $\text{ls}$  with  $\delta$ , we obtain a map  $\Phi: \text{RC}(B) \longrightarrow B$ .

## 4 Filling map

We determine the highest weight rigged configurations for  $B^{1,s}$  by using the virtual Kleber algorithm [22].

**Lemma 4.1** *Consider the KR crystal  $B^{1,s}$ . We have  $\text{RC}(B^{1,s}) = \bigoplus_{k=0}^s \text{RC}(B^{1,s}; k\bar{\Lambda}_1)$ . Moreover the highest weight rigged configurations in  $\text{RC}(B^{1,s}; k\bar{\Lambda}_1)$  are given by  $\nu^{(1)} = (s - k, s - k)$  and  $\nu^{(2)} = (s - k)$  with all labels 0.*

From Lemma 4.1 and the  $U_q(\mathfrak{g}_0)$ -crystal decomposition of  $B^{1,s}$  is multiplicity free, there exists a natural  $U_q(\mathfrak{g}_0)$ -crystal isomorphism  $\iota: \text{RC}(B^{1,s}) \longrightarrow B^{1,s}$ . For type  $D_4^{(3)}$ , we note that  $k\bar{\Lambda}_1$  can be considered as the partition  $(k)$ .

**Definition 4.2** *Let  $B^{1,s}$  be a KR crystal of type  $D_4^{(3)}$  and consider the classical component  $B(k\bar{\Lambda}_1) \subseteq B^{1,s}$ . The filling map  $\text{fill}: B^{1,s} \longrightarrow (B^{1,1})^{\otimes s}$  is given by adding  $\lfloor \frac{s-k}{2} \rfloor$  copies of the horizontal domino  $\begin{bmatrix} \bar{1} & 1 \end{bmatrix}$  and an additional  $\begin{bmatrix} \emptyset \end{bmatrix}$  if  $s - k$  is odd.*

Let  $T^{1,s}$  denote the image of  $B^{1,s}$  under  $\text{fill}$  written as a  $1 \times s$  rectangle. We note that  $T^{1,s}$  inherits a classical crystal structure from  $(B^{1,1})^{\otimes s}$ .

**Example 4.3** *Consider the element*

$$b = \begin{bmatrix} 3 & 0 & \bar{2} & \bar{2} & \bar{1} \end{bmatrix} \in B(5\bar{\Lambda}_1) \subseteq B^{1,9},$$

then we have

$$\text{fill}(b) = \begin{bmatrix} 3 & 0 & \bar{2} & \bar{2} & \bar{1} & \bar{1} & 1 & \bar{1} & 1 \end{bmatrix}.$$

Now suppose  $b \in B^{1,8}$ , then we have

$$\text{fill}(b) = \begin{bmatrix} 3 & 0 & \bar{2} & \bar{2} & \bar{1} & \bar{1} & 1 & \emptyset \end{bmatrix}.$$

We give a  $U'_q(\mathfrak{g})$ -crystal structure to  $T^{1,s}$  by following [10, 30] as the conditions for  $e_0$  and  $f_0$  are preserved under the filling map.

**Proposition 4.4** *The filling map  $\text{fill}: B^{1,s} \rightarrow T^{1,s}$  given in Definition 4.2 is a  $U'_q(\mathfrak{g})$ -crystal isomorphism.*

We also can show the following.

**Proposition 4.5** *Let  $B = B^{1,s}$ . Then  $\Phi = \text{fill} \circ \iota$  with  $\text{fill}$  as in Definition 4.2 on highest weight elements.*

## 5 Virtualization Map

**Lemma 5.1** *The virtualization map  $v: B^{1,s} \rightarrow B^{2,s}$  for types  $D_4^{(3)} \hookrightarrow D_4^{(1)}$  is given column-by-column by*

$$\begin{array}{cccc}
 \boxed{1} \mapsto \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} & \boxed{2} \mapsto \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} & \boxed{3} \mapsto \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} & \boxed{0} \mapsto \begin{array}{|c|} \hline 3 \\ \hline 3 \\ \hline \end{array} \\
 \boxed{3} \mapsto \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline \end{array} & \boxed{2} \mapsto \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline \end{array} & \boxed{1} \mapsto \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} & \boxed{\emptyset} \mapsto \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array}
 \end{array}$$

Using Lemma 5.1 and the analogue of  $\Phi$  in type  $D_4^{(1)}$  [18, 25], we can show the following.

