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Rigged configurations of type $D_4^{(3)}$ and the filling map

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Abstract. We give a statistic preserving bijection from rigged configurations to a tensor product of Kirillov–Reshetikhin crystals $\otimes_{i=1}^N B^{1,s_i}$ in type $D_4^{(3)}$ by using virtualization into type $D_4^{(1)}$. We consider a special case of this bijection with $B = B^{1,s}$, and we obtain the so-called Kirillov–Reshetikhin tableaux model for the Kirillov–Reshetikhin crystal.

Résumé. Nous donnons une bijection préservant les statistiques entre les configurations grées et les produits tensoriels de cristaux de Kirillov–Reshetikhin $\otimes_{i=1}^N B^{1,s_i}$ de type $D_4^{(3)}$, via une virtualisation en type $D_4^{(1)}$. Nous considérons un cas particulier de cette bijection pour $B = B^{1,s}$ et obtenons ainsi les modèles de tableaux appelés Kirillov–Reshetikhin pour le cristal Kirillov–Reshetikhin.

Keywords: rigged configuration, Kirillov–Reshetikhin crystal, bijection

1 Introduction

Rigged configurations were first introduced by Kerov, Kirillov, and Reshetikhin in [14, 15] as combinatorial objects that index solutions to the Bethe Ansatz for the Heisenberg spin chains. Rigged configurations were shown to be in bijection with semi-standard tableaux and classical highest weight elements of a tensor power of the vector representation in type $A_n^{(1)}$. This bijection was then extended to Littlewood–Richardson tableaux [16], to non-exceptional types [20], and to type $E_6^{(1)}$ [19]. This bijection Φ between rigged configurations and the tensor powers has been further expanded to include classically highest weight elements in a tensor product of certain Kirillov–Reshetikhin (KR) crystals [16, 21, 18, 27, 28].

Rigged configurations have been shown to display remarkable representation theoretic properties. A (classical) crystal structure was first given for simply-laced types [26], which was then extended to all finite types [27] and affine types [24]. While Φ is defined recursively, making it difficult to work with, it preserves certain natural statistics (cocharge and energy), giving a bijective proof of the $X = M$ conjecture of [5, 6]. Furthermore, the combinatorial R -matrix transforms into the identity map on rigged configurations under Φ . Rigged configurations are also well-behaved under virtualization [21, 22, 27], a process of realizing a non-simply-laced type crystal inside of a simply-laced type, and embeddings $B(\lambda) \hookrightarrow B(\mu)$ where $\lambda \leq \mu$ component-wise, leading to a model for $B(\infty)$ [24].

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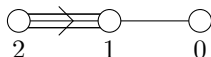


Fig. 2.1: Dynkin diagram of type $D_4^{(3)}$.

KR crystals in non-exceptional types were given a combinatorial model in [4] using Kashiwara–Nakashima tableaux [13]. The bijection Φ has also lead to a new tableaux model for KR crystals, coined Kirillov–Reshetikhin (KR) tableaux, using filled rectangular tableaux [18, 25, 27]. The map between Kashiwara–Nakashima tableaux [13] and the KR tableaux is called the filling map.

The goal of this work is to extend Φ to type $D_4^{(3)}$ and describe the filling map. For this extended abstract, we will be focusing on the KR crystals $B^{1,s}$ and the rigged configurations associated with tensor products of the form $\bigotimes_{i=1}^N B^{1,s_i}$. In particular, we show Φ is a classical crystal isomorphism, and we describe the filling map for $B^{1,s}$. We do so by showing the filling map and bijection commute with the virtualization map, proving more special cases of many of the conjectures stated in [27].

This extended abstract is organized as follows. In Section 2, we give background on crystals, virtualization, and rigged configurations. In Section 3, we describe the bijection Φ . In Section 4, we describe the filling map. In Section 5, we describe the virtualization map and our main results. In Section 6, we give possible extensions to $B^{2,s}$ and some open questions. We conclude in Section 7 with some examples using Sage [29].

2 Background

2.1 Crystals

For this extended abstract, let \mathfrak{g} be the Kac–Moody algebra of type $D_4^{(3)}$ with index set $I = \{0, 1, 2\}$, generalized Cartan matrix $A = (A_{ij})_{i,j \in I}$, weight lattice P , root lattice Q , fundamental weights $\{\Lambda_i \mid i \in I\}$, simple roots $\{\alpha_i \mid i \in I\}$, and simple coroots $\{h_i \mid i \in I\}$. There is a canonical pairing $\langle \cdot, \cdot \rangle: P^\vee \times P \rightarrow \mathbb{Z}$ defined by $\langle h_i, \alpha_j \rangle = A_{ij}$, where P^\vee is the dual weight lattice. Let \mathfrak{g}_0 denote the classical subalgebra of type G_2 with index set $I_0 = \{1, 2\}$, weight lattice \bar{P} , root lattice \bar{Q} , fundamental weights $\{\bar{\Lambda}_1, \bar{\Lambda}_2\}$, and simple roots $\{\bar{\alpha}_1, \bar{\alpha}_2\}$.