**Theorem 5.2** *Consider a tensor product of KR crystals  $B = \bigotimes_{i=1}^N B^{1,s_i}$  of type  $D_4^{(3)}$ . The virtualization map  $v$  commutes with the map  $\Phi$ .*

We need to define the *complement rigging map*  $\theta: \text{RC}(B) \rightarrow \text{RC}(B_r)$  by sending  $(\nu, J) \mapsto (\nu, J')$ , where  $J'$  is obtained by  $x' = p_i^{(a)} - x$  for all labels  $x$  and  $B_r$  are the factors of  $B$  in reverse order. That is to say  $\theta$  maps each label  $x$  to its colabel. We can define  $\tilde{\delta} := \theta \circ \delta \circ \theta$ , and using the virtualization map, Proposition 4.5, and the results of [25], we can show the following.

**Lemma 5.3** *We have  $\delta \circ \tilde{\delta} = \tilde{\delta} \circ \delta$ .*

Using the results on the combinatorial  $R$ -matrix in [30], we can show the following.

**Lemma 5.4** *Consider  $B = B^{1,s} \otimes B^{1,1}$ . We have  $\Phi^{-1} \circ R \circ \Phi$  is the identity map on  $\text{RC}(B)$ .*

Then following [28, Sec. 8], the map  $rs := \theta \circ ls \circ \theta$  preserves statistics using [30]. From Lemma 5.4, the  $R$ -matrix preserves statistics. Thus iterating  $rs$  and  $R$ -matrices, we preserve statistics to  $\bigotimes_{i=1}^{N'} B^{1,1}$ . Then we use the results of [23, 25] and Theorem 5.2 to obtain our main result.

**Theorem 5.5** *Let  $B = \bigotimes_{i=1}^N B^{1,s_i}$  of type  $D_4^{(3)}$ . The map  $\Phi: \text{RC}(B) \rightarrow B$  is a  $U_q(\mathfrak{g}_0)$ -crystal isomorphism and  $\Phi \circ \theta$  sends cocharge to energy.*

From Proposition 4.4, Lemma 5.1, Theorem 5.5, and the filling map for type  $D_n^{(1)}$  given in [18], we can show the following.

**Theorem 5.6** *Let  $B = B^{1,s}$ . Then  $\Phi = \text{fill} \circ \iota$  with  $\text{fill}$  as in Definition 4.2 as  $U_q(\mathfrak{g}_0)$ -crystal morphisms.*

Thus we can define a  $U'_q(\mathfrak{g})$ -crystal structure on  $\text{RC}(B)$  by extending  $\Phi$  to be a  $U'_q(\mathfrak{g})$ -crystal isomorphism. Thus we have a special case in type  $D_4^{(3)}$  of the conjectures given in [27].

## 6 Extensions and questions

The  $U_q(\mathfrak{g}_0)$ -crystal decomposition of  $B^{2,s}$  and the highest weight rigged configurations will appear in the full version of this work. The author hopes to use this to determine the filling map for  $B^{2,s}$ .

There is a map  $lt: \text{RC}(B^{2,1} \otimes B^*) \rightarrow \text{RC}(B^{1,1} \otimes B^{1,1} \otimes B^*)$  called *left-top* which adds a singular string of length 1 to  $\nu^{(1)}$ . In the full version, this is used to extend the  $U_q(\mathfrak{g}_0)$ -crystal isomorphism  $\Phi$  to tensor products also containing  $B^{2,1}$ .

**Example 6.1** *Continuing from Example 3.2, we obtain*

$$\Phi(\nu, J) = \boxed{3} \otimes \boxed{2 \ 3} \otimes \boxed{\frac{1}{2}}.$$

The computations for the Kleber algorithm can be modified to determine the  $U_q(\mathfrak{g}_0)$ -crystal decomposition of  $B^{r,s}$  of type  $G_2^{(1)}$ . However there is a difficulty with determining what the map  $\delta$  should be. This would need to be overcome to define the filling map for type  $G_2^{(1)}$ .

There is a conjecture [22, Conj. 3.7] that we can realize  $B^{1,s}$  of type  $D_4^{(3)}$  as a virtual crystal in  $B^{1,s}$  of type  $D_4^{(1)}$ . Therefore obtaining a direct description of  $e_0$  and  $f_0$  on rigged configurations could lead to an answer to this conjecture using the results of [18]. The author hopes to have this description and prove this conjecture in this special case in the full version of this work.

## 7 Examples using Sage

The bijection  $\Phi$  and the rigged configurations have been implemented by the author in Sage [29]. We begin by setting up the Sage environment to give a more concise printing.

```
sage: RiggedConfigurations.global_options(display="horizontal")
```

We construct our the rigged configuration from Example 2.2 by specifying the partitions and corresponding labels.

```
sage: nu = RC(partition_list=[[4,1], [4]], rigging_list=[[3,1], [-2]]); nu
5[ ][ ][ ][ ] 3 -2[ ][ ][ ][ ] -2
1[ ] 1
```

We apply the full bijection and print the output using Sage's ASCII art.

```
sage: ascii_art(nu.to_tensor_product_of_kirillov_reshetikhin_tableaux())
3 # 2 3 # 1
-2
```

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