An *abstract $U_q(\mathfrak{g})$ -crystal* is a nonempty set \mathcal{B} together with a *weight function* $\text{wt}: \mathcal{B} \rightarrow P$, *crystal operators* $e_a, f_a: \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$, and maps $\varepsilon_a, \varphi_a: \mathcal{B} \rightarrow \mathbb{Z} \sqcup \{-\infty\}$ for $a \in I$, subject to the conditions

1. $\varphi_a(b) = \varepsilon_a(b) + \langle h_a, \text{wt}(b) \rangle$ for all $a \in I$,
2. if $e_a b \in \mathcal{B}$, then $\varepsilon_a(e_a b) = \varepsilon_a(b) - 1$, $\varphi_a(e_a b) = \varphi_a(b) + 1$, and $\text{wt}(e_a b) = \text{wt}(b) + \alpha_a$.
3. if $f_a b \in \mathcal{B}$, then $\varepsilon_a(f_a b) = \varepsilon_a(b) + 1$, $\varphi_a(f_a b) = \varphi_a(b) - 1$, and $\text{wt}(f_a b) = \text{wt}(b) - \alpha_a$.
4. $f_a b = b'$ if and only if $b = e_a b'$ for $b, b' \in \mathcal{B}$ and $a \in I$,
5. if $\varphi_a(b) = -\infty$ for $b \in \mathcal{B}$, then $e_a b = f_a b = 0$.

We define for all $b \in \mathcal{B}$

$$\varepsilon_a(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid e_a^k b \neq 0\}, \quad \varphi_a(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid f_a^k b \neq 0\}. \quad (2.1)$$

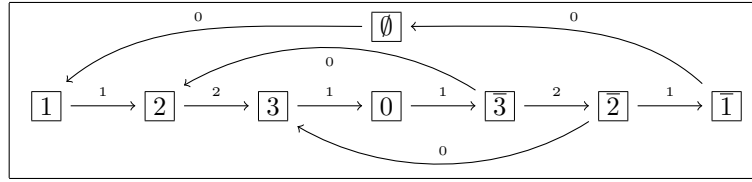


Fig. 2.2: The KR crystal $B^{1,1}$ of type $D_4^{(3)}$ which is isomorphic to $B(\overline{\Lambda}_1) \oplus B(0)$ as $U_q(\mathfrak{g}_0)$ -crystals.

An abstract $U_q(\mathfrak{g})$ -crystal with ε_a and φ_a defined as above is called a *regular* crystal.

Let \mathcal{B}_1 and \mathcal{B}_2 be abstract $U_q(\mathfrak{g})$ -crystals. The tensor product of crystals $\mathcal{B}_2 \otimes \mathcal{B}_1$ is defined to be the Cartesian product $\mathcal{B}_2 \times \mathcal{B}_1$ with the crystal structure

$$e_i(b_2 \otimes b_1) = \begin{cases} e_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1), \\ b_2 \otimes e_i b_1 & \text{if } \varepsilon_i(b_2) \leq \varphi_i(b_1), \end{cases} \quad \varepsilon_i(b_2 \otimes b_1) = \max(\varepsilon_i(b_2), \varepsilon_i(b_1) - \langle h_i, \text{wt}(b_2) \rangle)$$

$$f_i(b_2 \otimes b_1) = \begin{cases} f_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \\ b_2 \otimes f_i b_1 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \end{cases} \quad \varphi_i(b_2 \otimes b_1) = \max(\varphi_i(b_1), \varphi_i(b_2) + \langle h_i, \text{wt}(b_1) \rangle)$$

$$\text{wt}(b_2 \otimes b_1) = \text{wt}(b_2) + \text{wt}(b_1).$$

Remark 2.1 Our tensor product convention is the opposite to that given in [12].

Let \mathcal{B}_1 and \mathcal{B}_2 be two abstract $U_q(\mathfrak{g})$ -crystals. A *crystal morphism* $\psi: \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a map $\mathcal{B}_1 \sqcup \{0\} \rightarrow \mathcal{B}_2 \sqcup \{0\}$ with $\psi(0) = 0$ such that for $b \in \mathcal{B}_1$

1. if $\psi(b) \in \mathcal{B}_2$, then $\text{wt}(\psi(b)) = \text{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$;
2. we have $\psi(e_i b) = e_i \psi(b)$ provided $\psi(e_i b) \neq 0$ and $e_i \psi(b) \neq 0$;
3. we have $\psi(f_i b) = f_i \psi(b)$ provided $\psi(f_i b) \neq 0$ and $f_i \psi(b) \neq 0$.

A crystal embedding or isomorphism is a crystal morphism such that the induced map $\mathcal{B}_1 \sqcup \{0\} \rightarrow \mathcal{B}_2 \sqcup \{0\}$ is an embedding or bijection respectively. A crystal morphism is *strict* if it commutes with all crystal operators.

If an abstract $U_q(\mathfrak{g})$ -crystal \mathcal{B} is isomorphic to the crystal basis of an integrable $U_q(\mathfrak{g})$ -module, we simply say \mathcal{B} is a $U_q(\mathfrak{g})$ -crystal. In particular, an irreducible highest weight $U_q(\mathfrak{g}_0)$ -module with highest weight λ admits a crystal basis [11], which we denote by $B(\lambda)$. Moreover there is a unique element $u_\lambda \in B(\lambda)$ such that $\text{wt}(u_\lambda) = \lambda$ and $e_a u_\lambda = 0$ for all $a \in I_0$. For each dominant integral weight $\lambda = k_1 \overline{\Lambda}_1 + k_2 \overline{\Lambda}_2$, we can associate a partition $(k_1 + k_2, k_2)$. We can realize $B(\lambda)$ as semistandard tableaux of shape λ filled with entries in $B(\overline{\Lambda}_1)$ whose crystal structure is given by embedding into $B(\overline{\Lambda}_1)^{\otimes |\lambda|}$ using the reverse far-eastern reading word. The resulting tableaux were explicitly described by Kang and Misra [9].

2.2 Kirillov–Reshetikhin crystals

An important class of finite dimensional $U'_q(\mathfrak{g})$ -representations are Kirillov–Reshetikhin (KR) modules $W^{r,s}$ indexed by $r \in I_0$ and $s \in \mathbb{Z}_{>0}$. KR modules are characterized by their Drinfeld polynomials [2, 3]

and correspond to the minimal affinization of $B(s\bar{\Lambda}_r)$ [1]. The KR modules $W^{1,s}$ admit a crystal basis called *Kirillov–Reshetikhin (KR) crystals* and denoted by $B^{1,s}$. As $U_q(\mathfrak{g}_0)$ -crystals, we have $B^{1,s} \cong \bigoplus_{k=1}^s B(k\bar{\Lambda}_1)$, and $B^{1,s}$ is a perfect crystal [10]. This means we can use a semi-infinite tensor product of $B^{1,s}$ to realize highest weight $U_q(\mathfrak{g})$ -crystals, see [7] for details.

There is a statistic called *energy* defined on $B = \bigotimes_{i=1}^N B^{1,s_i}$ [5]. First we define the *local energy function* on $B^{1,s} \otimes B^{1,t}$ as follows. The combinatorial R -matrix is the unique $U'_q(\mathfrak{g})$ -crystal isomorphism $R: B^{1,s} \otimes B^{1,t} \rightarrow B^{1,t} \otimes B^{1,s}$ [10]. Let $c' \otimes c = R(b \otimes b')$.

$$H(e_i(b \otimes b')) = H(b \otimes b') + \begin{cases} -1 & i = 0 \text{ and } e_0(b \otimes b') = b \otimes e_0 b' \text{ and } e_0(c' \otimes c) = c' \otimes e_0 c, \\ 1 & i = 0 \text{ and } e_0(b \otimes b') = e_0 b \otimes b' \text{ and } e_0(c' \otimes c) = e_0 c' \otimes c, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

The local energy function is defined up to an additive constant [8], and so we normalize H by the condition $H(1^s \otimes 1^t) = 0$ where 1^k is row of length k filled with 1. Next we define $D_{B^{1,s}}: B^{1,s} \rightarrow \mathbb{Z}$ by

$$D_{B^{1,s}}(b) = H(b \otimes b^\sharp) - H(1^s \otimes b^\sharp), \quad (2.3)$$

where b^\sharp is the unique element such that $\varphi(b^\sharp) = s\Lambda_0$. Then we define

$$D(b_N \otimes \cdots \otimes b_1) = \sum_{1 \leq i < j \leq N} H_i R_{i+1} R_{i+2} \cdots R_{j-1} + \sum_{j=1}^N D_{B^{1,s_j}} R_1 R_2 \cdots R_{j-1}, \quad (2.4)$$

where R_i and H_i are the combinatorial R -matrix and local energy function, respectively, acting on the i -th and $(i + 1)$ -th factors and $D_{B^{1,s_j}}$ acts on the rightmost factor. We say the *energy* of an element $b \in B$ is $D(b)$.

2.3 Rigged configurations

Let $\mathcal{H}_0 = I_0 \times \mathbb{Z}_{>0}$. Consider a multiplicity array $L = (L_i^{(a)} \in \mathbb{Z}_{\geq 0} \mid (a, i) \in \mathcal{H}_0)$ and a dominant integral weight λ of \mathfrak{g}_0 . A $(L; \lambda)$ -*configuration* is a sequence of partitions $\nu = \{\nu^{(a)} \mid a \in I\}$ such that

$$\sum_{(a,i) \in \mathcal{H}_0} i m_i^{(a)} \bar{\alpha}_a = \sum_{(a,i) \in \mathcal{H}_0} i L_i^{(a)} \bar{\Lambda}_a - \lambda, \quad (2.5)$$

where $m_i^{(a)}$ is the number of parts of length i in the partition $\nu^{(a)}$. We denote the set of (L, λ) -configurations by $C(L, \lambda)$. The *vacancy numbers* of $\nu \in C(L; \lambda)$ are defined as

$$p_i^{(a)} = \sum_{j \geq 1} \min(i, j) L_j^{(a)} - \sum_{(b,j) \in \mathcal{H}_0} A_{ab} \min(i, j) m_j^{(b)}. \quad (2.6)$$

A *rigged configuration* of classical weight λ is a $(L; \lambda)$ -configuration ν , along with a sequence of multisets of integers $J = \{J_i^{(a)} \mid (a, i) \in \mathcal{H}_0\}$ such that $|J_i^{(a)}| = m_i^{(a)}$ (the size of $J_i^{(a)}$) and $\max J_i^{(a)} \leq p_i^{(a)}$. (Often each $J_i^{(a)}$ will be sorted in weakly decreasing order.) So to each row of length i , we have an integer $x \in J_i^{(a)}$ and we call the pair (i, x) a *string*. The integers $x \in J_i^{(a)}$ are called *label, rigging*, or

quantum number. The colabel of a string (i, x) is defined as $p_i^{(a)} - x$. A rigged configuration is *highest weight* if $\min J_i^{(a)} \geq 0$ for all $(a, i) \in \mathcal{H}_0$ and is *valid* if $\max J_i^{(a)} \leq p_i^{(a)}$. We say a string (i, a) is *singular* if $p_i^{(a)} = x$ and is *quasi-singular* if $p_i^{(a)} = x - 1$ and $\max J_i^{(a)} \neq p_i^{(a)}$.

Example 2.2 Rigged configurations will be depicted with vacancy numbers on the left and labels on the right. For example,

$$5 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline 1 & & & \\ \hline \end{array} 3 \quad -2 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} -2$$

is a rigged configuration of weight $2\bar{\Lambda}_1 + \bar{\Lambda}_2$ with L is given by $L_1^{(1)} = L_2^{(1)} = L_1^{(2)} = 1$ with all other $L_i^{(a)} = 0$. See Section 7 on how to construct this example in Sage.

Denote by $\text{RC}^*(L; \lambda)$ the set of valid highest weight rigged configurations. Rigged configurations have an abstract $U_q(\mathfrak{g}_0)$ -crystal structure [27]. To obtain the weight, we first note that we can compute the classical weight by

$$\overline{\text{wt}}(\nu, J) = \sum_{(a,i) \in \mathcal{H}_0} i(L_i^{(a)}\bar{\Lambda}_a - m_i^{(a)}\bar{\alpha}_a). \tag{2.7}$$

We can extend this to $\text{wt}: \text{RC}(L; \lambda) \rightarrow P$ by $\text{wt}(\nu, J) = k_0\Lambda_0 + \overline{\text{wt}}(\nu, J)$, where k_0 is such that $\langle \text{wt}(\nu, J), c \rangle = 0$ with c the canonical central element of \mathfrak{g} (i.e., we make $\text{wt}(\nu, J)$ be level 0). Explicitly, if $\overline{\text{wt}}(\nu, J) = c_1\bar{\Lambda}_1 + c_2\bar{\Lambda}_2$, then we have $k_0 = -2c_1 - 3c_2$. Next we recall the crystal operators.

Definition 2.3 Let \mathfrak{g}_0 be a Lie algebra of finite type and L a multiplicity array. Let (ν, J) be a valid rigged configuration. Fix $a \in I_0$ and let x be the smallest label of $(\nu, J)^{(a)}$, the strings associated to $\nu^{(a)}$.

1. If $x \geq 0$, then set $e_a(\nu, J) = 0$. Otherwise, let ℓ be the minimal length of all strings in $(\nu, J)^{(a)}$ which have label x . The rigged configuration $e_a(\nu, J)$ is obtained by replacing the string (ℓ, x) with the string $(\ell - 1, x + 1)$ and changing all other labels so that all colabels remain fixed.
2. If $x > 0$, then add the string $(1, -1)$ to $(\nu, J)^{(a)}$. Otherwise, let ℓ be the maximal length of all strings in $(\nu, J)^{(a)}$ which have label x and replace the string (ℓ, x) by the string $(\ell + 1, x - 1)$. In both cases, change all other labels so that all colabels remain fixed. If the result is a valid rigged configuration, then it is $f_a(\nu, J)$. Otherwise $f_a(\nu, J) = 0$.

Remark 2.4 The condition for highest weight rigged configurations matches with the usual crystal theoretic definition; i.e., that $e_a(\nu, J) = 0$ for all $(\nu, J) \in \text{RC}^*(L; \lambda)$.

Example 2.5 Let (ν, J) be the rigged configuration from Example 2.2. Then

$$e_1(\nu, J) = 0, \quad e_2(\nu, J) = 2 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline 1 & & & \\ \hline \end{array} 0 \quad -1 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} -1,$$

$$f_1(\nu, J) = 3 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline -1 & & & -1 \\ \hline -1 & & & -1 \\ \hline \end{array} 1 \quad -1 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} -1, \quad f_2(\nu, J) = 0.$$

Let $\text{RC}(L; \lambda)$ denote the set generated from $\text{RC}^*(L; \lambda)$ by the crystal operators. Let $\text{RC}(L)$ be the closure under the crystal operators of the set $\text{RC}^*(L) = \bigsqcup_{\lambda \in P^+} \text{RC}^*(L; \lambda)$.

Theorem 2.6 ([27]) *Let \mathfrak{g}_0 be a Lie algebra of finite type. For $(\nu, J) \in \text{RC}^*(L; \lambda)$, let $X_{(\nu, J)}$ be the closure of (ν, J) under e_a, f_a for $a \in I_0$. Then $X_{(\nu, J)} \cong B(\lambda)$ as $U_q(\mathfrak{g}_0)$ -crystals.*

There is a statistic called *cocharge* on rigged configurations given by

$$\text{cc}(\nu, J) = \frac{1}{2} \sum_{a, b \in I_0} \sum_{i, j \in \mathbb{Z}_{>0}} (\alpha_a | \alpha_b) \min(i, j) m_i^{(a)} m_j^{(b)} + \sum_{(a, i) \in \mathcal{H}_0} \sum_{x \in J_i^{(a)}} x. \tag{2.8}$$

Moreover cocharge is invariant under e_a and f_a for $a \in I_0$ [27].

2.4 Virtual crystals

Let $\widehat{\mathfrak{g}}$ be the Kac–Moody algebra with index set \widehat{I} of type $D_4^{(1)}$ and $\widehat{\mathfrak{g}}_0$ be of type D_4 . We consider the diagram folding $\phi: \widehat{I} \searrow I$ defined by $\phi(0) = 0, \phi(2) = 1$, and $\phi(1) = \phi(3) = \phi(4) = 2$. The folding ϕ restricts to a diagram folding of type $\widehat{\mathfrak{g}}_0 \searrow \mathfrak{g}_0$, and by abuse of notation, we also denote this folding by ϕ .

Remark 2.7 *To simplify our notation, for any object X or \overline{X} of \mathfrak{g}_0 , we denote the corresponding object of $\widehat{\mathfrak{g}}_0$ by \widehat{X} .*

Furthermore, the folding ϕ induces an embedding of weight lattices $\Psi: \overline{P} \longrightarrow \widehat{P}$ given by

$$\overline{\Lambda}_a \mapsto \sum_{b \in \phi^{-1}(a)} \widehat{\Lambda}_b, \quad \overline{\alpha}_a \mapsto \sum_{b \in \phi^{-1}(a)} \widehat{\alpha}_b. \tag{2.9}$$

This gives an embedding of crystals as sets $v: B(\lambda) \longrightarrow B(\Psi(\lambda))$, and let $V(\lambda)$ denote the image of v . We can define a crystal structure on V which is induced from the crystal $B(\Psi(\lambda))$ by

$$\begin{aligned} e^v &:= \prod_{b \in \phi^{-1}(a)} \widehat{e}_b, & f^v &:= \prod_{b \in \phi^{-1}(a)} \widehat{f}_b, \\ \varepsilon_a^v &:= \widehat{\varepsilon}_x, & \varphi_a^v &:= \widehat{\varphi}_x, \\ \text{wt} &:= \Psi^{-1} \circ \widehat{\text{wt}}, \end{aligned} \tag{2.10}$$

where we fix some $x \in \phi^{-1}(a)$. We say the pair $(V(\lambda), B(\Psi(\lambda)))$ is a *virtual crystal* and the isomorphism v is the *virtualization map*.

Proposition 2.8 ([27]) *Let \mathfrak{g}_0 be of finite type. Then we have $B(\lambda) \cong V(\lambda)$ as $U_q(\mathfrak{g}_0)$ -crystals.*

In particular, we can define a virtualization map on rigged configurations by

$$\widehat{\nu}^{(b)} = \nu^{(a)}, \tag{2.11a}$$

$$\widehat{J}_i^{(b)} = J_i^{(a)} \tag{2.11b}$$

for all $b \in \phi^{-1}(a)$ [27].

3 The bijection Φ

Consider a tensor product of KR crystals $B = \bigotimes_{i=1}^N B^{r_i, s_i}$. We write $\text{RC}(B)$ for $\text{RC}(L)$ with $L_i^{(a)}$ equal to the number of factors $B^{a, i}$ occurring in B . In this section, we describe the map $\Phi: \text{RC}(B) \longrightarrow B$.

3.1 The basic algorithm δ

We begin by describing the basic step $\delta: \text{RC}(B^{1,1} \otimes B^*) \rightarrow \text{RC}(B^*)$, where B^* is some tensor product of KR crystals. Each step δ returns some element $b \in B^{1,1}$, which we use to create B . We note that this is the special case of the algorithm given in [17] for type $D_4^{(3)}$.

Set $\ell_0 = 1$. Do the following process for $a = 1$. Find the minimal integer $i \geq \ell_{a-1}$ such that $\nu^{(a)}$ has a singular string of length i . If no such i exists, then set $b = a$ and $\ell_a = \infty$ and terminate. Otherwise set $\ell_a = i$ and repeat the above process for $a = 2$.

Suppose the process has not terminated. We remove the selected (singular) string of length ℓ_1 from consideration. If there are no singular or quasi-singular strings in $\nu^{(a)}$ larger than ℓ_2 or if $\ell_2 = \ell_1$ and there is only one string of length ℓ_1 in $\nu^{(1)}$, then set $b = 3$ and terminate. Otherwise find the smallest $i \geq \ell_2$ that satisfies one of the following three mutually exclusive conditions:

- (S) $J^{(1,i)}$ is singular and $i > 1$;
- (P) $J^{(1,i)}$ is singular and $i = 1$;
- (Q) $J^{(1,i)}$ is quasi-singular.

If (P) holds, set $b = \emptyset$, and $\ell_3 = i$ and terminate. If (S) holds, set $\ell_3 = i - 1$, $\bar{\ell}_3 = i$, say case (S) holds for $a = n$, and continue. If (Q) holds, find the minimal $j > i$ such that (S) holds. If no such j exists, set $b = 0$ and terminate. Else set $\bar{\ell}_3 = j$ and say case (Q, S) holds and continue.

Suppose the process has not terminated, and let $a = 2$. If $\ell_a = \bar{\ell}_{a+1}$, then set $\bar{\ell}_a = \ell_a$, afterwards reset $\ell_a = \bar{\ell}_a - 1$, and say case (S2) holds for a . Otherwise find the minimal index $i \geq \bar{\ell}_{a+1}$ such that $\nu^{(a)}$ has a singular string of length i . If no such i exists, set $b = \bar{a} + \bar{1}$ and terminate. Otherwise set $\bar{\ell}_a = i$ and repeat this for $a = 1$ (there must exist at least two singular strings if $\bar{\ell}_3 = \bar{\ell}_1$ and case (S2) does not hold). If the process has not terminated, set $b = \bar{1}$.

Set all undefined ℓ_a and $\bar{\ell}_a$ for $a = 1, 2, 3$ to ∞ .

3.2 Change in the rigged configuration

The rigged configurations change under δ as follows. We first remove a box from $\bar{\ell}_a$ in $\nu^{(a)}$ for $a = 1, 2$, and if case (S2) holds for a , we remove another box from that particular row, otherwise we remove a box from ℓ_a . If case (S) holds, then remove two boxes from $\bar{\ell}_3$ and make the resulting string singular. If case (Q) holds, remove a box from ℓ_3 and make the resulting string singular. If case (Q, S) holds, then we remove both boxes corresponding to ℓ_3 and $\bar{\ell}_3$, but we make the smaller one (i.e. the row corresponding to ℓ_3) singular and the larger one quasi-singular. Also make all the changed strings in $\nu^{(2)}$ singular.

Remark 3.1 We can determine the inverse algorithm by roughly doing the opposite of the above; in particular, selecting largest (quasi)singular strings at most as long as before.

Example 3.2 Using the rigged configuration (ν, J) from Example 2.2 and $B = B^{1,1} \otimes B^{1,2} \otimes B^{2,1}$. Applying the map δ , we get $b = 3$ and

$$\delta(\nu, J) = 3 \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} 3 \quad -2 \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} -2$$

3.3 Extending to arbitrary rectangles

We now extend Φ to $B = \otimes_{i=1}^N B^{1,s_i}$ by defining the map

$$\text{ls}: \text{RC}(B^{1,s} \otimes B^*) \longrightarrow \text{RC}(B^{1,1} \otimes B^{1,s-1} \otimes B^*),$$

which is known as *left-split*. On the rigged configurations, the map ls is the identity (but perhaps increases the vacancy numbers) and a strict crystal embedding. Thus iterating ls with δ , we obtain a map $\Phi: \text{RC}(B) \longrightarrow B$.

4 Filling map

We determine the highest weight rigged configurations for $B^{1,s}$ by using the virtual Kleber algorithm [22].

Lemma 4.1 *Consider the KR crystal $B^{1,s}$. We have $\text{RC}(B^{1,s}) = \bigoplus_{k=0}^s \text{RC}(B^{1,s}; k\bar{\Lambda}_1)$. Moreover the highest weight rigged configurations in $\text{RC}(B^{1,s}; k\bar{\Lambda}_1)$ are given by $\nu^{(1)} = (s - k, s - k)$ and $\nu^{(2)} = (s - k)$ with all labels 0.*

From Lemma 4.1 and the $U_q(\mathfrak{g}_0)$ -crystal decomposition of $B^{1,s}$ is multiplicity free, there exists a natural $U_q(\mathfrak{g}_0)$ -crystal isomorphism $\iota: \text{RC}(B^{1,s}) \longrightarrow B^{1,s}$. For type $D_4^{(3)}$, we note that $k\bar{\Lambda}_1$ can be considered as the partition (k) .

Definition 4.2 *Let $B^{1,s}$ be a KR crystal of type $D_4^{(3)}$ and consider the classical component $B(k\bar{\Lambda}_1) \subseteq B^{1,s}$. The filling map $\text{fill}: B^{1,s} \longrightarrow (B^{1,1})^{\otimes s}$ is given by adding $\lfloor \frac{s-k}{2} \rfloor$ copies of the horizontal domino $\begin{bmatrix} \bar{1} & 1 \end{bmatrix}$ and an additional $\begin{bmatrix} \emptyset \end{bmatrix}$ if $s - k$ is odd.*

Let $T^{1,s}$ denote the image of $B^{1,s}$ under fill written as a $1 \times s$ rectangle. We note that $T^{1,s}$ inherits a classical crystal structure from $(B^{1,1})^{\otimes s}$.

Example 4.3 *Consider the element*

$$b = \begin{bmatrix} 3 & 0 & \bar{2} & \bar{2} & \bar{1} \end{bmatrix} \in B(5\bar{\Lambda}_1) \subseteq B^{1,9},$$

then we have

$$\text{fill}(b) = \begin{bmatrix} 3 & 0 & \bar{2} & \bar{2} & \bar{1} & \bar{1} & 1 & \bar{1} & 1 \end{bmatrix}.$$

Now suppose $b \in B^{1,8}$, then we have

$$\text{fill}(b) = \begin{bmatrix} 3 & 0 & \bar{2} & \bar{2} & \bar{1} & \bar{1} & 1 & \emptyset \end{bmatrix}.$$

We give a $U'_q(\mathfrak{g})$ -crystal structure to $T^{1,s}$ by following [10, 30] as the conditions for e_0 and f_0 are preserved under the filling map.

Proposition 4.4 *The filling map $\text{fill}: B^{1,s} \rightarrow T^{1,s}$ given in Definition 4.2 is a $U'_q(\mathfrak{g})$ -crystal isomorphism.*

We also can show the following.

Proposition 4.5 *Let $B = B^{1,s}$. Then $\Phi = \text{fill} \circ \iota$ with fill as in Definition 4.2 on highest weight elements.*

5 Virtualization Map

Lemma 5.1 *The virtualization map $v: B^{1,s} \rightarrow B^{2,s}$ for types $D_4^{(3)} \hookrightarrow D_4^{(1)}$ is given column-by-column by*

$$\begin{array}{cccc} \boxed{1} \mapsto \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} & \boxed{2} \mapsto \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} & \boxed{3} \mapsto \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} & \boxed{0} \mapsto \begin{array}{|c|} \hline 3 \\ \hline 3 \\ \hline \end{array} \\ \boxed{3} \mapsto \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline \end{array} & \boxed{2} \mapsto \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline \end{array} & \boxed{1} \mapsto \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} & \boxed{\emptyset} \mapsto \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \end{array}$$

Using Lemma 5.1 and the analogue of Φ in type $D_4^{(1)}$ [18, 25], we can show the following.

Theorem 5.2 *Consider a tensor product of KR crystals $B = \bigotimes_{i=1}^N B^{1,s_i}$ of type $D_4^{(3)}$. The virtualization map v commutes with the map Φ .*

We need to define the *complement rigging map* $\theta: \text{RC}(B) \rightarrow \text{RC}(B_r)$ by sending $(\nu, J) \mapsto (\nu, J')$, where J' is obtained by $x' = p_i^{(a)} - x$ for all labels x and B_r are the factors of B in reverse order. That is to say θ maps each label x to its colabel. We can define $\tilde{\delta} := \theta \circ \delta \circ \theta$, and using the virtualization map, Proposition 4.5, and the results of [25], we can show the following.

Lemma 5.3 *We have $\delta \circ \tilde{\delta} = \tilde{\delta} \circ \delta$.*

Using the results on the combinatorial R -matrix in [30], we can show the following.

Lemma 5.4 *Consider $B = B^{1,s} \otimes B^{1,1}$. We have $\Phi^{-1} \circ R \circ \Phi$ is the identity map on $\text{RC}(B)$.*

Then following [28, Sec. 8], the map $rs := \theta \circ ls \circ \theta$ preserves statistics using [30]. From Lemma 5.4, the R -matrix preserves statistics. Thus iterating rs and R -matrices, we preserve statistics to $\bigotimes_{i=1}^{N'} B^{1,1}$. Then we use the results of [23, 25] and Theorem 5.2 to obtain our main result.

Theorem 5.5 *Let $B = \bigotimes_{i=1}^N B^{1,s_i}$ of type $D_4^{(3)}$. The map $\Phi: \text{RC}(B) \rightarrow B$ is a $U_q(\mathfrak{g}_0)$ -crystal isomorphism and $\Phi \circ \theta$ sends cocharge to energy.*

From Proposition 4.4, Lemma 5.1, Theorem 5.5, and the filling map for type $D_n^{(1)}$ given in [18], we can show the following.

Theorem 5.6 *Let $B = B^{1,s}$. Then $\Phi = \text{fill} \circ \iota$ with fill as in Definition 4.2 as $U_q(\mathfrak{g}_0)$ -crystal morphisms.*

Thus we can define a $U'_q(\mathfrak{g})$ -crystal structure on $\text{RC}(B)$ by extending Φ to be a $U'_q(\mathfrak{g})$ -crystal isomorphism. Thus we have a special case in type $D_4^{(3)}$ of the conjectures given in [27].

6 Extensions and questions

The $U_q(\mathfrak{g}_0)$ -crystal decomposition of $B^{2,s}$ and the highest weight rigged configurations will appear in the full version of this work. The author hopes to use this to determine the filling map for $B^{2,s}$.

There is a map $lt: \text{RC}(B^{2,1} \otimes B^*) \rightarrow \text{RC}(B^{1,1} \otimes B^{1,1} \otimes B^*)$ called *left-top* which adds a singular string of length 1 to $\nu^{(1)}$. In the full version, this is used to extend the $U_q(\mathfrak{g}_0)$ -crystal isomorphism Φ to tensor products also containing $B^{2,1}$.

Example 6.1 Continuing from Example 3.2, we obtain

$$\Phi(\nu, J) = \boxed{3} \otimes \boxed{2 \ 3} \otimes \boxed{\frac{1}{2}}.$$

The computations for the Kleber algorithm can be modified to determine the $U_q(\mathfrak{g}_0)$ -crystal decomposition of $B^{r,s}$ of type $G_2^{(1)}$. However there is a difficulty with determining what the map δ should be. This would need to be overcome to define the filling map for type $G_2^{(1)}$.

There is a conjecture [22, Conj. 3.7] that we can realize $B^{1,s}$ of type $D_4^{(3)}$ as a virtual crystal in $B^{1,s}$ of type $D_4^{(1)}$. Therefore obtaining a direct description of e_0 and f_0 on rigged configurations could lead to an answer to this conjecture using the results of [18]. The author hopes to have this description and prove this conjecture in this special case in the full version of this work.

7 Examples using Sage

The bijection Φ and the rigged configurations have been implemented by the author in Sage [29]. We begin by setting up the Sage environment to give a more concise printing.

```
sage: RiggedConfigurations.global_options(display="horizontal")
```

We construct our the rigged configuration from Example 2.2 by specifying the partitions and corresponding labels.

```
sage: nu = RC(partition_list=[[4,1], [4]], rigging_list=[[3,1], [-2]]); nu
5[ ][ ][ ][ ] 3 -2[ ][ ][ ][ ] -2
1[ ] 1
```

We apply the full bijection and print the output using Sage's ASCII art.

```
sage: ascii_art(nu.to_tensor_product_of_kirillov_reshetikhin_tableaux())
3 # 2 3 # 1
-2
```

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